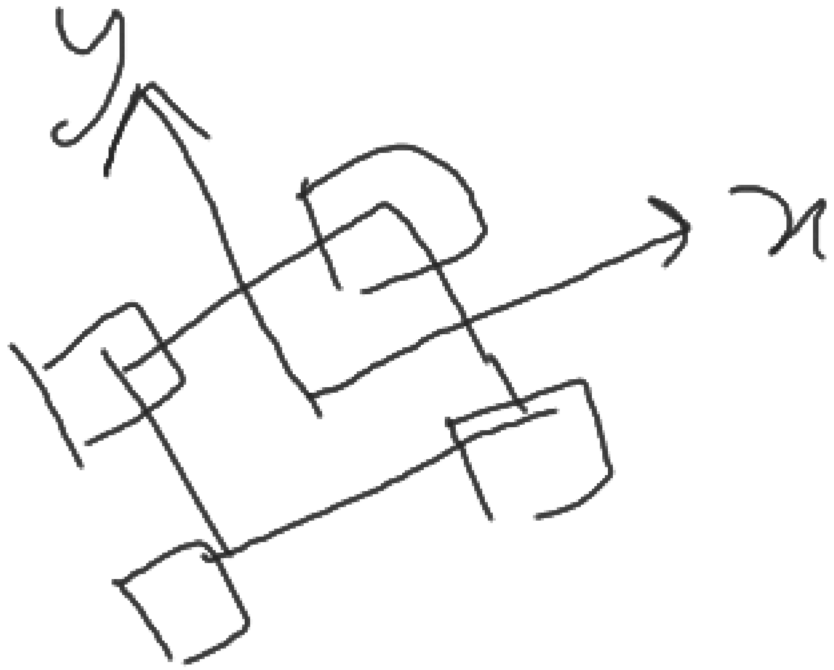
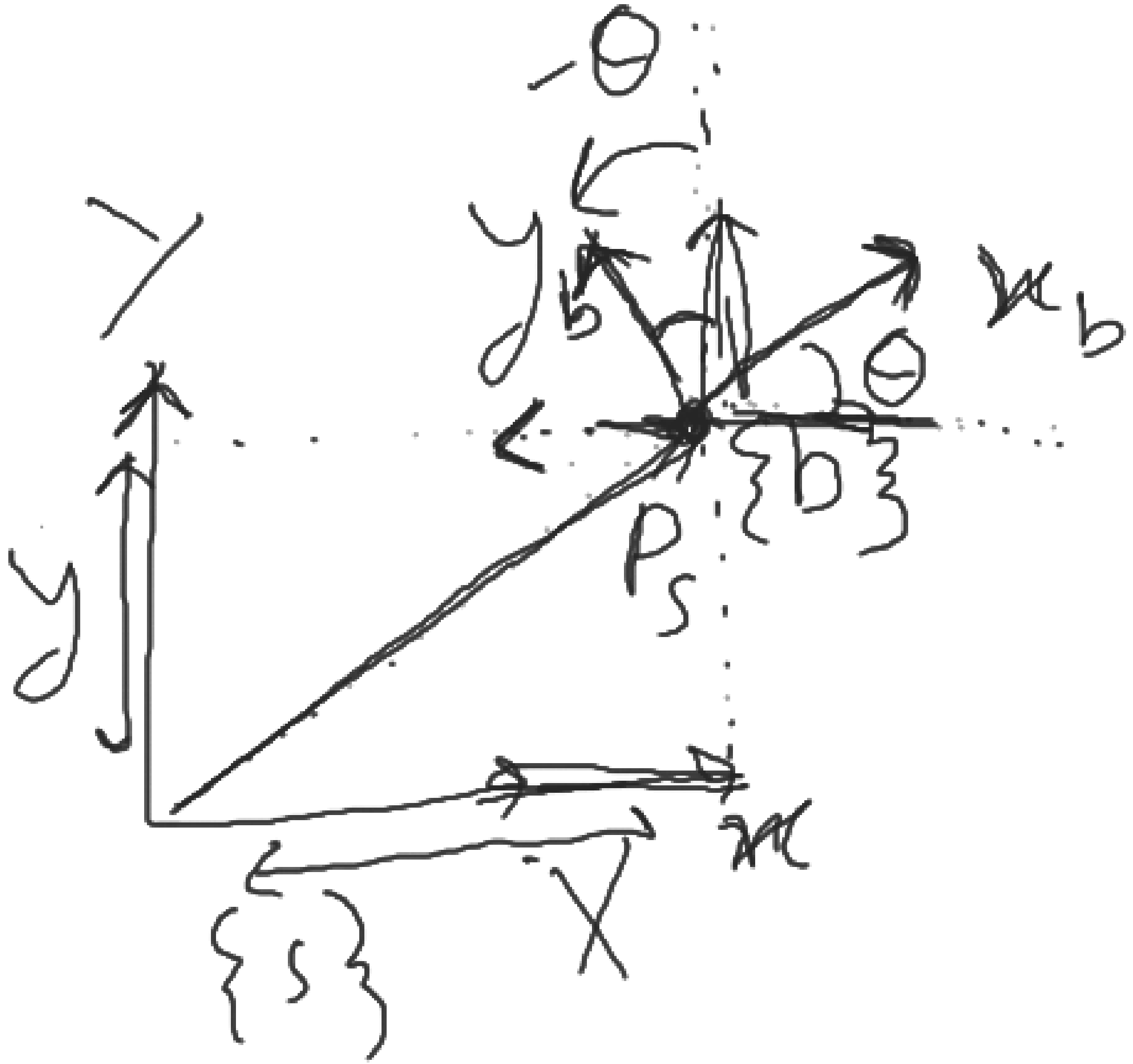
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} x_b \\ y_b \end{bmatrix}$$





$$P_s = x\hat{i} + y\hat{j}$$

$$\boxed{\begin{aligned} x_b &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ y_b &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{aligned}}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\hat{i}$

$\hat{j}$

Constraints:

