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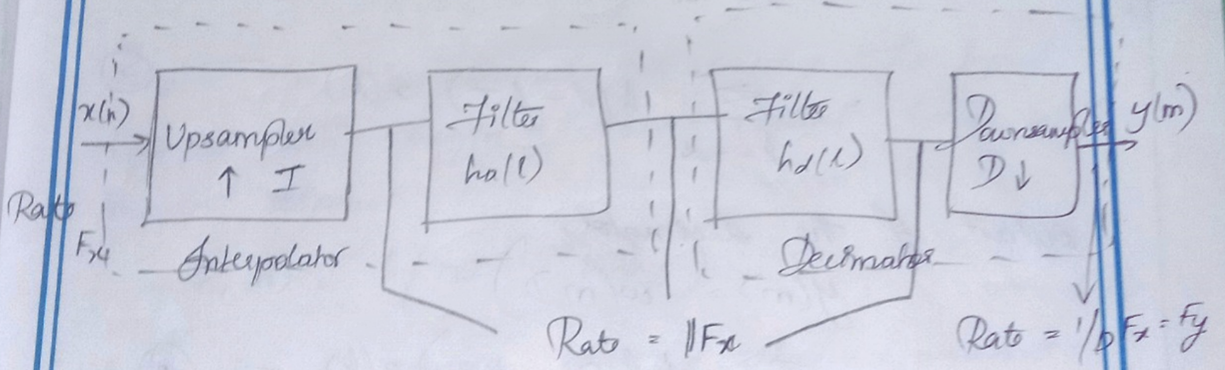
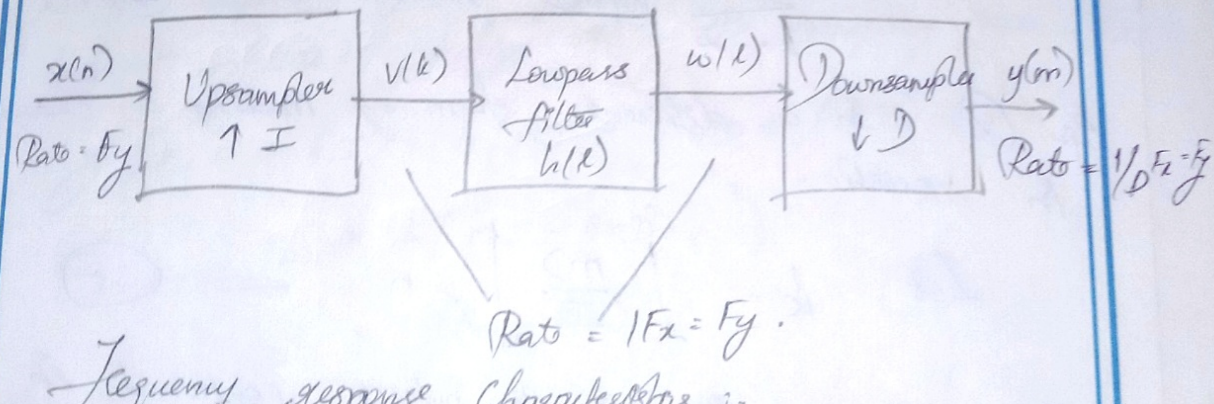
CEC332 - ADVANCED DIGITAL SIGNAL PROCESSING

① Sampling rate conversion by a rational factor I/D in multirate signal processing :-





Frequency response Characteristics :-

$$H(\omega_v) = \begin{cases} I, & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0, & \text{Otherwise} \end{cases} \quad ②$$

$$\omega_v = \frac{2\pi F}{F_v} = \frac{2\pi F}{IF_x} = \frac{\omega_x}{I}.$$

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Time Domain Representation by,

$$v(l) = \begin{cases} x(l/I), & l = 0, \pm I, \pm 2I \cdots \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Output of the linear time invariant filter is

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-k)\, v(k)$$

$$= \sum_{k=-\infty}^{\infty} h(l-kI)\, x(k) \quad — (3)$$

Output of the decimator,

$$y(m) = w(mD)$$

$$= \sum_{k=-\infty}^{\infty} h(mD-kI)\, x(k) \quad — (4)$$

eqn (4) in a different form by making a change in variable.

