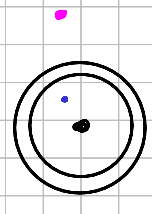
Binomial distribution :→

Ex: all three trials are independent $\left\{ \begin{array}{c} \xrightarrow{p/q \;\; (q \to 1-p)} \\ \longrightarrow \\ \longrightarrow \end{array} \right\}$



S: denotes the success in the trial

F: denotes the failure

X: Random Variable (No of Successes in the 3 trials)

$(X = x) = \{0, 1, 2, 3\}$

$$P[X=0] = p(F_1 \cap F_2 \cap F_3) = q \cdot q \cdot q \cdot = q^3$$

$$= {}^3C_0 \, p^0 \, q^{3-0}$$

$$P[X=1] = p\left[(S_1 \cap F_2 \cap F_3) \cup (F_1 \cap S_2 \cap F_3) \cup (F_1 \cap F_2 \cap S_3)\right]$$

$$= pq^2 + pq^2 + pq^2 = 3pq^2$$

$$= {}^3C_1 \, p^1 \, q^{3-1}$$

$$P[X=2] = {}^3C_2 \, p^2 \, q^{3-2}$$

$$p[X=3] = {}^3C_3 \, p^3 \, q^{3-3}$$

$$P[X=x] \begin{cases} = {}^nC_x \, p^x \, q^{n-x} & ; \; x = 0, 1, \dots, n \\ = 0 & ; \; \text{otherwise} \end{cases}$$

→ p.m.f. of a binomial distribution

Note :→

① The binomial distribution is probability distribution of sum of n independent bernouli variate

(11) $X \sim B(n, p)$

Ex : Let p be the probability of getting head in a toss of the coin. If this coin is tossed six times (assume each trial is independent and coin is fair)

find the probability of getting.

(1) exactly 3 heads

(2) less than three heads

(3) more than three heads

(4) more than 6 heads → an impossible event → 0

$$P[X=x] = {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$$

∵ coin is fair $p = 1-q = \frac{1}{2}$

$$P[X=x] = {}^{n}C_{x}\, p^{n} = {}^{n}C_{x}\left(\tfrac{1}{2}\right)^{n}$$

(1) $P[X=3] = {}^{6}C_{3}\left(\tfrac{1}{2}\right)^{6} = \frac{1}{64}\,\frac{6\times5\times4}{3\times2\times1}$

$= 20/64 = \frac{5}{16}$

(2) $P[X<3] = P[X=0] + P[X=1] + P[X=2]$

$$= \frac{1}{64}\left[{}^{6}C_{0} + {}^{6}C_{1} + {}^{6}C_{2}\right]$$

Q The probability of man hitting a target is $1/4$. He fires 5 times. What is the probability that he hits the target atleast twice.

Ans:⇒ $$p[x=x] = {}^{n}C_x \, p^x \, q^{n-x} \qquad p=1/4 \quad q=3/4$$

$$p[x=2]+p[x=3]+p[x=4]+p[x=5]$$

$$= {}^5C_2\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^3 + {}^5C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^2 + {}^5C_4\left(\tfrac{1}{4}\right)^4\tfrac{3}{4} + {}^5C_5\left(\tfrac{1}{4}\right)^5$$

Moments of Binomial Distribution

$$[\mu_1, \mu_2, \mu_3, \mu_4]$$
$$[\mu_1', \mu_2', \mu_3', \mu_4']$$

$$\mu_1' = \mu_x \rightarrow \text{mean}$$

$$\mu_1' = E[X] = \sum_{x=0}^{n} x \cdot P[X=x]$$

$$= \sum_{x=0}^{n} x \cdot {}^nC_x \, p^x q^{n-x}$$

as $x=0$ result in zero

$$= \sum_{x=1}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \cdot \frac{n}{x} \, {}^{n-1}C_{x-1} \, p^x q^{n-x}$$

$$\Rightarrow \quad {}^nC_x = \frac{\lfloor n}{\lfloor n-x \, \lfloor x} = \frac{n \, \lfloor n-1}{x \, \lfloor x-1 \, \lfloor n-1-(x-1)}$$

$$= \sum_{x=1}^{n} n \, {}^{n-1}C_{x-1} \, p^x \, q^{n-x} \quad \left\{ \begin{array}{l} n-x \\ = (n-1)-(x-1) \end{array} \right.$$

$$= \sum_{x=1}^{n} np \, {}^{n-1}C_{x-1} \, p^{x-1} \, q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} \, q^{(n-1)-(x-1)}$$

$$= np \left\{ {}^{n-1}C_0 \, p^0 \, q^{n-1} + {}^{n-1}C_1 \, p^1 \, q^{n-2} + {}^{n-1}C_2 \, p^2 q^{n-2} \right.$$