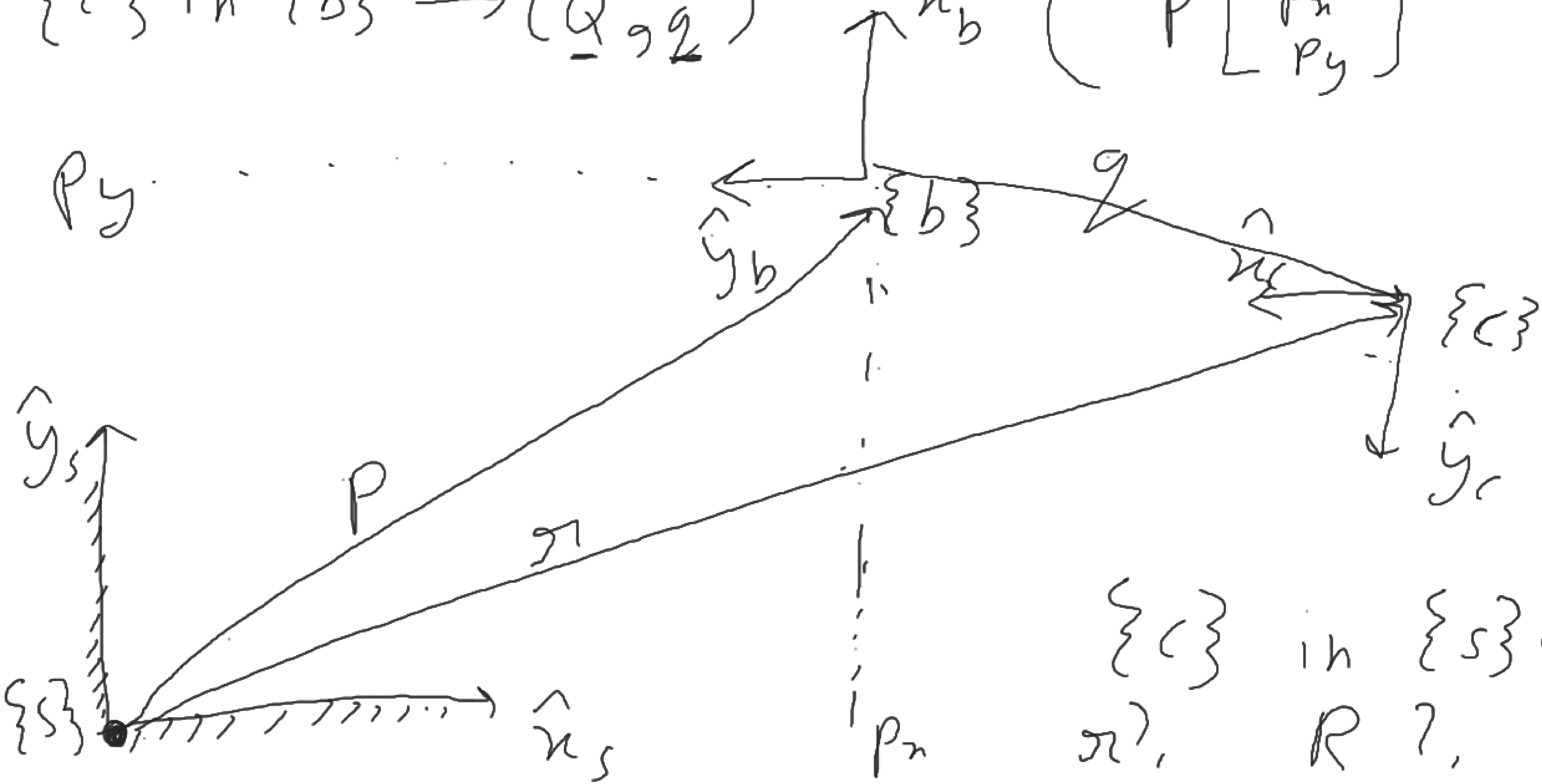
✓ ① Each column → Unit vector

✓ ② Columns must be orthogonal

$\{b\}$ in $\{s\} \longrightarrow (\underline{P}, \underline{p})$

$\{c\}$ in $\{b\} \longrightarrow (\underline{Q}, \underline{q})$

$$\left\{ \begin{array}{l} P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ p \begin{bmatrix} p_x \\ p_y \end{bmatrix} \end{array} \right.$$