$$\text{Let } k = \left\lfloor \frac{mD}{I} \right\rfloor - n \quad — (5)$$

$\lfloor r \rfloor$ denotes the largest integer contained in r

eqn (4) becomes,

$$y(m) = \sum_{n=-\infty}^{\infty} h\left(mD - \left\lfloor \frac{mD}{I} \right\rfloor I + nI\right) x\left(\frac{mD}{I} - n\right) \quad — (6)$$

$$mD - \left\lfloor \frac{mD}{I} \right\rfloor I = mD \text{ modulo } I$$
$$= (mD)_I$$

eqn ⑥ can be expressed as,

$$y(m) = \sum_{n=-\infty}^{\infty} h(nI + (mD)_I)\, x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right) \quad \text{—⑦}$$

output $y(m)$ is obtained by passing the input sequence $x(n)$ through a time-variant filter with impulse response.

$$g(n, m) = h(nI + (mD)_I) \quad -\infty < m, n < \infty \quad \text{—⑧}$$

$h(k)$ — impulse response of the time-invariant lowpass filter operating at the sampling rate IF_x.

$$g(n, m + kI) = h(nI + (mD + kDI)_I)$$
$$= h(nI + (mD)_I) \quad \text{—⑨}$$
$$= g(n, m)$$

$g(n, m)$ is periodic in the variable m with period I.

(4)

Frequency domain relationship can be obtained by the result of the interpolate and decimation process.

Spectrum at the output of the linear filter with impulse response $h(l)$ is.

$$V(\omega_v) = H(\omega_v)\, X(\omega_v I) .$$

$$= \begin{cases} I X(\omega_v I), & 0 \le |\omega_v| \le \min\left(\pi/D, \pi/I\right) \\ 0, & \text{Otherwise} \end{cases} \quad \text{——— (10)}$$

Spectrum of the output sequence $y(m)$ by the decimating sequence $V(n)$ factor of D is.

$$Y(\omega_y) = \frac{1}{D} \sum_{k=0}^{D-1} V\left(\frac{\omega_y - 2\pi k}{D}\right) \quad \text{——— (11)}$$

$\omega_y = D\omega_v$ linear filter prevents aliasing as implied by (10) the spectrum of the output sequence given by (11) reduces to

$$Y(\omega_y) = \begin{cases} \frac{I}{D} X\left(\frac{\omega_y}{D}\right), & 0 \le |\omega_y| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0, & \text{Otherwise.} \end{cases} \quad \text{——— (12).}$$

(2) Autocorrelation :-

The important second order characteristics like autocorrelation and autocovariance can be conveniently represented in the form of matrices.

Let us consider a discrete time random process represented by vector,

$$X = [x(0), x(1) \cdots x(p)]^T$$

Its outer product can be obtained by multiplying with a Hermitian transpose which is defined as the conjugate transpose of a matrix.

$$XX^H = [x(0), x(1), \cdots x(p)]^T [x^*(0), x^*(1) \cdots x^*(p)]$$

$$= \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(p) \end{bmatrix} [x^*(0)\ x^*(1) \cdots x^*(p)]$$

$$= \begin{bmatrix} x(0)x^*(0) & x(0)x^*(1) & \cdots & x(0)x^*(p) \\ x(1)x^*(0) & x(1)x^*(1) & \cdots & x(1)x^*(p) \\ \vdots & & & \\ x(p)x^*(0) & x(p)x^*(1) & \cdots & x(p)x^*(p) \end{bmatrix}$$

Taking expectation on both sides, we get.

$$E[x x^H] = E\begin{bmatrix} x(0)x^*(0) & x(0)x^*(1) & \cdots & x(0)x^*(p) \\ x(1)x^*(0) & x(1)x^*(1) & \cdots & x(1)x^*(p) \\ \vdots & & & \\ x(p)x^*(0) & x(p)x^*(1) & \cdots & x(p)x^*(p) \end{bmatrix}$$

$$r_{xx}(-k) = r_{xx}^*(k),$$

$$R_{xx} = E[x x^H] = \begin{bmatrix} r_{xx}(0) & r_{xx}^*(1) & \cdots & r_{xx}^*(p) \\ r_{xx}(1) & r_{xx}^*(0) & \cdots & r_{xx}^*(p-1) \\ \vdots & & & \\ r_{xx}(p) & \cdots & r_{xx}(p-1) \cdots & r_{xx}(0) \end{bmatrix}$$

is a $(p+1) \times p(+1)$ matrix of autocorrelation values which is called autocorrelation matrix of process $x(n)$.

