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1.
a) There are 4 execution paths for the given program.

⇒ 1,2,3,4,9,11,12,13,17
⇒ 1,2,5,6,7,9,11,12,13,17
⇒ 1,2,3,4,9,11,14,15,16,17
⇒ 1,2,5,6,7,9,11,14,15,16,17

b)
Path 1

Edge	Symbolic State	Path Condition
$1 \rightarrow 2$	$x \rightarrow X_0$, $y \rightarrow Y_0$	True
$2 \rightarrow 3$	$x \rightarrow X_0+7$, $y \rightarrow Y_0$	$X_0 + Y_0 > 15$
$3 \rightarrow 4$	$x \rightarrow X_0+7$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15$
$4 \rightarrow 9$	$x \rightarrow X_0+9$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15$
$9 \rightarrow 11$	$x \rightarrow X_0+9$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 > 27$
$11 \rightarrow 12$	$x \rightarrow 3X_0+27$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 > 27$
$12 \rightarrow 13$	$x \rightarrow 3X_0+27$, $y \rightarrow 2Y_0-24$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 > 27$
$13 \rightarrow 17$	$x \rightarrow 3X_0+27$, $y \rightarrow 2Y_0-24$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 > 27$

Path 2

Edge	Symbolic State	Path Condition
$1 \rightarrow 2$	$x \rightarrow X_0$, $y \rightarrow Y_0$	True
$2 \rightarrow 3$	$x \rightarrow X_0+7$, $y \rightarrow Y_0$	$X_0 + Y_0 > 15$
$3 \rightarrow 4$	$x \rightarrow X_0+7$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15$
$4 \rightarrow 9$	$x \rightarrow X_0+9$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15$
$9 \rightarrow 11$	$x \rightarrow X_0+9$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \leq 27$
$11 \rightarrow 14$	$x \rightarrow X_0+9$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \leq 27$
$14 \rightarrow 15$	$x \rightarrow 4X_0+36$, $y \rightarrow Y_0-12$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \leq 27$
$15 \rightarrow 16$	$x \rightarrow 4X_0+36$, $y \rightarrow 3Y_0 + 4X_0$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \leq 27$
$16 \rightarrow 17$	$x \rightarrow 4X_0+36$, $y \rightarrow 3Y_0 + 4X_0$	$X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \leq 27$

Path 3

Edge	Symbolic State	Path Condition
$1 \rightarrow 2$	$x \rightarrow X_0$, $y \rightarrow Y_0$	True
$2 \rightarrow 5$	$x \rightarrow X_0$, $y \rightarrow Y_0$	$X_0 + Y_0 \leq 15$
$5 \rightarrow 6$	$x \rightarrow X_0$, $y \rightarrow Y_0+10$	$X_0 + Y_0 \leq 15$
$6 \rightarrow 7$	$x \rightarrow X_0-2$, $y \rightarrow Y_0+10$	$X_0 + Y_0 \leq 15$
$7 \rightarrow 9$	$x \rightarrow X_0$, $y \rightarrow Y_0+10$	$X_0 + Y_0 \leq 15 \land 2X_0 + 2Y_0 > 1$
$9 \rightarrow 11$	$x \rightarrow X_0$, $y \rightarrow Y_0+10$	$X_0 + Y_0 \leq 15 \land 2X_0 + 2Y_0 > 1$
$11 \rightarrow 12$	$x \rightarrow 3X_0$, $y \rightarrow Y_0 +10$	$X_0 + Y_0 \leq 15 \land 2X_0 + 2Y_0 > 1$
$12 \rightarrow 13$	$x \rightarrow 3X_0$, $y \rightarrow 2Y_0 +20$	$X_0 + Y_0 \leq 15 \land 2X_0 + 2Y_0 > 1$
$13 \rightarrow 17$	$x \rightarrow 3X_0$, $y \rightarrow 2Y_0 +20$	$X_0 + Y_0 \leq 15 \land 2X_0 + 2Y_0 > 1$