$$\left. + \, {}^{n-1}C_{n-1} \, p^{n-1} \, q^{(n-1)-(n-1)} \right]$$

$$= np \cdot 1 \quad = np$$

$$E[X] = np = \mu_x = \mu_1'$$

$$\mu_2' = E[X^2] = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x \, q^{n-x}$$

$$\underline{x^2 = x(x-1) + x}$$

$$E[X^2] = \sum_{x=0}^{n} [x(x-1) + x] \, {}^nC_x \, p^x \, q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \, {}^nC_x \, p^x \, q^{n-x} + \sum_{x=0}^{n} x \, {}^nC_x \, p^x \, q^{n-x}$$

$$\Downarrow \; np = \mu_1'$$

$$= \sum_{x=0}^{n} \overset{\downarrow\, r}{x(x-1)}\, {}^{n}C_x\, p^x q^{n-x} + \mu_1'$$

for $x = 0, 1 \quad r = 0$

$$= \sum_{x=2}^{n} x(x-1)\, {}^{n}C_x\, p^x q^{n-x} + \mu_1'$$

$$= \sum_{x=2}^{n} \cancel{x(x-1)} \frac{n(n-1)}{\cancel{x(x-1)}}\, {}^{n-2}C_{x-2}\, p^x q^{n-x} + \mu_1'$$

$$= n(n-1)\, p^2 \underbrace{\boxed{\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{(n-2)-(x-2)}}}_{\Downarrow\ \underline{1}} + \mu_1'$$

$$= n(n-1)p^2 + np$$

$$= n^2p^2 - np^2 + np$$

$$\mu_2 = \mu_2' - [\mu_1']^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p) = npq$$

* for a binomial distribution

$\boxed{\mu_x > \sigma_x^2}$ ✓ $\qquad \mu_x = \boxed{np}$

or, $\sigma_x^2 > \mu_x$ ✗ $\qquad \sigma_x^2 = npq$

$\underline{\underline{\theta}}$ $\left.\begin{array}{l} \mu_x = 4 \\ S.D. = \sigma_x = \frac{2}{\sqrt{3}} \end{array}\right\}$ find $P[X \leq 1]$

$P[X \geq 3]$

here X is a binomial Random variable.

$\mu_x = 4$ $\sigma_x^2 = \left(\frac{2}{\sqrt{3}}\right)^2 = 4/3$

$\frac{\mu_x}{\sigma_x^2} = \frac{np}{npq} = \frac{4}{4/3} \Rightarrow q = 1/3$

$p = 1 - q = 1 - 1/3 = 2/3$

$\mu_x = np = 4 \Rightarrow n \, 2/3 = 4$

$n = \frac{4 \times 3}{2} = 6$

$P[X \leq 1] = P[X=0] + P[X=1]$

$= {}^6C_0 \, p^0 q^{6-0} + {}^6C_1 \, p^1 q^{6-1}$

$= q^6 + 6pq^5$

$= (1/3)^6 + 6 \, 2/3 \left(\frac{1}{3}\right)^5$

$= \frac{1 + 6 \cdot 2}{3^6} = \frac{13}{3^6}$

$\underline{\underline{\theta}}$ $X \sim B(n, p)$. Find the p if $\Rightarrow n = 6$

& $9 P[X=4] = P[X=2]$

$n = 6$ $\quad 9\,P[X=4] = P[X=2]$

$$\Rightarrow 9\ {}^6C_4\, p^4 q^2 = {}^6C_2\, p^2 q^4$$

$$\Rightarrow 9\ \frac{6\times\cancel{5}}{\cancel{2}}\, p^{\cancel{4}2} \cancel{q^2} = \frac{6\times\cancel{5}}{\cancel{2}}\, \cancel{p^2}\, q^{\cancel{4}2}$$

$$9p^2 = q^2$$

$$\Rightarrow 9p^2 = (1-p)^2 = 1 - 2p + p^2$$

$$8p^2 + 2p - 1 = 0$$

$$8p^2 + 4p - 2p - 1 = 0$$

$$(2p+1)(4p-1) = 0$$

$$p = -1/2 \text{ or } 1/4$$

$$p = 1/4 \qquad q = 3/4$$

Fitting of Binomial Distribution:→

Q Four coins were tossed and number heads noted. This is repeated 200 times. The no of tosses showing 0, 1, 2, 3 and 4 heads were found distributed as under.

# of heads	0	1	2	3	4
# of tosses	15	35	90	40	20

Fit a binomial distribution to these observation assuming that nature of coins are not known.