Autocovariance matrix is represented as,

$$C_{xx} = E\left[(x-m_x)(x-m_x)^H\right] \quad — (2)$$

$m_x = [m_x \; m_x \cdots m_x]^T$ is a vector having.

$$C_{xx} = R_{xx} - m_x m_x^H \quad — (3)$$

For a process with zero mean,

$$C_{xx} = R_{xx} \quad — (4).$$

Importance of autocorrelation matrix of WSS process :-

i) It is a Hermitian matrix (any matrix is said to be Hermitian if its entries are equal to its own conjugate transpose. For any matrix A,

$A = (A^*)^T$.

ii) All the terms along each of the diagonal elements are equal.

R_{xx} is a Hermitian Toeplitz matrix.

Properties of autocorrelation matrix :-

Property 1

1. Autocorrelation matrix of a WSS random process $x(n)$ is a Hermitian Toeplitz matrix.

$R_{xx} = \text{Toep}\{r_{xx}(0), r_{xx}(1), \ldots r_{xx}(p)\}$.

The converse is not true, i.e every Hermitian Toeplitz matrix is not a valid autocorrelation matrix.

Ex :- $R_{xx} = \begin{bmatrix} -8 & 2 \\ 2 & -8 \end{bmatrix}$.

Property 2 :-

The autocorrelation matrix of a WSS random process is non-negative definite, $R_{xx} > 0$;

Property 3 :-

Eigen values λ_k of the autocorrelation matrix of a WSS processes are real and non-negative.

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(3) Kalman Filter :-

→ Consider the problem of designing a causal wiener filter to estimate a process $d(n)$ from a set of noisy observations $x(n) = d(n) + v(n)$.

Primary limitation with the solution that was derived is that it requires that $d(n)$ and $x(n)$ be jointly wide-sense stationary processes.

Consider, a causal wiener filter for estimating a process $x(n)$ from the noisy measurement

$$y(n) = x(n) + v(n) \quad — (1)$$

$x(n)$ in the form of AR(1) process can be written as,

$$x(n) = a(1)x(n-1) + w(n) \quad — (1a).$$

where $w(n)$ and $v(n)$ are uncorrelated white noise processes. The optimum linear estimate of $x(n)$ is.

$$\hat{x}(n) = a(1)\hat{x}(n-1) + k[y(n) - a(1)\hat{x}(n-1)] \quad \text{--- (2)}$$

k - constant, Reffered to as the kalman gain. minimizes the mean square error.

$$E\{x(n) - \hat{x}(n)^2\}.$$

Eqn (1), (2) and (1a)

$$\left.\begin{array}{l} x(n) = A\,X(n-1) + w(n) \\ y(n)\ C^T x(n) + v(n) \end{array}\right\} \quad \text{--- (3)}$$

$$\hat{x}(n) = A\hat{x}(n-1) + k[y(n) - C^T A\hat{x}(n-1)] \quad \text{--- (4)}$$

A is a $p \times p$ state transition matrix.

$w(n) = [w(n), 0, \ldots 0]^T$ is a vector noise process.

C = Unit vector of length P.

k = kalman gain vector.

$x(n) = A(n-1)\, x(n-1) + w(n)$ — (5)

where $A(n-1)$ is a time varying $p \times p$ state transition matrix.

