1) If $t_1(n) \in O(g_1(n))$ and $O(g_2(n))$, then $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$. Prove the assertions.

i) $t_1(n) \in O(g_1(n))$ means there exist constants $c_1 > 0$ and n_1 such that for all $n \geq n_1$:

$$t_1(n) \leq c_1 g_1(n)$$

ii) $t_2(n) \in O(g_2(n))$ means there exist constants $c_2 > 0$ and n_2 such that for all $n \geq n_2$:

$$t_2(n) \leq c_2 g_2(n).$$

We need to show that $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$.

Proof:

Consider: $t_1(n) + t_2(n)$ and the definition of $\max\{g_1(n), g_2(n)\}$.

Let $g(n) = \max\{g_1(n), g_2(n)\}$.

By definition of maximum, for all n:

$g(n) = \max\{g_1(n), g_2(n)\} \geq g_1(n)$

$g(n) = \max\{g_1(n), g_2(n)\} \geq g_2(n)$

Given the bounds of $t_1(n)$ and $t_2(n)$:

$t_1(n) \leq c_1 g_1(n) \leq c_1 g(n)$ for all $n \geq n_1$

$t_2(n) \leq c_2 g_2(n) \leq c_2 g(n)$ for all $n \geq n_2$

To bound $t_1(n) + t_2(n)$, consider

$t_1(n) + t_2(n)$

For $n \geq \max(n_1, n_2)$:

$t_1(n) + t_2(n) \leq c_1 g(n) + c_2 g(n) = (c_1 + c_2) g(n)$

Thus, there exists a constant $c = c_1 + c_2$ and $n_0 = \max(n_1, n_2)$ such that for all $n \geq n_0$: $t_1(n) + t_2(n) \leq c g(n)$

By the definition of Big-O notation, this implies:

$t_1(n) + t_2(n) \in O(g(n))$

Since $g(n) = \max\{g_1(n), g_2(n)\}$, we have:

$t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$

Hence, we have proven the assertion.

2) Find the time complexity of the below recurrence equation:

i) $T(n) = \begin{cases} 2T(n/2) + 1 & \text{if } n > 1 \\ 2T(n/2) + 1 & \text{if } n \geq 1. \end{cases}$

We can use Master's theorem for divide & conquer recurrences of the form.

$T(n) = aT(n/b) + f(n)$ where, $a = 2$, $b = 2$, $f(n) = 1$

$\log_a^b = \log_2^2 = 1$

$f(n) = 1 \Rightarrow n^k \log_a^b = n^1$

$f(n) = 1 = O(n^0)$ and $0 < 1 \Rightarrow$ Case 1:

$f(n) = O(n^0)$ and $0 < \log_2^2$

$T(n) = \Theta(n^k \log_a^b) = \Theta(n^1) = \Theta(n)$

$\therefore T(n) = \Theta(n)$

ii) $T(n) = \begin{cases} 2T(n-1) & \text{if } n > 0 \end{cases}$

$T(n) = 2T(n-1)$ if $n > 0$

By unrolling the recurrence

$T(n) = 2T(n-1)$

$T(n-1) = 2T(n-2)$

$T(n-2) = 2T(n-3)$

Continuing this, we see

$T(n) = 2T(n-1) = 2 \cdot 2T(n-2) = 2^2 \cdot 2T(n-3) =$

$2^3 T(n-3)$

In general, after (k) steps, we have:

$T(n) = 2^k T(n-k)$

Base case, when $k = n$

$T(n) = 2^n T(0)$

Gn. $T(0) = 1 \Rightarrow T(n) = 2^n(1) = 2^n$

$\therefore T(n) = O(2^n)$

5) Big O notation : Show that $f(n) = n^2 + 3n + 5$ is $O(n^2)$.