$n = 4 \quad N = 200$

$\text{mean} = 2.075 = np$

$p = \dfrac{2.075}{4} = 0.518$

$q = 1 - 0.518 = 0.482$

$p(x) = {}^nC_x \, p^x \, q^{n-x}$

$p(x+1) = {}^nC_{(x+1)} \, p^{x+1} \, q^{n-x-1}$

$$\frac{p(x+1)}{p(x)} = \frac{{}^nC_{(x+1)}}{{}^nC_x} \left(\frac{p^{x+1}}{p^x} \quad \frac{q^{n-x-1}}{q^{n-x}} \right)$$

$$= \frac{\cancel{\lfloor n}}{\lfloor x+1 \; \lfloor n-x-1} \times \frac{\lfloor x \; \lfloor n-x}{\cancel{\lfloor n}} \quad \frac{p}{q}$$

$$= \frac{n-x}{x+1} \times \frac{p}{q}$$

$$\boxed{\frac{p(x+1)}{p(x)} = \frac{n-x}{x+1} \times \frac{p}{q}}$$

$$p(0) = P[X=0] = q^n$$

$$p(x+1) = \frac{n-x}{x+1} \frac{p}{q} p(x)$$

$$p(1) = \frac{n-0}{1} \frac{p}{q} p(0)$$

$$p(2) = \frac{n-1}{1+1} \frac{p}{q} p(1)$$

$$\Rightarrow f(x) = N \cdot P[X=x] = N\,{}^nC_x\, p^x q^{n-x}$$

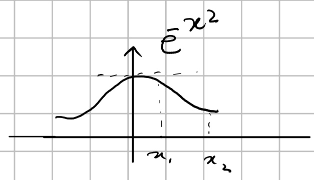
X	$\frac{n-x}{x+1} \frac{p}{q}$ ϕ	$P(x)$	$f(x)$
0	$\frac{4-0}{0+1}(1 \cdot 07)$ $= 4 \cdot 3$	$P(0) = 0 \cdot 05$	12
1		$P(1) =$	

Normal Distribution :→ $N(\mu, \sigma^2)$

A continuous Random variable X is said to be normal distribution with parameters μ $(-\infty < \mu < \infty)$ and σ^2 $(\sigma^2 > 0)$ if it takes on any real value and its probability density function

$$f_X(X=x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let :
→ $g(x) = e^{-x^2}$:→



(i) This probability density function depends on two const parameters $(\mu$ & $\sigma^2)$

(ii) If R.V. X follows Normal distribution with μ and σ^2 then we may write "X is distributed to $N(\mu, \sigma^2)$"
$X \sim N(\mu, \sigma^2)$

(iii) If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is also a normally distributed Random variable with mean $= 0$ and variance $= 1$.

⇒ This Z is standard normal variable.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad ; \quad -\infty < Z < \infty$$

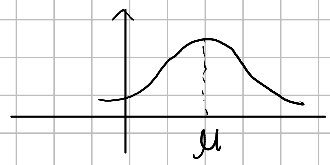
PDF

ⓐ $X \sim N(40, 25)$:→ $f_X(X=x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{(x-40)^2}{50}}$ ✓

ⓑ $X \sim N(0, 2)$:→ $f_X(X=x) \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{4}}$ ✓

$$f_X(x) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-46}{6}\right)^2}$$

$$\left.\begin{aligned} \mu &= 46 \\ \sigma &= 6 \end{aligned}\right\} \Rightarrow$$



* Properties of Normal distribution :→

1. Normal distribution is bell-shaped
2. The curve is symmetrical about $x = \mu$.
3. For a Normal distribution Mean = Median = Mode.
4. Being probability density ; it cannot -ve.
5. Normal distribution is unimodal.
6. Central moments of Normal distributions are

$$\mu_1 = 0 \qquad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

$$\mu_2 = \sigma^2$$

$$\mu_3 = 0 \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3$$

$$\mu_4 = 3\sigma^4$$

Note :→ All odd order central moments are zero.

⇒ It is Mesokurtic.

7. For Normal curve

$$Q_3 - \text{Median} = \text{Median} - Q_1$$

8. Quartile Deviation $Q.D. = \frac{Q_3 - Q_1}{2} = \frac{2}{3}\sigma$

(9) Mean Deviation $M.D. = \frac{4}{5}\sigma$

(10) $Q.D : M.D : S.D. = 10 : 12 : 15$

(11) If $X_1, X_2, X_3, \dots X_n$ are n Normal Random variable with means $\mu_1, \mu_2, \dots \mu_n$ and variance $\sigma_1^2, \sigma_2^2, \dots \sigma_n^2$ then the linear combination

$$aX_1 + a_2X_2 + \dots + a_nX_n$$ is also

a Normal distributed Random variable

with $$\mu = a_1\mu_1 + a_2\mu_2 + \dots + a_n\mu_2$$

$$\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \dots + a_n^2\sigma_n^2$$

