Topics in Physics

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Contents

Chapter 1

Lagrangian Mechanics

1.1 Lagrange Equation

Let us assume that we have a system of N particles and its configuration can be described by $3N$ coordinates $\mathbf{r}_i$, with $i = 1, 2, ..., N$ and $\mathbf{r}$ is a three dimensional vector. In Newtonian formalism the motion of particles is described by the following set of equations:

$$m_i \ddot{\mathbf{r}}_i = F_i^{(e)} + \sum_j F_{ij}, \tag{1.1}$$

where $F_i^{(e)}$ is the external force acting on i^{th} particle and F_{ij} is the force acting on it due to j^{th} particle.

Although Newtonian formalism looks straightforward but it has many disadvantages like (1) we have to consider all the forces acting including that of due to constraints which are actually a part of the solution and not known a priory (2) we need to deal with vector quantities and that can be inconvenient in many situations (3) Newton's equations of motion are not the same in all coordinate systems check those for the motion of an object in a plane in Cartesian and polar coordinates.

In order to overcome these shortcomings Lagrangian formalism is commonly used for mechanics of a system of particles and it proceeds in the following way.

Let there are k number of constraints (holonomic) given by k equations of the type:

$$f_i(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, t) = 0, \text{with} \quad i = 1, 2, ..., k. \tag{1.2}$$

Using these constraints we can eliminate some of the freedom and can define a set of "generalized coordinates" q_i to describe the configuration of the system. Note that these generalized coordinates need not to have the usual dimensions. For a system of k constraints equations and N particles we have $3N - k$ generalized coordinates. In terms of generalized coordinates we can write :

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_\nu, t)$$
$$d\mathbf{r} = \sum_{\nu=1}^{3N-k} \frac{\partial \mathbf{r}_i}{\partial q_\nu} dq_\nu,$$
$$\dot{\mathbf{r}}_i = \sum_{\nu=1}^{3N-kk} \frac{\partial \mathbf{r}_i}{\partial q_\nu} \dot{q}_\nu + \frac{\partial \mathbf{r}_i}{\partial t}. \tag{1.3}$$

In order to get the Lagrange equation of motion we use a principle called d'Alembert's principle which states that the "virtual work" [1] done by external forces on a system in equilibrium is zero.

$$\sum_i F_i^{(e)}.\delta\mathbf{r}_i = 0. \tag{1.4}$$

This equation directly follows from the equation :

$$F_i = F_i^{(e)} + f_i, \tag{1.5}$$

where the second terms is due to constraints. If we write:

$$\sum_i F_i.\delta\mathbf{r}_i = \sum_i F_i^{(e)}.\delta\mathbf{r}_i + \sum_i f_i.\delta\mathbf{r}_i = 0. \tag{1.6}$$

The second term on RHS is zero due to the fact that constraint forces and displacement are perpendicular to each other so the D'Alembert principle is proved. It is more useful to write :

$$\sum_i (F_i^{(e)} - \dot{\mathbf{p}}_i).\delta\mathbf{r}_i \tag{1.7}$$

or

$$\sum_i m_i \ddot{\mathbf{r}}_i.\delta\mathbf{r}_i = \sum_i \dot{\mathbf{p}}_i.\delta\mathbf{r}_i \tag{1.8}$$

We can simplify LHS in the following way:

$$\sum_i m_i \ddot{\mathbf{r}}_i \delta\mathbf{r}_i = \sum_i \sum_\nu m_i \ddot{\mathbf{r}}_i.\frac{\partial \mathbf{r}_i}{\partial q_\nu}\delta q_\nu \tag{1.9}$$

In the RHS of the above equation we exchange the summation and try to simplify the following:

$$\sum_i m_i \ddot{\mathbf{r}}_i.\frac{\partial \mathbf{r}_i}{\partial q_\nu} = \sum_i \frac{d}{dt}(m_i \dot{\mathbf{r}}_i).\frac{\partial \mathbf{r}_i}{\partial q_\nu}$$
$$= \sum_i \left[\frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i.\frac{\partial \mathbf{r}_i}{\partial q_\nu} \right) - m_i \dot{\mathbf{r}}_i \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_\nu} \right) \right] \tag{1.10}$$

[1] Virtual work corresponds to virtual displacement which occurs at a fixed time t, consistent with the external and constraint forces, rather than in a time interval dt as is the case for real displacement.

Now we use the following:

$$\frac{\partial \mathbf{r}_i}{\partial q_\nu} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_\nu}, \tag{1.11}$$

which directly comes from equation (1.3) and we also use

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_\nu}\right) = \frac{\partial}{\partial q_\nu}\left(\frac{d\mathbf{r}_i}{dt}\right) = \frac{\partial \mathbf{v}_i}{\partial q_\nu}. \tag{1.12}$$

With these simplifications we get:

$$\sum_i m_i \ddot{\mathbf{r}}_i . \frac{\partial \mathbf{r}_i}{\partial q_\nu} = \sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i . \frac{\partial \mathbf{v}_i}{\partial \dot{q}_\nu}\right) - m_i \mathbf{v}_i . \frac{\partial \mathbf{v}_i}{\partial q_\nu}\right] \tag{1.13}$$

or

$$\sum_i m_i \ddot{\mathbf{r}}_i . \frac{\partial \mathbf{r}_i}{\partial q_\nu} = \sum_i \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_\nu}\left(\frac{1}{2} m_i \mathbf{v}_i^2\right)\right\} - \frac{\partial}{\partial q_\nu}\left(\frac{1}{2} m_i \mathbf{v}_i^2\right)\right] \tag{1.14}$$

Since the virtual work is given by :

$$\sum_i F_i . \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i . \delta \mathbf{r}_i, \tag{1.15}$$

we can write :

$$\sum_i F_i . \delta \mathbf{r}_i = \sum_\nu \sum_i F_i . \frac{\partial \mathbf{r}_i}{\partial q_\nu} \delta q_\nu = \sum_\mu F_\mu . \delta_\mu, \tag{1.16}$$

where the generalized force is defined as:

$$F_\nu = \sum_\nu F_i . \frac{\partial \mathbf{r}_i}{\partial q_\nu}, \tag{1.17}$$

and using this equation (1.14) can be written as:

$$\sum_\nu \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right) - \frac{\partial T}{\partial q_\nu} - Q_\nu\right] \delta q_\nu = 0 \tag{1.18}$$