Big O $\Rightarrow$ $f(n) < c \cdot g(n)$

$f(x) = n^2 + 3n + 5$

Let us assume $g(x) = gx^2$

$= n^2 + 3n + 5 \leq c \cdot g(n)^2$ $\Rightarrow$ divide both sides by n^2 $\Rightarrow 1 + 3/n + 5/n^2 \leq c$

when $n = 1 \Rightarrow 1^2 + 3 + 5 = g(1)^2$

$= 9$

$3/n \leq 3$ for all $n \geq 1$

$5/n^2 \leq 5$ for all $n \geq 1$

Hence for $n \geq 1$:

$1 + 3/n + 5/n^2 \leq 1 + 3 + 5 = 9$

$\therefore c = 9$

when $n = 2$

$= 2^2 + 3(2) + 5 = 9(4)$

$= 4 + 6 + 5 = 36 \Rightarrow 15 < 36$

when $n = 3$

$= 3^2 + 3(3) + 5 = 9(9) = 9 + 9 + 5 = 81$

$= 23 < 81$

$\therefore f(x) \leq c \cdot g(x)$

$\therefore$ Big (O) is satisfied

b) Big Omega Notation: Prove that $g(n) = n^3 + 2n^2 + 4n$ in $\Omega(n^3)$

We have to demonstrate that constant $c > 0$ and $n_0 > 0$ such that for all $n \geq n_0$

$$g(n) \geq c \cdot n^3$$

$$n^3 + 2n^2 + 4n \geq c \cdot n^3$$

Simplify by dividing n^3 on both sides

$$1 + 2/n + 4/n^2 \geq c$$

As n grows larger, $2/n$ & $4/n^2$ becomes smaller.

For all $n \geq 1$: $2/n > 0$

$4/n^2 > 0$

Hence for $n \geq 1$:

$1 + 2/n + 4/n^2 \geq 1$

$\therefore c = 1$, for $n \geq 1$:

$1 + 2/n + 4/n^2 \geq 1$

We have shown that for $c = 1$ & $n_0 = 1$ inequality holds:

$n^3 + 2n^2 + 4n \geq n^3$ for all $n \geq 1$

$\therefore g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$ with $c = 1$ and $n_0 = 1$

$\therefore$ Hence Proved.

7) Big theta notation: ~~Prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$~~

Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not.

To show $h(n) = 4n^2 + 3n$ is $O(n^2)$, we need to find constants $c_1 > 0$ and $n_1 \geq 0$ such that for all $n \geq n_1$:

$$h(n) \leq c_1 \cdot n^2$$

$$4n^2 + 3n \leq c_1 \cdot n^2$$

Divide both sides by n^2

$$4 + 3/n \leq c_1$$

As n grows larger, $3/n$ becomes smaller. For $n \geq 1$:

$$4 + 3/n \leq 4 + 3 = 7.$$

$\therefore$ choose $c_1 = 7$ & $n_1 = 1$ $\therefore n \geq 1$

$$4n^2 + 3n \leq 7n^2$$

$\therefore h(n) = 4n^2 + 3n$ is $\Omega(n^2)$

$\Rightarrow$ Find constants $c_2 > 0$ & $n_2 \geq 0$ for all $n \geq n_2$:

$$h(n) \geq c_2 \cdot n^2$$

$$4n^2 + 3n \geq c_2 \cdot n^2$$

$$4 + 3/n \geq c_2 \quad (\text{divide by } n^2$$

n grows larger, 3/n becomes smaller. For $n \geq 1$:

$4 + 3/n \geq 4$

$\therefore c_2 = 4$ & $n_2 = 1$ $\quad n \geq 1$.

$4n^2 + 3n \geq 4n^2$

This shows $h(n) = 4n^2 + 3n$ is $\Omega(n^2)$

$\because h(n) = 4n^2 + 3n$ is both $O(n^2)$ & $\Omega(n^2)$ we conclude

$h(n) = 4n^2 + 3n = \Theta(n^2)$.

8) Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = n^2$ show whether $f(n) = \Omega(g(n))$ is true or false and justify your answer.

To determine whether $f(n) = n^3 - 2n^2 + n$ is $\Omega(g(n))$ where $g(n) = -n^2$, we need to show that there exist $c > 0$ and $n_0 \geq 0$ such that $n > n_0$:

$f(n) \geq c \cdot g(n)$

$n^3 - 2n^2 + n \geq c \cdot (-n^2) \Rightarrow n^3 - 2n^2 + n \geq -cn^2$

$n^3 + (c-2)n^2 + n \geq 0$

Consider $c = 3$

$n^3 + (3-2)n^2 + n = n^3 + n^2 + n$

For all $n \geq 1$:

$n^3 + n^2 + n \geq 0$

$\therefore$ n^3, n^2 and n are all +ve for $n \geq 1$, inequality holds true.