Area :→

(I) $$P[\mu - \sigma < X < \mu + \sigma] = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$$

$$= 0.6827$$

$$\Rightarrow P[-1 < Z < 1] = \int_{-1}^{1} \phi(z)\,dz = 0.6827$$

(II) $$P[\mu - 2\sigma < X < \mu + 2\sigma] = \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\,dx = .9544$$

$$P[-2 < Z < 2] = \int_{-2}^{2} \phi(z)\,dz = .9544$$

(III) $$P[-3 < Z < 3] = \int_{-3}^{3} \phi(z)\,dz = .9973$$

Gamma Function :→ If $n>0$, the integral

$$\gamma[n] = \int_0^\infty x^{n-1} e^{-x}\, dx \text{ ; is called gamma function}$$

$$\gamma[n] = \boxed{\int_0^\infty x^{n-1} e^{-x}\, dx = (n-1)!}$$

1. if $n>1$; $\gamma[n] = (n-1)\,\gamma[n-1]$
2. if n is a +ve integer $\gamma[n] = \underline{|n-1}$
3. $\gamma[1/2] = \sqrt{\pi}$

→ **First Central moment of Normal R.V. :→**

$$\mu_1 = 0$$

$$E[x-\mu] = \int_{-\infty}^{\infty} (x-\mu)\, f_X(x)\, dx = \int_{-\infty}^{\infty} x f_X(x)\, dx - \mu \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= \mu - \mu = 0$$

2nd Central moment :→

$$E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z \Rightarrow \boxed{\begin{matrix} x-\mu = z\sigma \\ \frac{dx}{\sigma} = dz \end{matrix}}$$

$$= \int_{-\infty}^{\infty} z^2\sigma^2 \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \cdot \cancel{\sigma}$$

$$= \int_{-\infty}^{\infty} \sigma^2 z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} \, dz$$

even

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2} \, dz$$

Let ; $z^2 = 2t \Rightarrow z = \sqrt{2}\sqrt{t}$

$$dz = \sqrt{2}\,\frac{1}{2}\, t^{-1/2}\, dt$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t\, e^{-t}\, \frac{1}{\sqrt{2}}\, t^{-1/2} dt$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}}\, \frac{2}{\sqrt{2}} \int_{0}^{\infty} t^{3/2-1} e^{-t}\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{3/2-1} e^{-t}\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, \gamma\left[3/2\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, 1/2\, \gamma\left[1/2\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\, \frac{1}{2}\, \sqrt{\pi} = \sigma^2$$

$$\mu_2 = \sigma^2$$

Third Central moment :→

$$\boxed{\mu_3 = 0}$$

$$\mu_3 = \int_{-\infty}^{\infty} (x-\mu)^3 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^3 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z \qquad \Rightarrow \left.\begin{aligned} x-\mu &= z\sigma \\ dx &= \sigma\, dz \end{aligned}\right\}$$

$$= \int_{-\infty}^{\infty} z^3 \sigma^3 \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma^3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^3 e^{-\frac{1}{2}z^2} dz = 0$$

Fourth Central moment :→

$$\mu_4 = 3\sigma^4$$

$$\mu_4 = E\left[(x-\mu)^4\right] = \int_{-\infty}^{\infty} (x-\mu)^4 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{x-\mu}{\sigma} = z \qquad \Rightarrow \begin{aligned} x-\mu &= z\sigma \\ dx &= \sigma\, dz \end{aligned}$$

$$\mu_4 = \int_{-\infty}^{\infty} z^4 \sigma^4 \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma^4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 e^{-\frac{1}{2}z^2} dz$$

$$= \frac{2\sigma^4}{\sqrt{2\pi}} \int_0^\infty z^4 e^{-\frac{1}{2}z^2} \, dz$$

$$z^2 = 2t$$

$$2z\,dz = 2\,dt \Rightarrow dz = \frac{dt}{z}$$

$$\mu_4 = \frac{2\sigma^4}{\sqrt{2\pi}} \int_0^\infty (2t)^2 \, e^{-t} \, \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^4}{\sqrt{2\pi}} \frac{4}{\sqrt{2}} \int_0^\infty t^2 \, e^{-t} \, t^{-1/2} \, dt$$

$$= \frac{4\sigma^4}{\sqrt{\pi}} \int_0^\infty t^{5/2-1} \, e^{-t} \, dt$$

$$= \frac{4\sigma^4}{\sqrt{\pi}} \gamma\left[5/2\right]$$

$$\left\{ \begin{aligned} \gamma[5/2] &= \gamma[3/2] \cdot 3/2 \\ &= 3/2 \cdot \frac{1}{2} \gamma[1/2] \\ &= \frac{3}{4}\sqrt{\pi} \end{aligned} \right.$$

$$= \frac{\cancel{4}\sigma^4}{\cancel{\sqrt{\pi}}} \frac{3}{\cancel{4}} \cancel{\sqrt{\pi}}$$

$$\boxed{\mu_4 = 3\sigma^4}$$

Moment coefficient of skewness $= 0 = \beta_1$

moment coefficient of kurtosis $= 3 = \beta_2$

$\Rightarrow$ mesokurtic.

