Cosmological Microwave Background Radiation

Likelihood Analysis - Part 2

Jayanti Prasad

Inter-University Centre for Astronomy & Astrophysics (IUCAA)
Pune, India (411007)

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Plan of the talk

- Cut-sky likelihood function
- Likelihood of the Polarization field
- Karhunen-Loveve Techniques

Likelihood of Correlated Gaussian fields

- CMBR temperature anisotropy field is represented as

$$T(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n}) \tag{1}$$

where

$$a_{lm} = \int d\hat{n} T(\hat{n}) Y^*_{lm}(\hat{n}) \quad \text{and} \quad < a_{lm} a^*_{l'm'} >= \delta_{ll'} \delta_{mm'} C_l \tag{2}$$

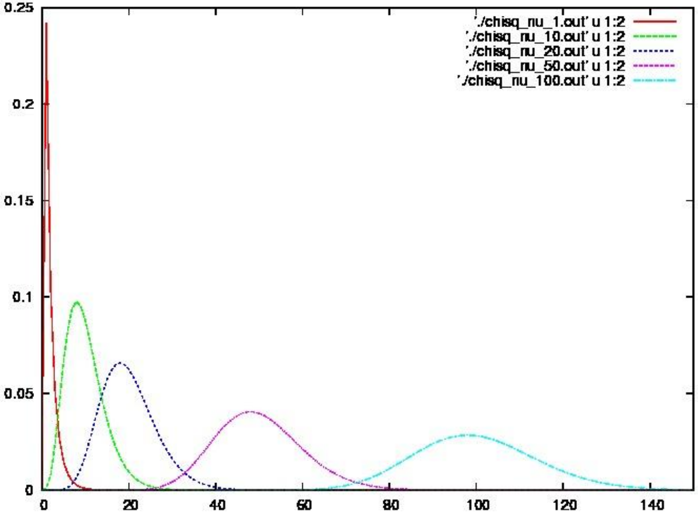
where the distribution of a_{lm} is Guassian with mean zero and variance given by the angular power spectrum C_l, which follows χ^2 distribution.

$$f(x;n) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} x^{(n/2)-1} \exp[-x/2] & \text{for } x > 0; \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

note that

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{y_i - y(x_i)}{\sigma_i} \right]^2 \tag{4}$$

which is the "square sum of Gaussian errors".



http://www.iucaa.ernet.in/ jayanti/gsl/chisq.c

- CMBR temperature and polarization fields, which individually follow Gaussian distribution, are correlated.

$$\mathbf{a}_{lm} = (a^T_{lm}, a^E_{lm}, a^B_{lm}) \tag{5}$$

- In this case we have six power spectra, two of these BB and BE are zero for a "parity-invariant ensemble" so we are left with $C_l^{TT}, C_l^{EE}, C_l^{BB}, C_l^{TE}$.
- The covariance matrix $\mathbf{C}_l$ and estimator $\hat{\mathbf{C}}_l$ are given as:

$$\mathbf{C_l} \equiv \left\langle \mathbf{a}_{lm}\, \mathbf{a}^\dagger_{lm} \right\rangle \quad \text{and} \quad \hat{\mathbf{C}}_l \equiv \frac{1}{2l+1} \sum_m \mathbf{a}_{lm}\, \mathbf{a}^\dagger_{lm} \tag{6}$$

- The probability $P(\{\mathbf{a}_{lm}|\ \mathbf{C}_l)$of a set $\mathbf{a}_{lm}$ at a given l is given by:

$$\begin{aligned} -2ln\left[P(\{\mathbf{a}_{lm}|\ \mathbf{C}_l)\right] &= \sum_{m=-l}^{m=l} \left[\mathbf{a}_{lm}\mathbf{C}_l^{-1}\, \mathbf{a}^\dagger_{lm} + ln|2\pi\ \mathbf{C}_l| \right] \\ &= (2l+1)\left(Tr[\ \hat{\mathbf{C}}_l\mathbf{C}_l^{-1}] + ln|\ \mathbf{C}_l| \right) + const \end{aligned} \tag{7}$$