Since generalized coordinates are linearly independent so we can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right) - \frac{\partial T}{\partial q_\nu} - Q_\nu = 0 \tag{1.19}$$

If we assume that the force can be derived from some potential V then :

$$F_i = -\nabla_i(V), \tag{1.20}$$

and so the generalized force can be written as:

$$Q_\nu = -\sum_i F_i \frac{\partial \mathbf{r}_i}{\partial q_\nu} = -\sum_i \nabla_l(V) \frac{\partial \mathbf{r}_i}{\partial q_\nu} = -\frac{\partial V}{\partial q_\nu}. \tag{1.21}$$

Note that we have used the following simplification above :

$$\left(\frac{\partial V}{\partial x_i}\hat{i} + \frac{\partial V}{\partial y_i}\hat{j} + \frac{\partial V}{\partial z_i}\hat{k}\right) \cdot \left(\frac{\partial x_i}{\partial q_\nu}\hat{i} + \frac{\partial y_i}{\partial q_\nu}\hat{j} + \frac{\partial z_i}{\partial q_\nu}\hat{k}\right)$$
$$= \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_\nu} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_\nu} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_\nu} = \frac{\partial V}{\partial q_\nu} \tag{1.22}$$

Using this equation (1.19 can be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right) - \frac{\partial (T-V)}{\partial q_\nu} = 0, \tag{1.23}$$

since V is independent from velocity so the above equation can be written as:

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_\nu}\right) - \frac{\partial (T-V)}{\partial q_\nu} = 0 \tag{1.24}$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\nu}\right) - \frac{\partial L}{\partial q_\nu} = 0} \tag{1.25}$$

Here we define Lagrangian function $L = T - V$ and the above equation is called Lagrange Equations. In general, the Lagrangian function L is a function of generalized positions q_ν, generalized velocities $\dot{q}_\nu$ and time i.e., $L(q_\nu, \dot{q}_\nu, t)$.

1. For a single particle moving uniformly along x direction we have :

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{1.26}$$

so the Lagrange equation can be written as:

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial V(x)}{\partial x} = 0, \tag{1.27}$$

or

$$m\ddot{x} = -\frac{\partial V(x)}{\partial x}, \tag{1.28}$$

which is nothing but the Newton's equation of motion.

2. Note that we can always add a total time derivative in the Lagrangian without affecting the equation of motion.

$$L'(q_\nu, \dot{q}_\nu, t) = L(q_\nu, \dot{q}_\nu, t) + \frac{dF}{dt}. \tag{1.29}$$

3. Lagrangian for a charged particle in Electromagnetic field $(\phi, \mathbf{A})$ is given by:

$$L = \frac{1}{2}m\dot{r}^2 - q\phi + q\mathbf{A}.\dot{\mathbf{r}}, \tag{1.30}$$

for which we get the following equation of motion:

$$m\ddot{\mathbf{r}} = -q[E + \dot{\mathbf{r}} \times \mathbf{B}], \tag{1.31}$$

where

$$E = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{1.32}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.33}$$

4. One of the main motivations to use Lagrangian formalism is to eliminate the constraint forces from the equation of motion.

5. Another advantage of Lagrangian formalism is that we have to deal with only scalar functions like T and V and not vectors like positions, velocities and accelerations of particles.

6. Central force problem For a particle in moving in a place under a Central force given by potential $V(r)$ the Lagrangian is given by:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{1.34}$$

from which we compute :

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \tag{1.35}$$

and so the Lagrange equations are :

$$\begin{aligned} m\ddot{r} &= -\frac{\partial V}{\partial r} \\ mr^2\ddot{\theta} &= 0 \end{aligned} \tag{1.36}$$

By solving the above equations we can find the orbit of the particle.

1.2 Hamiltonian

If Lagrangian does not depend on time explicitly then :

$$\frac{\partial L}{\partial t} = 0 \tag{1.37}$$

but that does not mean that:

$$\frac{dL}{dt} = 0, \tag{1.38}$$

since L may still depend on t by $q(t)$ and $\dot{q}(t)$ so we find another constant of motion (invariant) called Hamiltonian in the following way:

$$\frac{dL}{dt} = \sum_\nu \left[\frac{\partial L}{\partial \dot{q}_\nu} \dot{q}_\nu \ddot{q}_\nu + \frac{\partial L}{\partial q_\nu} \dot{q}_\nu \dot{q}_\nu \right] - \frac{\partial L}{\partial t} \tag{1.39}$$

or

$$\frac{dL}{dt} = \sum_\nu \left[\frac{\partial L}{\partial \dot{q}_\nu} \dot{q}_\nu \ddot{q}_\nu + \dot{q}_\nu \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\nu} \right) \right] = \frac{d}{dt} \sum_\nu \frac{\partial L}{\partial \dot{q}_\nu} \dot{q}_\nu \tag{1.40}$$

or

$$\frac{d}{dt} \left(\sum_\nu \frac{\partial L}{\partial \dot{q}_\nu} \dot{q}_\nu - L \right) = 0 = \frac{d}{dt} H(q, p, t), \tag{1.41}$$

where the Hamiltonian H is defined as:

$$H(q, p, t) = p\dot{q} - L, \tag{1.42}$$

with

$$p_\nu = \frac{\partial L}{\partial \dot{q}_\nu} \tag{1.43}$$

1.3 Lagrange Equation from variational principle

In the last section we derived Lagrange equation from the d'Alembert's principle by the method of virtual displacement, which consider the configuration of the system only at some fixed time. However, it is possible to derive the Lagrange equation considering the motion of the path along the complete path by a principle called the variational principle. This can be states in the form of Hamilton's principle:

> The motion of a system from time t_1 to time t_2 is such that the line integral (action) of the Lagrangian has the stationary value for the actual motion. Or if
>
> $$I = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt, \tag{1.44}$$
>
> then $\delta I = 0$.

Here stationary means the value of the integral is the same for the neighboring paths (within first order infinitesimal). Note that Hamilton's principle describe the motion of those mechanical system for which the force can be derived from a scalar potential that may be a function of position, velocity and time. In order to use this principle to derive Lagrange equation we need to use functional differentiation which is done in the following way. Let $y(x)$ is a path and different paths are parameterized by some parameter α and for a particular path we compute :

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x) dx, \tag{1.45}$$

with $\dot{y} = dy/dx$. Now we compute :

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} \right) dx \tag{1.46}$$

The second path in RHS :

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha} dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} dx \tag{1.47}$$

and so

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) \right] \frac{\partial y}{\partial \alpha} dx \tag{1.48}$$

For stationary case we have :

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) \right] \frac{\partial y}{\partial \alpha}|_0 dx = 0 \tag{1.49}$$

Using the fundamental lemma of the calculus of variation we get :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \tag{1.50}$$

From this the Lagrange equation directly follows.

