CS6840: Advanced Complexity Theory Jan 31, 2012

Lecture 15 : Introduction to Toda's Theorem

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THEME: Between P and PSPACE.
LECTURE PLAN:Introduce Toda's Theorem, give some relevant definitions and construct a proof strategy for Toda's Theorem

There are two different way to generalize hard problems like optimization problems.

1. Non-determinism or quantification
2. Counting

It was an important open question in 1980's about relative power of those two approach, alternation and counting. There are various classes are defined independently in those two different approach. But how are those class comparable? In 1991, Toda came up with the following relation among the two different world.

Theorem 1. $PH \subseteq \mathsf{P}^{\#\mathsf{P}}$

This theorem take the whole quantification world to counting world. That is we can solve any problem in PH in polynomial with an oracle access to a #P-Complete problem. Before going to the proof of Toda's theorem lets define the following.

Definition 2. $\exists$. Operator.

Let us consider the definition of NP.

$$L \in \mathsf{NP} \text{ if } \exists B \in \mathsf{P} \text{ such that}$$

$$x \in L \iff \exists y \text{ such that } (x, y) \in B$$

So we can think $\exists$ as an operator and let us define it as follows

Let C be any class of languages. Define,

$$L \in \exists.C \text{ if } \exists B \in C \text{ such that } x \in L \iff \exists y \text{ such that } (x, y) \in B$$

So $\mathsf{NP} = \exists.\mathsf{P}$

Definition 3. BP. Operator
Let us consider definition of the class BPP

$L \in \mathsf{BPP}$ if $\exists B \in \mathsf{P}$ such that

$$x \in L \Longrightarrow \mathsf{Pr_y}[(x,y) \in B] \geq \frac{3}{4}$$
$$x \notin L \Longrightarrow \mathsf{Pr_y}[(x,y) \in B] \leq \frac{1}{4}$$

So similarly we can define BP. operator as

Let C be a class of languages. $L \in BP.C$ if $\exists B \in$ C such that

$$x \in L \Longrightarrow \mathsf{Pr}[(x,y) \in B] \geq \frac{3}{4}$$
$$x \notin L \Longrightarrow \mathsf{Pr}[(x,y) \in B] \leq \frac{1}{4}$$

So $\mathsf{BPP} = BP.\mathsf{P}$

Now lets define the class $\oplus\mathsf{P}$:

Definition 4. L $\in \oplus\mathsf{P}$ if there is a branching machine M such that

$$x \in L \Longrightarrow \#Acc_M(x) \text{ is odd}$$

$\oplus\mathsf{P}$ can be considered as the class of decision problems corresponding to the least significant bit of a $\#\mathsf{P}$-problem. Now the natural question is "Is $\mathsf{NP} \subseteq \oplus\mathsf{P}$?".

We will see that NP problems are reduced to $\oplus\mathsf{P}$ in some extend. Lets now define Randomize Reduction as follows

Definition 5. $A \leq B$ via a randomize reduction function σ if

1. σ is computable by choosing a random string $y \in \{0,1\}^{p(n)}$.
2. $x \in L \iff \sigma_y(x) \in B$ with high probability.

We will see that NP problems are reduced to some $\oplus\mathsf{P}$ problem via randomize reduction. More specifically $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$. Now we will develop the proof strategy for Toda's theorem. We will proof the theorem in the following steps ...

$$\mathsf{NP} \subseteq BP.(\oplus\mathsf{P}) \subseteq \mathsf{BPP}^{\oplus\mathsf{P}} \subseteq \mathsf{P}^{\#\mathsf{P}} = \mathsf{P}^{\mathsf{PP}}$$

Then by induction on the number of quantifiers we will show that the entire $PH \subseteq \mathsf{P}^{\#\mathsf{P}}$.

So our first step is to show $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$.

To prove this claim let us define the following languages ....

Definition 6.

$$\oplus\mathsf{SAT} = \{\phi : \#\text{ of satisfying assignment of } \phi \text{ is odd}\}$$

Let us define the following promise problem USAT as,

Definition 7. For all boolean formulas ϕ,

$$\phi \in \mathsf{USAT} \implies \#SAT(\phi) = 1$$
$$\phi \notin \mathsf{USAT} \implies \#SAT(\phi) = 0$$

This is a promise problem as we will consider only those ϕ which falls into above two cases. Now we introduce a very important result called *Valiant-Vazirani lemma* proved by Valiant and Vazirani in 1986.

Lemma 8. *(Valiant-Vazirani Lemma)*
There exists a randomize reduction σ_y such that

$$\phi \in \mathsf{SAT} \iff Pr_y[\sigma_y(\phi) \in \mathsf{USAT}\ ] \geq \frac{3}{4}$$
$$\phi \notin \mathsf{SAT} \iff Pr_y[\sigma_y(\phi) \in \mathsf{USAT}\ ] \leq \frac{1}{4}$$

We will prove this lemma in the next lecture and using this lemma we will prove our first step $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$.