Q If X_1 and X_2 are two independent Random variable and each is distributed as $N(0,1)$.
then find the distribution of (a) $\boxed{X_1+X_2}$
(b) $\boxed{X_1-X_2}$

$$\mu_1 = \mu_2 = 0$$
$$\sigma_1^2 = \sigma_2^2 = 1$$

$$X_1+X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2) \sim N(0,2)$$
$$X_1-X_2 \sim N(\mu_1-\mu_2, \sigma_1^2+\sigma_2^2) \sim N(0,2)$$

Q $X \sim N(0,1)$; what are its first four central moment.

$$\mu_1 = 0$$
$$\mu_2 = \sigma^2 = 1$$
$$\mu_3 = 0$$
$$\mu_4 = 3\sigma^4 = 3$$

Q
$$\left.\begin{aligned} X_1 &\sim N(40,25) \\ X_2 &\sim N(60,36) \end{aligned}\right\}$$ find $2X_1+3X_2 = Z$
$3X_1-2X_2 = Y$

$$Z \sim N(2\times40+3\times60, \sigma_1^2+\sigma_2^2)$$
$$Y \sim N(3\times40-2\times60, \sigma_1^2+\sigma_2^2)$$

Mode of Normal Distribution

Let $X \sim N(\mu, \sigma^2)$, then p.d.f. of X is given by

$$f_X(X=x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad -\infty < x < \infty$$

taking logarithm of equation :-

$$\log f(x) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \underbrace{\log_e e}_{=1}$$

Differentiating w.r.t. x ;

$$\frac{1}{f(x)} f'(x) = 0 - \frac{1}{2\sigma^2} 2(x-\mu) = -\left(\frac{x-\mu}{\sigma^2}\right)$$

$$\Rightarrow \quad \underline{f'(x)} = -\left(\frac{x-\mu}{\sigma^2}\right) f(x) = 0$$

$$\Rightarrow \quad \boxed{x = \mu}$$

$$f''(x) = -\frac{(x-\mu)}{\sigma^2} f'(x) - \frac{f(x)}{\sigma^2}$$

$$f''(\mu) = 0 - \frac{f(\mu)}{\sigma^2} < 0$$

So, the Mode of Normal distribution is

$$\boxed{\text{Mode} = \mu = \text{Mean}}$$

Median of Normal Distribution :→

Let M be the median of the normally distributed variable x.

We know that median divides the distribution into two equal parts.

$$\therefore \int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

now putting $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = z\sigma$

$\Rightarrow dx = \sigma\,dz$

as $x \to -\infty \Rightarrow z \to -\infty$

$x = \mu \Rightarrow z = 0$

$$\text{so} \Rightarrow \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

we know that $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$

$$\Rightarrow \int_{-\infty}^{0} \phi(z)\,dz = \frac{1}{2}$$

$$\text{so} \int_{\mu}^{M} f(x)\,dx = 0 \Rightarrow \boxed{M = \mu}$$

Mean = Median = Mode = μ for Normal distribution

Mean Deviation about Mean :→

$$\boxed{M.D. = \int_{-\infty}^{\infty} |x-\mu|\, f(x)\, dx}$$

$$M.D. = \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

now putting $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = \sigma z$

$dx = \sigma\, dz$

$$M.D = \int_{-\infty}^{\infty} |\sigma z| \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{|z|\, e^{-z^2/2}}\, dz$$

$\Downarrow$ even function of z

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} |z|\, e^{-z^2/2}\, dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z\, e^{-z^2/2}\, dz$$

$z^2 = 2t \Rightarrow 2z\,dz = 2\,dt$

$\Rightarrow z\,dz = dt$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} \cancel{z}\, e^{-t} \frac{dt}{\cancel{z}}$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t}\, dt$$

$$\int_{0}^{\infty} e^{-t} = \left.\frac{e^{-t}}{-1}\right|_{0}^{\infty}$$

$$= 1 - \lim_{t\to\infty} e^{-t}$$

$$= 1$$

$$\boxed{M.D. = \frac{2\sigma}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\cdot\sigma \approx \frac{4}{5}\sigma \approx 0.8\sigma}$$