$w(n)$ — vector of zero mean white noise process with

$$E\{w(n) w^H(k)\} = \begin{cases} Q_w(n) & ; k = n \\ 0 & ; k \neq n \end{cases} \quad \text{— (6)}$$

$$E\{v(n) v^H(k)\} = \begin{cases} Q_v(n) & ; k = n \\ 0 & ; k \neq n \end{cases} \quad \text{— (8)}$$

$y(n) = C(n)\, x(n) + v(n)$ — (7)

Generalizing eqn (4),

$$\hat{x}(n) = A(n-1)\hat{x}(n-1) + K(n)\left[y(n) - C(n)A(n-1)\hat{x}(n-1) \right] \quad \text{— (9)}$$

with appropriate kalman gain matrix $K(n)$

$e(n/n) = x(n) - \hat{x}(n/n)$

$e(n/n-1) = x(n) - \hat{x}(n/n-1)$ — (10)

$P(n/n)$ and $P(n/n-1)$ — error covariance matrices. $P(n/n) = E\{e(n/n)\, e^H(n/n)\}$.

$P(n/n-1) = E\{e(n/n-1)\, e^H(n/n-1)\}$ —

Step 1 :-

$$(5) \Rightarrow x(n) = A(n-1)\, x(n-1) + w(n).$$

Since $w(n)$ is a zero mean white noise processes.

$$\hat{x}(n/n-1) = A(n-1)\, \hat{x}(n-1/n-1).$$

$$e(n/n-1) = x(n) - \hat{x}(n/n-1).$$

$$= A(n-1)\, x(n-1) + w(n) - A(n-1)\, \hat{x}(n-1/n-1)$$

$$= A(n-1)\left[x(n-1) - \hat{x}(n-1/n-1)\right] + w(n).$$

$$e(n/n-1) = A(n-1)\, e(n-1/n-1) + w(n) \quad \text{--- (12)}$$

Then $E\{e(n/n-1)\} = 0$.

$$P(n/n-1) = A(n-1)\, P(n-1/n-1)\, A^H(n-1) + Q_w(n) \quad \text{--- (13)}$$

where $Q_w(n)$ is the covariance matrix for the noise process $w(n)$.

This is the first step of Kalman filter estimation.

(12)

Step :- 2

A linear estimate of $x(n)$ that is based on $\hat{x}(n/n-1)$ and $y(n)$ is of the form

$$\hat{x}(n/n) = k'(n)\,\hat{x}(n/n-1) + k(n)\,y(n) \quad \text{— (14)}$$

where $k(n)$ and $k'(n)$ are the matrices.

$E\{v(n)\} = 0$ and $E\{e(n/n-1)\} = 0$ then,

$$k'(n) = I - k(n)\,c(n).$$

$$x^{*}(n/n) = [I - k(n)c(n)]\,\hat{x}(n/n-1) + k(n)y(n) \quad \text{— (1}$$

The desired expression for the kalman gain -

$$k(n) = p(n/n-1)\,c^{H}(n)\,[c(n)\,p(n/n-1)\,c^{H}(n) + Qv(n)].$$

desired expression for the error covariance matrix is.

$$\boxed{P(n/n) = [I - k(n)\,C(n)]\,P(n/n-1)}$$

(4) LMS algorithm for adaptive filter

* The Least - mean - square (LMS) algorithm is part of the group of stochastic gradient algorithms. The update from steepest descent is straight forward while the dynamic estimates may have large variance.

The LMS algorithm is based on the concept of steepest - descent that update the weight vector :-

$$w(n+1) = w(n) + \mu(x(n)e(n)).$$

where μ is the stepsize

$$\frac{\partial E[e^2(n)]}{\partial w(n)} \text{ is replaced by } \frac{\partial e(n)}{\partial w(n)}$$

$$e(n) = d(n) - W^T(n)\,x(n)$$

where the coefficient vector $w(n)$ can be randomly initialized or some cases, it is typically chosen to be the zero vector.