Path 4

Edge	Symbolic State	Path Condition
1 → 2	$x \to X_0$, $y \to Y_0$	True
2 → 5	$x \to X_0$, $y \to Y_0$	$X_0 + Y_0 \le 15$
5 → 6	$x \to X_0$, $y \to Y_0+10$	$X_0 + Y_0 \le 15$
6 → 7	$x \to X_0-2$, $y \to Y_0+10$	$X_0 + Y_0 \le 15$
7 → 9	$x \to X_0$, $y \to Y_0+10$	$X_0 + Y_0 \le 15$
9 → 11	$x \to X_0$, $y \to Y_0+10$	$X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$
11 → 14	$x \to X_0$, $y \to Y_0 +10$	$X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$
14 → 15	$x \to 4X_0$, $y \to Y_0 +10$	$X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$
15 → 16	$x \to 4X_0$, $y \to 3Y_0 + 4X_0 +30$	$X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$
16 → 17	$x \to 4X_0$, $y \to 3Y_0 + 4X_0 +30$	$X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$

c)
Path 1→ $X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 > 27$ **(Feasible)**

$X_0 = 5$, $Y_0 = 12$

Path 2 → $X_0 + Y_0 > 15 \land 2X_0 + 2Y_0 \le 27$ **(Not Feasible)**

Path 3 → $X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 > 1$ **(Feasible)**

$X_0 = 3$, $Y_0 = 4$

Path 4 → $X_0 + Y_0 \le 15 \land 2X_0 + 2Y_0 \le 1$ **(Feasible)**

$X_0 = 0$, $Y_0 = 0$

2.
a) For the at-most-one constraint on variables a1, a2, a3, and a4, the following set of clauses can be used:
$\neg a1 \lor \neg a2$, $\neg a1 \lor \neg a3$, $\neg a1 \lor \neg a4$, $\neg a2 \lor \neg a3$, $\neg a2 \lor \neg a4$, $\neg a3 \lor \neg a4$

b)

b) Given first-order logic

$$(\forall x . \exists y . P(x) \vee Q(y)) \Leftrightarrow (\forall x . P(x)) \vee (\exists y . Q(y))$$

starting with LHS

$$(\forall x . \exists y \cdot P(x) \vee Q(y))$$

This means for every x, there exists y such that either $P(x)$ or $Q(y)$ is true.

on RHS:

$$(\forall x . P(x)) \vee (\exists y . Q(y))$$

This means either $\forall x . P(x)$ is true or $\exists y . Q(y)$ is true.

By considering the cases where either $P(x)$ is true or $Q(y)$ is true, we can see that the LHS is equivalent to the RHS.

c)

c) Given first-order logic

$$(\forall x . \exists y . P(x,y) \vee Q(x,y)) \Leftrightarrow (\forall x . \exists y . P(x,y)) \vee (\forall x . \exists y . Q(x,y))$$

Assume, for each x, we have $\exists y$. $P(x,y) \vee Q(x,y)$. Now, for each x, consider two cases:

→ If $\exists y . P(x,y)$ is true for that x, then $\forall x . \exists y . P(x,y)$ is true

→ If $\exists y . Q(x,y)$ is true for that x, then $\forall x . \exists y . Q(x,y)$ is true

Therefore, in either case, the consequent holds, and the implication is valid.

d)

d) Analyze each model to determine wether it satisfies or violates the given first-order logic formula ϕ:

Formula ϕ:

$$\exists x \exists y \exists z (P(x,y) \land P(z,y) \land P(x,z) \land \neg P(z,x))$$

a) $M_1 = \langle S_1, P_1 \rangle$, where $S_1 = N$ and $P_1 = \{(x,y) \mid x,y \in N$ $x<y\}$

This model has a linear ordering on natural numbers where P_1 represents pairs (x,y) such tha $x<y$. To check if M_1 satisfies ϕ, we need to find values for x, y & z that satisfy the formula

let $x=0$, $y=1$ and $z=2$

$P(0,1) \land P(2,1) \land P(0,2) \land \neg P(2,0)$

Since, $(0,1)$, $(2,1)$ $(0,2)$ are in P_1 (since, $0<1, 2<3, 0<2$) and $(2,0)$ is not in P_1 (as $2 \nless 0$), the formula is satisfied. Therefore $M_1 \models \phi$.

b) Model $M_2 = \langle S_2, P_2 \rangle$, where $S_2 = N$ and $P_2 = \{(x, x+1) \mid x \in N\}$:

This model has pairs $(x, x+1)$, representing consecutive natural Numbers. To check if M_2 satisfies ϕ, we need to find values for x, y and z that satisfy the formula.