Thus $c = 3$ and $n_0 = 1$

$f(n) = n^3 - 2n^2 + n \geq 3(-n^2) = -3n^2$

$f(n) \geq c \cdot g(n)$ for all $n > n_0$ we conclude

$f(n) = n^3 - 2n^2 + n$ is $\Omega(-n^2)$

$\therefore$ the statement $f(n) = \Omega(g(n))$ is true.

9) Determine whether $h(n) = n \log n + n$ is in $\Theta(n \log n)$ prove a rigorous proof for your conclusion.

$h(n) = n \log n + n$ is $O(n \log n)$ we need to find constants $c_1 > 0$ and $n_1 \geq 0$ such that for all $n \geq n_1$:

$h(n) \leq c_1 \cdot n \log n$

Let's consider the expression $h(n) = n \log n + n$:

$n \log n + n \leq c_1 \cdot n \log n$

We can factor out $n \log n$:

$n \log n + n = n(\log_2 n + 1)$

So, we need:

$n(\log n + 1) \leq c_1 \cdot n \log n$

Divide both sides by n (for $n > 0$):

$\log n + 1 \leq c_1 \log n$

Now, we can find c_1:

$\log n + 1 \leq c_1 \log n$

Divide both sides by $\log n$ (for $\log n > 0$):

$1 + \frac{1}{\log n} \leq c_1$

As n grows larger, $\frac{1}{\log n}$ becomes smaller. ∴ for sufficiently large n, $\frac{1}{\log n} \leq 1$, & thus:

$1 + \frac{1}{\log n} \leq 2$

Hence, we can choose $c_1 = 2$ and $n_1 \geq e$ (to ensure $\log n \geq 1$). This shows that: $n \log n + n \leq 2n \log n$

Thus, $h(n) = n \log n + n$ is $O(n \log n)$

Proof for $\Omega(n \log n)$:

To show that $h(n) = n \log n + n$ is $\Omega(n \log n)$, we need to find constants $c_2 > 0$ and $n_2 \geq 0$ such that for all $n \geq n_2$: $h(n) \geq c_2 \cdot n \log n$

Consider the expression $h(n) = n \log n + n$;

$n \log n + n \geq c_2 \cdot n \log n$

We can factor out $n \log n$:

$n \log n + n = n(\log n + 1)$

So, we need:

$n(\log n + 1) \geq c_2 \cdot n \log n$

Divide both sides by n (for $n > 0$):

$\log n + 1 \geq c_2 \log n$

Now, we can find c_2:

$\log n + 1 \geq c_2 \log n$

Divide both sides by $\log n$ for ($\log n > 0$):

$$1 + \frac{1}{\log n} \geq c_2$$

As n grows larger, $\frac{1}{\log n}$ becomes smaller. $\therefore$ for sufficiently large n, $\frac{1}{\log n} \leq \frac{1}{2}$ and thus,

$$1 + \frac{1}{\log n} \geq \frac{3}{2}$$

Hence, we choose $c_2 = \frac{3}{2}$ and $n_2 \geq e^2$, this shows

$n \log n + n \geq \frac{3}{2} n \log n$

Thus, $h(n) = n \log n + n$ is $\Omega(n \log n)$

∵ we have shown that $h(n) = n \log n + n$ is both $O(n \log n)$ and $\Omega(n \log n)$, we conclude that $h(n) = n \log n + n$ is $\Theta(n \log n)$

10) Solve the following recurrence relations and find the order of growth for solutions.

$T(n) = 4T(n/2) + n^2, \quad T(1) = 1$

$T(n) = aT(n/b) + f(n)$ where, $a=4, \ b=2, \ f(n)=n^2$

$f(n) = n^k = \log_n^p$

$= n^2 \log_n^p$

$\log_a^b = \log_4^2 = 2$

$f(n) = n^2$

$f(n) = \Theta(n^2)$

If $f(n) = \Theta(n^k \log_a^b)$

$T(n) = \Theta(n^k \log \log n)$

Thus,

$T(n) = \Theta(n^2 \log n)$

11) Given an array of [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9] integers, find the maxi. and mini. product that can be obtained by multiplying 2 integers from

the array.

~~given array:~~

Sort the array:

[-9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

Candidates for maxi. product

=> 10 X 11 = 110

=> -9 x -8 = 72

Candidates for min. product

=> -9 x -8 = 72

=> -9 X 11 = -99

=> -8 X 11 = -88

Max. product = 110

Min. product = -99

12) Demonstrate Binary Search method to search key = 23, from the array arr[] = {2, 5, 8, 12, 16, 23, 38, 56, 72, 91}.