$\{c\}$ in $\{s\}$?,

r?, R ?,

$Pq + \pi \rightarrow$ Converting q to $\{s\}$

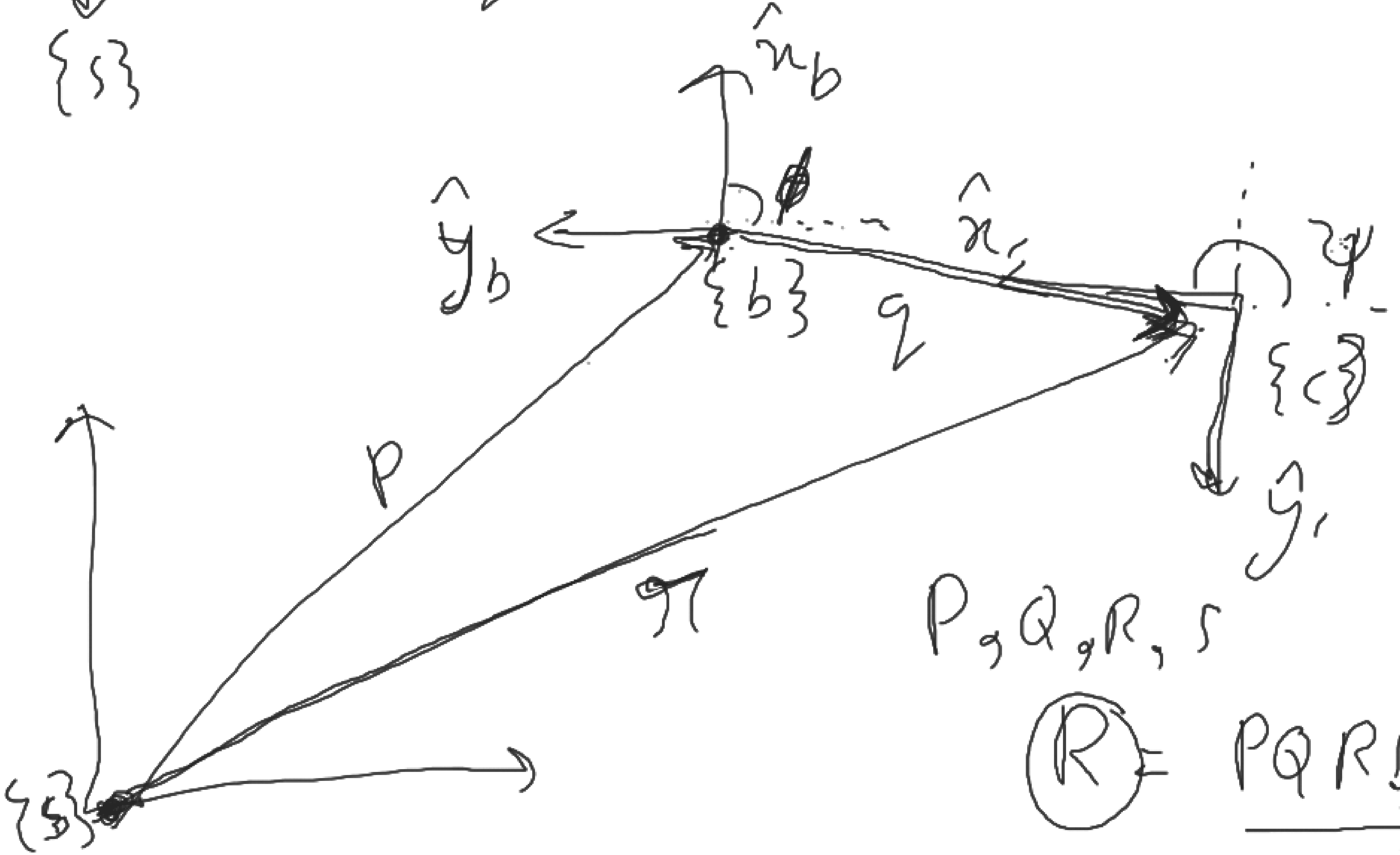
$\underbrace{Pq}_{\downarrow \{s\}}$

(P, p)

(Q, q)

$$\underline{P} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$



$$\begin{tikzcd} \{S\} \arrow[r, "\underline{P}"] \arrow[rr, bend right, "?"] & \{b\} \arrow[r, "\underline{Q}"] & \{c\} \end{tikzcd}$$

$$R = \underline{P}\,\underline{Q} \begin{bmatrix} x \\ y \end{bmatrix}$$

P, Q, R, S

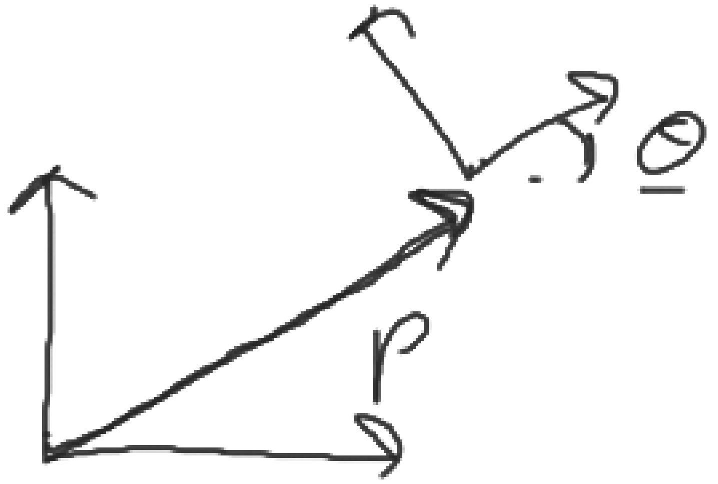
$$R = PQRS \longrightarrow$$

Uses:

1. Represent Orientation $\rightarrow$ θ, $R = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

2. Change Reference Frame $\rightarrow$ $\boxed{P_{ab} q_b = q_a}$

P, p



$$\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

$R \rightarrow 2 \times 2$

$p \rightarrow 2$

3×3

$$\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Homogeneous Transformation}$$

$\boxed{T_{bs}}$

$$T_{bs} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & p_x \\ \sin\theta & \cos\theta & p_y \\ 0 & 0 & 1 \end{bmatrix}}$$ 3x3

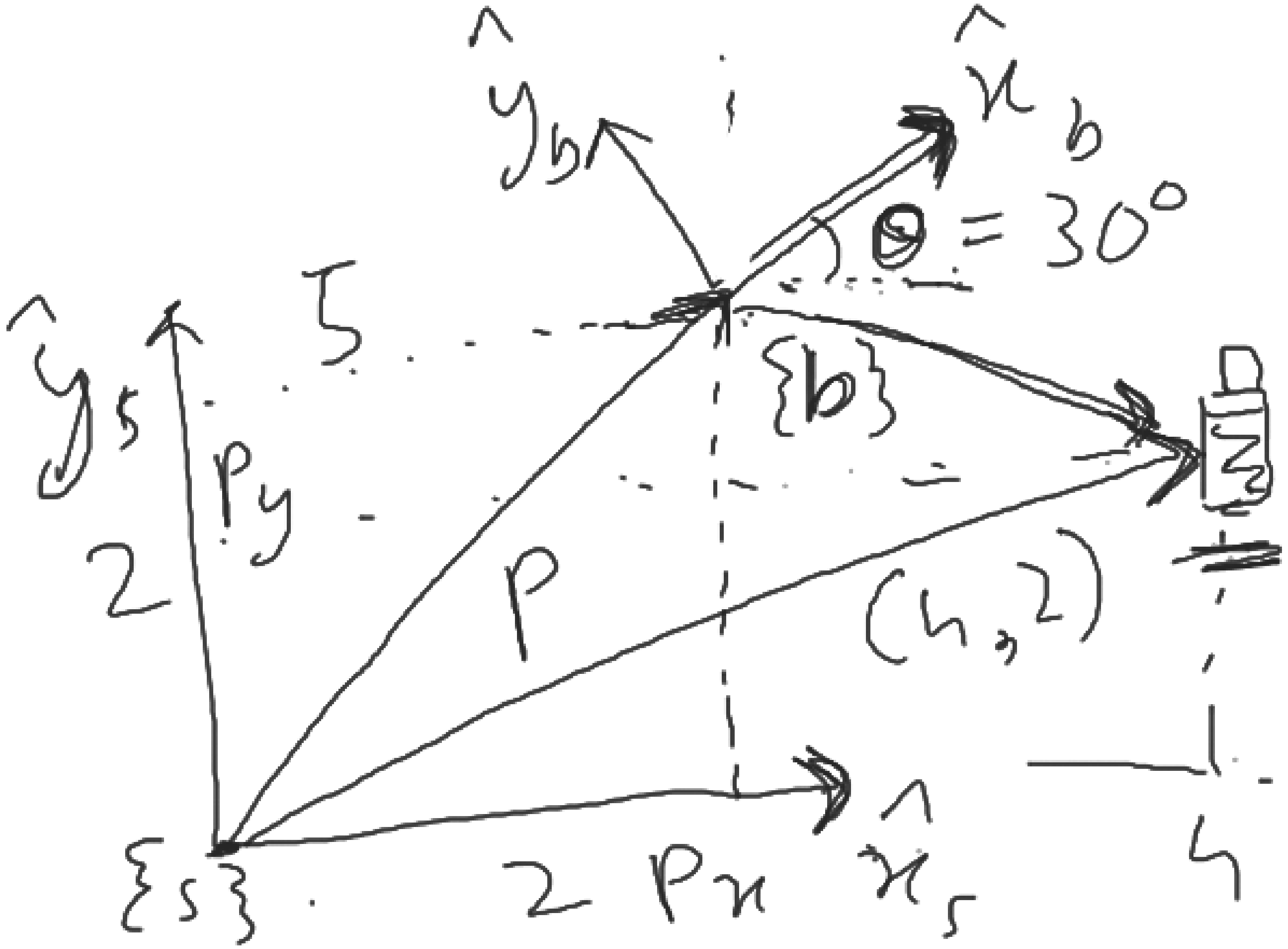
$\{b\} \rightarrow$ Robot

$\{s\} \rightarrow$ Room

$1 \rightarrow$ Scaling factor

$$q_s = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$ w.r.t room

$q_b = ?$, w.r.t robot



(4, 2)

