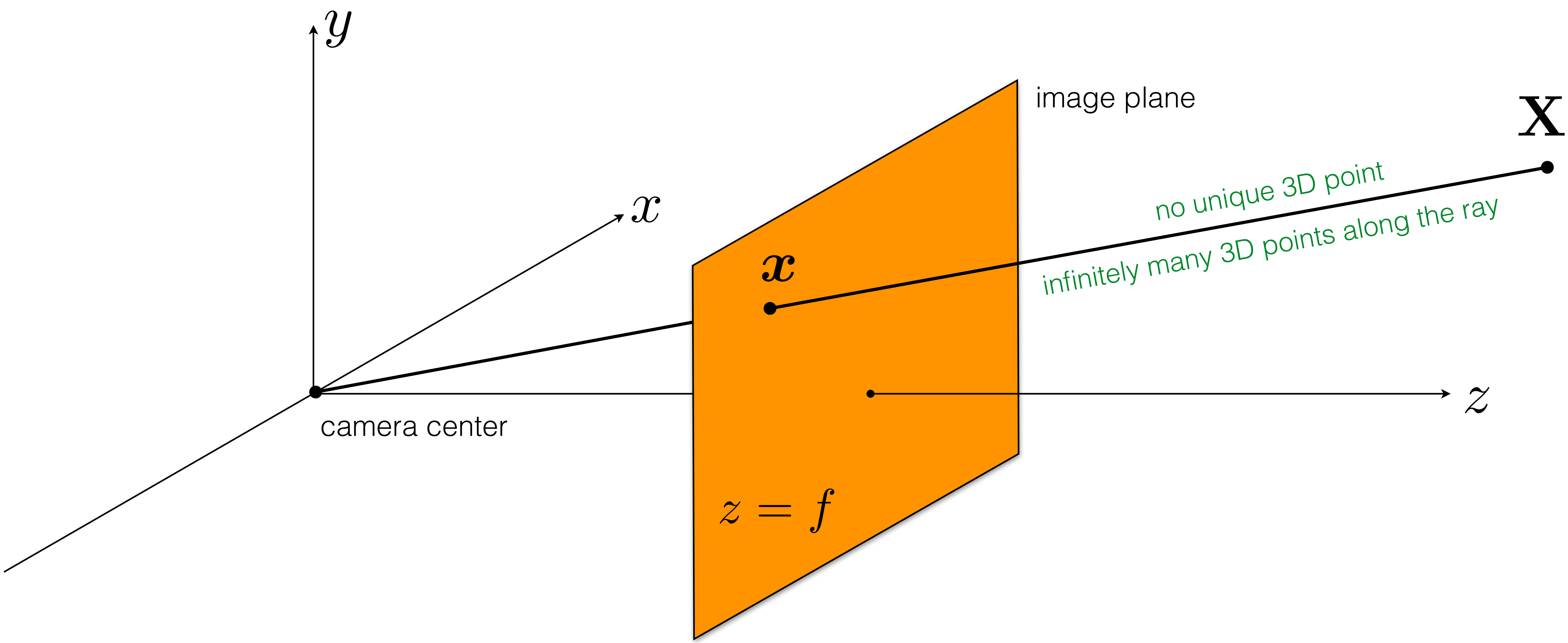
$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

known known

*Can we compute **X** from a single correspondence **x**?*



y
x
image plane
$\mathbf{X}$
no unique 3D point
infinitely many 3D points along the ray
$\boldsymbol{x}$
z
camera center
$z = f$