Pseudocode:

```
def bs(arr, key):
    start = 0
    end = len(arr) - 1
    while start <= end:
        mid = (start + end) // 2
```

```
if arr[mid] == key:
    return mid
elif arr[mid] < key:
    start = mid+1
else:
    end = mid-1
return -1.
```

i) Start = 0, end = 9

mid = 4

16 < 23, so search in the right half.

ii) 23, 38, 56, 72, 91

start = 5, end = 9

mid = 7

56 > 23, search in the left half.

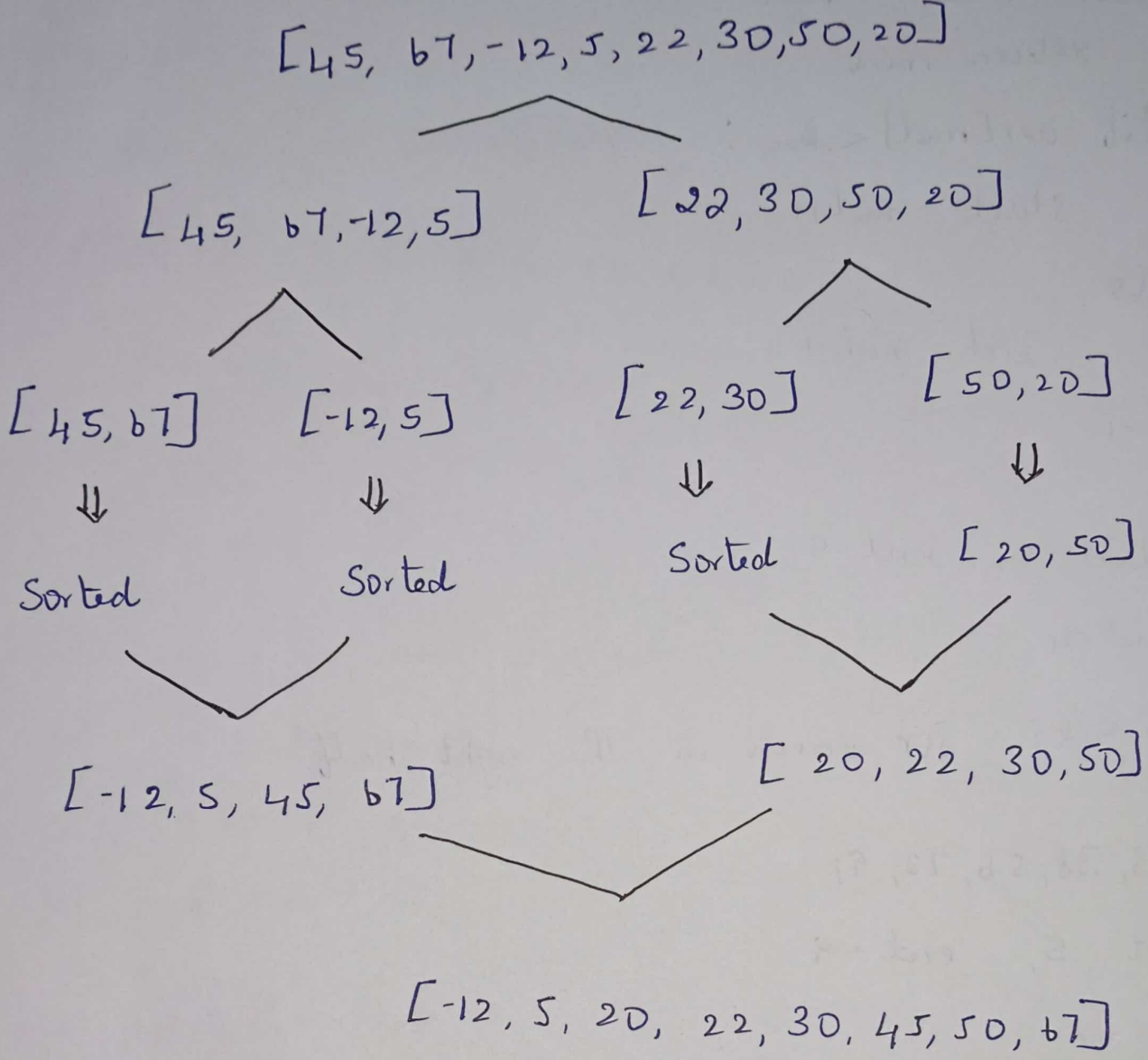
iii) 23, 38

start = 5, end = 6

mid = 5

23 = 23, key found at index 5.

