Pulsars

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1 Dispersion

Dipole moment per unit volume induced by an electric field E is given by

$$\vec{P} = \varepsilon_0(\varepsilon - 1)\vec{E} \tag{1}$$

Polarization current due to time variation of dipole moment is given by

$$\vec{j}_P = \frac{\partial \vec{P}}{\partial t} = \varepsilon_0(\varepsilon - 1)\frac{\partial \vec{E}}{\partial t} \tag{2}$$

From the Maxwell equations:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_p + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon}{c^2}\frac{\partial E}{\partial t} \tag{3}$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Refractive index n is defined as $\sqrt{\varepsilon}$. Whenever there is a medium $c \longrightarrow c/n$

When a plane electromagnetic waves $\vec{E} = E_0 \sin \omega t \hat{z}$ passes through plasma electrons set into motion

$$m_e \frac{d^2 \vec{z}}{dt^2} = -eE_0 \sin \omega t \hat{z} \tag{4}$$

which can be solved as

$$z = -\frac{eE_0}{m_e \omega^2} \sin \omega t \tag{5}$$

and so the dipole moment

$$\vec{p} = -\frac{e^2 E_0}{m_e \omega^2} \sin \omega t \hat{z} \tag{6}$$

and so the polarization P (dipole moment per unit volume)

$$\vec{P} = -\frac{n_e e^2}{m_e \omega^2}\vec{E} \tag{7}$$

where n_e is the number density of electrons in the plasma.

Using equation (1) we get

$$\varepsilon = 1 + \frac{\vec{P}}{\varepsilon_0 \vec{E}} == 1 - \frac{\omega_p^2}{\omega^2} \tag{8}$$

where $\omega_p = \sqrt{n_e e^2/\varepsilon_0 m_e}$ is called the plasma frequency.

Let us consider a plane wave:

$$\vec{E} = E_0 \exp i \left(\vec{k}.\vec{x} - \omega t\right) \tag{9}$$

where the phase velocity v_p is defined as

$$v_p = \frac{\omega}{k} \tag{10}$$

but in a dielectric medium $v_p = c/n$ where $n = \sqrt{\varepsilon}$ so

$$\frac{\omega}{k} = c\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{-1/2} \tag{11}$$

or

$$k = \frac{\omega}{c}\sqrt{\left[1 - \frac{\omega_p^2}{\omega^2}\right]} \tag{12}$$

2 Propagation of a plane electromagnetic wave in a dispersive medium

If a plane electromagnetic wave $E = E_0 \exp(-\omega t)$ travels in a dispersive medium (plasma) for L distance then its phase gets changed by the following amount

$$\phi_p = k \times L \tag{13}$$

from equation (12)

$$\phi_p = \frac{\omega}{c}\sqrt{\left[1 - \frac{\omega_p^2}{\omega^2}\right]} L \approx \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\right] L \tag{14}$$

We can compute time delay corresponding to the above phase shift

$$\tau_\omega = \frac{d\phi_p}{d\omega} = \frac{L}{c}\left[1 + \frac{\omega_p^2}{2\omega^2}\right] \tag{15}$$

If there are two waves of frequencies ω_1 and ω_2 then the differential delay

$$\tau_{\omega_2} - \tau_{\omega_1} = \frac{L}{c}\frac{\omega_p^2}{2}\left[\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right] = \frac{e^2}{8\pi^2 \varepsilon_0 m_e c}\left[\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right] n_e L \tag{16}$$

or

$$\tau_{21} = \frac{r_e c}{2\pi}\left[\frac{1}{\nu_2^2} - \frac{1}{\nu_1^2}\right] DM \tag{17}$$

where

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = 2.81794 \times 10^{-15}\ mt \tag{18}$$

is the classical electron radius and

$$DM = \int_0^l n_e(x)dx \tag{19}$$

is called the dispersion measure (DM).

$$\tau_{21} \approx \left[\left(\frac{MHz}{\nu_2}\right)^2 - \left(\frac{Mhz}{\nu_1}\right)^2\right] \frac{DM}{cm^{-3}\ pc} \times 4150\ sec \tag{20}$$

3 De-dispersion

As a result of dispersion electromagnetic waves with higher frequencies (channels) reach earlier than waves with lower frequencies. We can correct for this effect by delaying waves of different frequencies by appropriate amount. This is called the dedispersion.