Chapter 2

Two Body Central force problem

Here our aim will be to find the equation for the orbit of a body revolving around another body under the influence of gravitational attraction. This problem is generally called the Kepler problem and has great importance in astronomy and mechanics in general. In the first part we will find the equation of motion under the general central force and and in the second part will explicitly use the form of gravitational potential. In polar coordinates the Lagrangian of the system can be written as:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \tag{2.1}$$

from this we can write down the Lagrange equations of θ and r in the following way:

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0, \tag{2.2}$$

or

$$mr^2\dot{\theta} = l, \quad \text{or} \quad \frac{d\theta}{dt} = \frac{l}{mr^2}, \tag{2.3}$$

where l is a constant of motion which we identify with the angular momentum. From equation (2.2) we can also write that :

$$\frac{d}{dt}(\frac{1}{2}r^2 d\theta) = \frac{dA}{dt} = l, \tag{2.4}$$

where A is the area. This means that in our case areal velocity is constant which is one of the postulates of Kepler's laws. Now we can write the Lagrange equation for the radial vector r:

$$\frac{d}{dt}(m\ddot{r}) = mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r}, \tag{2.5}$$

substituting the value of $\dot{\theta}$ from equation (2.3) we get :

$$m\ddot{r} + \frac{\partial}{\partial r}\left(\frac{l^2}{2mr^2} + V(r)\right) = 0 \tag{2.6}$$

multiplying the above equation by $\dot{r}$ we can write :

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)\right] = 0, \tag{2.7}$$

and after integration we get :

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = E, \tag{2.8}$$

where E is the constant of integration and can be identified with the total energy of the system (another constant of motion). We can also write:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}}\left[(E-V) - \frac{l^2}{2mr^2}\right]^{1/2}. \tag{2.9}$$

We can eliminate t form equation (2.3) and (2.9) and get:

$$d\theta = \frac{dr/r^2}{\sqrt{\left[\frac{2mE}{l^2} + \frac{2mk}{l^2}\frac{1}{r} - \frac{1}{r^2}\right]}} \tag{2.10}$$

Now we can make a change of variable $u = 1/r$ and $du = -dr/r^2$ so with some calculation we get

$$d\theta = -\frac{du}{q^2 - x^2}, \tag{2.11}$$

with

$$q^2 = \left(\frac{mk}{l^2}\right)^2, \tag{2.12}$$

and

$$x = u - \frac{mk}{l^2}. \tag{2.13}$$

We can integrate equation (2.11) in standard way and get :

$$\theta = \theta_0 + \cos^{-1}\left[\frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk}}}\right] \tag{2.14}$$

which directly gives :

$$u = \frac{mk}{l^2}\left[1 + e\cos(\theta - \theta_0)\right], \tag{2.15}$$

with

$$e = \sqrt{1 + \frac{2El^2}{mk}}. \tag{2.16}$$

Substituting back the value of r we get :

$$\boxed{\frac{1}{r} = \frac{mk}{l^2}[1 + e\cos(\theta - \theta_0)]} \tag{2.17}$$

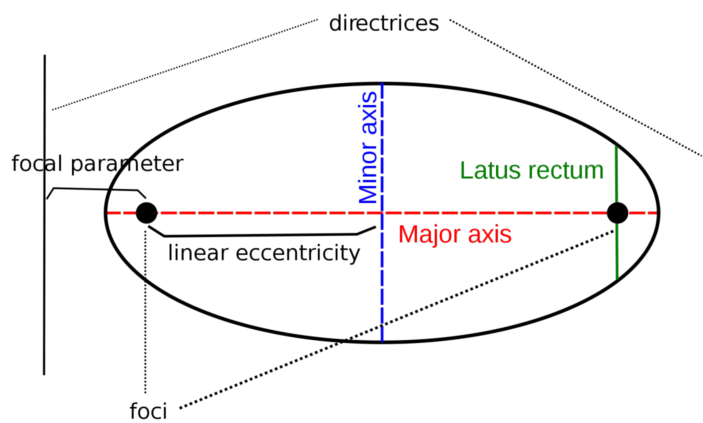

Figure 2.1: Parameters of an ellipse

This is the expression for the Keplerian orbit and the shape of actual orbit depends on the constants E and e. The most interesting case is $E < 0$ and $e < 1$ which describes an elliptic orbit. We can also write equation (2.17) in the following form:

$$\frac{p}{r} = 1 + \cos(\theta - \theta_0), \quad \text{with} \quad p = \frac{l^2}{mk}. \tag{2.18}$$

In the above equation $2p$ is called latus rectum and from the geometry of ellipse the semi major a and minor axis b can be written in terms of that.

$$a = \frac{p}{1-e^2} = \frac{k}{2|E|}, \quad \text{and} \quad b = \frac{p}{\sqrt{1-e^2}} = \frac{l}{\sqrt{2m|E|}}. \tag{2.19}$$

Note that for an ellipse the eccentricity is related to a and b in the following way:

$$e = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2.20}$$

Some of the important points to note are as follows:

1. There are two points at distance r_{min} and r_{max} at which the radial velocity of the particle is zero and those can be found by solving :

$$E - \frac{l^2}{2mr^2} + \frac{k}{r} = 0. \tag{2.21}$$

2. If the angular separation of the the points at which radial velocity is zero is integer multiple of π we get close orbits otherwise we do not.

$$\Delta\theta = \int_{r_{\min}}^{r_{max}} \frac{dr/r^2}{\sqrt{\left[\frac{2mE}{l^2} + \frac{2mk}{l^2}\frac{1}{r} - \frac{1}{r^2}\right]}}, \tag{2.22}$$

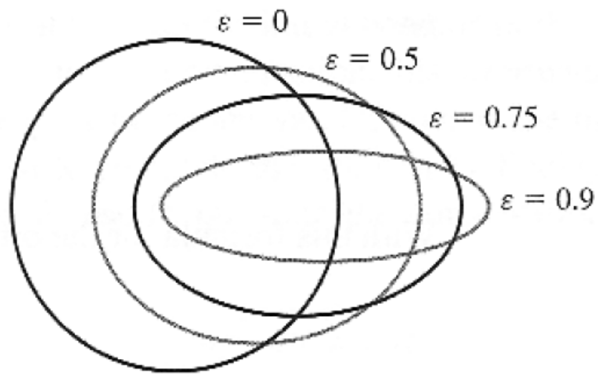

Figure 2.2: Eccentricity of an ellipse

with $\Delta\theta = n\pi$, where $n = 1, 2, ...,$. It can be shown that for potential $V(r) = kr^{\alpha}$, with $\alpha = -1, 2$ we get close orbits.