Area of Normal Distribution :→

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

If we assume $\frac{x-\mu}{\sigma} = z$

$x-\mu = z\sigma$

$dx = \sigma\,dz$

Let us assume $x = x_1$ $\quad z = z_1 = \frac{x_1-\mu}{\sigma}$

$x = x_2$ $\quad z = z_2 = \frac{x_2-\mu}{\sigma}$

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \sigma\,dz$$

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{z_1}^{z_1} \phi(z)\,dz$$

$$\boxed{\int_{x_1}^{x_2} f(x)\,dx = \int_{z_1}^{z_2} \phi(z)\,dz}$$

$\phi(z)$ → Standard Normal $\sim N(0,1)$

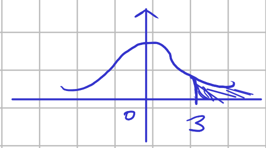
Prob : If X is normally distributed with mean $= 80$ and standard deviation $= 5$ then find

(i) $P[X > 95]$ (ii) $P[X < 72]$

$x = 95$ $\quad z = \frac{x-\mu}{\sigma} = \frac{95-80}{5} = 3$

$$P[x > 95] = P[z > 3]$$

$$= \int_{3}^{\infty} \phi(z)\,dz$$



$$= 0.00135$$

(ii) $P[x < 72]$

$$z = \frac{x - \mu}{\sigma} = \frac{72 - 80}{5}$$

$$= -8/5 = -1.6$$

$$P[x < 72] = P(z < -1.6)$$

$$= \int_{-\infty}^{-1.6} \phi(z)\,dz$$

(iii) $P[85 < x < 97] =$

$$z_1 = \frac{85-80}{5} = 1 \qquad z_2 = \frac{97-80}{5} = \frac{17}{5} = 3.4$$

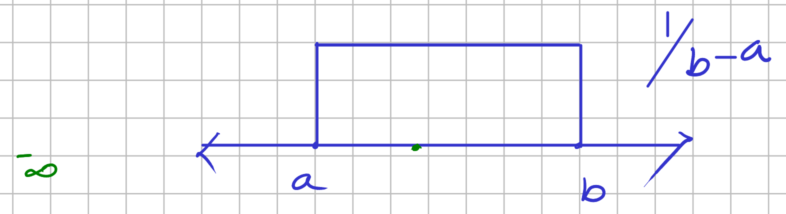
$$P[85 < x < 97] = P[1 < z < 3.4]$$

$$= \int_{-\infty}^{3.4} \phi(z)\,dz - \int_{-\infty}^{1} \phi(z)\,dz$$

$$=$$

Continuous Uniform Distribution :→ A random variable X is said to follow continuous uniform distribution over a interval (a, b) if its p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; \ a < x < b \\ 0 & ; \ \text{otherwise} \end{cases}$$



$$\text{C.D.F.} = P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

Case 1 $x < a$ $\Rightarrow$ $F_X(x) = P(X \leq a) = \int_{-\infty}^{x} 0 \, dx = 0$

Case 2 $a < x < b$ $\Rightarrow$ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \int_{-\infty}^{a} 0 \, dx + \int_{a}^{x} \frac{1}{b-a} \, dx$$

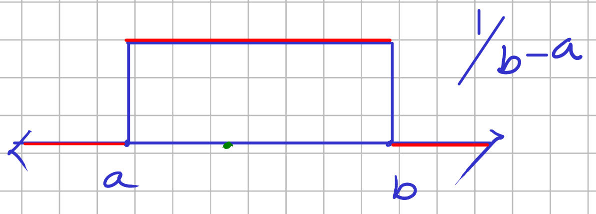
$$= 0 + \frac{1}{b-a}(x - a) = \frac{x-a}{b-a}$$

Case 3 $b < x < \infty$:⇒ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \int_{-\infty}^{a} 0 \, dx + \int_{a}^{b} \frac{1}{b-a} \, dx + \int_{b}^{x} 0 \, dx$$

$$= 0 + 1 + 0 = 1$$

pdf $\Rightarrow$



$f_X(x)$ ✓

cdf

$F_X(x)$

$\mu_x = \mu_1' =$ First order moment about origin

$$\mu_x = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \frac{1}{b-a}\int_a^b x\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\Big|_a^b\right] = \frac{1}{b-a}\,\frac{b^2-a^2}{2}$$

$$\boxed{\mu_x = \frac{b+a}{2}}$$

$\mu_1 = 0$

$\mu_2' = 2^{nd}$ moment about origin

$$= \int_a^b x^2 f_X(x)\, dx = \frac{1}{(b-a)}\left[\frac{x^3}{3}\Big|_a^b\right]$$

$$= \frac{1}{b-a}\,\frac{b^3-a^3}{3}$$

$$= \frac{1}{3}\,\frac{1}{\cancel{b-a}}\,\cancel{(b-a)}\,(b^2+ab+a^2)$$

$$\boxed{\mu_2' = \frac{a^2+ab+b^2}{3}}$$

$\sigma^2 = \mu_2 =$ 2nd moment about mean

$$\mu_2 = -(\mu_x)^2 + \mu_2'$$

$$= \mu_2' - (\mu_x)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

Ex: If X is a uniform R.V. with mean 2 and variance 12, find (a) $P[X<3]$
(b) $P[X>3]$