If we express the jth time sample of the lth channel by R_{jl} then dedispersed time series is given by:

$$R_j = \sum_l R_{j+k(l),l} \tag{21}$$

where k is the number of time samples added to channel l which is given by

$$k(l) = \frac{4.15msec}{t_{samples}} \left(\frac{DM}{pc\ cm^{-3}}\right) \left[\left(\frac{GHz}{\nu_l}\right)^2 - \left(\frac{GHz}{\nu_1}\right)^2\right] \tag{22}$$

where

$$\nu_l = \nu_1 - (l-1)\Delta\nu_{chanels} \tag{23}$$

Note that ν_1 is the highest frequency channel.

4 Pulse broadening

The observed width w of a pulse is different from its intrinsic width w_{int} due to propagation effects (dispersion, scattering) and variation of sensitivity of instruments with bandwidth $d\nu$ and observing frequency ν.

$$w = \left[w_{int}^2 + \Delta t_{DM}^2 + \Delta t_{\delta DM}^2 + \Delta t_{\Delta\nu}^2 + \tau_d^2\right]^{1/2} \tag{24}$$

where various terms are as follows:

1. Dispersion smearing Δt_{DM}:

$$\Delta t_{DM} = 8.3\mu\ sec \times \left(\frac{DM}{pc\ cm^{-3}}\right) \left(\frac{\Delta\nu}{MHz}\right) \left(\frac{GHz}{\nu}\right)^3 \tag{25}$$

2. Dispersion Error $\Delta_{\delta DM}$: Dedispersing for a DM which is away from the true DM by amount δDM leads to smearing by
$$\delta t_{\delta DM} = \Delta t_{DM} \left(\frac{\delta DM}{DM} \right) \tag{26}$$

3. Filter response of individual frequency $\Delta t_{\Delta\nu}$: this term is inversely proportional to channel width
$$\Delta t_{\Delta\nu} \sim \frac{1}{\Delta\nu} = \left(\frac{MHz}{\Delta\nu} \right) \mu\, sec \tag{27}$$

4. Broadening due to multipath scattering τ_d: this effect can be modeled as a convolution of the intrinsic pulse profile with a function approximately exponential in form and having a characteristic timescale τ_d. This term may depend on direction and distance. However, it can be considered a statistical function of DM.
$$\log \tau_d = -3.72 + 0.411 \log DM + 0.937 (\log DM)^2 - 4.4 \log \left(\frac{\nu}{GHz} \right) \mu\, sec \tag{28}$$
with scatter about the mean fit of $\sigma_{\tau_d} \approx 0.65$

Δt_{DM} decreases with $\Delta\nu$ however $\Delta t_{\Delta\nu}$ increases. The minimum time resolution achievable is
$$\Delta t_0 = \left[2 \left(\Delta t DM_{min} \right)^2 + \tau_d^2 \right]^{1/2} \tag{29}$$
where
$$\Delta t_{DM_{min}} = \left[8.3 DM \left(\frac{GHz}{\nu} \right)^3 \right]^{1/2} \mu \text{ sec} \tag{30}$$
is the minimum dispersion smearing.

note: The spacing of trial DMs is determined by the maximum smearing one is willing to accept in the final dedispersed time series. This residual dispersion smearing is given by $\Delta t_{\Delta DM}$, calculated using equations (25) and (26) with $\delta\nu$ MHz as the total bandwidth.

