Recursion Tree Method

$T(n)$ TC of D and C with n input.

$f(n)$ is TC ~~of~~ to divide the algo.

$g(n)$ TC to solve the subproblem small (P).

$$T(n) = \begin{cases} g(n) = \Theta(1) & n = 1 \\ a\,T(n/b) + f(n) & n > 1 \end{cases}$$

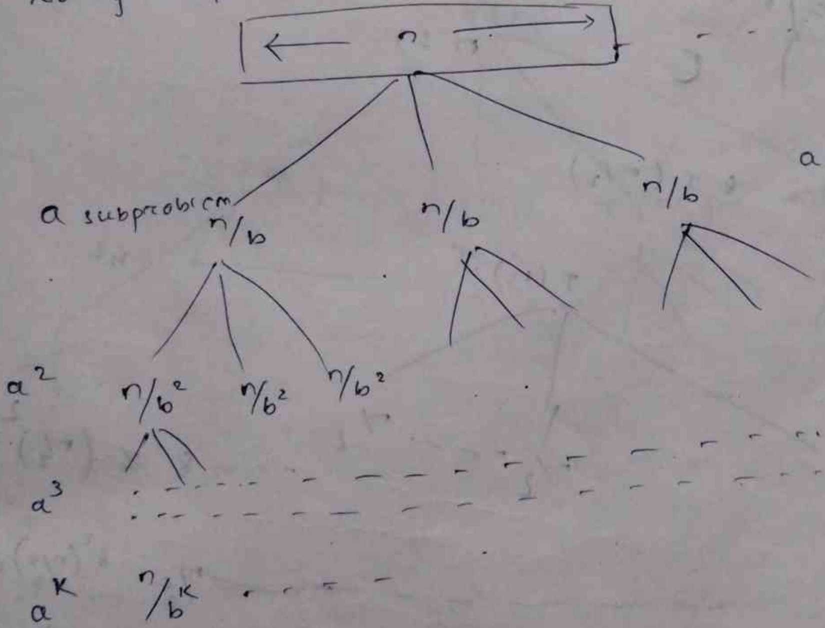
Recurrence Relⁿ

- substituting method.
- Tree method.
- Master Method.
- Change of variable.
- Homogeneous equation.

Tree Method

$$T(n) = a\,T(n/2) + f(n)$$

No. of subproblems.



$f(n)$.

$a * f(n/n)$.

$a^2 * f(n/b^2)$

$a^{k-1} * f(n/b^{k-1})$

$a^k * f(n/b^k)$

$$T(n) = a^k\,T(n/b^k) + \sum_{i=n}^{k-1} a^i f(n/b^i)$$

$$\frac{n}{b^k} = 1 \Rightarrow n = b^k \Rightarrow \cancel{n = \log_b k}$$

$$k = \log_b n$$

depth of the tree $= \cancel{k}+1 = \log_2 n + 1$

space complexity $= \Theta(\log_n n)$

$$T(n) = a^k T(1) + \sum_{i=0}^{k-1} a^i f(n/b^i)$$

$$= a^{\log_b n} \theta(1) + (\sum (a^i f(n/b^i))$$

$$\boxed{T(n) = n^{\log_b a} \cdot \theta(1) + \sum_{i=0}^{\log_b n} a^i f(n/b)}$$

No of function calls $= 1 + a + a^2 + \cdots + a^k$

$$= \frac{a^{k+1} - 1}{a - 1} = \frac{a^{\log_b n + 1} - 1}{a - 1} = a^{\cancel{\log}}_n$$

$$a \cdot n^{\log_b a} - 1$$

$$= \theta(n^{\log})$$

(1) $T(n) = \begin{cases} 8T(n/2) + n^2 & n > 1 \\ C & n = 1 \end{cases}$

$T(n) = 8T(n/2)$

$T(n)$ — n^2

$n/2$ $n/2$ - - - $n/2$ - - - $8 * (n/2)^2 = 2^3$

$n/2^2$ $8^2 (n/2^2) = 2^2 n^2$

$8^3 \left(\frac{n}{2^3}\right)^2 = 2^3 n^2$

$n/2^{k-1}$ $n/2^{k-1}$ - - - $8^{k-1}\left(\frac{n}{2^{k-1}}\right) = 2^{k-1} n^2$

