Cosmic Microwave Background Radiation

Lecture 1 : Physics of CMB

Jayanti Prasad

Inter-University Centre for Astronomy & Astrophysics (IUCAA)
Pune, India (411007)

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BITS PILANI

Plan of the Talk

- Standard Model of Cosmology

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- Theoretical Framework
  - Statistical Mechanics of photons
  - Boltzmann Equation
  - Recombination

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- Perturbations
  - Metric Perturbations
  - Boltzmann equation for photons
  - Line of sight integration

Hot Big Bang Cosmology : Standard Model of Cosmology

- Large scale uniformity - Homogeneity and Isotropy - Hubble expansion

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- Homogeneous and Isotropic space time - FRW metric:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{2}$$

only two parameters - scale factor $a(t)$ and spatial curvature k.

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- Inflation

Friedman Equations

- For FRW metric, Einstein equation (1) can be written in terms of a pair of equations called Friedman equations

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{4}$$

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- Energy density of any species is given by the density parameter $\Omega\rho/\rho_c$ where ρ_c is called the critical density and is defined as:

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \tag{6}$$

Distances

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$$\lambda(t) = \frac{a(t)}{a(t_0)}\lambda(t_0) \tag{7}$$

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- Comoving size of the Universe is given by η:

$$\eta = \int \frac{cdt}{a(t)} = \int_0^a \frac{da'}{a'} \frac{cda'}{a' H(a')} \tag{9}$$

Numbers

- Hubble parameter h is measured in 100 Km/sec/ Mpc[1]

$$H_0 = \frac{h}{0.98 \times 10^{10}\text{year}} \quad \text{where} \quad 0.5 < h < 1.0 \qquad (10)$$

[1] 1 Mpc $= 3.0856 \times 10^{24}$ cm

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$$\begin{aligned}\rho_c = \frac{3H_0^2}{8\pi G} &= 1.88h^2 \times 10^{-29}\text{gm} \quad \text{cm}^{-3} \\ &= 2.775h^{-1} \times 10^{11} M_\odot/(h^{-1}\text{Mpc})^3\end{aligned} \tag{12}$$

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- Temperature:

$$T_{\mathrm{CMB}} = 2.725K \approx 2.35 \times 10^{-4} \ \text{ eV} \tag{13}$$

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Problem 1

Given that the equation of state parameter for a species is $w = P/\rho$ show that its energy density will change as $\rho(a) \propto a^{-3(1+w)}$, as the universe expands adiabatically.

Problem 2

Show that the Hubble parameter $H(a)$ depends on the energy densities of various species in the following way:

$$H^2 = H_0^2 \left[\Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_r \left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \Omega_k \left(\frac{a_0}{a}\right)^2\right] \qquad (14)$$

with $\Omega_k = 1 - \Omega_{\mathrm{Total}}$.

Observational Support of the Big Bang Model

- Hubble expansion

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- WMAP and Planck have further measured CMB anisotropies with great precision.

What we know about CMB ?

- CMB is a perfect blackbody radiation with temperature 2.725 degree Kelvin so its specific intenisty is given by

$$I_\nu = \frac{2h^3}{c^2}\frac{1}{e^{h\nu/k_BT}-1} \tag{15}$$

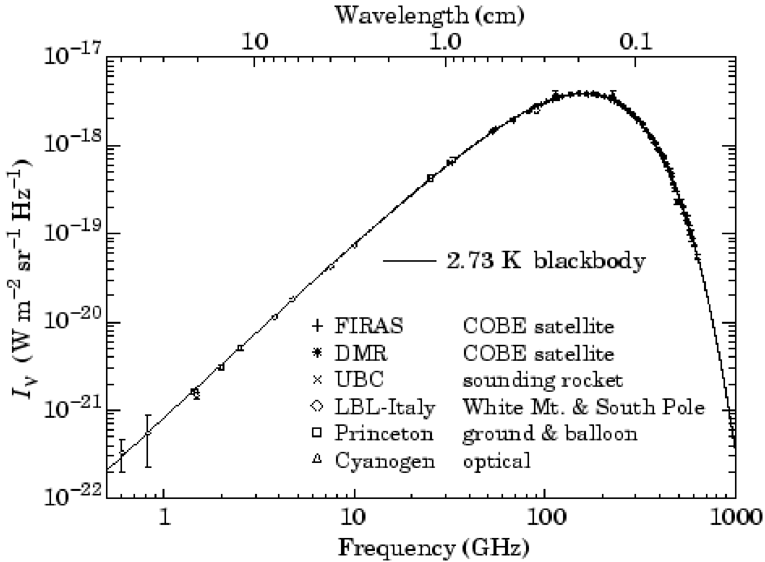
- Largest anisotropy 0^{-3} in the CMB sky is due to the motion of the solar system with respect to the rest frame of CMB (dipole) :