3. We can compute the orbital period with :

$$\frac{dA}{dt} = l, \tag{2.23}$$

or

$$\pi ab = l\tau \tag{2.24}$$

Chapter 3

Maxwell equations

Maxwell equations deal with physical quantities like electric and magnetic field intensities (in short electric E and magnetic fields B) which change with space and this demands the following mathematical tools to be introduced.

1. Gradient ∇T of a scalar function (field) $T(x, y, z)$ is defined in terms of the change in its value for an infinitesimal displacement $\vec{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$.

   $$dT = \nabla T.\vec{dl} = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).(dx\hat{x}+dy\hat{y}+dz\hat{z}) = \frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}+\frac{\partial T}{\partial z}. \tag{3.1}$$

   From the above equation it is clear that the maximum change occurs when $\vec{dl}$ and ∇T are parallel i.e., along the direction of ∇T. Although ∇, sometime which is written as $\vec{\nabla}$ also, does not have any independent existence (always need a function to act upon) still it is treated like a normal vector.

2. Divergence $\nabla.\vec{A}$ of a vector function $\vec{A}(x, y, z)$ is defined as:

   $$\nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{3.2}$$

   Unlike the gradient ∇T of a scalar quantity T (which is a vector quantity) divergence $\nabla.\vec{A}$ of a vector quantity $\vec{A}$ is a scalar quantity.

3. Curl $\nabla \times \vec{A}$ of a vector quantity $\vec{A}$ is defined by the cross product of ∇ with the vector $\vec{A}$.

4. We can also compute the following second order derivatives. Since ∇T is a vector quantity so we can compute $\nabla.(\nabla T)$ and $\nabla \times \nabla T$.

   $$\begin{aligned} \nabla.(\nabla T) &= \nabla^2 T \\ \nabla \times (\nabla T) &= 0, \end{aligned} \tag{3.3}$$

Note that ∇^2 is called laplacian and is defined as:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}. \tag{3.4}$$

Since $\nabla.\vec{A}$ is a scalar quantity so we can only take its gradient only. $\nabla \times \vec{A}$ is a vector quantity so we can take its divergence and curl:

$$\begin{aligned} \nabla.(\nabla \times A) &= 0 \\ \nabla \times (\nabla \times A) &= \nabla(\nabla.A) - \nabla^2 A. \end{aligned} \tag{3.5}$$

5. Line and surface integrals of a vector quantity $\vec{A}$ are defined as:

$$\int \vec{A}.\vec{dl} \quad \text{and} \quad \int \vec{A}.\vec{dS}. \tag{3.6}$$

Where suitable limits of integration should be used. In many cases we take integration over closed loops and surfaces so we write:

$$\oint \vec{A}.\vec{dl} \quad \text{and} \quad \oint \vec{A}.\vec{dS}. \tag{3.7}$$

Volume integral of a scalar quantity T is defined as

$$\oint T dV. \tag{3.8}$$

6. Two important theorems which will be used in Maxwell equations are as follows:

$$\begin{aligned} \oint \vec{A}.\vec{dS} &= \oint \nabla.\vec{A} dV \\ \oint \vec{A}.\vec{dl} &= \oint (\nabla \times \vec{A}).\vec{dS}. \end{aligned} \tag{3.9}$$

7. Some of the useful results are :

$$\nabla \left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} \quad \text{with} \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}. \tag{3.10}$$

and

$$\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = -4\pi\delta(\vec{r} - \vec{r}'). \tag{3.11}$$

and

$$\nabla. \left(\frac{\hat{r}}{r^2}\right) = -4\pi\delta^3(\vec{r}). \tag{3.12}$$

8. One interesting theorem says that any vector field always can be written as a sum of a gradient of scalar function and curl of a vector function:

$$\vec{F} = \nabla V + \nabla \times \vec{A} \tag{3.13}$$

3.1 Coulomb's law of electrostatics

Coulomb's law states that the electrostatic force acting between two electric charges Q and q is proportional to the product of the charges and inversely proportional to the distance between those:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}. \tag{3.14}$$

The force is repulsive for like charges and attractive for unlike charges and is always along the line joining the charges. In the above expression ϵ_0 is called the permitivity of vacuum and the value of the proportionality constant $1/4\pi\epsilon_0$ is $9 \times 10^{-9} Ntmt^2/C^2$. In many cases we deal with electric field $\vec{E}(\vec{r})$ which is defined in terms of the force acting on test charge of 1 C at point $\vec{r}$ due to charge Q.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}. \tag{3.15}$$

Note that the direction of $\vec{E}$ is always away from a test (+ve) charge. In practical life we hardly deal with an isolated charge and deal with a distribution of charges which may be discrete or continuous. For the case of discrete distribution the net $\vec{E}$ can be given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} Q_i \frac{\vec{r}_i - \vec{r}}{|\vec{r}_i - \vec{r}|^3}. \tag{3.16}$$

For the case of continuous charge distribution we consider $\vec{E}$ due to an infinitesimal charge dq at some point $\hat{r}$ and then integrate over relevant region. Note that charge distribution can be linear, planar or spatial for which we get :

$$dq = \lambda dl = \sigma dS = \rho dV, \tag{3.17}$$

where λ, σ and ρ are respective charge densities. Let us consider a charge inside a closed surface as shown in figure (3.3) and compute:

$$\oint \vec{E}.\vec{dS} = \frac{1}{4\pi\epsilon_0}\oint \frac{q}{r^2}\hat{r}.\vec{dS}. \tag{3.18}$$

Now substituting $\vec{dS} = r^2 d\Omega\hat{r}$ we get:

$$\oint \vec{E}.\vec{dS} = \frac{q}{4\pi\epsilon_0}\oint d\Omega = \frac{q}{\epsilon_0} \tag{3.19}$$