- Integrating out all the $\mathbf{a}_{lm}$ with the same $\hat{\mathbf{C}}_l$ (or normalizing with respect to $\hat{\mathbf{C}}_l$)gives a Wishart distribution for $\hat{\mathbf{C}}_l$:

$$P(\hat{\mathbf{C}}_l|\ \mathbf{C}_l) \propto \frac{|\ \hat{\mathbf{C}}_l|^{(2l-n)/2}}{|\ \mathbf{C}_l|^{(2l+n)/2}} \exp\left[-(2l+1)\, Tr[\ \hat{\mathbf{C}}_l \mathbf{C}_l^{-1}]/2\right] \propto \mathcal{L}(\ \mathbf{C}_l|\hat{\mathbf{C}}_l) \tag{8}$$

- It can be shown that the likelihood has a maximum $\mathbf{C}_l = \hat{\mathbf{C}}_l$, so $\hat{\mathbf{C}}_l$ is the maximum likelihood estimator.
- For only temperature ($n = 1$) the Wishart distribution reduces to χ^2 distribution with $(2l+1)$ degrees of freedom:

$$-2lnP(\hat{\mathbf{C}}_l|\ \mathbf{C}_l) = (2l+1)\left[\frac{\hat{C}_l}{C_l} + ln|C_l| - \frac{2l-1}{2l+1} ln(\hat{C}_l)\right] + const \tag{9}$$

Karhunen-Loveve Techniques

- In any experiment there are always some modes which are heavily contaminated by noise and somehow if we can identify and discard those, we can speed up computation. If we can keep only 10% of the modes, computational can speed up by a factor of 1000 !
- Karhunen-Loveve (KL) technique provides a prescription for that.

[Tegmark et al. (1997); Dodelson (2003)]

How Karhunen-Loveve Technique ?

We carry out a coordinate transformation on the data vector Δ

$$\Delta'_i \equiv R_{ij}\Delta_j, \tag{10}$$

and the new covariance matrix

$$C'_{ij} = \langle (R\Delta)_i (R\Delta)_j \rangle \quad \text{or} \quad C' = RCR^T \tag{11}$$

where the old covariance matrix is given by

$$C = \langle \Delta_i \Delta_j \rangle \equiv C_{S,ij} + C_{N,ij} \tag{12}$$

Note that C^N and C^S are real symmetric matrices so can be easily diagonalized.

How Karhunen-Loveve Technique works ?

This technique needs the following three rotations:

- R_1 : To diagonalized the noise covariance matrix C_N.
- R_2 : To make the diagonalized C_N, i.e., C'_N unity.
- R_3 : To diagonalized the new signal covariance matrix C'_S.

Example:

$$C_N = \begin{pmatrix} \sigma_n^2 & 0 \\ 0 & \sigma_n^2 \end{pmatrix}$$

,

$$C_S = \sigma_s^2 \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$$

$$R_1 = I, R_2 = \frac{1}{\sigma_n^2} I$$

and

$$R_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

which gives

$$\Delta'_1 = \frac{1}{\sqrt{2}\sigma_N}(\Delta_1 + \Delta_2), \Delta'_2 = \frac{1}{\sqrt{2}\sigma_N}(\Delta_1 - \Delta_2)$$

and

$$C'_S = \frac{\sigma_s^2}{\sigma_n^2} \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}$$

Summary & Discussion

- I have discussed the difficulties and approximations used to compute the CMBR likelihood function.
- My aim was to understand why and how the likelihood function is computed differently for different values of l and different type of power spectra.
- I have also discussed various estimators related to angular power spectrum and the likelihood function.
- I was not able to discuss the actual computation of the various components of the likelihood function in the WMAP likelihood code.
- It has been suggested (by Dodelson) that the Nelder Mead method or downhill simplex method (commonly known amoeba) can be be used for finding the best fit parameters.
- Another method called NEWUOA which is used for unconstrained optimization without derivatives, is also suggested (by Antony Lewis) to find the best fit parameters, optimization of the likelihood function.

Dodelson, S. 2003, Modern cosmology (San Diego, U.S.A.: Academic Press)

Tegmark, M., Taylor, A. N., & Heavens, A. F. 1997, Astrophys. J. , 480, 22