$$\frac{\Delta T}{T} = \frac{v}{c}\cos\theta \tag{16}$$

  for v=370 km/sec we get $\Delta T = 3.358\times 10^{-3}$ Kelvin.

- Ignoring the dipole anisotropy, CMB anisotropies are of the order of 10^{-3}.

CMB Black Body spectrum



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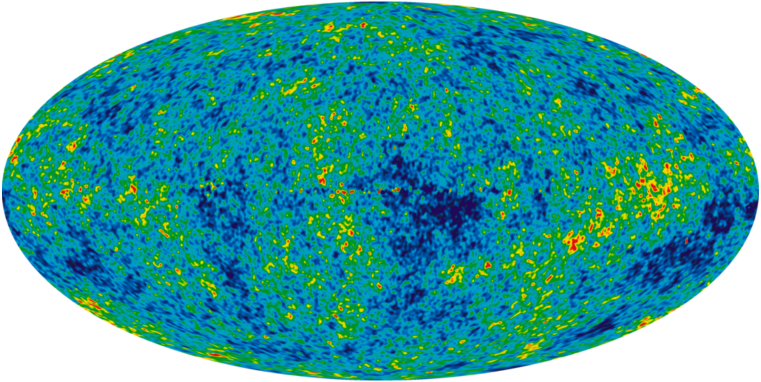

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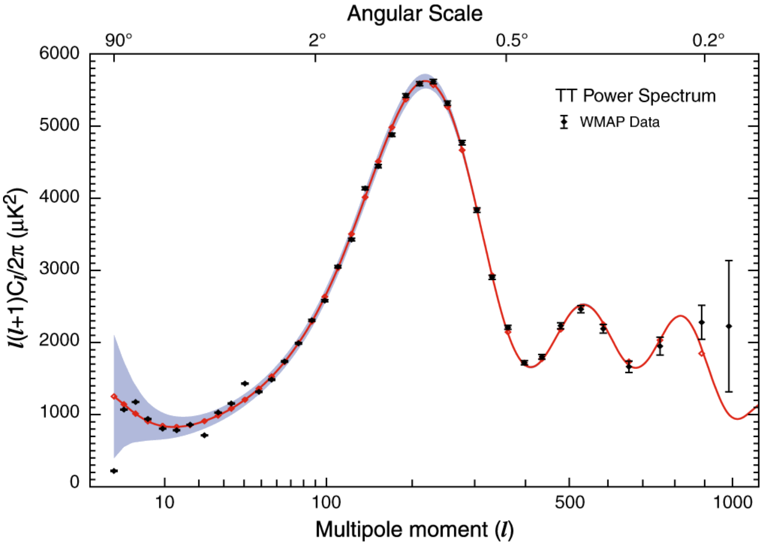

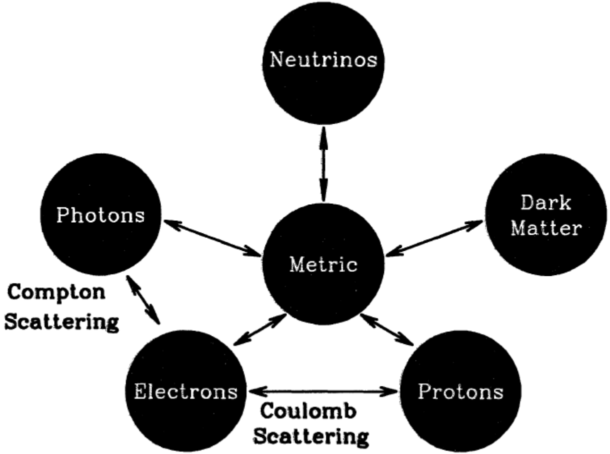

Figure 4.1. The ways in which the different components of the universe interact with each other. These connections are encoded in the coupled Boltzmann–Einstein equations.