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

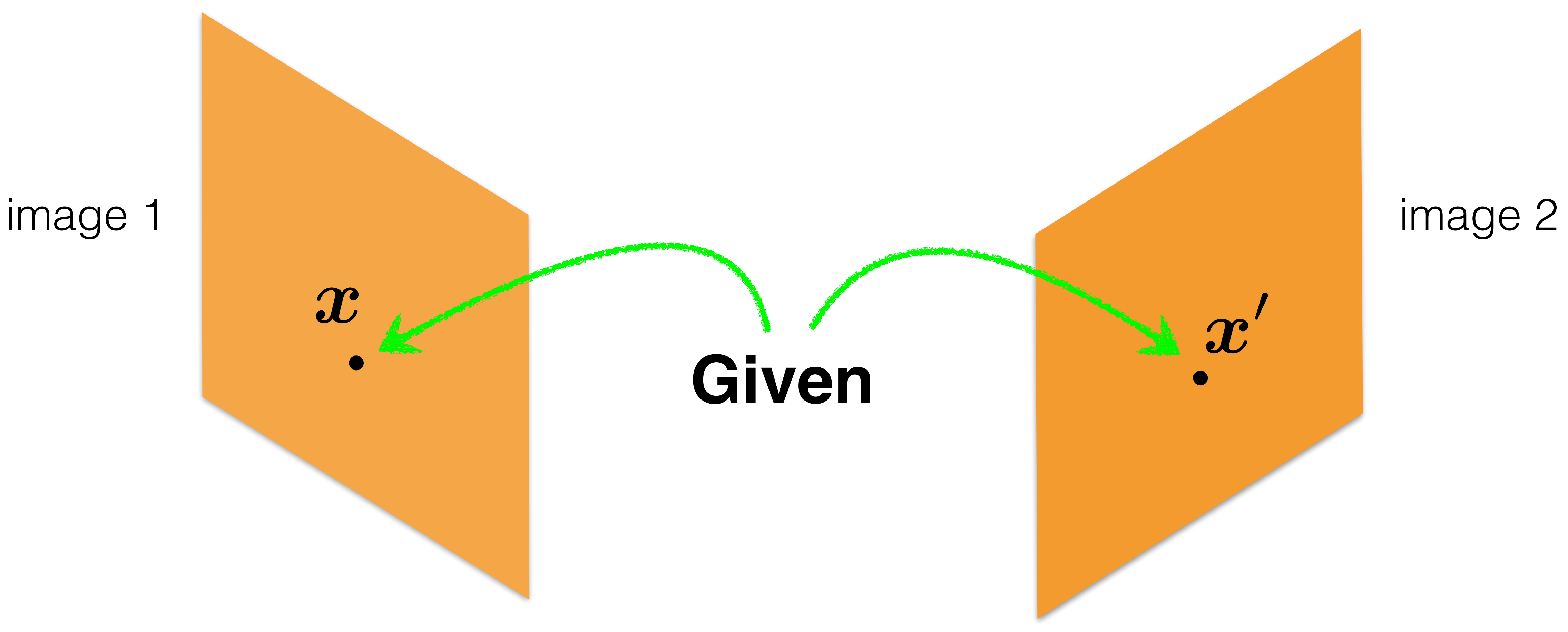
known known

Can we compute ***X*** *from* <u>*two*</u> *correspondences* ***x*** *and* ***x'****?*

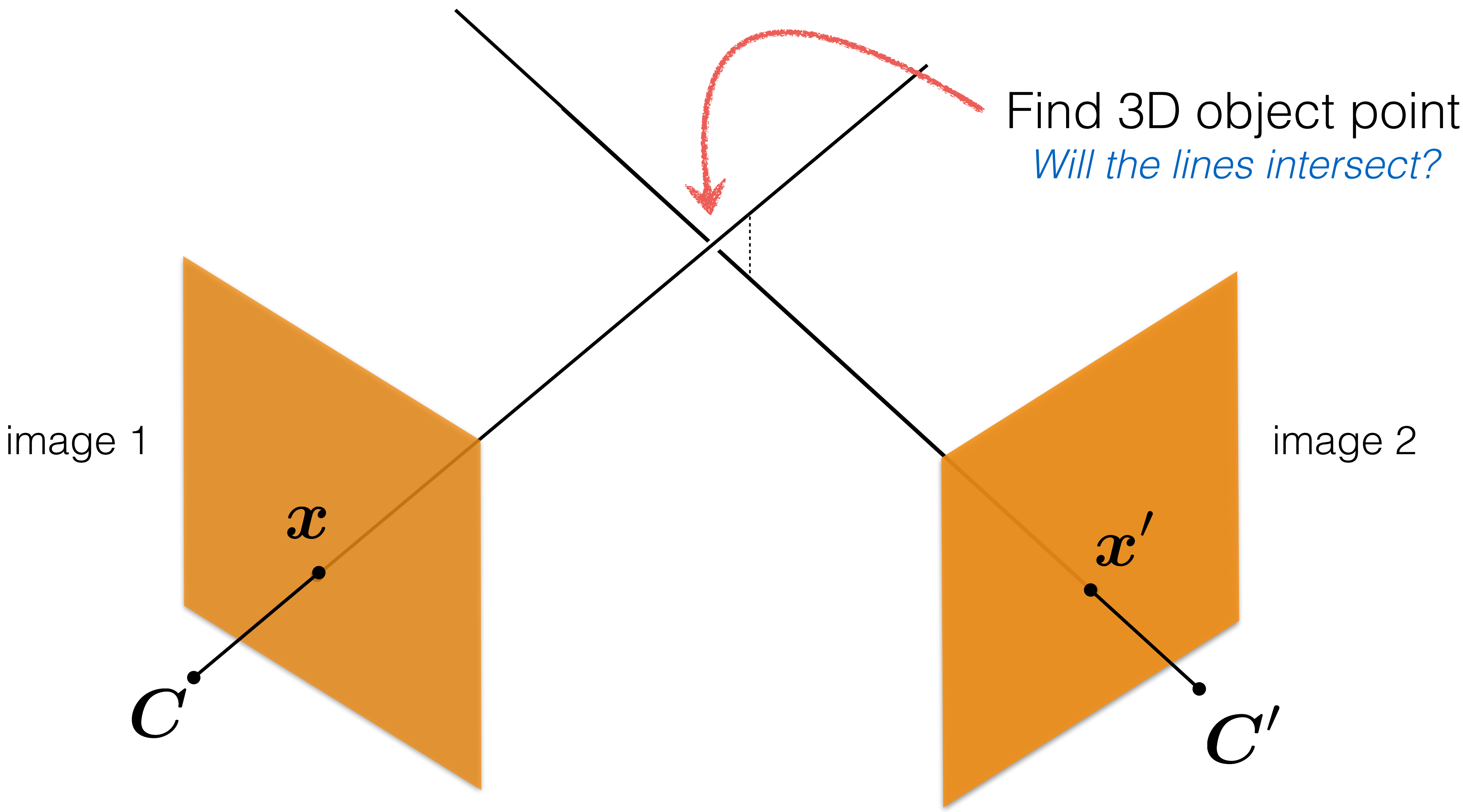
$$P = KR\left[I \mid -X_0\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \end{bmatrix}$$

