Cosmic Microwave Background Radiation

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October 17, 2015

1 CMB aniostropies

$$T(\hat{k},\mu,\eta) = T^0(\eta)[1+\Theta(\hat{k},\mu,\eta)], \quad \mu = \hat{k}.\hat{p} \tag{1}$$

Here $\hat{p}$ is the direction along which the photons move i.e., momentum.

CMB anisotropies which are represented by $\Theta(\hat{k},\mu,\eta_0)$ at present, can be computed by solving the following integral:

$$\Theta(k,\mu,\eta_0) = \int_0^{\eta_0} \left\{ -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + i\mu v_b - \frac{1}{2}\mathcal{P}_2\Pi\right]\right\} \times e^{ik\mu(\eta-\eta_0)-\tau} d\eta, \tag{2}$$

where

$$\Pi = \Theta_2 + \Theta_2^P + \Theta_0^P, \tag{3}$$

and $\mathcal{P}_2$ is the second Legendre polynomials.

Optical depth $\tau(\eta)$ at any time η is defined as:

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{4}$$

Note that the following expansion is used for the multipole expansion:

$$\Theta(\mu) = \sum_{l=0}^{\infty} \frac{2l+1}{i^l} \Theta_l \mathcal{P}_l(\mu), \tag{5}$$

with

$$\Theta_l = \frac{i^l}{2}\int_{-1}^{1} \mathcal{P}_l(\mu)\Theta(\mu)d\mu. \tag{6}$$

The following important identity is used to related the Legendre's polynomials with Bessel's function:

$$\boxed{\frac{i^l}{2}\int_{-1}^{1} \mathcal{P}_l(\mu) e^{ik\mu(\eta-\eta_0)} d\mu = j_l[k(\eta_0-\eta)]} \tag{7}$$

This means that the angular integration can be done in terms of the Bessel's functions and now we have only the radial integration to be done:

$$\Theta_l(k,\eta_0) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0-\eta)] d\eta, \tag{8}$$

where the source function $S(k,\eta)$ is defined as:

$$S(K,\eta) = g\left[\Theta_0 + \Psi + \frac{1}{4}\Pi\right] + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right] - \frac{1}{k}\frac{d}{d\eta}(gv_b) + \frac{3}{4k^2}\frac{d^2}{d\eta^2}(g\Pi), \tag{9}$$

with

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)}, \tag{10}$$

and

$$\int_0^\infty g(\eta)d\eta = 1. \tag{11}$$

The metric perturbations potentials Ψ and Φ are defined in (Newtonian Gauge) the following way:

$$g_{\mu\nu} = \begin{bmatrix} -(1+2\Psi) & 0 \\ 0 & a^2\delta_{ij}(1+2\Phi) \end{bmatrix} \tag{12}$$

Here Ψ is the perturbation in Newtonian potential and Φ in spatial curvature.

The observed CMB angular power spectrum $C_l \approx \Theta_l(\vec{x})$ at position $\vec{x} = 0$ is given by the Fourier transformation of $\Theta_l(k)$. In order to take into account the scale dependence of the initial fluctuations we must use :

$$C_l = \int \frac{d^k}{(2\pi)^3} P(k)\Theta_l^2(k) \tag{13}$$

For the Harrison-Zeldovich case:

$$P(k) = \frac{2\pi^2}{k^3}\left(\frac{k}{H_0}\right)^{n_s-1}. \tag{14}$$

Some of the important points are summarized below:

1. For large l

$$\Theta_l(k,\eta), \Theta_l^P(k,\eta) \approx j_l(k\eta) \tag{15}$$

2. For $l_{\max} = 1200$ it is sufficient to consider 100 values of k between $k_{\min} = 0.1H_0$ and $k_{\max} = 1000H_0$ and it is recommended to use non-uniform binning like:

$$k_i = k_{\min} + \left(\frac{i}{100}\right)^2 (k_{\max} - k_{\min}). \tag{16}$$

