

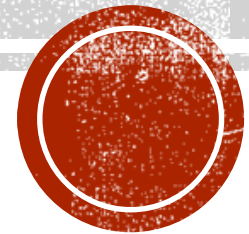
LINEAR EQUATIONS PROBLEM

Dr Divya Meena Sundaram

Sr. Assistant Prof. Grade 2

SCOPE

VIT-AP University



Linear Equations

Example

$$3x_1 - x_2 + x_3 = 2$$

$$2x_1 + x_2 = 1$$

$$x_1 + 2x_2 - x_3 = 3$$

The equations can be expressed as

$$\mathbf{AX} = \mathbf{B}$$

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\mathbf{A} \qquad \mathbf{X} \qquad \mathbf{B}$

$$\mathbf{X} = \mathbf{A}^{-1} \mathbf{B}$$

We need to find \mathbf{A}^{-1} using the formula:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \cdot \text{adj}(\mathbf{A}),$$

where:

1. $\det(\mathbf{A})$ is the determinant of \mathbf{A} ,
2. $\text{adj}(\mathbf{A})$ is the adjugate matrix (transpose of the cofactor matrix).

Step 1: Calculate $\det(A)$ **|A|**

The determinant of a 3×3 matrix is calculated as:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Substitute the values from A :

$$\det(A) = 3 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}.$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Calculate the 2×2 determinants:

$$1. \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (1)(-1) - (0)(2) = -1,$$

$$2. \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (0)(1) = -2,$$

$$3. \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2)(2) - (1)(1) = 4 - 1 = 3.$$

Substitute back:

$$\det(A) = 3(-1) - (-1)(-2) + 1(3).$$

$$\det(A) = -3 - 2 + 3 = -2.$$

$$\det(A) = -2 \quad \text{Since } \det(A) \neq 0, \text{ the matrix is invertible.}$$

Step 2: Find the cofactor matrix of A

To compute the cofactor matrix, calculate the minor for each element and apply the sign pattern.

The sign pattern for a 3×3 matrix is:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Cofactor calculations:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

1. Cofactor of $a_{11} = 3$: Minor:

$$\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1.$$

$$\text{Cofactor} = (+)(-1) = -1.$$

2. Cofactor of $a_{12} = -1$: Minor:

$$\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -2.$$

$$\text{Cofactor} = (-)(-2) = 2.$$

3. **Cofactor of $a_{13} = 1$:** Minor:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

$$\text{Cofactor} = (+)(3) = 3.$$

4. **Cofactor of $a_{21} = 2$:** Minor:

$$\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)(-1) - (1)(2) = 1 - 2 = -1.$$

$$\text{Cofactor} = (-)(-1) = 1.$$

5. **Cofactor of $a_{22} = 1$:** Minor:

$$\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = (3)(-1) - (1)(1) = -3 - 1 = -4$$

$$\text{Cofactor} = (+)(-4) = -4.$$

6. **Cofactor of $a_{23} = 0$:** Minor:

$$\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = (3)(2) - (-1)(1) = 6 + 1 = 7.$$

$$\text{Cofactor} = (-)(7) = -7.$$

7. **Cofactor of $a_{31} = 1$:** Minor:

$$\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = (-1)(0) - (1)(1) = -1.$$

$$\text{Cofactor} = (+)(-1) = -1.$$

8. **Cofactor of $a_{32} = 2$:** Minor:

$$\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = (3)(0) - (1)(2) = -2.$$

$$\text{Cofactor} = (-)(-2) = 2.$$

9. **Cofactor of $a_{33} = -1$: Minor:**

$$\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-1)(2) = 3 + 2 = 5.$$

$$\text{Cofactor} = (+)(5) = 5.$$

Cofactor matrix:

$$\text{Cofactor}(A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

Step 3: Find adjugate (transpose of cofactor matrix)

Transpose the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

$$\text{Cofactor}(A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

Step 4: Compute A^{-1}

Divide each element of $\text{adj}(A)$ by $\det(A) = -2$:

$$A^{-1} = \frac{1}{-2} \cdot \text{adj}(A).$$

$$A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = A^{-1}B = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$$

Therefore

$$x_1 = 2,$$

$$x_2 = -3,$$

$$x_3 = -7$$

The values for the unknowns should be checked by substitution back into the initial equations

$$x_1 = 2,$$

$$x_2 = -3,$$

$$x_3 = -7$$

$$3x_1 - x_2 + x_3 = 2$$

$$2x_1 + x_2 = 1$$

$$x_1 + 2x_2 - x_3 = 3$$

$$3 \times (2) - (-3) + (-7) = 2$$

$$2 \times (2) + (-3) = 1$$

$$(2) + 2 \times (-3) - (-7) = 3$$