Let $x=0$, $y=1$, $z=2$

$P(0,1) \land P(2,1) \land P(0,2) \land \neg P(2,0)$

Here, $(0,1)$, $(2,1)$, $(0,2)$ are in P_2 (as $1=0+1$, $3=2+1$, $2=0+2$), but $(2,0)$ is also in P_2 (as $2 \neq 1+1$), therefore, the formula is not satisfied. Thus, $\underline{M_2 \not\models \phi}$

c) Model $M_3 = \langle S_3, P_3 \rangle$, where $S_3 = P(N)$ and $P_3 = \{(A,B) \mid A, B \in P(N) \wedge A \subseteq B\}$:

This model represents the powerset of natural numbers with pairs (A,B) such that A is a subset of B. To check if M_3 satisfies ϕ, we need to find values that satisfy the formula for x, y and z.

Let $x = \{0\}$, $y = \{1\}$, $z = \{2\}$

$P(\{0\},\{1\}) \wedge P(\{2\},\{1\}) \wedge P(\{0\},\{2\}) \wedge \neg P(\{2\},\{0\})$

All conditions satisfied because $\{0\} \subseteq \{1\} \subseteq \{2\}$ and $\{2\} \not\subseteq \{0\}$.

Therefore, $\underline{M_3 \models \phi}$

e)

e) To extend the encoding for the "at-most-one" constraint to n variables using at most $O(n)$ clauses and variables, we can use a pairwise comparison approach.

Let $a_1, a_2, \dots a_n$ be the boolean variable we want to constrain. We introduce binary variables b_{ij} for each pair i, j such that $1 \le i \le j \le n$. The variable b_{ij} will be true if a_i is true and a_j is false, and false otherwise.

The clauses to ensure the "at-most-one" constraint are:

1. For each i, ensure that at least one of a_i is true:

$\bigvee_{j=1}^{n} a_j$

2. For each $i<j$, ensure that at least one of a_i or a_j is false:

$\bigvee_{i=1}^{n} \bigvee_{j=i+1}^{n} \neg a_i \vee \neg a_j$

3. For each $i<j$, ensure that if a_i is true, then a_j is false:

$\bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} (\neg a_i \vee \neg a_j \vee b_{ij})$

4. For each $i<j$, ensure that if a_j is true, then a_i is false

$\bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} (\neg a_j \vee \neg a_i \vee b_{ij})$

These clauses ensure that each variable a_i is true in atleast one assignment, and for each pair i,j, at most one of a_i or a_j is true.

This encoding uses $O(n)$ variables ($a_1, a_2, \ldots a_n$ and b_{ij}) and $O(n)$ clauses, making in linear in the number of variables. The pairwise comparison technique is a common method for encoding "at-most-one" constraints efficiently.

3.

a)

3)

a) To express the constraints for a magic square in First-order Logic (FOL), we can use a set of quantifier-free constraints.

→ let M be the $n \times n$ matrix representing the magic square

→ Let V be the set of distinct positive integers from 1 to n^2.

→ Let RowSum, ColSum and DiagSum be functions that compute the sum of elements in a row, column and diagonal, respectively.

Now, we can express the constraints in FOL

1. Distinct Values: Each cell in the magic square must contain a distinct positive integer.

$$\forall i,j : 1 < i < n \wedge 1 \leq j \leq n \Rightarrow M[i,j] \in V$$

2. Unique values: No two cells in the magic square can contain the same integer.

$\forall i,j,k,l: (1 \le i \le n \wedge 1 \le j \le n \wedge 1 \le k \le n \wedge 1 \le l \le n) \wedge (i \ne k \vee j \ne l) \Rightarrow M[i,j]$

3. Row sums: The sum of elements in each row must be equal.

$\forall i: 1 \le i \le n \Rightarrow RowSum(M[i,:]) = RowSum(M[1,:])$

4. Column sum: The sum of elements in each the column must be equal.

$\forall j: 1 \le j \le n \Rightarrow ColSum(M[:,j]) = ColSum(M[:,1])$

5. Diagonal sums: The sum of elements each diagonal must be equal.

$$DiagSum(M) = DiagSum(M[::-1,:])$$

6. Range of values: All integers from 1 to n^2 must appear in the matrix

$\forall x: x \in V \Rightarrow \exists i,j: 1 \le i \le n \wedge 1 \le j \le n \wedge M[i,j] = x$

These constraints capture the essential properties of a magic square.

Note: $M[i,:]$ represents the i^{th} row of matrix M, $M[:,j]$ represents the j^{th} column, and $M[::-1,:]$ represents the reverse of matrix M.