$\frac{n}{2^k}$

$2^k n^2$

$$T(n) = 8^k\, T(n/2^k) + n^2(1 + 2 + 2^2 + \dots + 2^{k-1})$$

$$= 8^k\, T(n/2^k) + n^2[2^k - 1]$$

$$\frac{n}{2^k} = 1 \qquad 2^k = 1$$

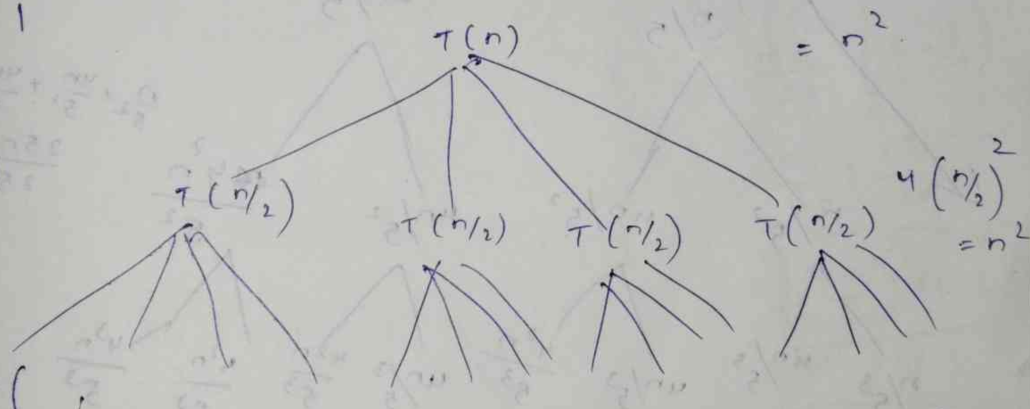
$$= 8^{\log_2 n}\, T(1) + n^2(n - 1)$$

$$= n^3\, T(1) + n^3 - n^2$$

$$= O(n^3).$$

(2) $T(n) = \begin{cases} 4T(n/2) + n^2 & n >= 1 \\ \text{constant} & n > 1 \end{cases}$

$$T(n) = 4T(n/2) + n^2.$$



4^3 $\qquad 4^3 (n/2^3)^2 = 4^3 \times \frac{n^2}{4^3} = n^2$

4^{k-1} — — — — — $n/2^{k-1}$ $\qquad 4^{k-1}\left(\frac{n}{2^{k-1}}\right)^2 = n^2$