Theoretical Framework

- Expansion of the universe, fluctuations in space time metric - General Relativity

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- Synthesis of light elements - Nuclear and particle physics.
- Origin of density fluctuations in Inflation - Quantum Field theory

Statistical Mechanics of photons

- The phase space density of photons is given by:

$$f(p) = \frac{1}{e^{pc/k_B T} - 1} \tag{17}$$

[Kolb & Turner (1990); Dodelson (2003); Weinberg (2008)]

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- Number density is given by:

$$n_\gamma = 2 \int \frac{d^3 p}{(2\pi\hbar)^3} f(p) = 8\pi \left(\frac{k_B T}{hc} \right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} \tag{18}$$

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- Energy density:

$$\rho_\gamma = 2 \int \frac{d^3p}{(2\pi\hbar)^3} (pc) f(p) = \frac{8\pi^5 k_B^4}{15 h^3 c^3} T^4 = a T^4 = \frac{4\sigma}{c} T^4 \tag{19}$$

with

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\xi(3) = 2.404 \text{ and } \int_0^\infty \frac{x^3 dx}{e^x - 1} = 6\xi(4) = \frac{\pi^2}{15} \tag{20}$$

[Kolb & Turner (1990); Dodelson (2003); Weinberg (2008)]

Problem 3

Given that the CMB is a black body distribution with temperature 2.725 K show that:

- number density of CMB photons is around 440 /cc and
- energy density $\Omega_\gamma \approx 2.47 \times 10^{-4}/h^2$
- photon to baryon ratio is around 10^9.

Boltzmann Equation

- Boltzmann equation describes the evolution of phase space density $f(t, \vec{x}, \vec{p})$ in the phase space:

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  where the RHS is the collision terms which represent the change in the phase space density due to emission, absorption and scattering.

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- We can write the LHS explicitly as:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} \tag{22}$$

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- In the absence of scattering, emission or absorption the Boltzmann equation is simply:

$$\frac{df}{dt} = 0 \tag{23}$$

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- The number density n_1 of particle type '1':
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  - decrease due to annihilation with particle '2'
- If the phase-space densities of particles 1, 2, 3 and 4 and f_1, f_2, f_3 and f_4 respectively then from the Boltzmann Equation:

$$\begin{aligned} a^{-3}\frac{d(a^3 n_1)}{dt} = & \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ & \times \delta_D(E_1 + E_2 - E_3 - E_4)\delta_D(p_1 + p_2 - p_3 - p_4)\mathcal{M}^2 \\ & \times \{f_3 f_4[1 \pm f_1][1 \pm f2] - f_1 f_2[1 \pm f_3][1 \pm f4]\} \end{aligned} \tag{25}$$

- We are interested in a limit in which the exponential term is far greater than the unity, so the Bose-Fermion difference can be ignored:

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- We can replace the first two lines of equation (25) by $<\sigma v>$ so we get:

$$a^{-3}\frac{d(a^3 n_1)}{dt} = n_1^{(0)} n_2^{(0)} < \sigma v > \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right] \tag{27}$$

where

$$n_i = g_i e^{\mu/T} \int \frac{d^3p}{(2\pi)^3} e^{-E/T} \tag{28}$$

and

$$n_i(0) = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}, & \text{if} T << m_i \\ , g_i \frac{T^3}{\pi^2} & \text{if} T >> m_i \end{cases} \tag{29}$$

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- If the reaction rate $n_2 < \sigma v >>> H$ then the RHS will be much larger and the particles can be in equilibrium.
- We can maintain the equality if the individual terms in RHS cancel each other.