$\theta_b = 0°$

$$T_{\cancel{b}s}\, q_{\cancel{b}} = q_s$$

$$= \begin{bmatrix} \cos 30 & -\sin 30 & 2 \\ \sin 30 & \cos 30 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Uses:

1. Represent an object w.r.t frame
2. Transform frame to another frame

2D $\longrightarrow$ 3D

Properties R

1. $R^T = R^{-1} \Rightarrow R^T R = I$
2. $\det R = 1$
3. $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
   Associative ✓
4. $R_1 R_2 \neq R_2 R_1$ ✗

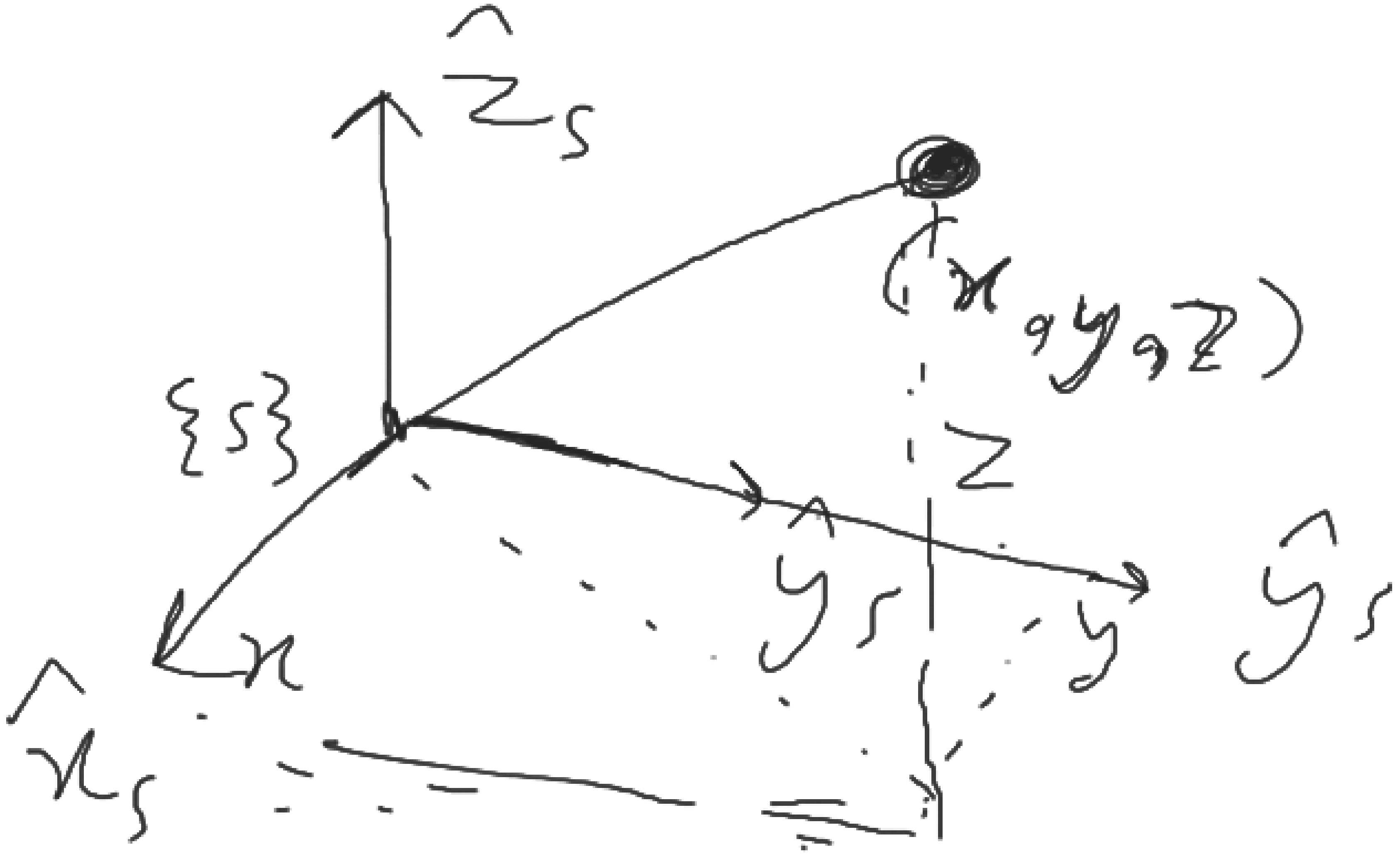
Properties T

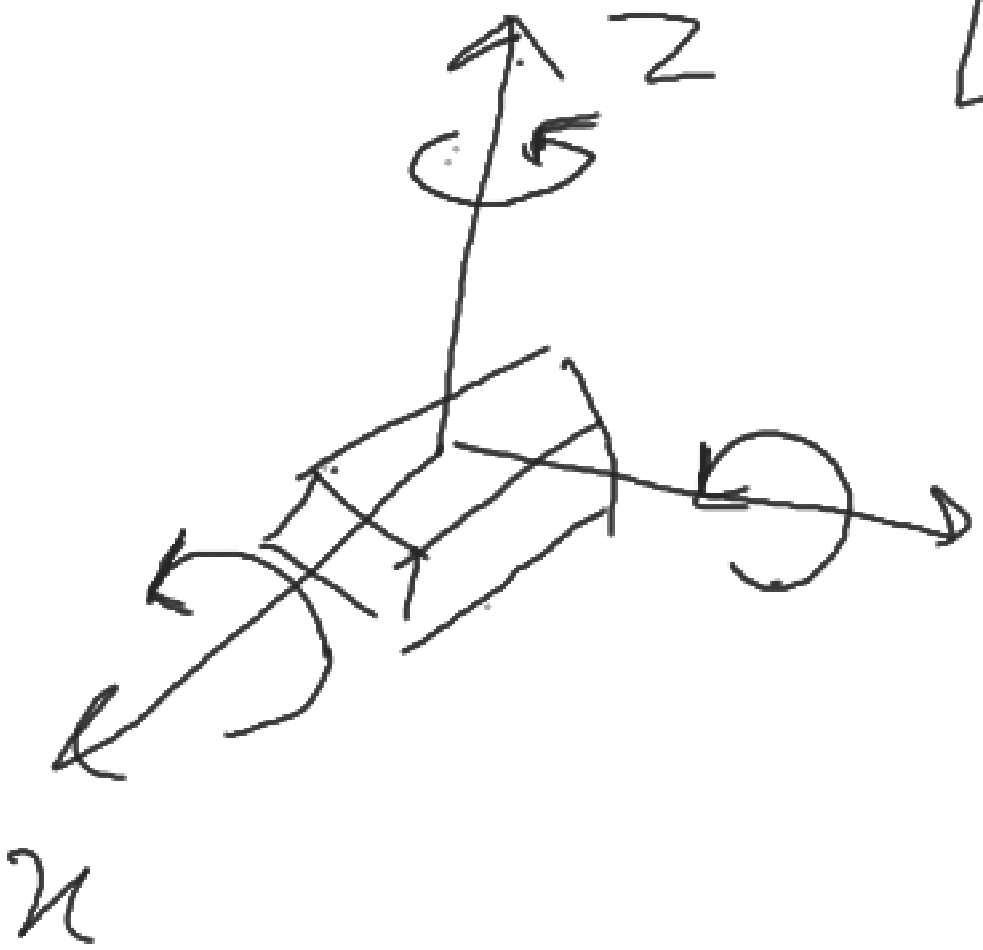
1. $T_{sb} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