13) Apply merge sort and order the list of 8 elements, d = (45, 67, -12, 5, 22, 30, 50, 20). Set up a recurrence relation for the no. of key comparisons made by mergesort.



Recurrence relation

$$T(n) = 2T(n/2) + n$$

$$\log_a^b = \log_2^2 = 1$$

$$f(n) = n \Rightarrow n^k \Rightarrow k = 1, \; p = 1$$

$$\log_a^b = k \Rightarrow \underline{\text{Case 1}} \Rightarrow p > -1 : \; \theta(n^k \log^{p+1} n)$$

$$T(n) = \theta(n \log n)$$

14) Find the no. of times to perform swapping for selection sort. Also estimate the time complexity for the order of notation set S = {12, 7, 5, -2, 18, 6, 13, 4}

i) [-2, 7, 5, 12, 18, 6, 13, 4]

ii) [-2, 4, 5, 12, 18, 6, 13, 7]

iii) [-2, 4, 5, 6, 18, 12, 13, 7]

iv) [-2, 4, 5, 6, 7, 12, 13, 18]

∴ Total no. of swaps performed = 7

Best case for selection sort involves $O(n^2)$ comparisons

Worst case also involves $O(n^2)$ comparisons and $O(n)$ swaps.

Average case also involves $O(n^2)$ comparisons and $O(n)$ swaps.

$\therefore T(n) = O(n^2)$

15) Find the index of the target value 10 using binary search from the following list of

[2, 4, 6, 8, 10, 12, 14, 16, 18, 20]