3. The dominant contribution comes from $\eta << \eta_0$ so roughly :

$$C_l \approx \int_0^\infty \frac{j_L^2(k\eta)}{k} dk. \tag{17}$$

   Since the Bessel functions have periodicity of 2π so the integrand with oscillation with period $k = 2\pi/\eta_0$ so we must have grid with resolution:

$$\Delta k = \frac{2\pi}{10\eta_0} \tag{18}$$

   This leads to the total number of points around 5000. We roughly want to have

$$\frac{0.9l}{\eta_0} \leq k \leq \frac{2l}{\eta_0} \tag{19}$$

4. Note that C_l is a smooth function of l so we need not to compute C_l for every l and we can interpolate. For example, at low-l we can have higher resolution and at high-l low : $l = 2, 3, 4, 8, 10, 12, 15, 20$ up to $l = 100$ we can compute every tenth and up to $l = 300$ every 25th and so on.

5. The source function is smooth but the Bessel function makes it oscillatory.

6. CMB power spectrum calculated from the low and high resolution grids are indistinguishable because the dominant part of the integration comes from recombination, with only a small correction from the late time oscillations caused by the Bessel function. The source function is non-zero at late times mostly due to the integrated Sachs-Wolfe effect, which is most important for low-l (low ks), whereas the oscillations in the Bessel function dominate for large ks, hence the details of the oscillations are unimportant.

7. The argument of the Bessel's functions should be from 0 to $k_{\max}\eta_0 = 3400$.

2 Cosmic Variance

Anisotropies in the Cosmological Microwave Background Radiation (CMBR) sky ($\Delta T/T_0 = \Theta$), where T_0 is the average temperature or monopole term, are in general expressed in terms of spherical harmonic:

$$\frac{\Delta T(\theta,\phi)}{T} = \sum_{l=0} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta,\phi) \tag{20}$$

sometime we will write $\hat{n}$ also for the direction (θ,ϕ) on sky. In equation (20) the coefficients a_{lm} can be written

$$a_{lm} = \int d\Omega Y^*_{lm}(\theta,\phi) \frac{\Delta T(\theta,\phi)}{T} \tag{21}$$

When we take a very small patch of sky which can be considered flat, equations (20) and (21) represent a Fourier transformation pair in two dimension and l and θ make Four transformation pair $\theta = 2\pi/l$. If the temperature anisotropies are Gaussian, then the multiple moments a_{lm}s are completely completely characterize by their power spectrum C_l.

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l \tag{22}$$

In general it is a common practice to represent power "per logarithmic interval in wavenumber" Δ^2_T which is defined as:

$$\Delta^2_T = \frac{l(l+1)}{2\pi} C_l T^2 \tag{23}$$

For every l we have $2l+1$ samples in m and this leads a fundamental limitation on the accuracy with which C_l can be measured called "cosmic variance"

$$\Delta C_l = \sqrt{\frac{2}{2l+1}} C_l \tag{24}$$

If only a fraction f_{sky} is observed the above expression get modified in the following way:

$$\Delta C_l = \sqrt{\frac{2}{(2l+1) f_{sky}}} C_l \tag{25}$$

If beam pattern is also taken into account then the above equation get modified. The observed temperature anisotropies are convolution of the actual anisotropies and the beam function **?**

$$\frac{\Delta T_{obs}(\hat{n})}{T_0} = \int d\Omega_{\hat{n}} \frac{\Delta T_{obs}(\hat{n}')}{T_0} W(\hat{n} - \hat{n}') \tag{26}$$

The window function W can be approxiated by a Gaussian

$$W(\hat{n} - \hat{n}') = \frac{1}{2\pi\theta_s^2} \exp\left[-\frac{|\hat{n}' - \hat{n}|^2}{2\theta_s^2} \right] \tag{27}$$

where $\theta_s = 0.425\theta_{FWHM}$ is the smoothing angle.

$$\Delta C_l = \sqrt{\frac{2}{(2l+1) f_{sky}}} \left[C_l + \frac{1}{w f_{sky}} e^{l^2 \sigma_b^2} \right] \tag{28}$$

where

$$w = \left(\frac{T_0}{\sigma_{pix} \theta_{FWHM}} \right)^2 \quad \text{with} \quad \sigma_{pix} = \frac{s}{\sqrt{t_{pix}}} \tag{29}$$

for detail see Jungman et al. (1996)