4^k — — — — — — — — — $\frac{n}{2^k}$

$$T(n) = 4^k T(n/2^k) + kn^2$$

$$\frac{n}{2^k} = 1 \qquad k = \log_2 n$$

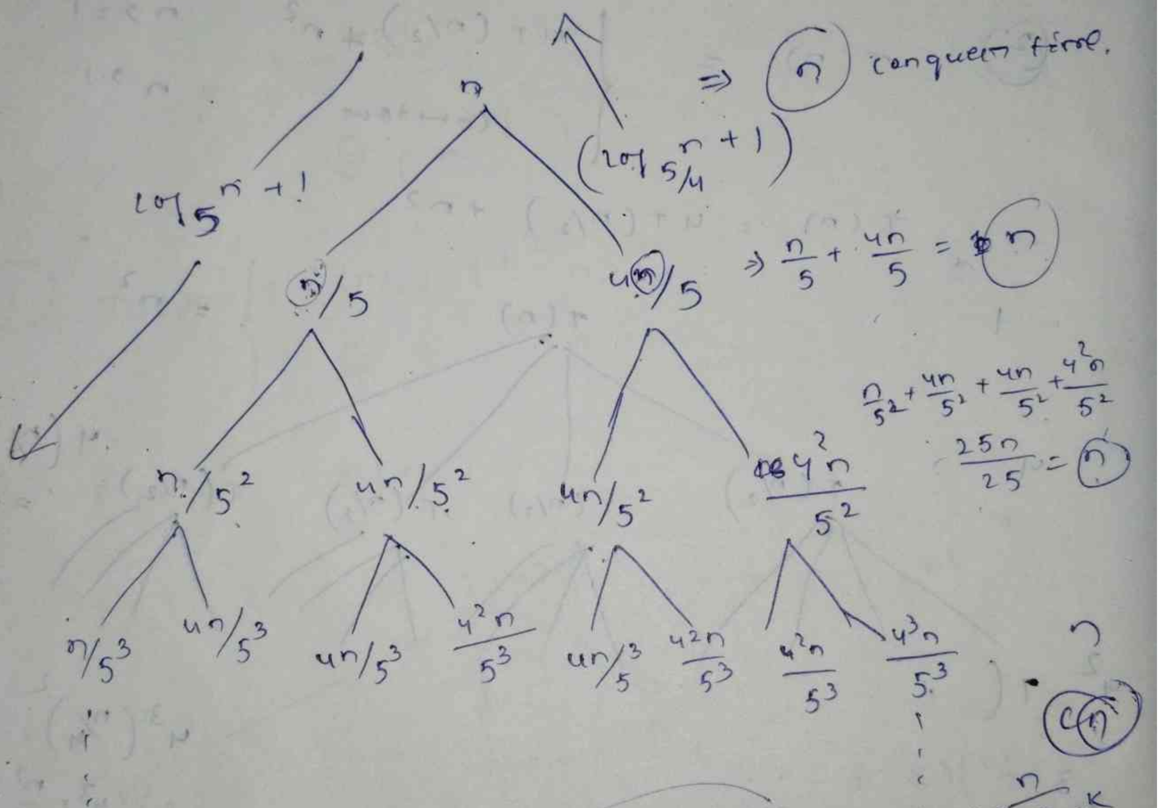
$$= 4^{\log_2 n} T(1) + (\log_2 n) n^2$$

$$= n^2 + n^2 \log_2 n$$

$$T(n) = O(n^2 \log_2 n)$$

(5)

$$T(n) = \begin{cases} T(n/5) + T(4n/5) + n & n > 1 \\ a & n = 1 \end{cases}$$



$$T(n) = 2^k \cdot a + (\log_{5/4} n) n$$

$$= 2^{\log_{5/4} n} \cdot a + n \log_{5/4} n$$

$$= \log_{5/4} 2 + n \log_{5/4} n$$

$$T(n) = O(n \log_5 n)$$

(4) $T(n) = \begin{cases} T(n/5) + T(3n/4) + n & n > 1 \\ \text{const} & n = 1 \end{cases}$

$$T(n) = T(n/5) + T(3n/4) + n$$

n conquer

depth $\log_{4/3} n$

$\frac{n}{5} + \frac{3n}{4} = \frac{19n}{20}$

n

$k = \log_5 n$ $\quad n/5 \qquad\qquad 3n/4$

$n/5^2 \qquad \frac{3n}{4\times5} \qquad \frac{3n}{4\times5} \qquad \frac{3^2 n}{4\times5}$

$\frac{1}{5}(n/5) \qquad \frac{3}{4}\left(\frac{n}{5}\right) \qquad \frac{1}{5}\left(\frac{3n}{4}\right) \qquad \frac{3}{4}\left(\frac{3n}{4}\right) \qquad \frac{n}{(4/3)^2}$

$\frac{1}{5}\left(\frac{n}{5}+\frac{3n}{4}\right) + \frac{3}{4}\left(\frac{n}{5}+\frac{n}{4}\right)$