This is called Gauss' law of electrostatics and says that the total "electric flux" crossing any closed are is proportional to total charge inside that area. There is a differential form also of this law which can found by using one of theorems we have discussed earlier:

$$\oint \vec{E}.\vec{dS} = \oint (\nabla.\vec{E}) dV = \frac{1}{\epsilon_0}\int \rho dV, \tag{3.20}$$

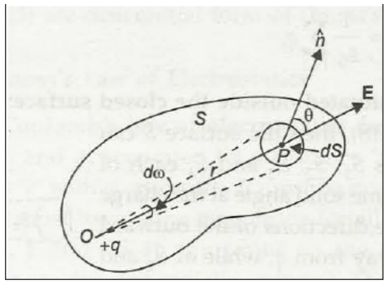

Figure 3.1: Total electric flux passing through an enclosed surface depend on the total electric charge inside that.

or

$$\boxed{\nabla.\vec{E} = \frac{\rho}{\epsilon_0}.} \tag{3.21}$$

Using

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}, \tag{3.22}$$

we can also write :

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} Q \nabla\left(\frac{1}{r}\right) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\right) = -\nabla V(\vec{r}), \tag{3.23}$$

where

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{3.24}$$

is a scalar quantity and called electrostatic potential and in terms of equation (3.21) can be written as:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \tag{3.25}$$

which is called the Poisson's equation.

3.2 Ampere's law of magnetostatics

This law is based on some of these observations:

- It was observed that two parallel current (steady) carrying conductors attract each other if the direction of the current is the same and repel if it is different.

- When a charge particle moves in a magnetic (static) filed a force acts on it which is given by:

$$\vec{F} = Q(\vec{v} \times \vec{B}), \tag{3.26}$$

  where the symbols are used in their obvious meaning.

These and many other observations have supported the idea that a current carrying conductor produces a static magnetic filed around it and Biot-Savart gave a law (like Coulomb's law) for computing the intensity of that. According to the Biot-Savort law magnetic field $\vec{B}$ produced by conductor of length $\vec{dl}$ carrying current I is given by:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\vec{dl} \times \hat{r}}{r^2}. \tag{3.27}$$

where $\frac{\mu_0}{4\pi}$ is a constant and have value $10^{-7}Nt/Amp^2$ and μ_0 is called permeability of vacuum. We can write the above equation in the following form also:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times \hat{r}}{r^2}dV = \nabla \times \vec{A} \tag{3.28}$$

with

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r'}|}dV'. \tag{3.29}$$

It can be shown that since B can be written as a curl of some quantity so:

$$\boxed{\nabla.\vec{B} = 0.} \tag{3.30}$$

From the Biot-Savort law we can find the magnetic field at distance r from a current (steady) carrying conductor of infinite length:

$$B = \frac{\mu_0 I}{2\pi r}, \tag{3.31}$$

which is directed anti clock wise if the direction of the current is upwards. Since now we know the magnetic field due to infinitely long conductor carrying current we can draw a closed circle around it and try to find the following line integral:

$$\oint \vec{B}.\vec{dl}. \tag{3.32}$$

Here $\vec{dl}$ is along the circumference of the circle. If we define current density $\vec{J}$ as:

$$I = \int \vec{J}.\vec{dS}, \tag{3.33}$$

then we can write :

$$\oint \vec{B}.\vec{dl} = \oint (\nabla \times \vec{B}).\vec{dS} = \mu_0 \int \vec{J}.\vec{dS} \tag{3.34}$$

or

$$\nabla \times \vec{B} = \mu_0 \vec{J}. \tag{3.35}$$

This equation is called Ampere's law and have some issue which was fixed by Maxwell.

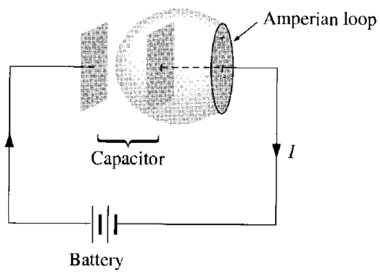

Figure 3.2: Magnetic field is produced not only by steady current, it is also produced by changing electric field.

In terms of vector potential we can write :

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} \tag{3.36}$$

or

$$\nabla(\nabla.\vec{A}) - \nabla^2 A = \mu_0 \vec{J} \tag{3.37}$$

or

$$\boxed{\nabla^2 A = -\mu_0 \vec{J}} \tag{3.38}$$

Equation (3.35) says that we draw a closed loop then the magnetic field at any point of it depends on the current density of the area the loop makes. In figure (3.2) we can draw two different surfaces for the same loop and one which (oval type) has zero current density ! This means that we will have two different values of B at any point of the loop depending on the choice of the surface. In order to fix this problem we need to use the following equation, called the continuity equation:

$$\nabla.\vec{J} = \frac{\partial \rho}{\partial t}, \tag{3.39}$$

which says that the loss inside close surface due to the flow of charge must be compensated with the production of more charge. Using :

$$\nabla.\vec{E} = \frac{\rho}{\epsilon_0} \tag{3.40}$$

we get :

$$\vec{J} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{3.41}$$

This current density which corresponds to time varying electric field is called "displacement current" density and the corrected Ampere's law should be written as:

$$\nabla \times \vec{B} = \mu_0(\vec{J} + \vec{J}_d) \tag{3.42}$$

or

$$\boxed{\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right).} \tag{3.43}$$

The problem can be understood in the following way also. We know

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{3.44}$$

so

$$\nabla.(\nabla \times \vec{B}) = \mu_0 \nabla.\vec{J} \tag{3.45}$$

must be zero and that means $\nabla.\vec{J} = 0$ which is not true since from continuity equation we know:

$$\nabla.\vec{J} = -\frac{\partial \rho}{\partial t} = \nabla.\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \tag{3.46}$$

If we add the extra term then we can get $\nabla.(\nabla \times \vec{B}) = 0$ without any contradiction.

3.3 Faraday's law of electromagnetic induction

From the last section we have seen that magnetism can be produced by electricity and Bio-Savart/Ampere's law give the expressions for the magnetic field produced by electric current. Maxwell's correction to the Ampere's law shows that magnetic field also can be produced by changing (time varying) electric field, as happens when a capacitor is charges or discharged. In this section we will discuss whether electricity also can be produced by magnetism.