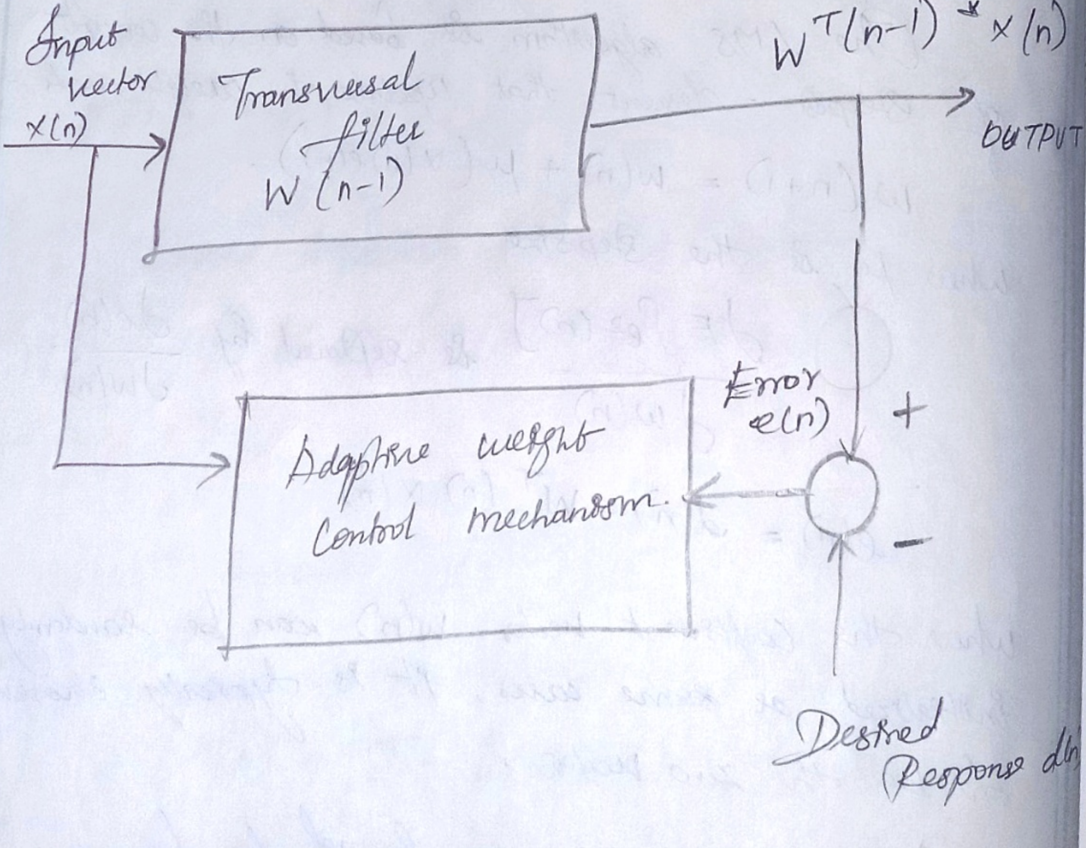
The LMS algorithm can be of two types :-

→ Filtering process

→ Adaptive process.

Filtering Process :-

This step entails calculating the output of FIR transversal filter by convolving input and tap weights. A transversal filter which is used to compute the filtered output and filtered error for a given input and desired signal.



LMS Filtering Process.