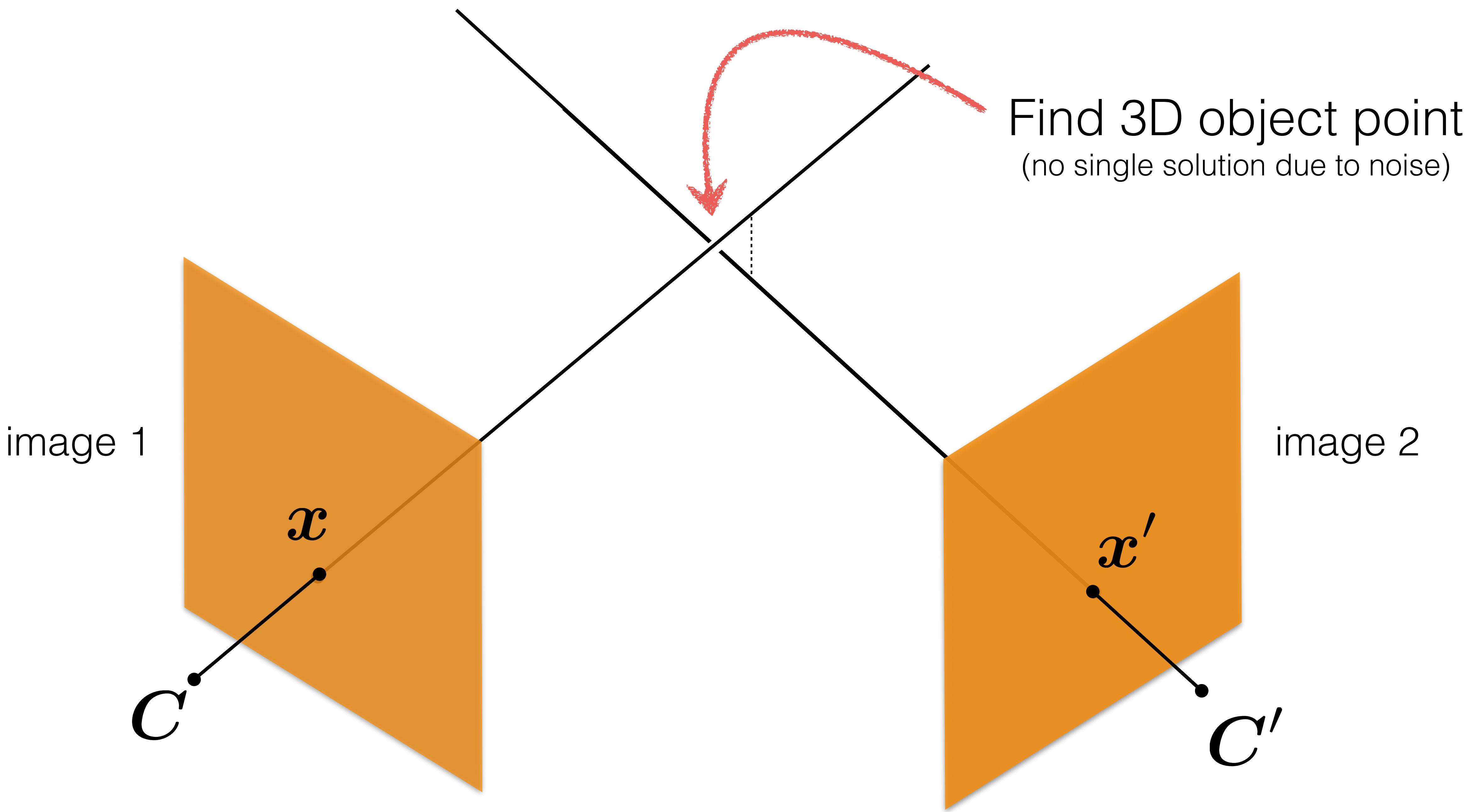
Triangulation



Triangulation



Triangulation



Triangulation

Given a set of (noisy) matched