Soln :→ $X \sim U(a,b)$

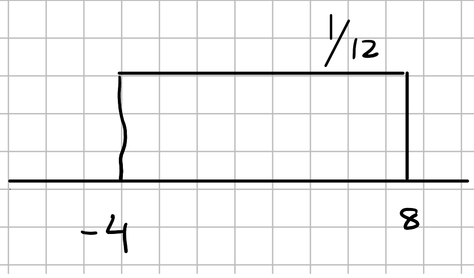
$$\frac{a+b}{2} = 2 \quad \Big| \quad \frac{(b-a)^2}{12} = 12$$

$a+b=4$ —(I) $\qquad b-a = 12$ —(II)

$b = 8$

$a = -4$

$$P(X<3) = \int_{-\infty}^{3} f_X(x)\,dx = \int_{-4}^{3} \frac{1}{12}\,dx = \frac{7}{12}$$



Q $X \sim U(0, 12)$ for thin find coefficient of variation

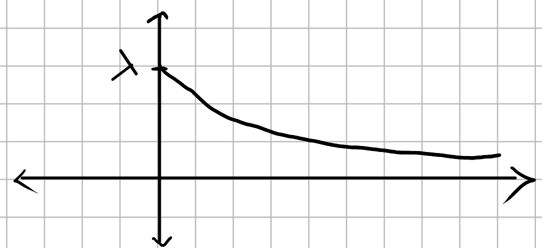
$$C.O.V. = \frac{\sigma}{\mu_x} \qquad \sigma^2 = 12 = \qquad \mu_x = 6$$

$$S.D. = \sigma = \sqrt{12}$$

$$COV = \frac{\sqrt{12}}{6} = \sqrt{\frac{12}{36}} = \sqrt{\frac{1}{3}}$$

Exponential distribution :→ A random variable X is said to be an exponential Random variable with a +ve parameter $\lambda > 0$, if it takes any non-negative real value and its pdf is given by

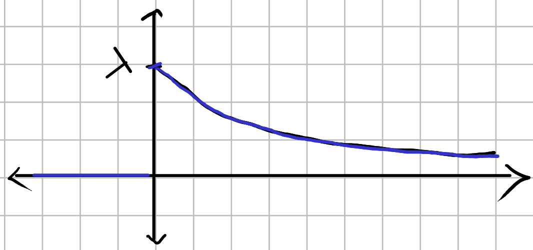
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leq x < \infty \\ 0 & \text{otherwise } -\infty < x < 0 \end{cases}$$



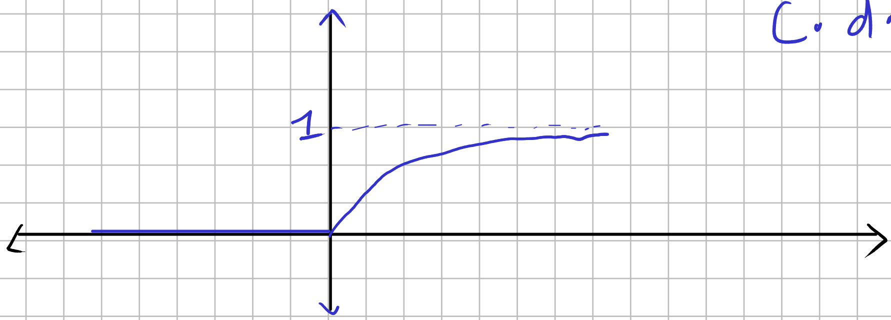
$$F_X(x) = P[X > x] = 0 \quad ; -\infty < x < 0$$

$$= 1 - e^{-\lambda x} \qquad 0 \leq x < \infty$$

p.d.f



c.d.f.