$\frac{19n}{20}\left(\frac{1}{5}+\frac{3}{4}\right)$

$\frac{19n}{20} \times \frac{19}{20}$

$= \boxed{\left(\frac{19}{20}\right)^2 n}$

$\boxed{\left(\frac{19}{20}\right)^3 n}$

$\frac{n}{5^k} = 1$

$\frac{n}{(4/3)^k} = 1$

$\left(\frac{19}{20}\right)^{k-1} n$

$$T(n) = 2^k\, T\, n/(4/3)^k + n\left[1 + \frac{19}{20} + \left(\frac{19}{20}\right)^2 + \cdots \left(\frac{19}{20}\right)^{k-1}\right]$$

$$T(n) = 2^k\, T\left(n/(4/3)^k\right) + n\left[\frac{1-\left[\frac{19}{20}\right]^k}{1-19/20}\right]$$

$$T(n) = 2^k + \left(n/(4/3)^k\right) + 20n\left[1 - \left(\frac{19}{20}\right)^k\right]$$

$$\frac{n}{(4/3)^k} = 1 \Rightarrow k = \log_{4/3} n$$

$$T(n) = 2^{\log_{4/3} n}\, T(1) + 20n\left(1 - \frac{19}{20}\right)^{\log_{4/3} n}$$

$$= n^{\log_{4/3} 2} + 20n - 20n\, n^{\log_{4/3} n}$$

$$= \theta(n).$$

$k = \log_5 n$ — minimum depth.

$k = \log_{4/3} n$ — maximum depth.

Substitution method:

(5) $T(n) = 2T(n/2) + n\log_2 n$

$$= 2\left[2T(n/2^2) + \left(\frac{n}{2}\right)\log_2\left(\frac{n}{2}\right)\right] + n\log n$$

$$= 2^2\, T(n/2^2) + n\log(n/2) + n\log n$$

$$= 2^2\left[2T(n/2^3) + \left(\frac{n}{2^2}\right)\log\left(\frac{n}{2^2}\right)\right.$$

$$= 2^3\, T\left(\frac{n}{2^3}\right) + n\log\left(\frac{n}{2^2}\right) + \log_2\left(\frac{n}{2}\right) + \log_2 n$$

$$= 2^k\, T\left(\frac{n}{2^k}\right) + n\left[\log_2\left(\frac{2n}{2^{k-1}}\right) + \log_2\left(\frac{n}{2^{k-2}}\right) + \dots + \log_2\left(\frac{n}{2}\right) + \log_2 n\right]$$

$\Rightarrow \frac{n}{2^k} = 1 \rightarrow n = 2^k$

$k = n\log n \quad [\because n = 2^k]$

$$\log\frac{n}{2^{k-1}} = \log\frac{n}{\frac{2^k}{2}} = \log_2\frac{2n}{2^k}$$

$$= \log_2 2$$

$$= 1$$

$$\log\left(\frac{n}{2^{k-2}}\right) = \log \frac{n}{\left(\frac{2^k}{2^2}\right)} = \log_2 \frac{2^2 \cancel{n}}{\cancel{2^k}}$$

$$= 2$$

$$\log \frac{n}{2^{k-3}} = \log \frac{n}{\frac{2^k}{2^3}} = \log 2^3 \frac{n}{2^k} = 3$$

$$= 2^k T(1) + n\left[1 + 2 + 3 + \cdots + \log_2 n\right]$$

$$= n \cdot a + n\left[\frac{\log_2 n (\log_2 n + 1)}{2}\right]$$

$$= \Theta\left(n (\log n)^2\right) \quad \text{(ans.)}$$

6 $$T(n) = 2T(n/2) + \frac{n}{\log_2 n}$$

$$= 2\left[2T\left(\frac{n}{2^2}\right) + \frac{n/2}{\log_2 (n/2)}\right] + \frac{n}{\log_2 n}$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + \frac{n}{\log (n/2)} + \frac{n}{\log_2 n}$$

$$= \underline{2^2 T\left(\frac{n}{2^2}\right)} + \frac{n}{\log (n/2)} + \frac{n}{\log_2 n}$$