points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

and camera matrices

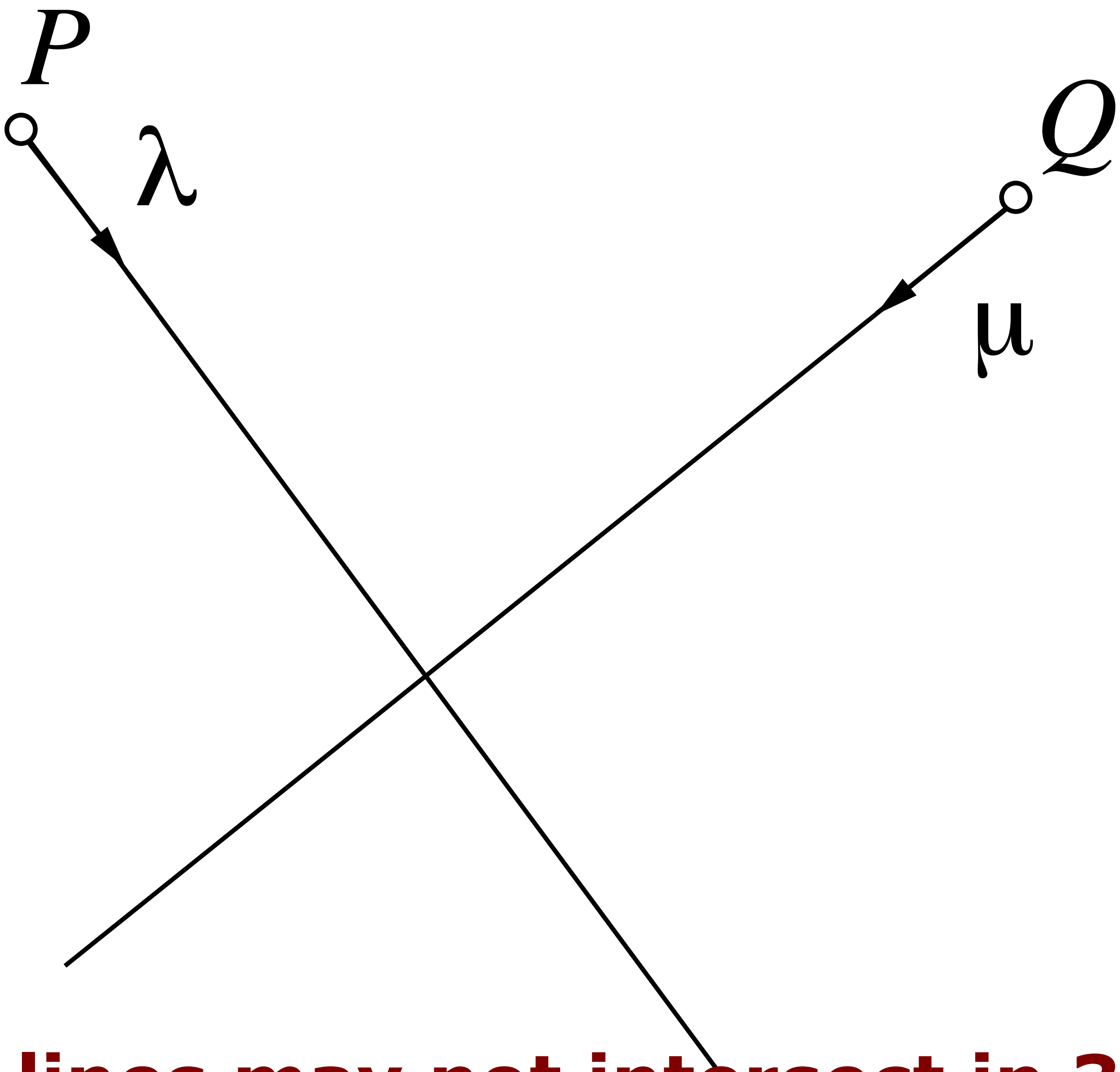
$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{X}$$