Michael Faraday in 1831 made one very important observation. He noticed that current is produced in a loop of conducting wire when a magnet moves with respect to it along the direction perpendicular to the plane of the loop. He found that :

- The effect of moving the magnet keeping the loop stationary or moving the loop and keeping the magnet stationary is the same.
- Faster the motion greater the current. When the motion stops current also stops.
- The direction of the current is different when the loop and magnetic come closer than when they move away from each other.

The current produced in the loop due to relation motion of the magnet is closed "induced current" and corresponding emf is close induced emf.

The direction of the induced current is given by a law called the **Lenz's law** which says that the direction of the induced current is such that it can opposes the caused which is producing it. Basically this means that in the process of induction the magnetic flux passing through the loop remains constant.

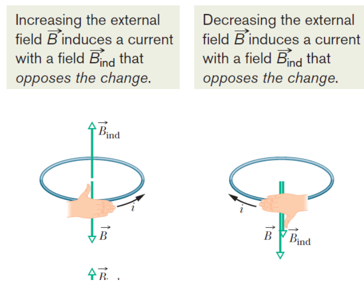

Figure 3.3: Direction of the induced current is such that it can oppose the applied magnetic field.

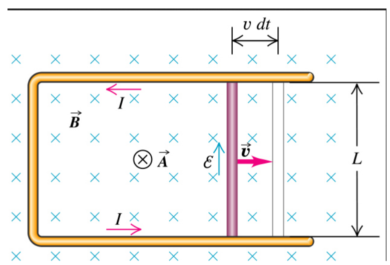

Figure 3.4: We can derive the Faraday's law by the law of conservation of energy. Lorentz force acting on moving charged particles produces the induced current.

When the magnetic flux is increasing then induced current will be produced in such a way that it can produce a magnetic field which has opposite direction then the applied field. For example when north pole of magnet is brought close to a loop then anti-clock wise current is produced. Two of the examples with increasing and decreasing applied magnetic filed are shown in figure (**??**).

Note that induced current/emf depends in the change in magnetic flux which may happen due to change in the area or magnetic field. For example when a loop moves in stationary magnetic field which confined to space than current is induced in the loop.

In order to compute the induced emf Faraday's laws of induction is used which says that the induced emf is proportional to the rate of change of the magnetic flux passing through a closed loop/circuit.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{3.47}$$

or

$$\oint \vec{E}.\vec{dl} = \frac{d}{dt}\oint \mathbf{B}.\vec{dS} \tag{3.48}$$

or

$$\oint (\nabla \times \vec{E}).d\vec{S} = \frac{d}{dt}\oint \mathbf{B}.d\vec{S} \tag{3.49}$$

or

$$\boxed{\nabla \times E = -\frac{dB}{dt}.} \tag{3.50}$$

Now we can write down the full set of Maxwell's equation:

$$\nabla.\vec{E} = \frac{\rho}{\epsilon} \quad \text{(Gauss' law of electrostatics)} \tag{3.51}$$

$$\nabla.\vec{B} = 0 \quad \text{(No name)} \tag{3.52}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt} \right) \quad \text{(Ampere's law)} \tag{3.53}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \quad \text{(Faraday's law).} \tag{3.54}$$

These are the Maxwell's equation in vacuum, however, in electric/magnetic media the equations can be written in the following way. In a dielectric medium electric polarization is induced which is quantified in terms of the polarization vector $\vec{P}$ and depend on the applied field $\vec{E}$.

$$\vec{P} = \chi \epsilon_0 \vec{E}, \tag{3.55}$$

and in magnetic medium magnetization is induced which is represented by the magnetization vector $\vec{M}$

$$\vec{M} = \chi_m \mu_0 \vec{H}. \tag{3.56}$$

In medium Maxwell's equations are written as:

$$\nabla.\vec{D} = 0 \tag{3.57}$$

$$\nabla.\vec{B} = 0 \tag{3.58}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \tag{3.59}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{3.60}$$

with

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = (1+\chi)\epsilon_0 \vec{E} = \epsilon \vec{E} \tag{3.61}$$

$$\vec{B} = \mu_0 \vec{H} + \vec{M} = (1+\chi_m)\mu_0 \vec{H} = \mu \vec{H}. \tag{3.62}$$

Note that $\vec{D}, \vec{B}, \vec{E}$ and $\vec{H}$ satisfy the following boundary conditions:

$$(\vec{D}_2 - \vec{D}_1).\hat{n} = \sigma \tag{3.63}$$

$$(\vec{B}_2 - \vec{B}_1).\hat{n} = 0 \tag{3.64}$$

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \tag{3.65}$$

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{j} \tag{3.66}$$

This means that normal component of $\vec{E}$ and tangential components of $\vec{H}$ are discontinuous and these discontinuities are equal to the free charge and current densities on the surface.

We can understand the meaning of $\vec{D}$ and $\vec{H}$ in the following way. When external electric field is applied on any dielectric material then polarization is developed in it. The polarization corresponds to some "bound charge" density ρ_b :

$$\rho_b = -\nabla.\vec{P} \tag{3.67}$$

and so the Gauass' law reads as:

$$\nabla.\vec{E} = \frac{1}{\epsilon_0}(\rho_f + \rho_f) \tag{3.68}$$

or

$$\nabla.(\vec{P} + \epsilon_0 \vec{E}) = \rho_f \tag{3.69}$$

or

$$\nabla.\vec{D} = \rho_f \tag{3.70}$$

where $\vec{D} = \vec{P} + \epsilon_0 \vec{E}$.

The same works for magnetic field also where "bound current" density is associated with the Magnetic vector $\vec{M}$ as follows:

$$\vec{J}_B = \nabla \times \vec{M} \tag{3.71}$$

and

$$\nabla \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_B) \tag{3.72}$$

or

$$\nabla \times (\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J}_f \tag{3.73}$$

or

$$\nabla \times \vec{H} = \vec{J}_f \tag{3.74}$$

where

$$\vec{B} = \mu_0 \vec{H} - \vec{M}. \tag{3.75}$$

Some extra equations are as follows:

$$\begin{aligned}
\vec{E} &= -\nabla V & (3.76)\\
\vec{B} &= \nabla \times \vec{A} & (3.77)\\
\nabla.\vec{J} &= \frac{d\rho}{dt} & (3.78)\\
\nabla^2 V &= \frac{\rho}{\epsilon_0} & (3.79)\\
\nabla^2 \vec{A} &= -\mu_0 \vec{j} & (3.80)
\end{aligned}$$

with

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{3.81}$$

3.4 Electromagnetic Waves

Before we introduce electromagnetic wave we must specify what does mean it by a wave. There is no clear definition of waves but one approximate definition is that it is a medium which propagates in some medium with a constant velocity.