$T_{sb}^{-1} = T_{bs}$

$T_{sb}^{-1} = T_{bs} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$

Euler Angles



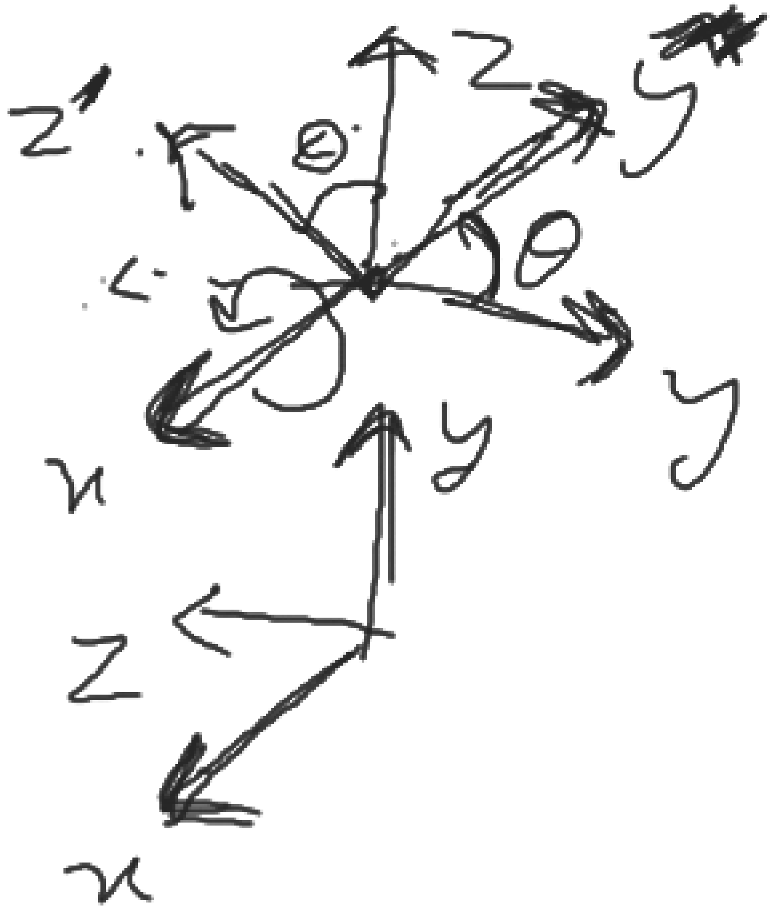


$[\beta_x, \beta_y, \beta_z]$

$(\theta_x, \theta_y, \theta_z)$

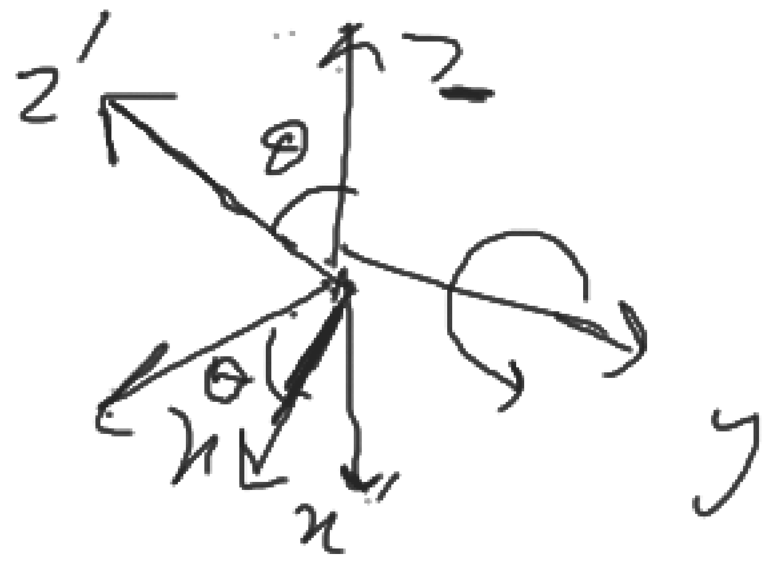
R_x

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \\ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \end{array}$$



R_y

$$\begin{array}{c} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \end{array}$$



(3×3)

R_z

$$\begin{array}{c} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \end{array}$$