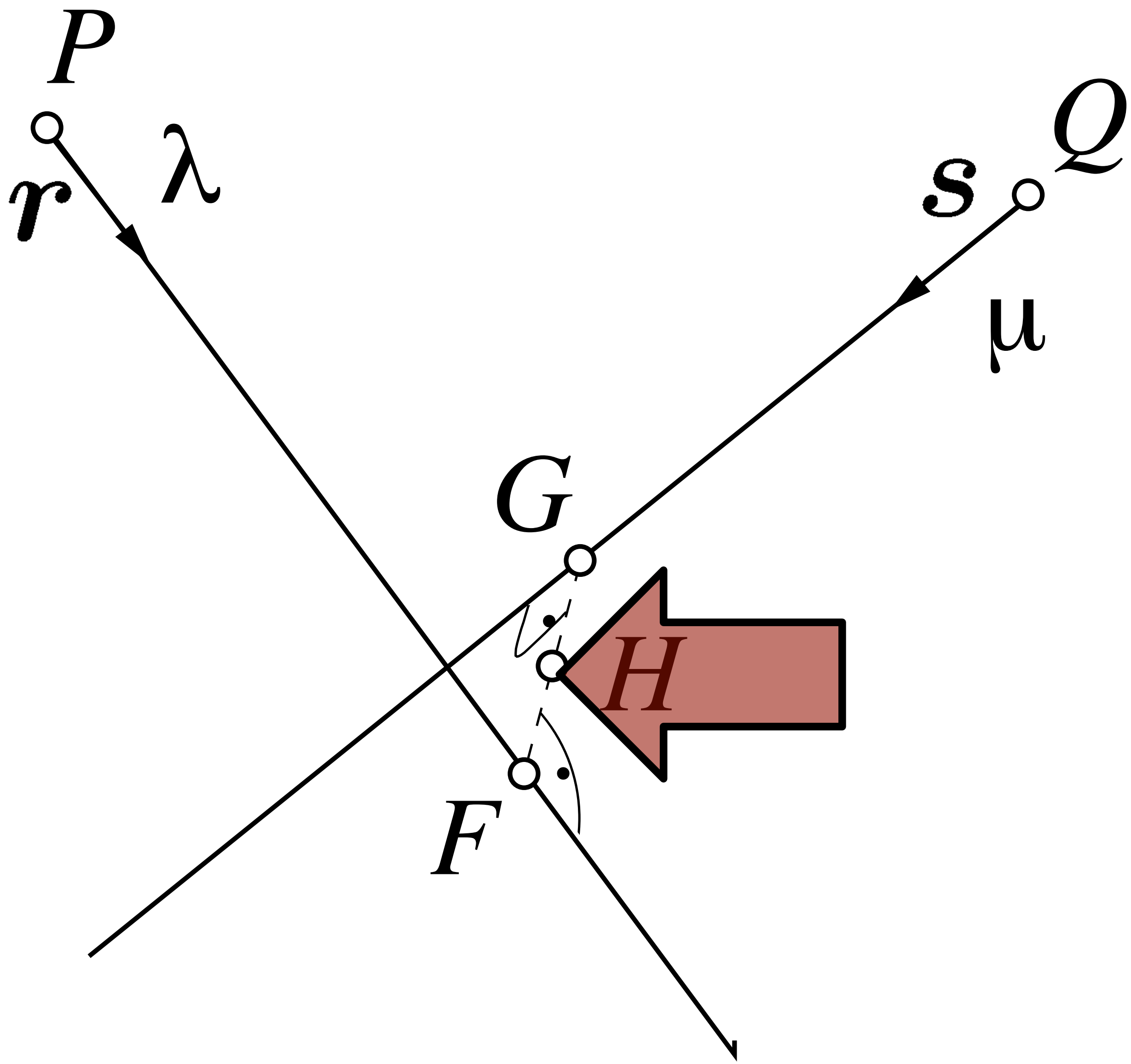
1.
Geometric Approach

The Problem



The lines may not intersect in 3D!