3 Map Making

The CMBR data vector d_t is given by

$$d_t = A_{ti}\Theta_i + n_t \tag{30}$$

where Θ_i is the temperature anisotropy at pixel i which represent direction (θ, ϕ) i.e., $\Theta_i = \Delta T(\theta, \phi)/T_0$ and A_{ti} is a matrix which has non-zero entry only when pixel i is observed at time t. Noise n_t can be given by its covariance matrix

$$\langle n_t n_{t'} \rangle = C_{d,tt'} \tag{31}$$

We can write equation (30) in a slightly different form

$$d_i = s_i + n_i \tag{32}$$

which says that the data d_i at pixel i is contributed by signal s_i and noise n_i. Now if we assume that signal and noise are uncorelated then we can define covariance matrices for s_i, n_i and d_i

$$C_{T,ij} = \langle s_i s_j \rangle\,; \quad C_{n,ij} = \langle n_i n_j \rangle \quad \text{and} \quad \langle d_i d_j \rangle = C_{T,ij} + C_{N,ij} \tag{33}$$

Here it is important that signal covariance matrix $C_{T,ij}$ is related to C_l in the following way

$$\boxed{C_{T,ij} = \sum_l \frac{2l+1}{4\pi} C_l \mathcal{W}_{ij}(l)}$$

where $\mathcal{W}(l)$ is related to the beam function in the following way

$$\mathcal{W}_{ij}(l) = \sum_{nn'} W_{in} W_{j,n'} P_l(\cos\theta) \tag{34}$$

Note that in the above equation pixel i, j, n and n' pixels reprsent directions $\hat{i}, \hat{j}, \hat{n}$ and $\hat{n}'$ respectively, and $W_{ij} = W(\hat{n} - \hat{n}')$.

Likelihood function, which is defined as the probability distribution for observing the data d_t for a given $C_{N,tt'}$, can be identified with the probability distribution for noise, which is assumed Gaussian

$$\mathcal{L}(d_t) = \frac{1}{(2\pi)^{N_t/2}\sqrt{|C_d}} \exp\left[-\frac{1}{2}(d_t - A_{ti}\Theta_i) C^{-1}_{d,tt'} (d_{t'} - A_{t'j}\Theta_j)\right] \tag{35}$$

Using Bayes theorem we can compute the probability for parameters i.e., in this case Θ_i (which we want to estimate) serve the purpose of parameters, from the likelihood function

$$P(\Theta_i|d_t) \propto P(d_t|\Theta_i) = \mathcal{L}(d_t) \tag{36}$$

By minimizing equation (35) with respect to Θ_i we get the maximum likelihood estimator as

$$\hat{\Theta}_i = C_{N,ij} P_{jt} C^{-1}_{d,tt'} d_{t'} \tag{37}$$

where

$$C_N \equiv (P' C_d^{-1} P)^{-1} \tag{38}$$

The mean of above estimator is Θ_i and variance is C_N. Cramer-Rao theorem says that the above maximum likelihood estimator is also the minimum variance estimator.

4 Band Power estimation

Once maps have been made next task is compute the band power for which the map Θ_i serves the purpose of data. We can draw Θ_i from a Gaussian distribution with covariance

$$\langle \Theta_i \Theta_j \rangle = C_{s,ij} = \sum_l \Delta^2_{T,l} W_{i,ij} 0 \tag{39}$$

where $\Delta^2_{T,l}$ depends on cosmological parameters through C_l and $W_{i,ij}$ is the window function. For the Gaussian model the likelihood function is given by

$$\mathcal{L}_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2}\sqrt{|C_\Theta|}} \exp\left[-\frac{1}{2}\Theta_i C^{-1}_{\Theta,ij}\Theta_j\right] \tag{40}$$

where N_p is the number of pixels in the map and C_Θ is defined as

$$C_\Theta = C_S + C_N \tag{41}$$

In this case the theoretical (model) parameters are B_a i.e., the band power, on which the covariance matrix depends and the likelihood is not Gaussian in parameters and so there is no simple analytic way to compute.