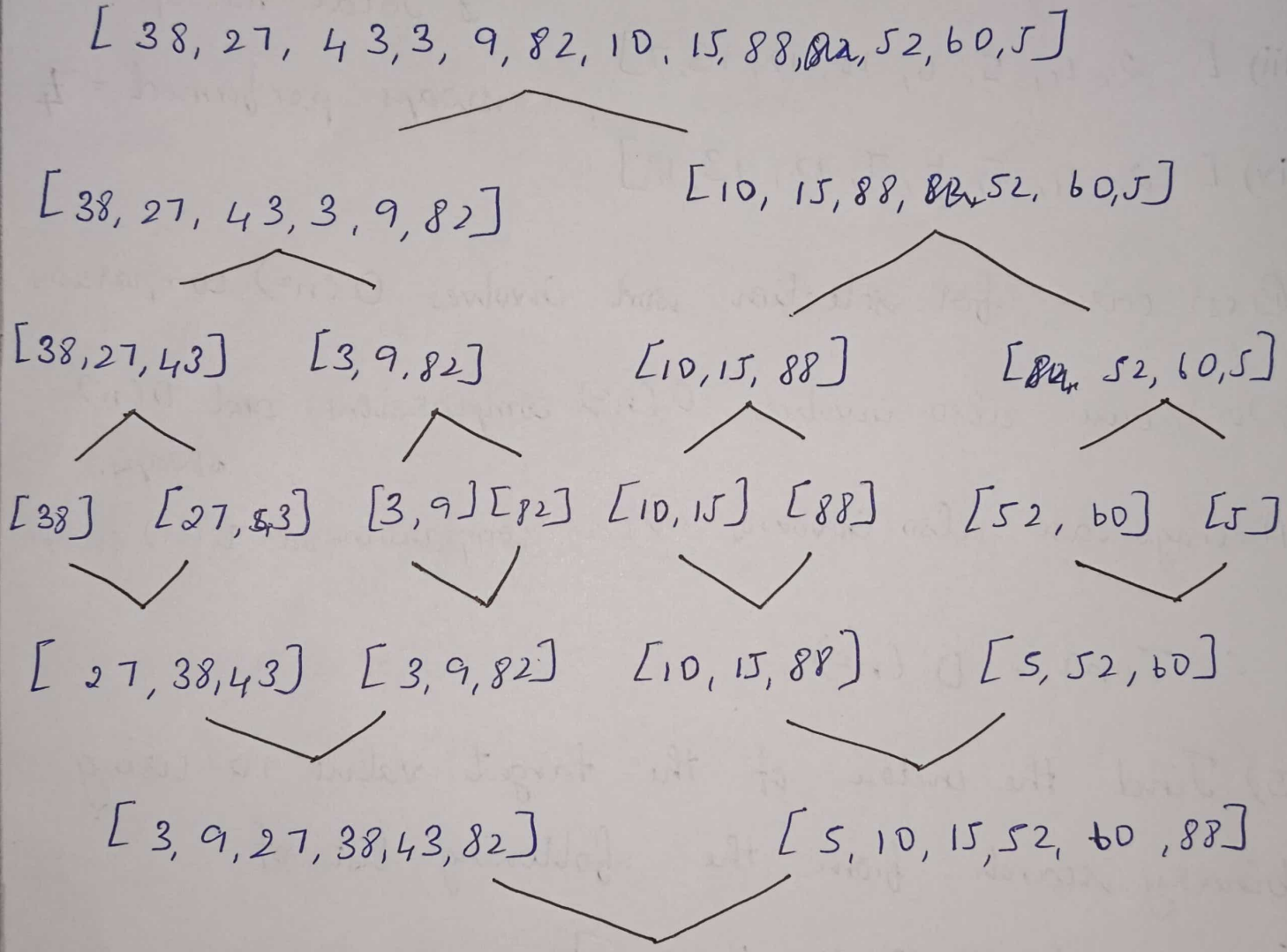
Initially ⇒ Start = 0, end = 9

$$mid = \frac{start + end}{2} = \frac{0+9}{2} = 4$$

list[mid] = list[4] = 10.

Target value is at index 4.

16) Sort the following elements using Merge sort divide and conquer strategy [38, 27, 43, 3, 9, 82, 10, 15, 88, 82, 52, 60, 5] and analyze complexity of algorithm.



Recurrence relation

$T(n) = 2T(n/2) + n$

$\log_a^b = \log_2^2 = 1$

$f(n) = n \Rightarrow n^k \Rightarrow k = 1, p = 1$

$\log_a^b = k \Rightarrow$ Case 1

$p > -1 : \Theta(n^k \log_n^{p+1})$

$T(n) = \Theta(n \log n)$

17) Sort the array 64, 34, 25, 12, 22, 11, 90 using bubble sort. What is the time complexity of Selection Sort in the best, worst and average cases ?

1st Pass $\Rightarrow$ [34, 64, 25, 12, 22, 11, 90]

[34, 25, 64, 12, 22, 11, 90]

[34, 25, 12, 64, 22, 11, 90]

[34, 25, 12, 22, 64, 11, 90]

[34, 25, 12, 22, 11, 64, 90]

2nd Pass $\Rightarrow$ [25, 34, 12, 22, 11, 64, 90]

[25, 12, 34, 22, 11, 64, 90]

[25, 12, 22, 34, 11, 64, 90]

[25, 12, 22, 11, 34, 64, 90]

3rd Pass $\Rightarrow$ [12, 25, 22, 11, 34, 64, 90]

[12, 22, 25, 11, 34, 64, 90]

[12, 22, 11, 25, 34, 64, 90]

4th Pass ⇒ [12, 11, 22, 25, 34, 64, 90]

5th Pass ⇒ [11, 12, 22, 25, 34, 64, 90]

Best case: $O(n^2)$

Worst case: $O(n^2)$

Average case: $O(n^2)$

18. Sort the array 64, 25, 12, 22, 11 using Selection Sort. What is the time complexity of Selecton in the best, worst and average cases?

1st Pass: [11, 25, 12, 22, 64]

2nd Pass: [11, 12, 25, 22, 64]

3rd Pass: [11, 12, 22, 25, 64]

4th Pass: 25 & 64 is swapped.

