Matched Filtering

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1 Matched Filtering

Matched filter is an optimal linear filter which maximizes the signal to noise ratio (SNR) for the data $d(t)$ which contains signal $s(t)$ and some additive stochastic noise $n(t)$.

$$d(t) = s(t) + n(t) \tag{1}$$

Let us apply a filter $h(t)$ (convolve) on the data :

$$d_h(t) = \int d(\tau)h(t-\tau)d\tau \tag{2}$$

or

$$\mathbf{d}_h = \mathbf{h}^\dagger \mathbf{d} \tag{3}$$

Let us define the noise covariance matrix $C_n =< \mathbf{n}\mathbf{n}^\dagger >$ and SNR as:

$$\text{SNR} = \frac{|s_h|^2}{< |s_n|^2 >} = \frac{|\mathbf{h}^\dagger \mathbf{s}|^2}{< |\mathbf{h}^\dagger \mathbf{n}|^2 >} \tag{4}$$

We can expand the denominator:

$$< |\mathbf{h}^\dagger \mathbf{n}|^2 >= \left\langle (\mathbf{h}^\dagger \mathbf{n})(\mathbf{h}^\dagger \mathbf{n})^\dagger \right\rangle = \mathbf{h}^\dagger C_n \mathbf{h} \tag{5}$$

Now the SNR can be written as:

$$\text{SNR} = \frac{|\mathbf{h}^\dagger \mathbf{s}|^2}{\mathbf{h}^\dagger C_n \mathbf{h}} = \frac{|(C_n^{1/2}\mathbf{h})^\dagger (C_n^{-1/2}\mathbf{s})|}{(C_n^{1/2}\mathbf{h})^\dagger (C_n^{1/2}\mathbf{h})} \tag{6}$$

Now we will use the following inequality, called the Cauchy-Schwartz inequality:

$$\boxed{|\mathbf{a}^\dagger \mathbf{b}|^2 \leq (\mathbf{a}^\dagger \mathbf{a})(\mathbf{b}^\dagger \mathbf{b})} \tag{7}$$

which gives us :

$$|(C_n^{1/2}\mathbf{h})^\dagger (C_n^{-1/2}\mathbf{s})| \leq [(C_n^{1/2}\mathbf{h})^\dagger C_n^{1/2}\mathbf{h}][(C_n^{-1/2}\mathbf{s})^\dagger C_n^{-1/2}\mathbf{s}] \tag{8}$$

or

$$\boxed{\text{SNR} \leq \mathbf{s}^\dagger C_n^{-1} \mathbf{s}} \tag{9}$$

The case for which equality holds, can be written as:

$$C_n^{1/2}\mathbf{h} = \alpha C_n^{-1/2}\mathbf{s} \tag{10}$$

or

$$\mathbf{h} = \alpha \; C_n^{-1}\mathbf{s} \tag{11}$$

where α is a proportionality constant. The constant α can be set by normalizing the filter output unity for a noise only case i.e.,

$$< |n_h|^2 >= 1 = \left\langle (\mathbf{h}^\dagger \mathbf{n})(\mathbf{h}^\dagger \mathbf{n}) \right\rangle \tag{12}$$

substituing $\mathbf{h} = \alpha\ C_n^{-1}\mathbf{s}$ and $C_n =< \mathbf{n}\mathbf{n}^\dagger >$ we get:

$$\alpha^2 \mathbf{s}^\dagger C_n^{-1}\mathbf{s} = 1 \tag{13}$$

or

$$\alpha = \frac{1}{\sqrt{\mathbf{s}^\dagger C_n^{-1}\mathbf{s}}} \tag{14}$$

so

$$\boxed{\mathbf{h} = \frac{1}{\sqrt{\mathbf{s}^\dagger C_n^{-1}\mathbf{s}}} C_n^{-1}\mathbf{s}} \tag{15}$$

2 Convolution

Convolution of two functions $f(t)$ and $g(t)$ is defined in the following way:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t')g(t-t')dt' \tag{16}$$

Some of the important properties of convolution are as follows:

1. Convolution operation is symmetric:
$$f(t) * g(t) = g(t) * f(t) \tag{17}$$

2. Fourier transformation of the convolution of two functions is just product of their Fourier transformations:
$$\mathcal{F}[f(t) * g(t)] = \mathcal{F}[f(t)] * \mathcal{F}[g(t)] \tag{18}$$

3. Convolution also can be considered as a inner product also:
$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t')g^*(t-t')dt' = \langle f(t'), g(t-t')\rangle \tag{19}$$

4. Cauchy-Schwarz Inequality: convolution of two bounded function is also bounded:
$$|f(t) * g(t)| \leq |f(t)||g(t)| \tag{20}$$