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{30}$$

This equation is called Nuclear Statistical Equilibrium (NSE) or Saha equation.

Applications of Boltzmann Equations

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- Above temperature 3500 K or 0.3 eV photons are hot enough to ionize any hydrogen atom which forms:

$$e^{-} + p^{+} \longleftrightarrow H + \gamma \tag{33}$$

however once temperature of photons falls below 0.3 eV they decouple and this event is called decoupling, recombination, or last scattering.

Recombination

- Up to temperature 1 eV, photons remain tightly coupled to electrons via Compton scattering and electrons to protons via Coulomb scattering.
- For $e^- + p = H + \gamma$ to be in equilibrium : we need

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \tag{34}$$

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- Defining :

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_e + n_H} \tag{35}$$

equation (34) can be written as:

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}\right] \tag{36}$$

where $\epsilon_0 = m_e + m_p - m_H$ is the Binding energy of hydrogen atom.

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- Using the fact that $n_\gamma \propto T^3$ equation (36) can be written as:

$$\frac{X_e^2}{1 - X_e} \approx 10^9 \left(\frac{m_e}{2\pi T}\right)^3 e^{-\epsilon_0/T} \approx 10^{15} \quad \text{when} \quad T = \epsilon_0 \quad (37)$$

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- Since the RHS becomes very large so the equation is satisfied only when X_e is close to unity or all the atoms are ionized.
- Saha equation (34) correctly predicts the epoch of recombination but for fails when electron fraction drops and the equation goes out of equilibrium and we need solve the full Boltzmann equation numerically :

$$a^{-3}\frac{d(a^3 n_e)}{dt} = n_e^{(0)} n_p^{(0)} < \sigma v > \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}}\right]$$
$$= n_b < \sigma v > \left\{(1 - X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} - X_e^2 n_b\right\} \tag{38}$$

or

$$\frac{dX_E}{dt} = \left\{(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}\right\} \tag{39}$$

with ionization rate β and the recombination rate $\alpha^{(2)}$ are given by:

$$\beta = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} \tag{40}$$

and

$$\alpha^{(2)} =< \sigma v > \tag{41}$$

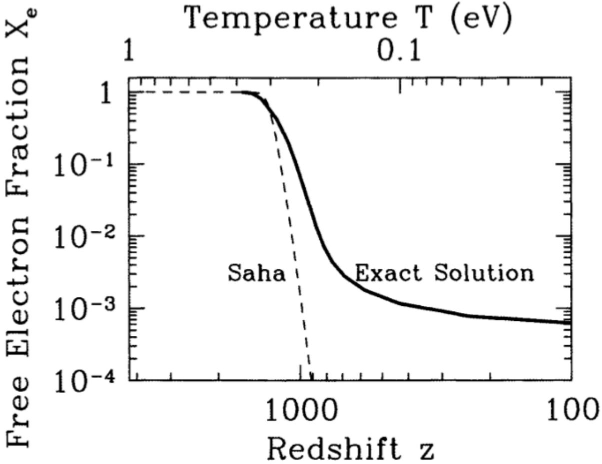

Figure 3.4. Free electron fraction as a function of redshift. Recombination takes place suddenly at $z \sim 1000$ corresponding to $T \sim 1/4$ eV. The Saha approximation, Eq. (3.37), holds in equilibrium and correctly identifies the redshift of recombination, but not the detailed evolution of X_e. Here $\Omega_b = 0.06, \Omega_m = 1, h = 0.5$.

Recombination

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- The change in the number density of free electrons is important from the point of view of observational cosmology since recombination at $z^* \approx 1000$ is directly related to the decoupling of CMB photons.
- Decoupling of CMB photons occurs roughly when the rate for photons to Compton scatter off electrons becomes smaller than the expansion rate.

$$n_e\sigma_T = X_e n_b \sigma_T = 7.477 \times 10^{-30}\ \mathrm{cm}^{-1} X_e \Omega_b h^2 a^{-3} \qquad (42)$$