If some disturbance which is represented by by $A(x, t)$ starts at time t and if intercepted at time t at position x then we must have :

$$A(x, t) = A(x - vt, 0) = f(x - vt) \tag{3.82}$$

Any physical quantity which depend on x and t through the combination $(x-vt)$ is a wave. In general solution of the following equation have this property and so is called the wave equation.

$$\boxed{\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial f(x,t)}{\partial t^2} = 0} \tag{3.83}$$

where v is the velocity with which the wave propagates. It is important to note that the wave equation is linear i.e., if $f_1(x, t)$ and $f_2(x, t)$ are two solutions then $f_1(x, t) + f_2(x, t)$ is also a solution. It is common to write waves in the following forms:

$$\begin{aligned} f(x,t) &= A\cos(kx \pm \omega t) \quad \text{or} \quad f(x,t) = A\sin(kx \pm \omega t) \\ f(x,t) &= Ae^{\pm i(kx-\omega t)}, \end{aligned} \tag{3.84}$$

with

$$k = \frac{\omega}{v}. \tag{3.85}$$

Two of the Maxwell's equation in vacuum (no charge and current densities) can be written as:

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{dt} \\ \nabla \times \vec{B} &= \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{3.86}$$

Taking curl of the above equations:

$$\begin{aligned} \nabla \times (\nabla \times \vec{E}) &= -\frac{\partial (\nabla \times \vec{B})}{dt} \\ \nabla \times (\nabla \times \vec{B}) &= \mu_0\epsilon_0 \frac{\partial (\nabla \times \vec{E})}{\partial t} \end{aligned} \tag{3.87}$$

and so

$$\begin{aligned} \nabla(\nabla.\vec{E}) - \nabla^2\vec{E} &= -\mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla(\nabla.\vec{B}) - \nabla^2\vec{B} &= -\mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \end{aligned} \tag{3.88}$$

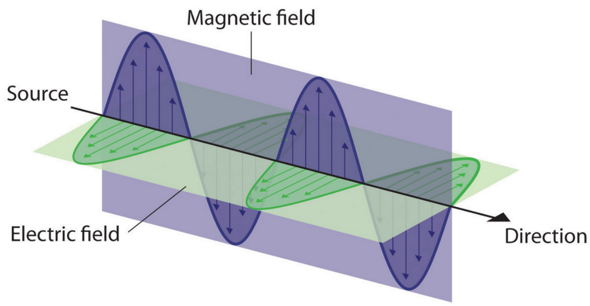

Figure 3.5: Propagation of electromagnetic waves.

or

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{3.89}$$

since $\nabla . \vec{E} = \nabla . \vec{B} = 0$.

From the above set of equations we can see that electric and magnetic field behave like waves and their combination is called electromagnetic waves. The speed of electromagnetic waves is given by :

$$c^2 = \frac{1}{\epsilon_0 \mu_0}. \tag{3.90}$$

Maxwell's equations tell use that $\vec{E}$ is parallel to $\nabla \times \vec{B}$ and $\vec{B}$ is parallel to $\nabla \times \vec{E}$ which is possible when $\vec{E}$ and $\vec{B}$ are perpendicular to each other. From this we can conclude that in electromagnetic wave propagation $\vec{E}$ and $\vec{E}$ both are perpendicular to the direction of the propagating and of each other.

An electromagnetic wave traveling in the +ve z-direction is given by:

$$\vec{E}(z,t) = \vec{E}_0 \cos(kz - \omega t) \tag{3.91}$$
$$\vec{B}(z,t) = \vec{B}_0 \cos(kz - \omega t) \tag{3.92}$$

with

$$\vec{E}_0 = E_{x0} \hat{x} + E_{y0} \hat{y} \tag{3.93}$$
$$\vec{B}_0 = B_{x0} \hat{x} + B_{y0} \hat{y}. \tag{3.94}$$

Since $\vec{E}$ and $\vec{B}$ are perpendicular so we can have $E_{y0} = B_{x0} = 0$ and this is well described by figure (4.1).

3.5 Relativistic Electrodynamics

In Relativistic electrodynamics $\vec{E}$ and $\vec{B}$ transform as component of a second rank anti-symmetric tensor called the "Field tensor" $F^{\mu\nu}$.

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{3.95}$$

The four current density is defined as:

$$j^\mu = (c\rho, \vec{J}). \tag{3.96}$$

and so the complete set of Maxwell's equations is written as:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 j^\mu \quad \text{or } \partial_\nu F^{\mu\nu} = \mu_0 j^\mu, \tag{3.97}$$

and the continuity equation is written as:

$$\partial_\mu J^\mu = 0. \tag{3.98}$$

The "Four Potential" A^μ is defined as:

$$A^\mu = (V/c, \vec{A}) \tag{3.99}$$

and so the field tensor can be written as:

$$F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A_\mu. \tag{3.100}$$

Note that if A^μ is a solution then :

$$A^\mu + \frac{\partial \lambda}{\partial x^\mu} \tag{3.101}$$

is also a solution and this is called "gauge freedom". In order to "fix gauge" we use

$$\nabla.\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \tag{3.102}$$

and this gauge is called Lorentz Gauge. In this Gauge Maxwell's equation become :

$$\Box^2 A^\mu = -\mu_0 j^\mu. \tag{3.103}$$

Chapter 4

Black Body Radiation

Note that the reference for this chapter is Weinberg 2013, Quantum Mechanics.

When we heat a body it starts radiating and when we raise the temperature
:

1. Overall brightness of the body increases.

2. It starts getting bluer from redder i.e., starts emitting more at shorter wavelength.

In practice radiation coming out of a hot body does depend on the material the body is made of. However, if we create a cavity with small opening then the radiation coming out the opening does not depend on the matter of which the cavity is made of. This radiation is called the “black body radiation” and has great importance in radiation physics, as great as “ideal gas” has in statistical and thermodynamics.