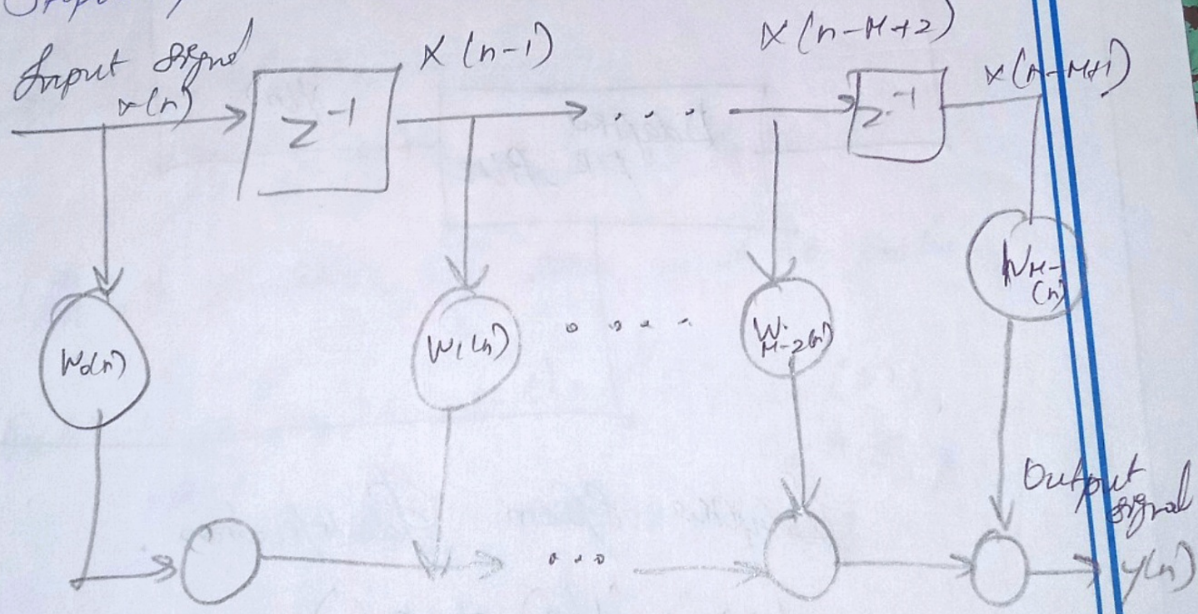
Adaptive process :-

This is simultaneous operation during implementation as it adjust tap weight based on the estimated error. The adjustment is done via.

$$w(n+1) = w(n) - \mu \frac{\partial(e^2(n))}{\partial(w(n))}.$$

$$w(n+1) = w(n) + 2\mu e(n)\, x(n).$$

$w(n+1)$ is altered based on error $e(n)$ and stepsize μ.

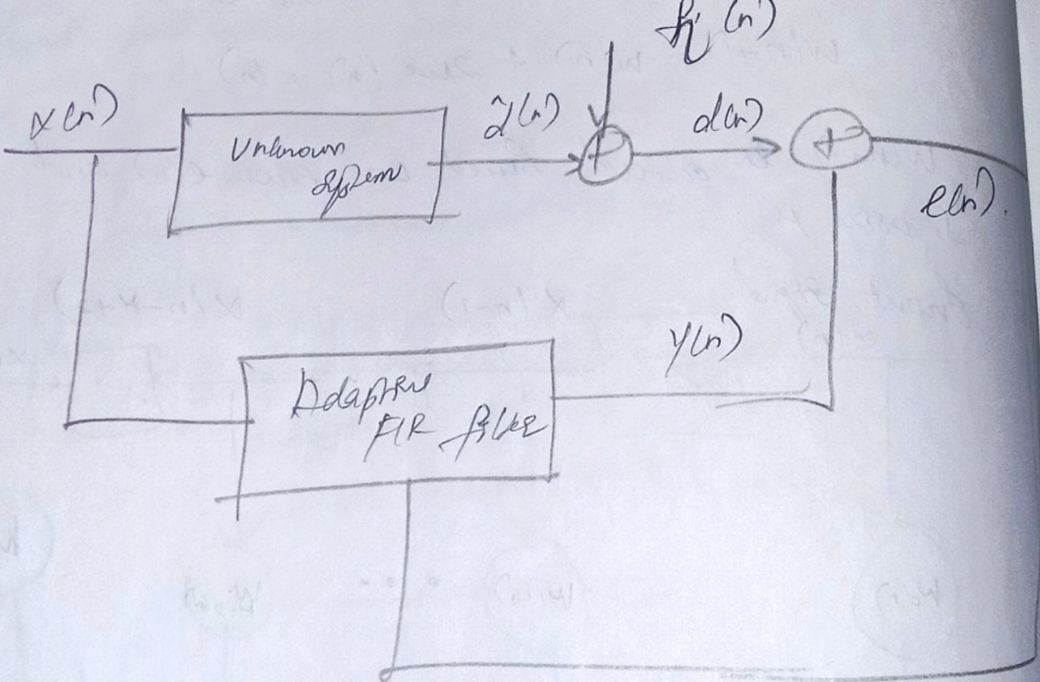


LMS Adaptation Process.

Application

* The primary goal of adaptive system is to approximately to determine a discrete estimation of transfer function of an unknown system.

* It helps in providing a linear representation that is the best approximation of unknown system.