$$= 2^2\left[2T\left(\frac{n}{2^3}\right) + \frac{(n/2^2)}{\log (n/2^2)}\right] + \frac{n}{\log (n/2)} + \frac{n}{\log_2 n}$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + \frac{n}{\log (n/2^2)} + \frac{n}{\log (n/2)} + \frac{n}{\log_2 n}$$

$$= 2^k T\left(\frac{n}{2^k}\right) + \frac{n}{\log (n/2^{k-1})} + \frac{n}{\log\left(\frac{n}{2^{k-2}}\right)} + \cdots + \frac{n}{\log (n/2)} + \frac{n}{\log n}$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{\log_b n}\right]$$

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} = \log_b n}$$

$$= n\,T(1) + n\left[\log_b n \quad \log_b n\right]$$

$$= \Theta(n \log(n \log n))$$ (answer)

(7) ~~series~~

(1) $s = 2^n * 1 + 2^{n-1} \cdot 2 + 2^{n-2} \cdot 3 + \cdots 2^1 \cdot n$

(2) $s_n = 2^n \cdot n + 2^{n-1}(n-1) + 2^{n-2}(n-2) + \cdots + 2^1 \cdot 1$

(3) $s_n = \frac{1}{2^n} \cdot 1 + \frac{1}{2^{n-1}} \cdot 2 + \frac{1}{2^{n-2}} \cdot 3 + \cdots + \frac{1}{2^1} \cdot n$

→ (4) $s_n = \frac{1}{2^n} \cdot n + \frac{1}{2^{n-1}} \cdot (n-1) + \frac{1}{2^{n-2}} \cdot (n-2) + \cdots + \frac{1}{2} \cdot 1$

(7) Solve the (4th) series

$$s_n = \frac{1}{2^n} \cdot n + \frac{1}{2^{n-1}}(n-1) + \frac{1}{2^{n-2}} \cdot (n-2) + \cdots \quad \text{—— (1)}$$

multiply the series with $\frac{1}{2}$

$$\frac{s_n}{2} = \frac{1}{2^{n+1}} \cdot n + \frac{1}{2^n} \cdot (n-1) + \frac{1}{2^{n-1}}(n-2) + \cdots \quad \text{—— (2)}$$

subtract ~~2 ~~of~~ 1 from 2~~ ~~(2-1)~~

(eq 1) − eq (2).

$$S_n - \frac{S_n}{2} = \frac{-1}{2^{n+1}} \cdot n + \left[\frac{1}{2^n}(1-n-1) + \frac{1}{2^{n-1}}(1) + \cdots + \frac{1}{2^2}(1) + \frac{1}{2}(1)\right]$$

$$= -\frac{1}{2^{n+1}} n + \left[\frac{\cancel{\frac{1}{2}}\left[1-\left(\frac{1}{2}\right)^n\right]}{\cancel{(1-\frac{1}{2})}}\right]$$

$$\frac{S_n}{2} = -\frac{n}{2^{n+1}} + \left(1 - \frac{1}{2^n}\right)$$

$$S_n = \frac{-n}{2^n} + 2 - \frac{1}{2^{n-1}}$$

[multiplying 2 both sides]

$$S_n = 2 - \frac{n}{2^n} - \frac{1}{2^{n-1}}$$

for larger value of n, $\frac{n}{2^n}$ and $\frac{1}{2^{n-1}}$ becomes close to zero. so it is $O(1)$.

$\therefore$ O (constant).

$S_n = 2$
$= O(1)$

Master method :-

1. $\frac{f(n)}{g(n)}$ = const. [$f(n)$ & $g(n)$ Asymptotically equal]

2. $\frac{f(n)}{g(n)}$ = const. or log function [$f(n)$ & $g(n)$ polynomially equal]

const < log fun^n < polynomial < exponential

neglected for Asymptotic comparision.

neglected for polynomial comparision.

