Gravitaional Waves

Jayanti Prasad (Visiting Fellow)
Inter-University Centre for Astronomy & Astrophysics
Pune, India

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1 Tidal Field

In a gravitational field tidal tensor $\mathcal{E}_{ij}$ is given by:

$$\mathcal{E}_{ij} = \frac{\partial \phi}{\partial x^i \partial x^j} \tag{1}$$

For a body falling towards the Earth :

$$\mathcal{E}_{ij} = -\frac{GM_E}{r^5}\left[3x^i x^j - \delta_{ij} r^2\right] \tag{2}$$

Power radiated in gravitational waves is given by:

$$\frac{dW}{dt} = -\frac{1}{2}\mathcal{E}_{ij}\frac{dI^{ij}}{dt}, \tag{3}$$

where I^{ij} is called quadrupole tensor:

$$I^{ij} = \int x^i x^j \rho(x) d^3x. \tag{4}$$

Note that the moment of inertia tensor is given by:

$$\mathcal{I}_{ij} = \int (r^2\delta_{ij} - x_i x^j)\rho(x) d^3x. \tag{5}$$

2 Binary System

Suppose that the two masses are m_1 and m_2, and they are separated by a distance r. The power given off (radiated) by this system is:

$$\begin{aligned} P = \frac{dE}{dt} &= -\frac{32}{5}\frac{G^4}{c^5}\frac{(m_1 m_2)^2(m_1 + m2)}{r5} \\ &= -\frac{32}{5}\left(\frac{Gm_1}{c^2 r}\right)^2\left(\frac{Gm_2}{c^2 r}\right)^2 (m_1+m_2)c^2\left(\frac{c}{r}\right)\text{Watt.} \end{aligned} \tag{6}$$

For the Sun-Earth system with the mass of the Suna and earth $m_1 = 2\times 10^{30}, m_2 = 6\times 10^{24}$ Kg and distance $r = 1.5\times 10^{11}$m we get power radiated around 200 W (compare that with the total power radiated by Sun 3.86×10^{26} watts. This emission of gravitational radiation will decay the orbit of earth by 10^{-15} mt per day.

The rate of decrease of distance for a binary system is given by:

$$\begin{aligned} \frac{dr}{dt} &= -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1 m_2(m_1+m_2)}{r3} \\ &= -\frac{64}{5}\left(\frac{Gm_1}{c^2 r}\right)\left(\frac{Gm_2}{c^2 r}\right)\left(\frac{G(m_1+m_2)}{c^2 r}\right)c. \end{aligned} \tag{7}$$

The lifetime of an orbit is given by:

$$\begin{aligned} t &= \frac{5}{256}\frac{c^5}{G^3}\frac{r^4}{m_1 m_2(m_1+m_2)} \\ &= \frac{5}{256}\left(\frac{Gm_1}{c^2 r}\right)^{-1}\left(\frac{Gm_2}{c^2 r}\right)^{-1}\left(\frac{G(m_1+m_2)}{c^2 r}\right)^{-1}\frac{r}{c}. \end{aligned} \quad (8)$$

Gravitational wave strain amplitude for the the plus and cross polarization for a binary system is given by:

$$\begin{aligned} h_+ &= -2\left(\frac{Gm_1}{c^2 R}\right)\left(\frac{Gm_2}{c^2 r}\right)(1+\cos^2\theta)\cos[2\omega(t-R)] \\ h_X &= -4\left(\frac{Gm_1}{c^2 R}\right)\left(\frac{Gm_2}{c^2 r}\right)(\cos\theta)\sin[2\omega(t-R)]. \end{aligned} \quad (9)$$

where the angular velocity is given by:

$$\omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}. \quad (10)$$

For the Earth-Sun system we get:

$$h_+ = -\frac{1}{R}1.7\times 10^{-10} mt. \quad (11)$$

For $R = 1$ lyr we get $h_+ \approx 10^{-26}$.

3 Gravitational waves in Tensor notation

Let us consider a perturbed Minkowski metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \approx (\eta_{\mu,\nu} + h_{\mu,nu})dx^\mu dx^\nu. \quad (12)$$

Here $\eta_{\mu,\nu}$ is the usual Minkowski metric and $h_{\mu,nu}$ represents the linearlized gravitational field. Upon lineralization, the coordinate invariance of full general relativity is replaced by global Lorentz invariance and local gauge invariance under infinitesimal coordinate transformaton $x^\mu \longrightarrow x^\mu + \xi^\mu$

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \quad (13)$$

In Lorentz Gauge Einstein Field equations reduce to:

$$\left(\frac{\partial 2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = 16 G_N T_{\mu\nu}. \quad (14)$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \quad (15)$$

In transverse-traceless gauge gravitational wave propagateing along z-direction can be written as:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

The lowest order contribution to gravitational waves comes from the quadruple distribution of matter:

$$h_{ij}(t,\vec{x}) = \frac{2G}{c^4}\frac{\ddot{Q}_{ij}}{r} \quad (17)$$

with

$$Q_{ij} = \int d^3x \left(x_i x_j - \frac{2}{3}x^2\delta_{Ij}\right)\rho(t,\vec{x}), \quad (18)$$

or

$$Q_{ij} = M\mathcal{R}_{ij} \quad (19)$$

or

$$h_{ij}(t,\vec{x}) = \left(\frac{2GM}{c^2 r}\right)\frac{\ddot{R}_{ij}}{c^2}. \tag{20}$$

Note that in above equations we should actually use retarded time $t - r/c$ in place of t.

Newtonian potential is given by:

$$\phi(x,y,z) = \frac{GM}{(x^2+y^2+z^2)^{1/2}}, \tag{21}$$

and from which we can get tidal tensor:

$$\mathcal{E}_{ij} = -\frac{GM}{r^5}(3x_i x_j - \delta_{ij} r^2). \tag{22}$$

4 Quadrouple formula

We can get wave equation with:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| << 1. \tag{23}$$

and so we get :

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = -\frac{8\pi G}{c^2}T_{\mu\nu}. \tag{24}$$