Adaptive System Identification.

$$d(n) = \hat{d}(n) + \eta(n).$$

Stability = $0 < \mu < 2/\lambda_{max}$.

i) MA process :-

→ Moving average process is another special case of ARMA (p,q) process, when p=0 with q=0 x(n) is generated by filtering white noise

$$H(z) = \sum_{k=0}^{q} b_q(k) z^{-k} \quad [\because A_p(z) = 1]$$

ARMA (0,q) process is known as a moving average process.

$$P_x(z) = \sigma_v^2 B_q(z) B_q^*(1/z^*) \quad (\alpha)$$

In terms of ω,

$$P_x(e^{j\omega}) = \sigma_v^2 |B_q(e^{j\omega})|^2.$$

$P_x(z)$ contains 2q zeros and no poles.

$$r_x(k) + \sum_{k=1}^{p} a_p(l) r_x(k-l) = \begin{cases} \sigma_v^2 c_q(k); & 0 \le k \le q \\ 0; & k > p. \end{cases}$$

$r_x(k) + 0 = \sigma_v^2 c_q(k)$. but

$$c_q(k) = \sum_{l=0}^{q-k} b_q(l+k) h^*(l).$$

$$\gamma_x(k) = \sigma_v^2\, b_q(k) * b_q^*(-k)$$

$$\boxed{\gamma_x(k) = \sigma_v^2 \sum_{l=0}^{q-k} b_q(l+|k|)\, b_q^*(l)}$$

ii) ARMA process

Auto Regressive moving average process where white noise $v(n)$ is filtered with causal linear shift invariant filter having rational system function with p poles and q zeros.

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q(k) z^{-k}}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}}.$$

Power spectrum of Random process is,

$$P_x(z) = \sigma_v^2 \frac{B_q(z)\, B_q^*\left(\frac{1}{z^*}\right)}{A_p(z)\, A_p^*(1/z^*)}$$

$$P_x(e^{j\omega}) = \sigma_v^2 \frac{|B_q(e^{j\omega})|^2}{|A_p(e^{j\omega})|^2}.$$

Let v(n) be a white noise available for x(n), x(n) be the required output to be generated

$$H(z) = \frac{\sum_{l=0}^{q} b_q(l)\, z^{-l}}{1 + \sum_{l=1}^{p} a_p(l) z^{-l}} = \frac{X(z)}{V(z)}$$

multiplying the above eqn $x^*(n-k)$ and taking expectation on both sides,

$$E\left[x(n)\, x^*(n-k)\right] = E\left[\sum_{l=1}^{p} a_p(l)\, x(n-l)\, x^*(n-k)\right]$$

$$= E\left[\sum_{l=0}^{q} b_q(l)\, v(n-l)\, x^*(n-k)\right]$$

$$\boxed{\gamma_x(k) + \sum_{l=1}^{p} a_p(l)\, \gamma_x(k-l) = \sum_{l=0}^{q} b_q(l)\, \gamma_{vx}(k-l)}$$

$$x(n) = v(n) * h(n)$$

$$x(n) = \sum_{m=-\infty}^{\infty} v(m)\, h(n-m)$$

$$\boxed{x^*(n-k) = \sum_{m=-\infty}^{\infty} v^*(m)\, h^*(n-k-m)}$$

$$\boxed{\gamma_{vx}(k-l) = \sigma_v^2\, h^*(l-k)}$$

$$= \sigma_v^2\, C_q(k) \quad \text{— (1)}$$

Matrix representation

$$r_x(0) + \sum_{l=1}^{p} a_p(l)\, r_x(-l) = \sigma_v^2 C_q(0)$$

$$r_x(0) + a_p(1) r_x(-1) + a_p(2) r_x(-2) + \cdots + a_p(p)\, r_x(-p) = \sigma_v^2 C_q(0)$$

Sub $k = q$.

$$r_x(q) + \sum_{l=1}^{p} a_p(l)\, r_x(q-l) = \sigma_v^2 C_q(q)$$

Represent these values in matrix.

$$\begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p) \\ r_x(1) & r_x(0) & \cdots & r_x(-p+1) \\ \vdots & \vdots & & \vdots \\ r_x(q) & r_x(q-1) & \cdots & r_x(q-p) \\ r_x(q+1) & r_x(q) & & r_x(q-p+1) \\ \vdots & & & \vdots \\ r_x(q+p) & r_x(p+q-1) & & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a(p)(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} C_q(0) \\ C_q(1) \\ \vdots \\ C_q(q) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

These are the basic equations of ARMA i.e Auto Regressive Moving average and MA (Moving average) process.