Cosmological parameters are estimated by finding the maxium of band power likelihood

$$\mathcal{L}_c(\hat{B}_a) = \frac{1}{(2\pi)^{N_c/2}\sqrt{|C_B|}} \exp\left[-\frac{1}{2}(\hat{B}_a - B_a)C^{-1}_{B,ab}(\hat{B}_b - B_B)\right] \tag{42}$$

Rough estimate of errors can be computed from the inverse of Fisher matrix

$$F_{c,ij} = \frac{\partial B_a}{\partial c_i} C^{-1}_{B,ab} \frac{\partial B_b}{\partial c_i} \tag{43}$$

All the above dicussion was taken from Hu & Dodelson (2002).

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5 Parameter estimation eroors

If the error bars $\delta\theta_i$ on the estimated parameters around mean $<\theta_i>=\theta_0$ are small we can expand the likelihood around maximum Bond et al. (1997)

$$\log L(\theta) = \log L(\theta_0) + \sum_{i,j} \frac{\partial \log L}{\partial \theta_i}\frac{\partial \log L}{\partial \theta_i}\delta\theta_i\delta\theta_j \tag{44}$$

or

$$\mathcal{L} = \mathcal{L}_m \exp\left[-\frac{1}{2}\sum_{i,j} F_{ij}\delta\theta_i\delta\theta_j\right] \tag{45}$$

where $\mathcal{L} = 2\log L$ and

$$F_{ij} = \sum_l (\Delta C_l)^2 \frac{\partial C_l}{\partial \theta_i}\frac{\partial C_l}{\partial \theta_j} \tag{46}$$

and ΔC_l is given by equation (25). In the above approximation covariance matric $C_{ij} =< \delta\theta_i\delta\theta_j >= F^{-1}_{ij}$ and 1σ error on θ_i is given by $\sqrt{C_{ii}}$.

The accuracy of cosmological parameter estimation (by covriance matrix) depends on the followings [Bond et al. (1997)]

1. The validity of the Gaussian approximation to likelihood.
2. The number and choice of parameters θ_i defining the theoretical model.
3. The parameters θ_0 of the target model.
4. The numerical accuracy of the derivative of C_l.
5. The inclusion of prior constraints on θ_0.
6. Systematic errors in the estimation of C_l caused by galactic and extra galactic background.

6 Power spectrum estimation

Temperature map is defined as

$$\frac{\Delta T(\hat{n})}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}) \tag{47}$$

We can construct an estimator for spherical harmonic coefficient

$$\hat{a}_{lm} = a_{lm}^T + a_{lm}^N \tag{48}$$

and correspondingly the power spectra

$$< |a_{lm}^T|^2 >= C_l^T \quad \text{and} \quad < |a_{lm}^N|^2 > \tag{49}$$

we can construct an unbiased estimator for power spectrum i.e., $< \hat{C}_l^T >= C_l^T$

$$\hat{C}_l^T = \frac{1}{2l+1} \sum_{m=-l}^{m=l} \langle \hat{a}_{lm}^* \hat{a}_{lm} \rangle - C_l^N \tag{50}$$

and variance in estimator

$$\left\langle \hat{C}_l \hat{C}_l \right\rangle - \left\langle \hat{C}_l \right\rangle^2 = \frac{2}{2l+1} (C_l^T + C_l^N)^2 \tag{51}$$

6.1 From power spectrum covariance to parameter covariance

In any parameter estimation exercise (likelihood) cosmological parameters θ_i are not used directly and in place of those C_l are used. So it becomes important to understand the transformation. Let us consider F is the fisher matrix for C_l (and covariance $C = F^{-1} = F_{\alpha,\beta}^{-1}$. Making infinitesimal transformation we can write

$$F_{ij}^\theta = \sum_{\alpha\beta} \frac{\partial \pi_\alpha}{\partial \theta_i} F_{\alpha\beta} \frac{\partial \pi_\beta}{\partial \theta_j} \tag{52}$$

where $\pi_\alpha = C_l$. simplifying

$$F_{ij}^\theta = \sum_l \frac{2l+1}{2(C_l^T + C_l^N)^2} \frac{\partial C_L}{\partial \theta_i} \frac{\partial C_l}{\partial \theta_j} \tag{53}$$

References

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