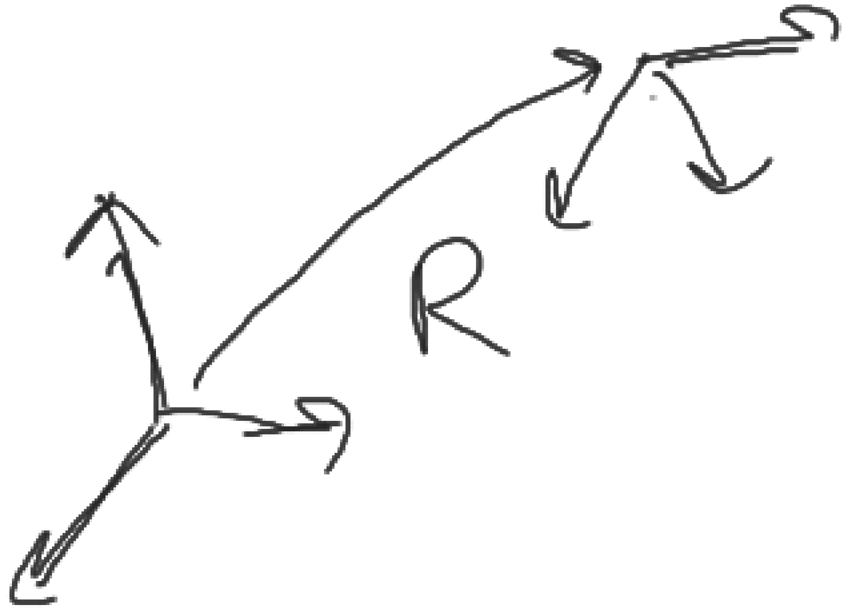
$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2D

(2×2)

Euler Angles

$$R = \underline{R_x}\,\underline{R_y}\,\underline{R_z} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$



$p \Rightarrow \hat{w} = \_\,\hat{i} + \_\,\hat{j} + \_\,\hat{k}$

$\mathrm{Rot}(\hat{w}, \theta) \rightarrow$ Fn plane

$= \begin{bmatrix} c\theta + \hat{w}_1^2(1 - c\theta) & & \\ & & \\ & & \end{bmatrix}$

2 ways Euler

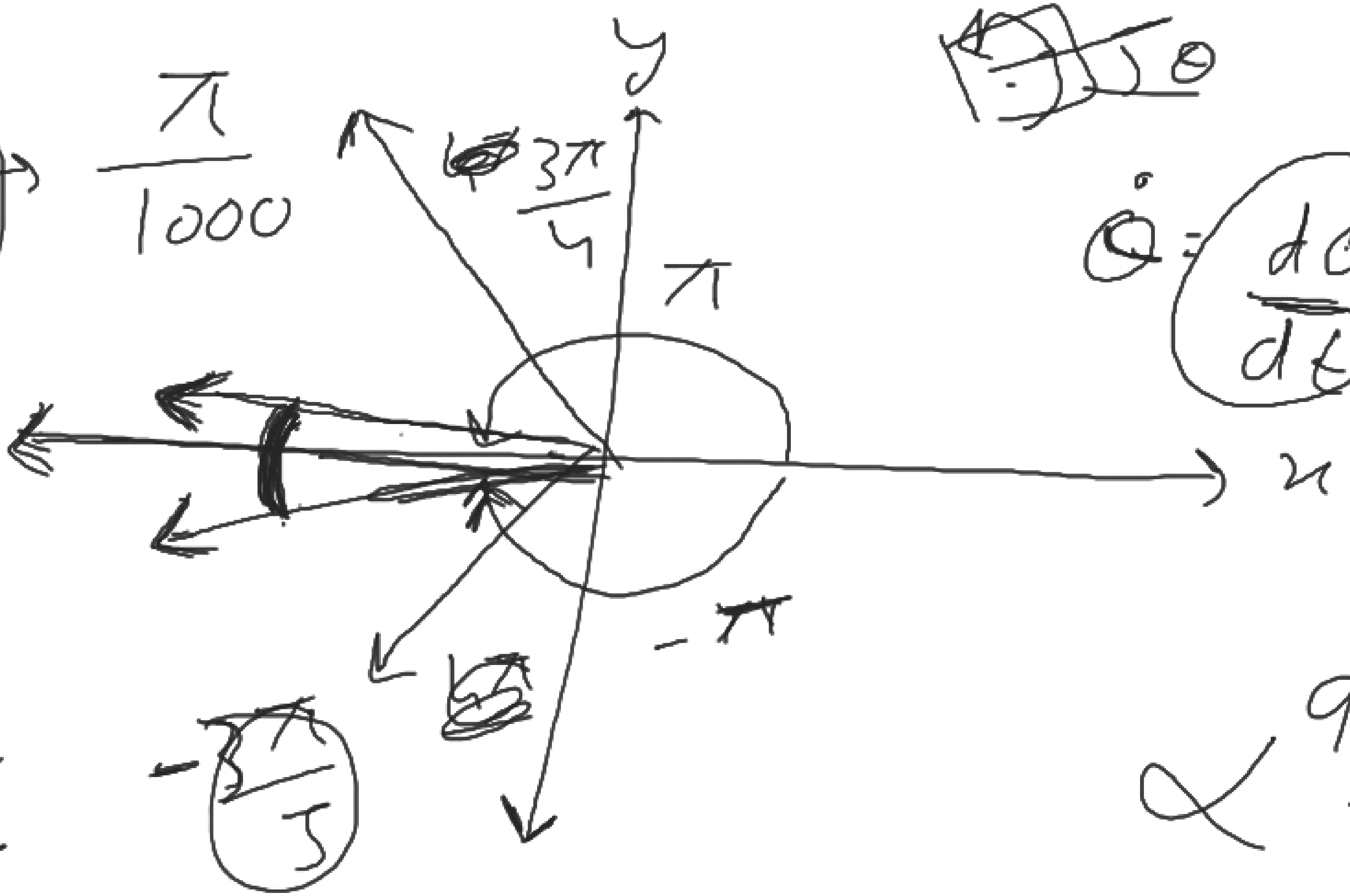
1. $R_x, R_y, R_z \rightarrow R_x R_y R_z = R$

2. $Rot(\hat{\omega}, \theta) \rightarrow$ [Find out]