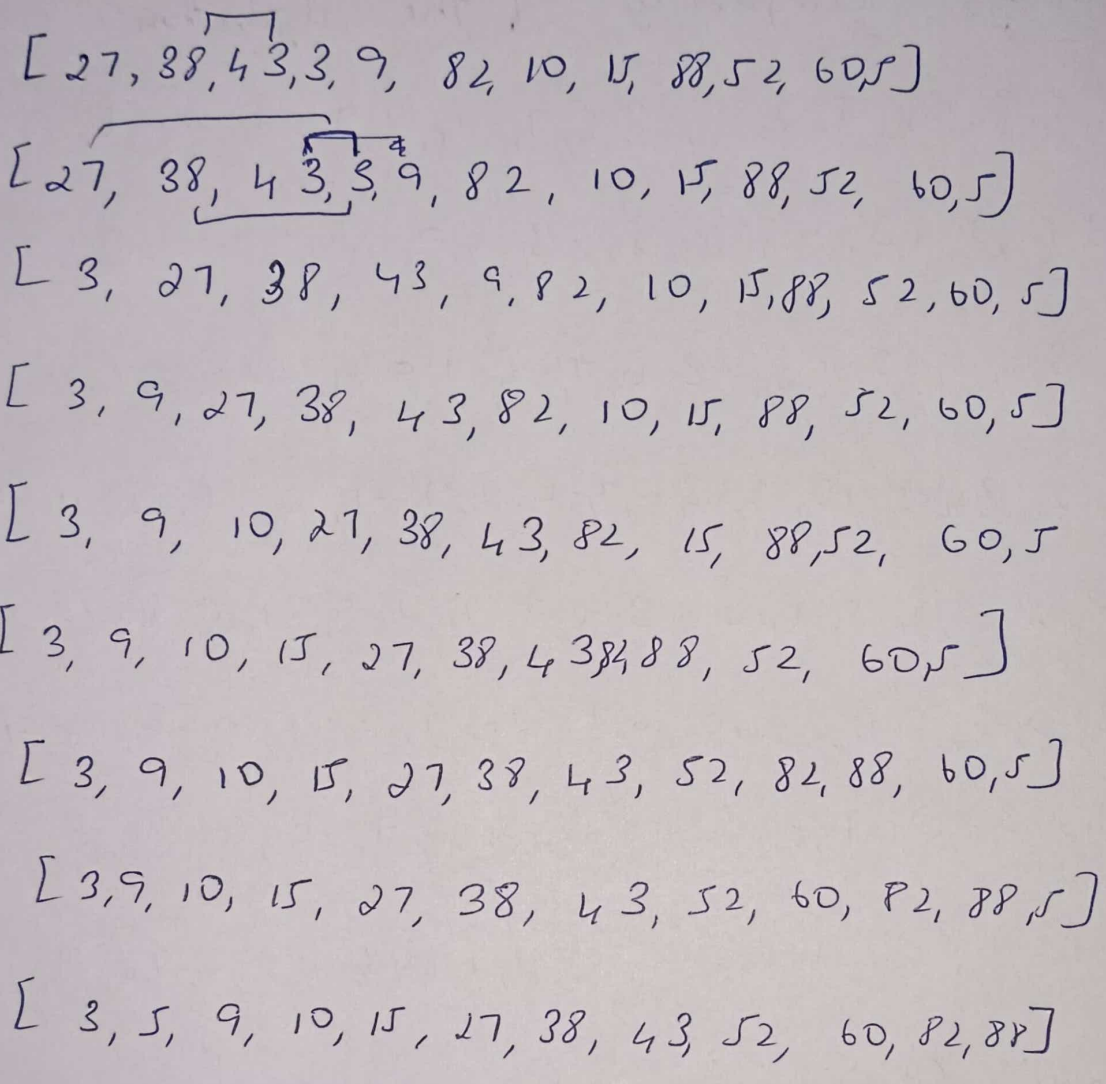
Sorted ⇒ [11, 12, 22, 25, 64]

Best case: $O(n^2)$

Worst case: $O(n^2)$

Average case: $O(n^2)$

19) Sort the flwng. elements using insertion sort using Brute Force Approach strategy [88, 27, 43, 3, 9, 82, 10, 15, 88, 52, 60, 5] and analyze complexity of the algo.



Best case: $O(n)$

Worst case: $O(n^2)$

Average case: $O(n^2)$

20. Given an array [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9] integers, sort the flwng. elements using insertion sort using Brute Force Approach strategy analyze complexity of the algorithm.

⇒ [-2, 4, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9]

[-2, 3, 4, 5, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9]

[-5, -2, 3, 4, 5, 10, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9]

[-5, -2, 2, 3, 4, 5, 10, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9]

[-5, -2, 2, 3, 4, 5, 8, 10, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9]

[-5, -3, -2, 2, 3, 4, 5, 8, 10, 6, 7, -4, 1, 9, -1, 0, -6, -8, -11, -9]

[-5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, -6, -8, -11, -9]

[-9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

Worst-case: $O(n^2)$

Best-case: $O(n)$

Average-case: $O(n^2)$