Compute Co-variance

$$\boxed{\operatorname{Cov}(x_1, x_2) = \frac{1}{n-1} \sum_{i=1}^{n} (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2)}$$

How to Compute Co-variance matrix

x_1	x_2	y
1	2	4
2	3	5
3	4	6
4	5	7
5	6	8

Step: 1)
data set

Calculate Covariance matrix??

to Calculate the Covariance matrix, we need to compute the covariance between each pair of features x_1 and x_2.

Let me Calculate Covariance matrix:

$\boxed{\text{Step 2}}$ Covariance matrix.

(a) Calculate mean's

$$\bar{x}_1 = \frac{1+2+3+4+5}{5} = 3$$

$$\bar{x}_2 = \frac{2+3+4+5+6}{5} = 4$$

(b) Centre the data
Subtract the mean from each Value.

• Centred x_1: $[-2, -1, 0, 1, 2]$

Centred x_2: $[-2, -1, 0, 1, 2]$

3. Compute 'Covariance':

Covariance between x_1 and x_2

$$Var(x_1) = \frac{1}{4}\left[(-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2\right] = 2.5$$

Covariance between x_1 and x_2

$$cov(x_1, x_2) = \frac{1}{4}\left[(-2)(-2) + (-1)(-1) + (0)(0) + (1)(1) + (2)(2)\right] = 2.5$$

Covariance matrix

$$\Sigma = \begin{pmatrix} 2.5 & 2.5 \\ 2.5 & 2.5 \end{pmatrix}$$

Find Eigen values and Eigen vectors

1. Eigen value equation

Solve characteristic equation

$$\det(\Sigma - \lambda I) = 0$$

I → identity matrix.

Substituting the covariance

$$\begin{vmatrix} 2.5-\lambda & 2.5 \\ 2.5 & 2.5-\lambda \end{vmatrix} = 0$$

Solve determinant: ⇒ $\lambda^2 - 5\lambda = 0$

$\lambda_1 = 0, \lambda_2 = 5$

Eigen vectors

$$(\Sigma - \lambda I)v = 0$$

$\lambda_1 = 0$ $\quad (\Sigma - 0I).V = 0$

$$\Sigma V = 0$$

$$\begin{pmatrix} 2.5 & 2.5 \\ 2.5 & 2.5 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

Solving v_1 and v_2

$v_1 = -v_2$

$\boxed{\lambda_1 = 0}$ $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\text{Eigen vetor}}$ $\boxed{\lambda_2 = 5}$

$(\Sigma - 5I) \cdot v = 0$

$$\begin{pmatrix} 2.5-5 & 2.5 \\ 2.5 & 2.5-5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{-2.5v_1 + 2.5v_2 = 0}$

$\boxed{v_1 = v_2}$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Eigen vector $\lambda_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

  $\lambda_2 = 5 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

~~PCA~~

~~(1) Feature reduction Technique.~~

Principal Components

- Principal components are the directions of maximum variance in the data, which correspond to the eigen vectors of the covariance matrix. These components are ranked by the magnitude of their corresponding eigenvalues:

- Eigen values $\lambda_1 = 0, \lambda_2 = 5$

Eigen vectors (Principal components)

For $\lambda_1 = 0, \; v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 5 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Interpretation

Interpretation

- First principal component corresponds to the eigen vector with the largest eigen value. In this case.

Partial Least Squares

	x_1	x_2	y
1	2.0	3.0	5.0
2	1.5	2.5	4.5
3	3.0	4.0	6.0
4	2.5	3.5	5.5

Step 1: Standardize predictors

(1) calculate the mean and variance of each predictor:

For x_1:

$$\bar{x}_1 = \frac{2.0 + 1.5 + 3.0 + 2.5}{4} = 2.0$$

$$Var(x_1) = \frac{(2.0-2.0)^2 + (1.5-2.0)^2 + (3.0-2.0)^2 + (2.5-2.0)^2}{3} = 0.25$$

Standardize x_1 and x_2

$$x_1^{(0)} = \frac{x_1 - \overline{x_1}}{\sqrt{Var(x_1)}}$$

$$x_2^{(0)} = \frac{x_2 - \overline{x_2}}{\sqrt{Var(x_2)}}$$

After Standadization data look like:

	$x_1^{(0)}$	$x_2^{(0)}$	y
1	0.0	0.0	5.0
2	-0.5	-0.5	4.5
3	1.0	1.0	6.0
4	0.5	0.5	5.5

Step 2: Perform PLS Iteration.

iteration 1:

ⓐ compute Z_1:

Compute $\hat{\theta}_{1j}$

$$\hat{\theta}_{11} = (x_1^{(0)}, y) = \frac{0.0 \cdot 5.0 + (-0.5) * 4.5 + 1.0 \times 6.0 + 0.5 \times 5.5}{4}$$

$$\hat{\theta}_{12} = \frac{(x_2^{(0)}, y) = 0.0 \cdot 5.0 + (-0.5) \cdot 4.5 + (1.0) * 6.0 + 0.5 * 5.5}{4} = \underline{0.75}$$

Compute z_1 :-

$$z_1 = \hat{\theta}_{11} \cdot x_1^{(0)} + \hat{\theta}_{12} \, x_2^{(0)}$$

$$z_1 = 0.75 \cdot x_1^{(0)} + 0.75 \cdot x_2^{(0)}$$

$$z_1 = 0.75 * (x_1^{(0)} + x_2^{(0)})$$

(b) Compute $\hat{\theta}_1$: $\hat{\theta}_1 = \frac{(z_1, y)}{z_1 \cdot z_1}$

Compute (z_1, y)

$$(z_1, y) = \frac{0.75 \cdot (0.0 + 0.0) \cdot 5.0 + (-0.5 - 0.5) \cdot 4.5 + 0.75 \cdot (1.0 + 1.0) \cdot 6.0}{4}$$

Compute (z_1, z_1) =

$$(z_1, z_1) = \frac{0.75 \cdot (0.0 + 0.0) \cdot (0.0 + 0.0) + 0.75 \cdot (-0.5 - 0.5) \cdot (-0.5 - 0.5) + 0.75}{4}$$

$= 0.75$

Therefor $\hat{\theta}_1 = \frac{4.875}{2.25} = 2.167$

(c) update fitted value $\hat{y}^{(1)}$

$$\hat{y}^{(1)} = \hat{y}^{(0)} + \hat{\theta}_1 z_1$$

$$\hat{y}^{(1)} = 5.0 + 2.167 z_1$$

(d) Orthogonalize the predictors:

update $x_1^{(0)}$ and $x_2^{(0)}$.

$$x_1^{(1)} = x_1^{(0)} - \frac{(z_1 \cdot x_1^{(0)})}{(z_1 \cdot z_1)} \cdot z_1$$

$$x_2^{(1)} = x_2^{(0)} - \frac{z_1 \cdot x_2^{(0)}}{(z_1 \cdot z_1)} \cdot z_1$$

if need repeat in iteration 2