(1) ex $f(n) = 10n^2$, $g(n) = 5n^2$

$\frac{f(n)}{g(n)}$ = constant. $\frac{10n^2}{5n^2} = 2$

if f(n) and g(n) asymptotically equal. and then f(n) and g(n) polynomially equal.

(2) ex $f(n) = 10n^2 * \log n$, $g(n) = 5n^2$

$\frac{f(n)}{g(n)} = \frac{10n^2 \log n}{5n^2} = 2 \log n$.

if f(n) and g(n) asymptotically not equal but then f(n) and g(n) polynomially equal. (This is not in master method)

(3) ex $f(n) = n^2$ ~~log n~~ $g(n) = n^3 \log n$

$\frac{f(n)}{g(n)} = \frac{n^3}{n^2 \log n} = \frac{n}{\log n}$.

- f(n) and g(n) asympt not equal (bigger than f(n))

- f(n) polynomially bigger than g(n)

(4) ex $f(n) = n^2 . 2^n$ $g(n) = n^2$.

- f(n) not only asymptotically bigger than g(n) as well as f(n) polynomiall bigger than g(n).

(5) ex $f(n) = n^2 . \log n^n$ $g(n) = n^2$

f(n) asymptotical bigger than g(n) but f(n) polynomially equal g(n).

Master theorem method :-

$T(n) = aT(n/b) + f(n)$

$3T(n/4) + n^0$
$2T(n/2) + n$
$T(n/4) + n^0$

case-1 if $f(n)$ is $O(n^{\log_b a - \epsilon})$ cond^n is true.

$\epsilon > 0$ real number.

then $\boxed{T(n) = \Theta(n^{\log_b a})}$

case-2 if $f(n)$ is $\Theta(n^{\log_b a})$ cond^n is true

$\boxed{T(n) = \Theta(n^{\log_b a} * \log_b n)}$

where $\log_b n$ is the depth of the recursion.

case-3 if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$ cond^n is true.

$\epsilon > 0$ real number

$\boxed{T(n) = \Theta(f(n))}$

① $T(n) = 7T(n/2) + n^3$

$3 \log_4 8^i$

$a = 7 \quad b = 2 \Rightarrow \log_2 a$

$\Rightarrow \log_2 7$

$n^{\log_b a}$

$n^{2.81}$

case 1 $\quad n^2 = O(n^{\log_2 7 - \epsilon})$ $\quad [n^2 \le n^{2.81 - \epsilon}]$

$T(n) = \Theta(n^{\log_2 7})$

$= \Theta(n^{2.81})$

ϵ
$2.81 - 0.81$
$2.81 + 0.$

② $T(n) = 8T(n/2) + n^2$

$f(n) = n^2 = O(n^{\log_2 8 - \epsilon})$

$= O(n^{3-\epsilon})$ (True)

$n^2 \le n^3$
$f(n) \quad g(n).$

③ $T(n) = 4T(n/2) + n^2$

① $f(n) = n^2 = O(n^{\log_2 4 - \epsilon})$

$= O(n^{2-\epsilon})$ so it is false.

if $\epsilon = 0$ then it is true.

② $f(n) = n^2 = \theta(n^{\log_2 4})$

$= \theta(n^2)$ equal

$\boxed{\text{Solution is } n^2 \cdot \log_2 n}$

④ $f(n) = 3T(n/2) + n^2$

case-1

$f(n) = n^2 = O(n^{\log_2 3 - \epsilon})$

$n^2 \le n^{\log_2 3}$ $[\log_2 3 = 1.62$

$n^2 \le n^{1.62}$

case-2

$n^2 = \theta(n^{1.62})$ false.

case-3

$f(n) = n^2 = \Omega(n^{1.62 + \epsilon})$ True.

$n^2 \ge n^{1.62}$

In Omega

$f(n)$ is bigger than the $g(n)$.

I.M.P

⑥ master method (in detail)

case-1 $f(n) = O(n^{\log_b a - \epsilon})$

$f(n) \le \frac{n^{\log_b a}}{n^\epsilon}$ polynomial.