Find the Point H



Geometric Solution

- Equation for two lines in 3D

$$\boldsymbol{f} = \boldsymbol{p} + \lambda \boldsymbol{r} \qquad \boldsymbol{g} = \boldsymbol{q} + \mu \boldsymbol{s}$$

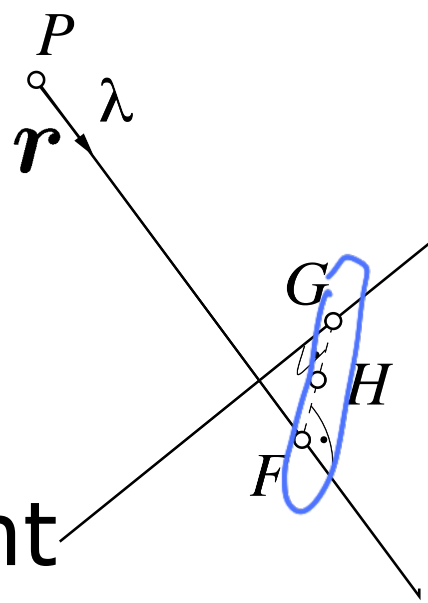
- with the points $\boldsymbol{p} = \boldsymbol{X}_{O'} \quad \boldsymbol{q} = \boldsymbol{X}_{O''}$
- and the directions (calibrated camera)

$$\boldsymbol{r} = R'^{\mathsf{T}}\ {}^{k}\mathbf{x}' \qquad \boldsymbol{s} = R''^{\mathsf{T}}\ {}^{k}\mathbf{x}''$$

$$\sim K^{-1} x \qquad \sim K_2^{-1} x''$$

- with ${}^{k}\mathbf{x}' = (x', y', c)^{\mathsf{T}} \quad {}^{k}\mathbf{x}'' = (x'', y'', c)^{\mathsf{T}}$

Geometric Solution



- The shortest connection requires that FG is orthogonal to both lines
- This leads to the constraint

$$(\boldsymbol{f}-\boldsymbol{g})\cdot\boldsymbol{r}=0 \qquad (\boldsymbol{f}-\boldsymbol{g})\cdot\boldsymbol{s}=0$$

which directly leads to

$$\begin{aligned}(\boldsymbol{q}+\lambda\boldsymbol{s}-\boldsymbol{p}-\mu\boldsymbol{r})\cdot\boldsymbol{s} &= 0\\ (\boldsymbol{q}+\lambda\boldsymbol{s}-\boldsymbol{p}-\mu\boldsymbol{r})\cdot\boldsymbol{r} &= 0\end{aligned}$$

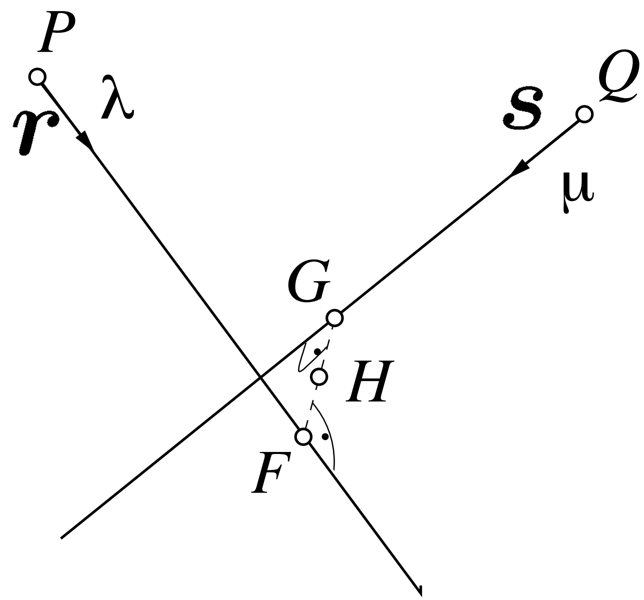
- Two equations, two unknowns

Geometric Solution

- By solving the equations

$$
\begin{aligned}
(\boldsymbol{q} + \lambda \boldsymbol{s} - \boldsymbol{p} - \mu \boldsymbol{r}) \cdot \boldsymbol{s} &= 0 \\
(\boldsymbol{q} + \lambda \boldsymbol{s} - \boldsymbol{p} - \mu \boldsymbol{r}) \cdot \boldsymbol{r} &= 0
\end{aligned}
$$

  we obtain λ, μ



- λ, μ directly yield F and G
- We compute H as the middle of the line connecting F and G

More Concretely...