Recombination

Dividing the recombination rate by expansion rate (radiation dominated):

$$\frac{H}{H_0} = \Omega_m^{1/2} a^{-3/2} [1 + a/e_{eq}]^{1/2} \tag{43}$$

gives:

$$\frac{n_e \sigma_T}{H} = 113 X_e \left(\frac{\Omega_b h^2}{0.02}\right) \left(\frac{.15}{\Omega_m h^2}\right)^{1/2} \left(\frac{1+z}{1000}\right)^{3/2} \left[1 + \frac{1+z}{3600} \frac{0.15}{\Omega_m h^2}\right]^{-1/2} \tag{44}$$

Problem 4

- Derive equation (44)
- Show that decoupling will eventually happen whether recombination takes place or not.
- Find the redshift of decoupling for $X_e = 10^{-2}$ and $X_e = 1.0$.
- How z_{eq}, z_{dec} and z_{rec} are related and find their values.

Perturbations

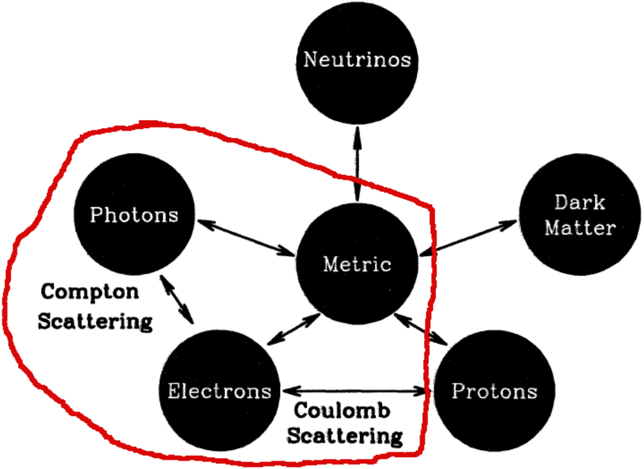

Figure 4.1. The ways in which the different components of the universe interact with each other. These connections are encoded in the coupled Boltzmann–Einstein equations.

- Geometric structure of a homogeneous and isotropic Universe is given by the Friedman-Robertson-Walker (FRW) metric and for spatially flat case this can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t)\delta_{ij} dx^i dx^j \tag{45}$$

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CMB Theory : Metric perturbations

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- Since $g_{\mu\nu}$ is a second rank symmetric tensor and so have 10 components.
- There is a theorem called the decomposition theorem which says that perturbations to the metric can be divided up into three types: scalar, vector, and tensor and of these type evolves independently.
- If some physical process in the early universe sets up tensor perturbations, these do not induce scalar perturbations and vice versa.

Scalar Perturbations

Scalar perturbation to metric are represented by (in conformal Newtonian Gauge) by two functions $\Psi(\vec{x},t)$ and which $\Phi(\vec{x},t)$ which represents perturbations in Newtonian potential and spatial curvature respectively.

$$ds^2 = -[1+2\Psi(\vec{x},t)]c^2dt^2 + a^2(t)\delta_{ij}[1+2\Phi(\vec{x},t)]dx^i dx^j \quad (46)$$

We can compute the Einstein tensor for the metric given above:

$$g_{\mu\nu} \longrightarrow \Gamma \longrightarrow (R, R_{\mu\nu}) \longrightarrow G_{\mu\nu}$$

Problem 5

Show that for the metric given by (46) Ricci Tensor:

$$R_{00} = -3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi - 3\Psi_{,00} + 3H(\Psi_{,0} - 2\Phi_{,0})$$

$$\begin{aligned} R_{ij} = \delta_{ij}\Big[&(2a^2H^2 + a\ddot{a})(1+2\Phi-2\Psi) + a^2H^2(6\Phi_{,0} - \Psi_{,0}) \\ &+ a^2\Phi_{,00} + k^2\Phi\Big] + k_i k_j(\Phi+\Psi) \end{aligned} \quad (47)$$

Scalar Perturbations

- In order to solve for the potential Ψ and Φ we use the Einstein equation. The temporal part:

$$\delta G_0^0 = 8\pi G T_0^0 \tag{48}$$

and for spatial component we use only the traceless part:

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right) G_j^i = 8\pi G \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right) T_j^i \tag{49}$$

Note that non-relativistic particles, such as baryons and dark matter, do not contribute anisotropic stress.

