LINEAR EQUATIONS PROBLEM

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SCOPE

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Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as $\boxed{AX = B}$

$$\underbrace{\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}_{\mathbf{B}}$$

$$\boxed{X = A^{-1}B}$$

We need to find A^{-1} using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A),$$

where:

1. $\det(A)$ is the determinant of A,
2. $\operatorname{adj}(A)$ is the adjugate matrix (transpose of the cofactor matrix).

Step 1: Calculate $\det(A)$ $|A|$

The determinant of a 3×3 matrix is calculated as:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Substitute the values from A:

$$\det(A) = 3 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}.$$

$$\boxed{A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}}$$

Calculate the 2×2 determinants:

1. $\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (1)(-1) - (0)(2) = -1$,
2. $\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (0)(1) = -2$,
3. $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2)(2) - (1)(1) = 4 - 1 = 3$.

Substitute back:

$$\det(A) = 3(-1) - (-1)(-2) + 1(3).$$

$$\det(A) = -3 - 2 + 3 = -2.$$

$$\det(A) = -2$$

Since $\det(A) \neq 0$, the matrix is invertible.

Step 2: Find the cofactor matrix of A

To compute the cofactor matrix, calculate the minor for each element and apply the sign pattern.

The sign pattern for a 3×3 matrix is:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

$$\boxed{A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}}$$

Cofactor calculations:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

1. **Cofactor of $a_{11} = 3$:** Minor:

$$\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1.$$

Cofactor = $(+)(-1) = -1.$

2. **Cofactor of $a_{12} = -1$:** Minor:

$$\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -2.$$

Cofactor = $(-)(-2) = 2.$

3. **Cofactor of $a_{13} = 1$**: Minor:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

Cofactor = $(+)(3) = 3$.

4. **Cofactor of $a_{21} = 2$**: Minor:

$$\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)(-1) - (1)(2) = 1 - 2 = -1.$$

Cofactor = $(-)(-1) = 1$.

5. **Cofactor of $a_{22} = 1$:** Minor:

$$\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = (3)(-1) - (1)(1) = -3 - 1 = -4$$

Cofactor = $(+)(-4) = -4$.

6. **Cofactor of $a_{23} = 0$:** Minor:

$$\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = (3)(2) - (-1)(1) = 6 + 1 = 7.$$

Cofactor = $(-)(7) = -7$.

7. **Cofactor of $a_{31} = 1$:** Minor:

$$\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(0) - (1)(1) = -1.$$

Cofactor = $(+)(-1) = -1$.

8. **Cofactor of $a_{32} = 2$:** Minor:

$$\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = (3)(0) - (1)(2) = -2.$$

Cofactor = $(-)(-2) = 2$.

9. **Cofactor of $a_{33} = -1$:** Minor:

$$\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-1)(2) = 3 + 2 = 5.$$

Cofactor = $(+)(5) = 5.$

Cofactor matrix:

$$\text{Cofactor}(A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

Step 3: Find adjugate (transpose of cofactor matrix)

Transpose the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

$$\boxed{\text{Cofactor}(A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}}$$

Step 4: Compute A^{-1}

Divide each element of $\mathrm{adj}(A)$ by $\det(A) = -2$:

$$A^{-1} = \frac{1}{-2} \cdot \mathrm{adj}(A).$$

$$A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

$$\boldsymbol{X = A^{-1}B}$$

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore $x_1 = 2,$ $x_2 = -3,$ $x_3 = -7$

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2, \qquad 3x_1 - x_2 + x_3 = 2$$
$$x_2 = -3, \qquad 2x_1 + x_2 = 1$$
$$x_3 = -7 \qquad x_1 + 2x_2 - x_3 = 3$$

$$3\times(2) - (-3) + (-7) = 2$$
$$2\times(2) + (-3) = 1$$
$$(2) + 2\times(-3) - (-7) = 3$$