mean $= \mu_x = \mu_1' = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

$$= \int_{-\infty}^{0} x \cdot 0 \, dx + \int_{0}^{\infty} x \lambda e^{-\lambda x} \, dx$$

$$= 0 + \lambda \int_{0}^{\infty} x e^{-\lambda x} \, dx$$

$$= \lambda \left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (1)\frac{e^{-\lambda x}}{(-\lambda)^2} + 0 \cdot \frac{e^{-\lambda x}}{(-\lambda)^2} \right]_0^{\infty}$$

$$\int uv\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \cdots$$

$$= \lambda \left[-\frac{x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \frac{\lambda}{\lambda^2} \left[0 + 1 \right] = \frac{1}{\lambda}$$

$$\mu_x = \frac{1}{\lambda}$$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^{\infty} x^2 \lambda\, e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} x^2 e^{-\lambda x}\,dx$$

$$= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{(-\lambda)^2} + 2 \frac{e^{-\lambda x}}{(-\lambda)^3} \right]_0^{\infty}$$

$$= \lambda \frac{2}{(-\lambda)^3} \left[-1 \right] = \frac{2}{\lambda^2}$$

$$\mu_2 = \mu_2' - (\mu_x)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mu_x = \frac{1}{\lambda} \qquad \sigma_\theta^2 = \frac{1}{\lambda^2}$$

Mean = λ variance

$$\Longrightarrow \begin{cases} \lambda < 1 \Rightarrow \mu_x < \sigma^2 \\ \lambda = 1 \Rightarrow \mu_x = \sigma^2 \\ \lambda > 1 \Rightarrow \mu_x > \sigma^2 \end{cases}$$

Ex Telephone calls arrives at a switchboard following an exponential distribution with $\lambda = 12$ per hour. If we are at switchboard what is the probability that the waiting time for a call is

i) at least 15 mintes

ii) not more than 10 minutes

Soln :→ $X \sim$ exponential Random v. with $\lambda = 12$

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

$$F_X(x) = P[X \leq x] = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

① $P[X \geq 15 \text{ minutes}] = P[X \geq \frac{1}{4}]$

$$= 1 - P[X \leq 1/4]$$

$$= 1 - (1 - e^{-12 \times 1/4}) = e^{-3}$$

② $P[X \leq 10 \text{ min}] = P[X \leq 1/6]$

$$= 1 - e^{-\lambda 1/6}$$

$$=$$

Ex :→ find mean and variance of X if

$$f_X(x) = 3e^{-3x} \; ; x \geq 0$$

$$\mu_x = \frac{1}{\lambda} = \frac{1}{3}$$

$$\sigma_x^2 = \frac{1}{\lambda^2} = \frac{1}{9}$$

$$\mu_x = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x\, 3e^{-3x}\,dx$$

$$= 3\int_{0}^{\infty} x e^{-3x}\,dx$$

$$= 3\left[\frac{xe^{-3x}}{-3} - \frac{e^{-3x}}{(-3)^2}\right]\Bigg|_0^{\infty}$$

$$= \frac{3}{3^2}\left[-0 + (1)\right] = 1/3$$

SAMPLING DISTRIBUTION:

* The process of generalizing sample results to the population is called statistical inference.

population :→ entire data under consideration/study

- (SIZE)
  - FINITE
  - INFINITE
- (TYPE)
  - REAL
  - HYPOTHETICAL

Sample :-> A sample is a part/fraction/subset of the population

Sample survey / complete survey (enumeration)

Complete Survey :-> Complete enumeration / Census

* we study each and every element of the population.

Sample Survey / Sample Enumeration :-> we take a part of population for the study.

Parameter vs Sample statistics

Population

$$\mu_x = \frac{\sum f_i x_i}{\sum f_i}$$

$$\sigma_x^2 = \frac{\sum f_i (x_i - \mu)^2}{\sum f_i}$$

Sample

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

$$s^2 = \frac{1}{n-1}\sum_{i}^{n}(x_i - \bar{x})^2$$

Ex

population

Typist	Number of Errors
A	4
B	2
C	3
D	1

$$\mu = \frac{4+2+3+1}{4} = \frac{10}{4} = 2.5 \text{ errors}$$

Sample No	sample (2)	Sample obs	Sample mean
1.	(A, A)	(4, 4)	4
2.	(A, B)	(4, 2)	3
3.	(A, C)	(4, 3)	3.5
4.	(A, D)	(4, 1)	2.5
5.	(B, B)	(2, 2)	2
6.	(B, C)	(2, 3)	2.5
7.	(B, D)	(2, 1)	1.5
8.	(B, A)	(2, 4)	3
9.	(C, A)	(3, 4)	3.5
10.	(C, B)	(3, 2)	2.5
11	(C, C)	(3, 3)	3
12.	(C, D)	(3, 1)	2
13.	(D, A)	(1, 4)	2.5
14	(D, B)	(1, 2)	1.5
15	(D, C)	(1, 3)	2
16	(D, D)	(1, 1)	1

$\bar{X}$ = mean of sample mean = μ = 2.5

Standard Error :→