(After divide small poly funⁿ from $n^{\log_b a}$)

still $n^{\log_b a}$ asymp. bigger than $f(n)$

$n^2 \le n^2$ ✗

$n^2 < \frac{n^2 \log}{n^\epsilon}$ ✗

$n^2 < n^3/n^\epsilon$ ✓

if $n^{\log_b a}$ is $\boxed{\text{polynomialy bigger}}$ than $f(n)$ i.e. $f(n)$

then the soln is the bigger func. i.e. $T(n) = \Theta(n^{\log_b a})$

case. 2

$n^{\log_b a}$ asymptotic. equal to $f(n)$ Then

$$T(n) = \Theta(n^{\log_b a} * \underbrace{\log_b n}_{\text{depth of the recursion}})$$

case-3

$f(n) = \Omega(n^{\log_b a + \epsilon}) \longrightarrow f(n)$ is bigger or equal

then soln is the bigger function

$f(n) \geq n^{\log_b a}$

$$\frac{f(n)}{n^{\epsilon}} \geq n^{\log_b a}$$

case 3

$f(n)$ is polynomialy bigger than $f(n)$ $n^{\log_b a}$

	$T(n) = a\,T(n/b) + f(n)$	$f(n)$	$n^{\log_b a}$
eg 5 (1)	$T(n) = 2\,T(n/2) + \sqrt{n}$ soln - $\Theta(n^1)$	$\sqrt{n}$ 0.5	$n^{\log_b a} = n^1$
(2)	$T(n) = T(n/2) + \sqrt{n}$ soln $\Rightarrow \Theta(\sqrt{n})$	$\sqrt{n}$ $n^{0.5}$	$n^{\log_2 1} = n^0$
(3)	$T(n) = 8\,T(n/4) + n^4$ $\Theta(n^4)$	n^4	$n^{\log_4 8} = n^{\frac{3}{2}}$
(4)	$T(n) = 8\,T(n/2) + n^2$ $a = 8$, $b = 2$, $f(n) = n^2$ $\Theta(n^3)$	n^2	$n^{\log_2 8} = n^3$
(5)	$T(n) = 8\,T(n/2) + n^3$ $\Theta(n^3 \log_b n)$	n^3	n^3
(6)	$T(n) = T(n/2) + 1$ $\Theta(\log_b n)$	n^0	$n^{\log_2 1} = n^0$

	$f(n)$	$n^{\log_b a}$
(7) $26\,T(n/3) + n^3$ sol^n $\Theta(n^3)$	n^3	$n^{\log_3 26} \simeq n^{2.9}$
(8) $26\,T(n/3) + n^2$ $\Theta\left(n^{\log_3 26}\right)$	n^2	$n^{\log_3 26} \simeq n^{2.9}$
(9) $2T(n/2) + n^{\log_2 2}\log_2 n$	$n\log_2 n$	$n^{\log_2 2} = n^1$

Here $f(n)$ and $n^{\log_b a}$ asymptotically not equal. so case 2 not applicable. but polynomially equal. so case-1 and case-3 not applicable.. so it can not be solve such type of problem by using Master method Apply substitution method.

i.m.p

ans. $n(\log n.\log n)$

master theorem fails for polynomially equal & Asymptotic not equal.

(10) $T(n) = 2T(n/2) + \frac{n}{\log n}$ $\rightarrow$ sol^n $\Theta(n \cdot \log n \log n)$

Master Theorem fails.

$T(n) = 4T(n/2) + n^2(\log n)^2$

$T(n) = 4T(n/2) + n^2 \log n$

sol^n $\Theta(n^2 \log^2 n)$

i.m.p.

some book also gives the master theorem in the following format.

$T(n) = aT(n/b) + n^{\log_b a} \cdot \log_b^k n$

then $\Theta(n^{\log_b a} \cdot \log^{k+1} n)$

Change of Variable Method :→