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Note that non-relativistic particles, such as baryons and dark matter, do not contribute anisotropic stress.

- The energy momentum tensor is given by:

$$T^0_0(\vec{x}, t) = -\sum g_i \int \frac{d^3p}{(2\pi)^3} E_i(p) f_i(\vec{p}, \vec{x}, t) = -\rho_\gamma(1 + 4\Theta_0) \tag{50}$$

where Θ_0 is the monopole part:

$$\Theta_0(\vec{x}, t) = \frac{1}{4\pi} \int d\Omega' \Theta(\hat{p}', \hat{x}, t) \tag{51}$$

Scalar Perturbations

- Temporal part of Einstein equation gives:

$$k^2\Phi+3\frac{\dot{a}}{a}\left(\dot{\Phi}-\Psi\frac{\dot{a}}{a}\right)=4\pi Ga^2[\rho_{dm}\delta_{dm}+\rho_b\delta_b+4\rho_\gamma\Theta_0+4\rho_\nu\mathcal{N}_0] \tag{52}$$

note that here dot represent derivative with conformal time and $\mathcal{N}_0$ is the monopole term for neutrinos.

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note that here dot represent derivative with conformal time and $\mathcal{N}_0$ is the monopole term for neutrinos.

- Second Einstein equation gives:

$$k^2(\Phi+\Psi)=-32\pi Ga^2(\rho_\gamma\Theta_2+\rho_\nu\mathcal{N}_2) \tag{53}$$

the two gravitational potentials are equal and opposite unless the pho- tons or neutrinos have appreciable quadruple moments.

Boltzmann Equation Photons

- Boltzmann equation is given by:

$$\frac{df}{dt} = C[f] \tag{54}$$

where $C[f]$ is the collision term which corresponds to Compton scattering of photons with electrons:

$$e^-(\vec{q}) + \gamma(\vec{p}) \longleftrightarrow e^-(\vec{q}') + \gamma(\vec{p}') \tag{55}$$

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$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \tag{56}$$

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- Now we need to compute dx/dt and dp/dt in the perturbed metric.

- Let us consider that the four momentum of photon is P^μ then :

$$P^\mu P^\mu = g_{00}(P^0)^2 + p^2 = -(1+2\Psi)(P^0)^2 + p^2 \qquad (57)$$

or

$$P^0 = \frac{p}{\sqrt{1+2\Psi}} \approx p(1-\psi) \qquad (58)$$

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- For spatial part we can write:

$$P^i = C\hat{p}^i \tag{59}$$

where C is a constant which we can compute in the following way:

$$p^2 = P^i P_i = C^2 g_{ij}\hat{p}^i\hat{p}^j = C^2 a^2(1 + 2\Phi) \tag{60}$$

and so

$$C = \frac{p}{a\sqrt{1 + 2\Phi}} \tag{61}$$

and

$$P^i = \frac{p}{a}(1 - \Phi)\hat{p}^i \tag{62}$$

- The velocity can be computed as:

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{1}{a}(1 + \Psi - \Phi)\hat{p}^i \tag{63}$$

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- For momentum we use Geodesic equation:

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{64}$$

which for 0 component:

$$\frac{dP^0}{d\lambda} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{65}$$

we can compute:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \tag{66}$$

Problem 6

Using equation (65) and metric given by equation (46) show that:

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial \Psi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \tag{67}$$

Using the expressions for dx^i/dt and dp^i/dt we can write:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{df}{dp}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] \tag{68}$$

- We can get the evolution equation for $\Theta(x, \hat{p}, t)$ from the evolution equation for $f(t, x, p)$ by expanding f around its zeroth order and keeping only the linear terms in $\Theta(x, \hat{p}, t)$:

$$f = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \tag{69}$$

where

$$f^{(0)} = \frac{1}{e^{p/T} - 1} \tag{70}$$

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where

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- Keeping only up to linear terms in Θ, the Boltzmann equation for photons become:

$$\frac{df}{dt} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right] \tag{71}$$