Euler → Gimbal Lock

$$\frac{999\pi}{1000} \rightarrow \frac{\pi}{1000}$$

$$-\frac{999\pi}{1000} \rightarrow -\frac{\pi}{1000}$$



$$\frac{2\pi}{1000}$$

Euler

$\in (-\pi, \pi)$

↳ $(-180, 180)$

$$\dot{\theta} = \frac{d\theta}{dt}$$

hard code

if $\theta > \pi$

$\pi - \theta$

$$\alpha \; \frac{999+999}{1000}$$

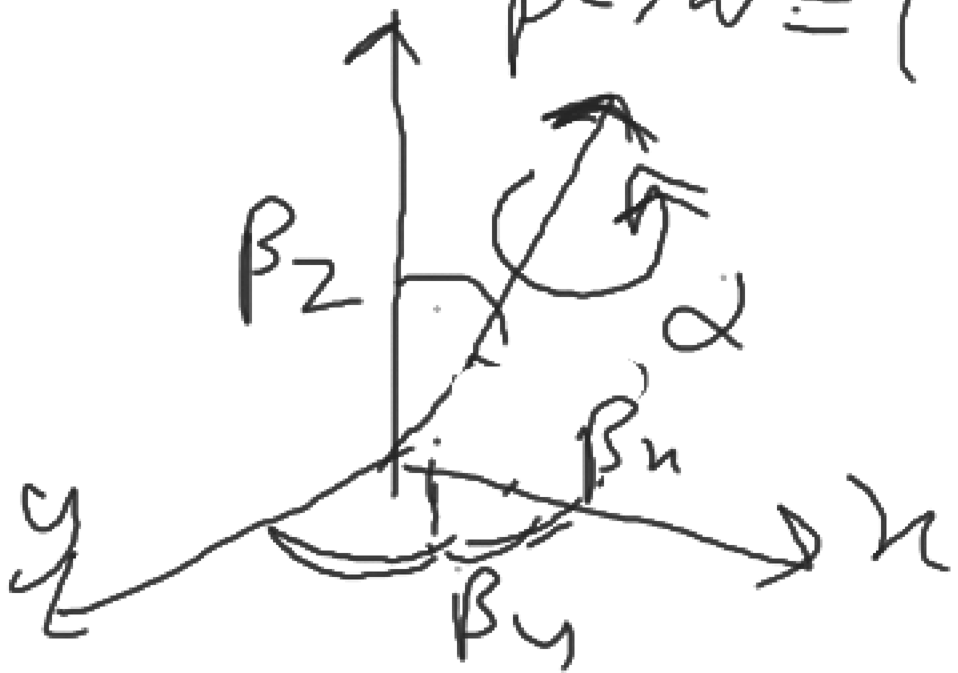
Continuous Representation

$\hookrightarrow$ Quaternion $\rightarrow$ differentiable

$$q = [q_w \quad q_x \quad q_y \quad q_z]$$

$$|q|^2 = q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1$$

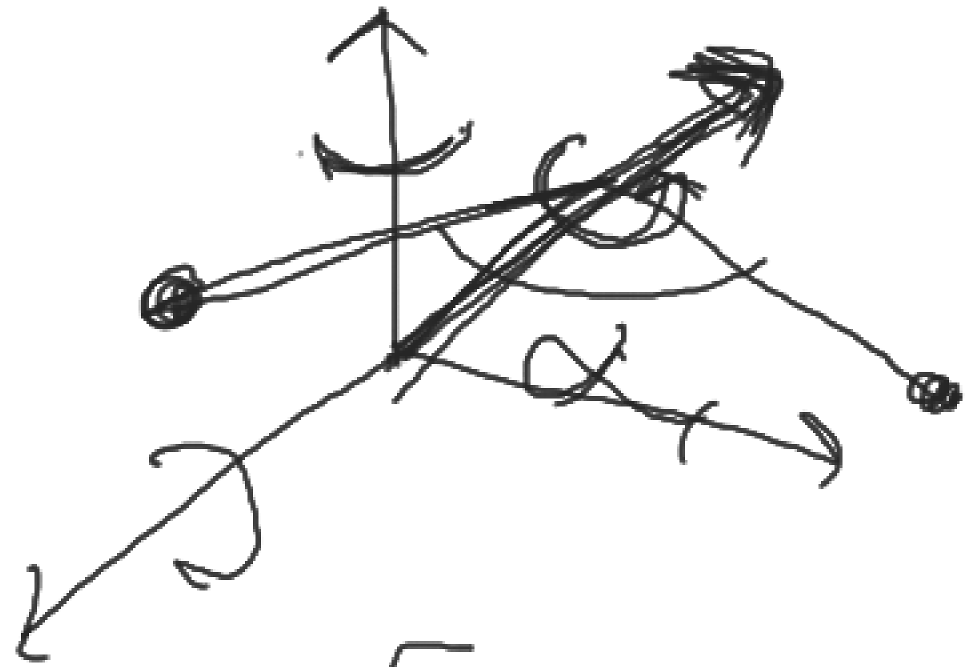
$p \Rightarrow \hat{w} = (\beta_x, \beta_y, \beta_z)$

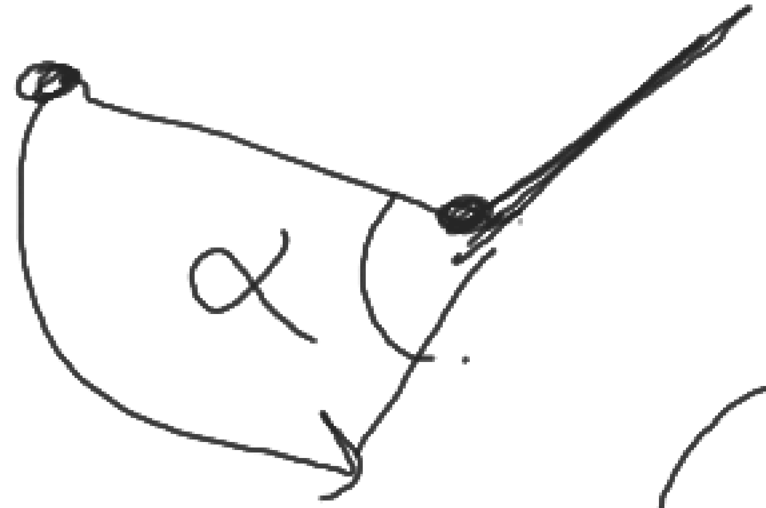


$$q_w = \cos\left(\alpha/2\right)$$

$$q_x = \sin\left(\frac{\alpha}{2}\right)\cos(\beta_x)$$

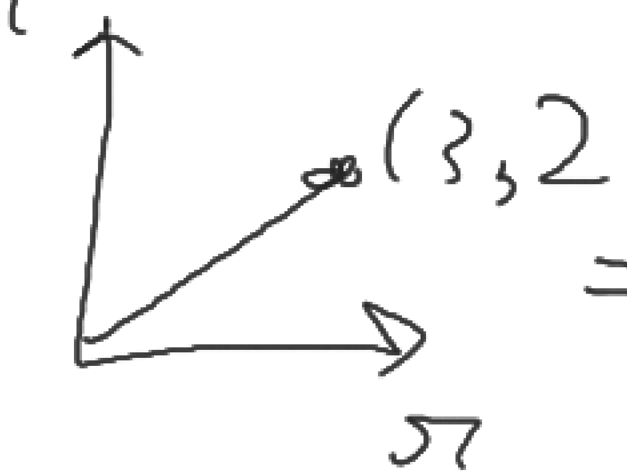
$$q_y = \sin\left(\frac{\alpha}{2}\right)\cos(\beta_y) \qquad q_z = \sin\left(\frac{\alpha}{2}\right)\cos(\beta_z)$$