- We can transforms

$$(\boldsymbol{X}_{O'} - \boldsymbol{X}_{O''})^\top \boldsymbol{r} + \lambda \boldsymbol{r}^\top \boldsymbol{r} - \mu \boldsymbol{s}^\top \boldsymbol{r} = 0$$
$$(\boldsymbol{X}_{O'} - \boldsymbol{X}_{O''})^\top \boldsymbol{s} + \lambda \boldsymbol{r}^\top \boldsymbol{s} - \mu \boldsymbol{s}^\top \boldsymbol{s} = 0$$

- into matrix form

$$\begin{bmatrix} \boldsymbol{r}^\top \boldsymbol{r} & -\boldsymbol{s}^\top \boldsymbol{r} \\ \boldsymbol{r}^\top \boldsymbol{s} & -\boldsymbol{s}^\top \boldsymbol{s} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} (\boldsymbol{X}_{O''} - \boldsymbol{X}_{O'})^\top \\ (\boldsymbol{X}_{O''} - \boldsymbol{X}_{O'})^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}$$

More Concretely...

- So that we can solve

$$\underbrace{\begin{bmatrix} \boldsymbol{r}^\top \boldsymbol{r} & -\boldsymbol{s}^\top \boldsymbol{r} \\ \boldsymbol{r}^\top \boldsymbol{s} & -\boldsymbol{s}^\top \boldsymbol{s} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \lambda \\ \mu \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} (\boldsymbol{X}_{O''} - \boldsymbol{X}_{O'})^\top \\ (\boldsymbol{X}_{O''} - \boldsymbol{X}_{O'})^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}}_{\boldsymbol{b}}$$

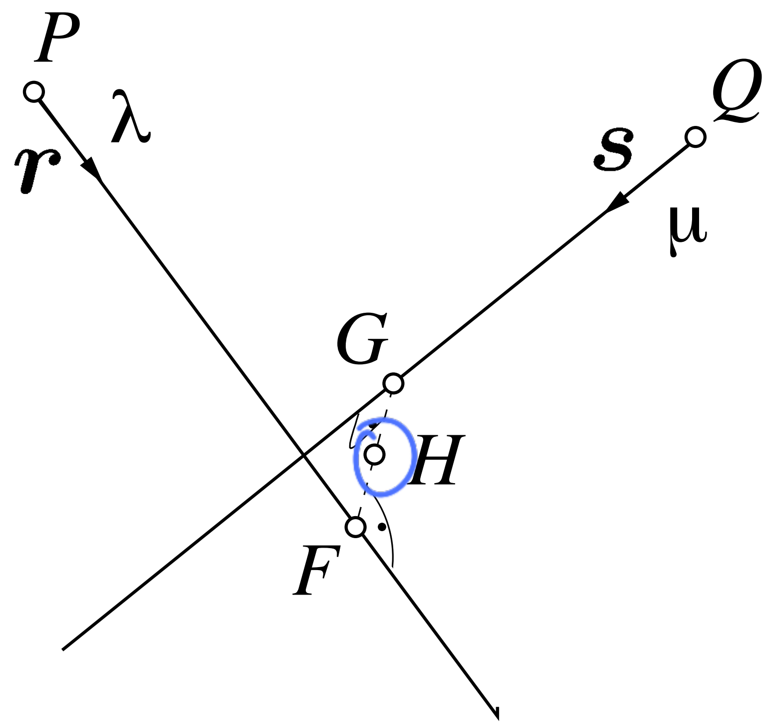
$x = A^{-1} b$

- with our standard $A\boldsymbol{x} = \boldsymbol{b}$ formulation
- Knowing $\underline{\lambda}, \underline{\mu}$ allows us to compute the intersecting point

Solution

- λ, μ directly yield F and G
- The 3D point H is the middle of the line connecting F and G
- The solution is:

$$\boldsymbol{H} = \frac{\boldsymbol{F} + \boldsymbol{G}}{2}$$



Geometric Solution

- Simple 3D geometry allows us to compute a solution
- Boils down to solving a system of two linear equations with two unknowns
- Does not take into account uncertainties, not statistically optimal

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

known known

Can we compute ***X*** *from <u>two</u> correspondences* ***x*** *and* ***x'****?*

yes if perfect measurements

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

known known

*Can we compute **X** from <u>two</u> correspondences **x** and **x'**?*

yes if perfect measurements

There will not be a point that satisfies both constraints because the measurements are usually noisy

$$\mathbf{x}' = \mathbf{P}'\boldsymbol{X} \quad \mathbf{x} = \mathbf{P}\boldsymbol{X}$$

Need to find the **best fit**

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(inhomogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

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How do we solve for unknowns in a similarity relation?