Compton scattering

- Scattering (Compton) between free electrons and photons also change the phase space density of photons:

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}), \tag{72}$$

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- The change in the phase density of photons due to Compton scattering is given by:

$$c[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\vec{p}) + \hat{p}.\vec{v}_b] \tag{73}$$

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- The full Boltzmann equation can be written as:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p}.\vec{v}_b] \tag{74}$$

This equation is called the Brightness equation [Kurki-Suonio (2010)]

- In terms of conformal time the full Boltzmann equation can be written as:

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a[\Theta_0 - \Theta + \hat{p}.\vec{v}_b] \qquad (75)$$

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- In Fourier space equation (75) becomes a algebraic equation:

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b] \qquad (76)$$

where :

$$\Theta(\hat{x}) = \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k}.\vec{x}]\tilde{\Theta}(\hat{k}) \qquad (77)$$

and the optical depth τ is defined as:

$$\tau(\eta) = \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a \qquad (78)$$

where $-n_e\sigma_T a = \dot{\tau}$ and the direction of propagation of photon $\mu = \hat{k}.\hat{p}$.

Note that if we take into account that the Compton scattering between photons and electrons depend on the direction also and temperature and polarization fields are coupled to each other, we get the following Boltzmann equation for photons:

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b - \frac{1}{2}P_2(\mu)\Pi\right] \quad (79)$$

where $P_2(\mu) = (3\mu^2 - 1)/2$ is the second Legendre polynomial and Π is defined as:

$$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0} \quad (80)$$

Boltzmann Equations

- Considering that the Universe at the time of decoupling consists photons, neutrinos, baryons and dark matter, we have the following set of seven equations for the evolution of $\Theta, \Theta_P, \delta, v, \delta_b, v_b$ and neutrino temperature $\mathcal{N}$

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b - \frac{1}{2}P_2(\mu)\Pi\right] \tag{81}$$

$$\dot{\tilde{\Theta}}_P + ik\mu\tilde{\Theta}_P = -\dot{\tau}\left[-\tilde{\Theta}_P + \frac{1}{2}(1 - P_2(\mu))\Pi\right] \tag{82}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{83}$$

$$\dot{v} + \frac{\dot{a}}{a} = -ik\Psi \tag{84}$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi} \tag{85}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = ik\Psi + \frac{\dot{\tau}}{R}[v_b + 3i\Theta_1] \tag{86}$$

$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi \tag{87}$$

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$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi \tag{87}$$

- We have two equations from Einstein's equation for potential Ψ and Φ:

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi Ga^2[\rho_{dm}\delta_{dm} + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0] \tag{88}$$

$$k^2(\Phi + \Psi) = -32\pi Ga^2(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2) \tag{89}$$

- In order to solve the set of 9 first order differential (Boltzmann-Einstein) equations we need initial conditions.

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- In fact when considering $\Psi = -\Phi$ we need just one initial condition i.e., for Φ.
- Inflation which explain large scale uniformity of the CMB sky also provides a mechanism to create perturbations in Φ.
- In the very early universe $k\eta << 1$ i.e., modes are outside horizon, these equations become quite simple since we can ignore terms which have k and higher power of k.

Line of sight integral

Problem 7

Show that the Boltzmann equation for photons can be solved as:

$$\Theta(k,\mu,\eta_0) = \int_0^{\eta_0} d\eta \tilde{S}(k,\mu,\eta) e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{90}$$

where

$$\tilde{S} = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right] \tag{91}$$

Hint: write

$$\dot{\Theta} + (ik\mu - \dot{\tau})\Theta = e^{-ik\mu\eta}\frac{d}{d\eta}[\Theta e^{ik\mu\eta-\tau}] \tag{92}$$

Rather than solving equation (90) for $\Theta(k,\mu,\eta_0)$ we solve for each multipole $\Theta_l(k,\eta_0)$ which becomes challenging since modes are coupled. [Seljak & Zaldarriaga (1996)]

Multipole expansion

- Non-relativistic particles like dark matter and baryons can be characterized by their densities $\delta(\vec{x}, t)$ and velocities $\vec{v}(\vec{x}, t)$ (which are equivalent to monopole and dipole).
- In Fourier space the evolution of densities and velocities for non-relativistic species depend on the magnitude of $\vec{k}$.