5 Effect of dispersion on the location t and width w of a pulse

From the dispersion delay relation discussed above, it is clear that for a given frequency ν and bandwidth $d\nu$ the dispersion delay is proportional to DM so
$$\frac{t_{DM_1}}{t_{DM_2}} = \frac{t_0 + \Delta t_{DM_1}}{t_0 + \Delta t_{DM_2}} = \frac{t_0 + C(DM_1/cm^{-3}pc)}{t_0 + C(DM_2/cm^{-3}pc)} \tag{31}$$
where
$$C = \left[\left(\frac{MHz}{\nu} \right)^2 \right] \times 4150\, sec \tag{32}$$
If we set $t_0 = 0$, then
$$\frac{t_{DM_1}}{t_{DM_2}} = \frac{DM_1}{DM_2} \tag{33}$$

From the pulse widening relation (equation (25))

$$\frac{w_{DM_1}}{w_{DM_2}} = \left[\frac{w_{int}^2 + \Delta t_{DM_1}^2}{w_{int} + \Delta t_{DM_2}^2}\right]^{1/2} = \left[\frac{w_{int}^2 + \mathcal{D}(DM_1/cm^{-3}pc)^2}{w_{int}^2 + \mathcal{D}(DM_2/cm^{-3}pc)^2}\right]^{1/2} \tag{34}$$

where

$$\mathcal{D} = \left[8.3\mu\, sec \times \left(\frac{\Delta\nu}{MHz}\right)\left(\frac{GHz}{\nu}\right)^3\right]^2 \tag{35}$$

If the true DM of a pulse is DM_0 and we dedisperse it at DM DM then the location of the pulse will be

$$\frac{t}{t_0} = \frac{DM}{DM_0} \text{ or } t = t_0\frac{DM}{DM_0} \tag{36}$$

6 Single Pulse Detection

Intensity $S(t)$ of a broadband source is modified in the following way:

$$I(t) = S(t) \star g_r \star g_d \star h_{DM} \star h_d(t) \star h_{RX} + N(t) \tag{37}$$

where

1. g_r: modulation due to refractive scintillation
2. g_d: modulation due to diffraction scintillation
3. h_{DM}: dispersion smearing
4. $h_d(t)$: pulse broadening due to multipath propagation
5. h_{RX}: averaging in the receiver and data acquisition system
6. $N(t)$: additive receiver noise

and $I(t)$ is the observed intensity.

7 Matched Filtering

Following steps are followed

1. Dedispersion
2. each dedispersed time series is searched for pulses with amplitude above some signal to noise (S/N) threshold. This search process is simply an exercise in matched filtering, with **the highest detected S/N achieved when the effective sampling time of the time series is equal to the detected width of the pulse**. Since the intrinsic widths of signals are typically unknown, as are the contributions to the width from dispersion and scattering, a large parameter space must be searched. This is mostly done by "smoothing" the time series by iteratively adding adjacent samples. In the absence of knowledge about the true pulse shape and width, the smoothing approach is a straightforward and efficient approximation to optimal detection.

   For a pulse of intrinsic amplitude S_i and width W_i for optimal detection

   $$\left(\frac{S}{N}\right)_i = \frac{1}{\sigma\sqrt{W_n}}\left(\frac{A_i}{\sqrt{W_I}}\right) \tag{38}$$

   where σ is the rms series (before smoothing), W_n correlation time and $A_i = S_i W_i$.

   For the system noise S_{sys} (in Jy) the rms radiometer noise

   $$\sigma = \frac{S_{sys}}{\sqrt{N_{pol}\Delta\nu W_n}} \tag{39}$$

   where N_{pol} is the number of polarizations and $\Delta\nu$ is bandwidth. S/N in the case of optimal detection is independent of W_n

   $$\frac{S}{N} \propto \frac{A_i}{\sqrt{W_i}} \tag{40}$$

so for a fixed pulse area, a narrower pulse yields a larger S/N. However, a low amplitude broad pulse is more easily detectable than a sharp narrow pulse if its area is sufficiently larger.