Direct Linear Transform

Remove scale factor, convert to linear system and solve with

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(inhomogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

Direct Linear Transform

Remove scale factor, convert to linear system and solve with SVD.

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \boldsymbol{X} = \mathbf{0}$$

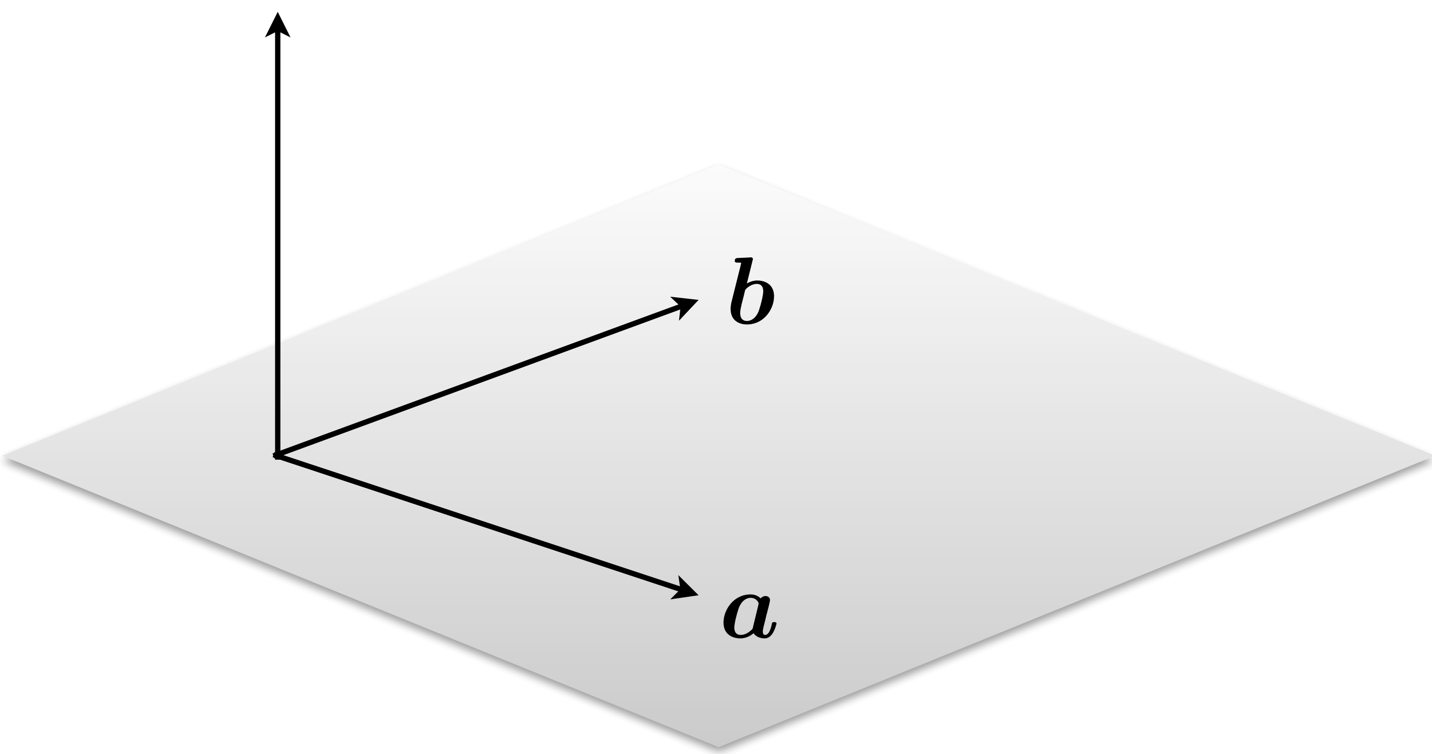
Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

Recall: Cross Product

Vector (cross) product

takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{a} \times \boldsymbol{b} = \left[\begin{array}{c} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{array} \right]$$

cross product of two vectors in the same direction is zero

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

$$\begin{bmatrix} x \\ y \\ 1\,\cancel{z} \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1\,\cancel{z} \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1\,\cancel{z} \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you [redacted] equations

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\left[\begin{array}{c} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

$$\underbrace{\left[\begin{array}{c} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \end{array}\right]} \boldsymbol{X} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

Now we can make a system of linear equations
(two lines for each 2D point correspondence)

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\boxed{\mathbf{A}\boldsymbol{X} = \mathbf{0}}$$

How do we solve homogeneous linear system?

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S V

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S V D

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S V D !