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- In Fourier space the evolution of densities and velocities for non-relativistic species depend on the magnitude of $\vec{k}$.
- The scalar velocities here are the components parallel to $\vec{k}$; these are the only ones that are cosmologically relevant.
- We need much more information to specify relativistic particle like photons since they have not only a monopole perturbation and a dipole but also a quadrupole, octopole, and higher moments as well.

Multipole expansion

- When there is azimuthal symmetry then we can write:

$$\Theta(k,\eta,\mu) = \sum_l (-i)^l (2l+1)\Theta_l(k,\eta) \tag{93}$$

where

$$\Theta_l(k,\eta) = \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\Theta(k,\eta,\mu) \tag{94}$$

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- Rather than using $\Theta(k,\eta,\mu)$ to specify the CMB anisotropy in Fourier space, we can use the multipole moments $\Theta_l(k,\eta)$ and study their evolution.
- Note that before recombination since photons and baryons were tightly coupled so only monopole $\Theta_0(k,\eta)$ terms were significant.

Inhomogeneities to anisotropies

- If recombination happens instantaneously then the CMB anisotropy $\Theta(\hat{n})$ is related to the inhomogeneity at the last scattering surface:

$$\Theta(\hat{n}) = \int dD \Theta(\mathbf{x}) \delta_D(D - D_*) \tag{95}$$

where D_* is the comoving distance of the recombination.

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$$\Theta(\hat{n}) = \int dD \Theta(\mathbf{x}) \delta_D(D - D_*) \tag{95}$$

where D_* is the comoving distance of the recombination.

- We can expand the inhomogeneity $\Theta(\mathbf{x})$ in Fourier space:

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Inhomogeneities to anisotropies

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- Translational and rotational invariance of $\Theta(\mathbf{x})$ leads:

$$< \tilde{\Theta}(\mathbf{k}) \tilde{\Theta}(\mathbf{k}') >= (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P_T(K) \tag{97}$$

where $P_T(K)$ is the power spectrum.

- We generally expand CMB anisotropy in spherical harmonics:

$$\Theta(\hat{n}) = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\hat{n}) \tag{98}$$

where $Y_{lm}(\hat{n})$ are spherical harmonics basis and follow the orthogonality relations:

$$\int d\hat{n} Y_{lm}(\hat{n}) Y_{l'm'}(\hat{n}) = 2\pi \delta_{ll'} \delta_{mm'} \tag{99}$$

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- In Fourier space:

$$\Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \tilde{\Theta}(\mathbf{k}) e^{i\mathbf{k}.D_* \hat{n}} \tag{100}$$

we can expand plane wave in spherical harmonics:

$$e^{i\mathbf{k}.D_* \hat{n}} = 4\pi \sum_{lm} i^l j_l(kD_*) Y^*_{lm}(\mathbf{k}) Y_{lm}(\hat{n}) \tag{101}$$

Problem 8

- Show that in the multipole expansion CMB multipole a_{lm} and the Fourier amplitude of the inhomogeneity $\tilde{\Theta}(k)$ are related in the following way:

$$a_{lm} = \int \frac{d^3k}{(2\pi)^3} \tilde{\Theta}(\mathbf{k}) 4\pi i^l j_l(kD_*) Y^*_{lm}(\mathbf{k}) \tag{102}$$

- Given that $\Delta^2_T(k) = k^3 P(k)/2\pi^2$ is slowly varying and $\int_0^\infty j_l^2(x) dlnx = 1/(2l(2l+1))$ show that the angular power spectrum C_l can be written in the following form:

$$< a_{lm} a_{l'm'} >= (2\pi)^3 \delta_{ll'} \delta_{mm'} C_l \tag{103}$$

with

$$C_l = \frac{2\pi}{l(l+1)} \Delta^2_T(l/D_*) \tag{104}$$

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