Black body radiation is emitted by a system which is in thermal equilibrium and the energy density of black body radiation $\rho(\nu, T)d\nu$ in frequency range $\nu, \nu+d\nu$ depends on the frequency ν and temperature T only and has a universal form, given by the Planck radiation formula:

$$\boxed{\rho(\nu, T)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{\frac{h\nu}{k_B T}} - 1}} \tag{4.1}$$

where $h = 6.6\times10^{-27}$erg sec is the Planck's constant and $k_B = 1.4\times10^{-16}$ erg$/K$ is the Boltzmann constant.

The total energy density is given by:

$$u(T) = \int \rho(\nu, T)d\nu. \tag{4.2}$$

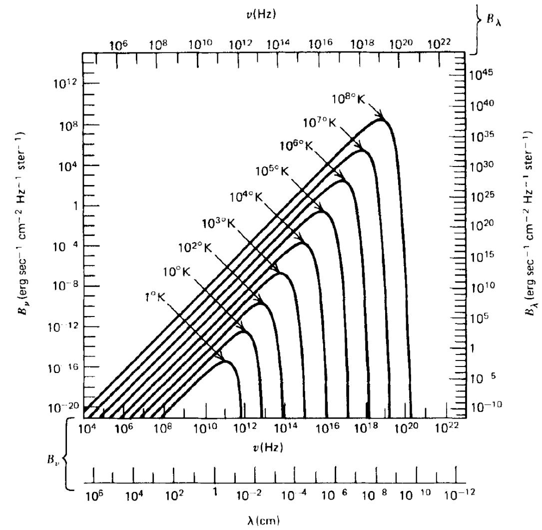

Figure 4.1: Black Body radiation.

4.1 Rayeligh's Law

One can explain black body radiation by considering the "cavity" in the form of a cubical box of size L and compute the normal modes:

$$k_n = \frac{2\pi n}{L}, \quad \text{with} \quad n = 1, 2, 3, ... \tag{4.3}$$

since we know :

$$\nu = \frac{c}{\lambda} = \frac{ck}{2\pi} \tag{4.4}$$

so we can write:

$$\nu_n = \frac{nc}{L}, \tag{4.5}$$

considering n as a vector we can compute the number of normal modes in the range ν and $\nu + d\nu$ of the space of vector $\mathbf{n}$ as :

$$N_\nu d\nu = 2 \times 4\pi |\mathbf{n}|^2 d\mathbf{n} = 8\pi (L/c)^3 \nu^2 d\nu. \tag{4.6}$$

The factor of two is for taking into account two possible polarizations.

In classical statistical mechanics the average energy of a harmonic oscillator $\bar{E}(T) = k_B T$ so

$$\rho(\nu, T) d\nu = \frac{\bar{E}(T) N_\nu d\nu}{L^3} = \frac{8\pi k_B T \nu^2 d\nu}{c^3}. \tag{4.7}$$

This is the Rayeligh's law which says that the energy density in frequnecy range $\nu, \nu + d\nu$ at temperature T is proportional $\nu^2 T$ and found to be observationally correct for small values of $h\nu/k_B T$ (at low frequencies). Rayeligh's law does not fit observations at higher frequencies and in fact we get :

$$\int \rho(\nu,T)d\nu = \infty \tag{4.8}$$

4.2 Planck's radiation formula

Since Rayeligh's law is inconsistent at higher frequencies so we need correction and this correction can be done using quantum nature of light. According to quantum theory radiation is made of "particles of light" called photons. Each photon can have energy $h\nu$ at frequency ν.

If we consider a bunch of photons distributed in different energy states then each states can have only $E_n = nh\nu$ energy, where n is an integer. According to statistical mechanics the probability of state of energy E_n is e^{-E_n/k_bT} at temperature T so the average energy:

$$\bar{E} = \frac{\sum E_n e^{-\beta E_n}}{\sum e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln\left(\sum e^{-\beta E_n}\right) \tag{4.9}$$

with $\beta = 1/k_bT$.

By the formula of the sum of geometric series:

$$\sum e^{-\beta E_n} = \sum e^{-nh\nu\beta} = \frac{1}{1-e^{-\beta h\nu}} \tag{4.10}$$

so the average energy is given by :

$$\bar{E} = \frac{h\nu e^{-\beta h\nu}}{1-e^{-\beta h\nu}} \tag{4.11}$$

With this expression for the average energy we get :

$$\rho(\nu,T)d\nu = \frac{8\pi h\nu^3 d\nu}{c^3}\frac{1}{e^{\beta h\nu}-1} \tag{4.12}$$

This is the Planck radiation formula.

If we consider $\beta h\nu << 1$ then we get :

$$\rho(\nu,T)d\nu = \frac{8\pi h\nu^3 d\nu}{c^3}\frac{1}{1+\beta h\nu + O((\beta h\nu)^2)-1} = \frac{8\pi\nu^2 d\nu}{\beta c^3} = \frac{8\pi k_B T\nu^2 d\nu}{c^3} \tag{4.13}$$

This is nothing but the Rayeligh's law.

4.3 Wein's Law

High frequnecy limit of the Planck's radiation formula is called the Wien's law which can be obtained in the following way:

Consider $\beta h\nu >> 1$ so

$$\frac{1}{e^{\beta h\nu}-1} \approx e^{-\beta h\nu} \tag{4.14}$$

and so

$$\rho(\nu,T)d\nu = \frac{8\pi h\nu^3 d\nu}{c^3}e^{-\beta h\nu}. \tag{4.15}$$

This is called the Wein's law.

4.4 Stephan's Boltzmann Law

Using :

$$\int_0^\infty \frac{x^3 dx}{e^x-1} = \frac{\pi^4}{15} \tag{4.16}$$

we can compute the energy density over all frequencies :

$$u(T) = \int \rho(\nu,T)d\nu = \int \frac{8\pi h\nu^3 d\nu}{c^3(e^{h\nu/k_BT}-1)} = \frac{8\pi k_B^4 T^4}{c^3h^3} \times \int_0^\infty \frac{x^3 dx}{e^x-1} = \frac{8\pi^5 k_B^4}{15c^3h^3}T^4 = aT^4 \tag{4.17}$$

Where:

$$a = \frac{8\pi^5 k_B^4}{15c^3h^3} \tag{4.18}$$

is the Stephan-Boltzmann constant.

Note that energy density $\rho(\nu,T)$ and specific intensity $I_\nu(\Omega)$ are related in the following way:

$$\rho(\nu,T) = \frac{1}{c}\int I_\nu d\Omega. \tag{4.19}$$