6) Design a linear phase FIR filter that satisfies the following specification.

Sampling frequency :- 8000 Hz.

Pass band : $0 \le F \le 75$

Transition band : $75 \le F \le 80$

Stopband : $80 \le F \le 4000$

Pass band ripple : $\delta_1 = 10^{-2}$; Stopband ripple $\delta_2 = 10^{-4}$

$$F_{stop} \le \frac{F_0}{2D}$$

$$80 \le \frac{8000}{2D}$$

$$D \le 50$$

$$D = 50 \qquad D = I = 50.$$

Let $D = D_1 D_2 = 50$

$$D_1 = 25 \; ; \; D_2 = 2$$

$I = I_1 I_2 = 50$. ; $I_1 = 2$; $I_2 = 25$

Design for single stage.

$$\delta_1 = 10^{-2}, \; \delta_2 = 10^{-4}$$

$$\Delta f = \frac{F_s - F_p}{F_0} = \frac{80 - 75}{8000} = 6.25 \times 10^{-4}$$

$$\boxed{\Delta f = 6.25 \times 10^{-4}}$$

Order of the filter is given as,

$$N = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{14.6\,\Delta f} + 1$$

$$= \frac{-10\log_{10}(10^{-2} \times 10^{-4}) - 13}{14.6 \times 6.25 \times 10^{-4}} + 1$$

$$= 5150.68 + 1$$

$$\boxed{N = 5151}$$

Design for Multirate Structure :-

$F_0 = 8000\ Hz$

$F_1 = \frac{F_0}{D_1} = \frac{8000}{25}$

$$\boxed{F_1 = 320\ Hz}$$

$F_2 = F_1/D_2 = 320/2$ $\boxed{F_2 = 160\ Hz}$

$F_p = 75\ Hz$

$F_{stop} = 80\ Hz$ as given in specifications of t

Design of Decimator Stage 1 ($i = 1$) :-

Passband : $0 \leq F \leq 75$

Stopband : $F_1 - F_{stop} \leq F \leq \frac{F_{i-1}}{2}$

$$F_1 - F_{stop} \leq f \leq F_0/2.$$

$320 - 80 \leq F \leq 8000/2 = 240 \leq F \leq 4000$

Transition band : $75 \le F \le 80$.

Since this is decimation as well as interpolation with four stage.

$$\delta_2 = \delta_2 / 4 = 10^{-4}/4 = 2.5 \times 10^{-5}$$

$$\Delta f_1 = \frac{240 - 75}{F_0} = \frac{165}{8000} = 0.020625$$

$$N_1 = \frac{-10\log_{10}(\delta_1 \delta_2) - 13}{14.6 \Delta f_1} + 1$$

$$N_1 = \frac{-10\log_{10}(10^{-2} \times 2.5 \times 10^{-5}) - 13}{14.6(0.020625)} + 1$$

$$= \frac{53.020}{0.301125} + 1$$

$$N_1 = 176.07$$

$$\boxed{N_1 = 176}$$

Design of Decimator for Stage 2 (i=2)

Passband : $0 \le F \le 75$

Stopband : $F_i - F_{stop} \le F \le F_i - 1/2$

tions of filte

≤ 4000

$F_2 - F_{stop} \le F \le F_1/2$

$160 - 80 \le F \le 320/2$

$80 \le F \le 160$

But the given transition band is $75 \le F \le 80$.
Hence band edge frequency of the stop band should taken as 80 Hz instead of 80 Hz.

Stopband :- $80 \le F \le 4000$

$$\Delta f_2 = \frac{80-75}{F_1} = \frac{5}{320} = 0.015625$$

$$N_2 = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{14.6\,\Delta f_2} + 1$$

$$= \frac{-10\log_{10}(10^{-2}\times 2.5\times 10^{-5}) - 13}{14.6\times 0.015625} + 1$$

$$\boxed{N_2 = 233}$$