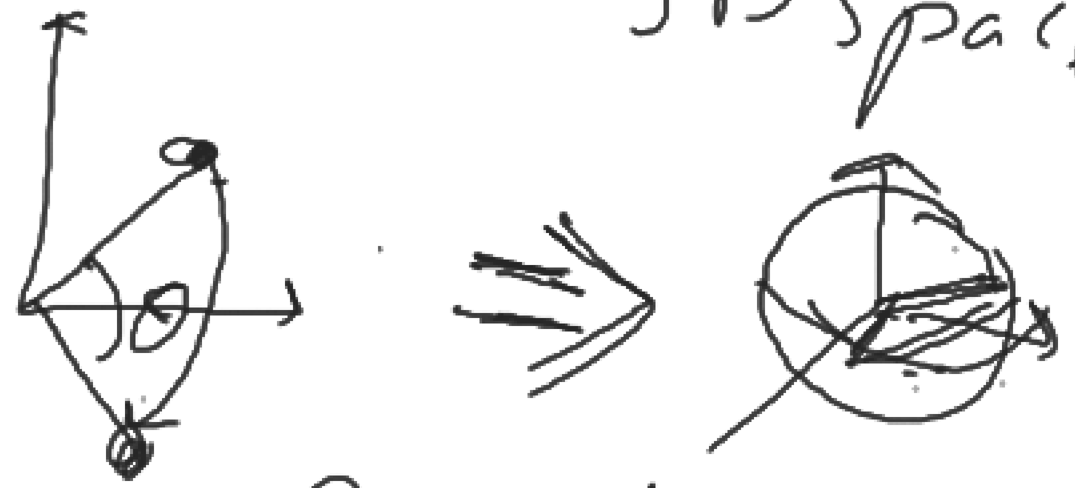


$$[q_w, q_x, q_y, q_z] \rightarrow \text{3D rotation in a 4D space}$$

$$q_w + \underline{q_x i} + \underline{q_y j} + \underline{q_z k} \rightarrow \text{2D rotation in a 3D space}$$



$$\Rightarrow \underbrace{3}_{R} + \underbrace{2i}_{C}$$



→ Project 3D to 2D

Euler $\rightarrow$ Quaternion

$\hat{w} = \hat{x}$, $\hat{w} = \hat{y}$

$\hat{w} = \hat{z}$

$R_x(\theta_x)$

$q_w = \cos\frac{\alpha}{2}$

$q_x = \sin\frac{\alpha}{2}$ $\quad \cos(0) \rightarrow 1$

$q_y = \sin\frac{\alpha}{2} \cdot \cos(\beta_y) = 0$

$q_z = 0$

$R_x(\theta) = [\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, 0, 0]$

$R_y(\theta_y)$

$\rightarrow [\cos\frac{\alpha}{2}, 0, \sin\frac{\alpha}{2}, 0]$

$q_w = \cos\frac{\alpha}{2}$

$q_x = \sin\frac{\alpha}{2} \cos(\beta_x) = 0$

$q_y = \sin\frac{\alpha}{2} \cos(0) = \sin\frac{\alpha}{2}$

$q_z = 0$

$R_z(\theta_z)$

$\hookrightarrow [\cos\frac{\alpha}{2}, 0, 0, \sin\frac{\alpha}{2}]$

$\frac{9\pi}{10} \to \frac{11\pi}{10} \left(-\frac{9\pi}{10}\right)$

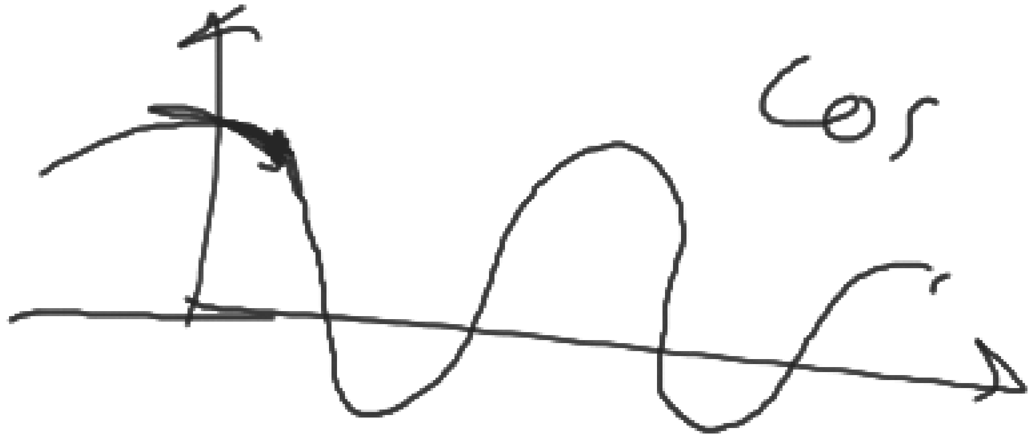
$$\boxed{R_x(\theta_x)} = \cos\left(\frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right), 0, 0$$

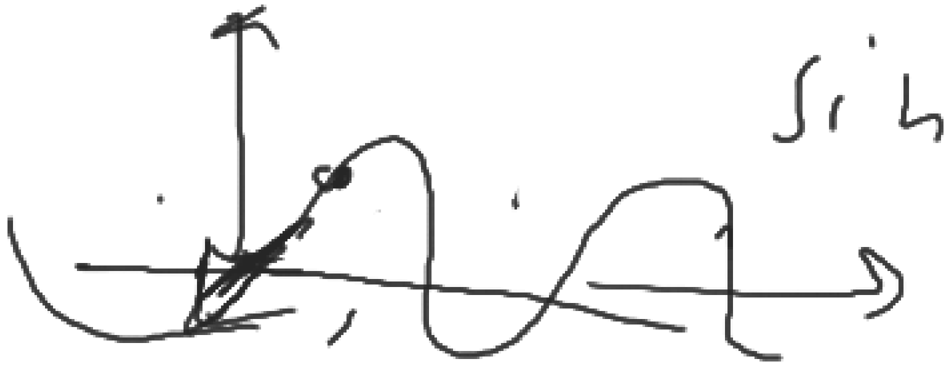
$$\boxed{R_y(\theta_y)} = \cos\left(\frac{\alpha}{2}\right), 0, \sin\left(\frac{\alpha}{2}\right), 0$$

$$\boxed{R_z(\theta_z)} = \cos\left(\frac{\alpha}{2}\right), 0, 0, \sin\left(\frac{\alpha}{2}\right)$$

$\frac{9\pi}{10}$ $-\frac{9\pi}{10}$

[_ _]





Mobile Robots & Drones

Quaternions $\longleftrightarrow$ Euler