THE NATURE OF THE PROCESS CONCEPTION OF FUNCTION

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From the research we have so far reviewed (e.g., Dreyfus and Eisenberg 1982, Dreyfus and Vinner, 1987; 1982; Sfard, 1989; Vinner, 1983; Breidenbach, et al., in press), we find that there is a certain amount of agreement among these authors so we will not be too far from using a common language if we adopt, for describing a function conception, the terms prefunction, action, process and object conceptions, as were defined in Breidenbach et al. (in press), and are summarized here:

For *prefunction* we consider that the subject really does not display very much of a function concept. Whatever the term means to such a subject, this meaning is not very useful in performing the tasks that are called for in mathematical activities related to functions.

An *action* is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions given via algebraic expressions by replacing each occurrence of the variable in one expression by the other expression and then simplifying, but he or she would probably be unable to compose two functions in more general situations, e.g., when they were given by different expressions on different parts of their domains, or if they were not given by expressions at all, but by algorithms.

A *process* conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as 1-1 or onto become more accessible as the subject's process conception strengthens.

A function is conceived of as an *object* if it is possible to perform actions on it, in general actions that transform it. We will not be concerned with object conceptions of functions in this paper but it is important to point out that an object conception is constructed by encapsulating a process. Very often in mathematics it is necessary to go back and forth between an object and process conception of the same entity. According to the theory on which the present research is based, when going from an object to a process, it is only possible to return to the process which originally gave rise to the object.

OBJECTIVE OF THE PAPER

In terms of a single individual whose understanding of functions is being constructed, one may consider that an action conception of function is a sort of *pre-process* conception. This implies, of course, that many individuals will be in transition from action to process and, as with all cognitive transitions, the progress is never in a single direction. This makes it quite difficult, in any but extreme cases, to determine with certainty that a function concept of an individual is limited to action or that he or she has a process conception. As a result, any interpretations of individual responses must be in broad strokes and close call situations should not be a basis for very much in the way of conclusions. In this paper we will see that this point of view is valid even with individuals who have made significant progress in the transition from action to process conception of function.

The individuals studied in this paper are undergraduate students who, as a result of an instructional treatment, were elevated towards a process conception of function from starting points varying from very primitive conceptions to action conceptions of function. The instructional treatment was based on the same theory as that employed in this paper and is described in detail in Breidenbach et al., (in press). A very few students began with something of a process conception and in general, these were strengthened. We began the present research with the question: How far beyond an action conception and how much into process conception was each student at the end of the instructional treatment?

As a result of analyzing interviews of these students, we have found that the process conception of function is very complex. As was observed by other researchers in dealing with other concepts (e.g., the concept of *slope,* (Arcavi et al., 1988); the concept of *multiplication* (Fischbein, et al., 1985)) we have found that the process conception of function is more complex than we expected. It is, in fact, composed of a number of facets or factors, many of which arose in the interviews. All of the facets that we saw are mathematically important and students can have different levels of progress on each. The idea of the strength of an individual's process conception of function is thus not a reference to a linear measurement, but rather to a partially ordered measurement scale. We therefore address our question not in terms of a progress scale but rather in terms of the following four factors. These factors, we must repeat, were not determined prior to the interviews' analysis, but rather they come from our interpretations of the observations of students' apparent ways of thinking about functions in a wide range of mathematical situations.

1. Restrictions students possess about what a function is. The three main restrictions observed are: (a) the *manipulation restriction* (you must be able to perform explicit manipulations or you do not have a function), (b) the *quantity restriction* (inputs and outputs

must be numbers), (c) the *continuity restriction* (a graph representing a function must be continuous).

2. Severity of the restriction. Some students feel, for example, that before they are willing to refer to a situation as a function, they personally have to know how to manipulate an explicit expression to get the output for a given input. Other students are satisfied with the presence of an expression even though they admit that they don't know how to deal with it.

3. Ability to construct a process when none is explicit in the situation, and students' autonomy in such a construction.

4. Uniqueness to the right condition; confusion with 1-1. We argue here that this issue is related to a process conception. According to our theoretical perspective, the confusion that is prevalent among students can only be resolved in terms of the process conception. The process notion entails a unique finishing point, whereas the idea of 1-1 is about uniqueness of starting point.

Our description of the nature of the process conception of function possessed by an individual is not in terms of any attempt to linearly order the quality of the conception by mathematical sophistication, utility or any other criteria. This would not be useful and, based on our experience, including the investigations of this paper, probably impossible. This is because the same individual will behave very differently at different times, in both constant and varying contexts. Instead, we try to analyze individual episodes with individual subjects. This leads to a mosaic that for some individuals allows us to judge their process conception as relatively strong or relatively weak. But for many subjects we have only a contradictory collection of conclusions and so we must revert to the totality of episodes for an individual. We hope that, ultimately, investigations such as this will contribute to the development of more precise criteria for evaluation.

SUBJECTS AND TASKS

A group of 22 students participated in a course in Discrete Mathematics, in which the instructional treatment used was based on a constructivist theory of learning and involved computer activities in the programing language ISETL designed to foster development of the students' conceptions of function. Several observations of the students concerning their apparent understanding of the concept of function were taken before, during, and after this instructional treatment (hereafter pre-observations, observations, and post-observations).

This paper reports on an analysis of one of the post-observations which consisted of interviews of 13 of the 22 students about their present reaction to a questionnaire used in one of the pre-observations. This questionnaire consisted of a list of 24 descriptions of situations (Table 1); the students were asked to respond in writing if they could make use of one or more functions to describe them. In the interviews, each of the 13 students was asked to give their definition of function and then they were queried on a combination of about 6-8 of the 24 situations. They were asked to explain, if they could remember, their thinking in giving the original response, if they still felt the same way or wanted to change their answer, and to explain their present feeling about the situation.

These situations deal with 8 different contexts, in the following distribution:

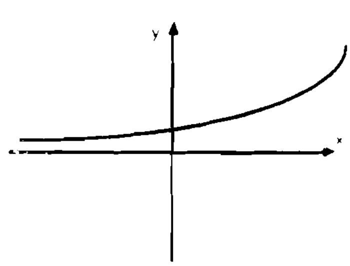
1. Two ISETL procedures (S_6 and S_{22}).
2. Three finite sequences: two with integer values (S_2 and S_{23}) and one with boolean, true/false values (S_{10}).
3. Two character strings (S_5 and S_{21}).
4. Four graphs: a single-valued continuous curve (with respect to both axes; S_3), a vertical line (with respect to the x-axis; S_7), a discrete graph (S_{16}), and a multi-valued continuous curve (only with respect to the x-axis; S_{24}).
5. Four sets of ordered pairs (S_1, S_8, S_{14}, and S_{17}).
6. One table (S_{20}).
7. Five equations: One single-variable equation (S_9), three two-variable equations (S_4, S_{13}, and S_{15}), and one set of parametric equations (S_{18}).
8. Three statements: two describing physical situations (S_{11} and S_{19}) and one specifying a set of records (S_{12}).

Table 1: Situations Presented to Students

S_1. $\{[x,2x+1]: x \text{ in the set of all integers}\}$

S_2. $[2n+n^3 : n \text{ in } [1\ldots100]]$

S_3.



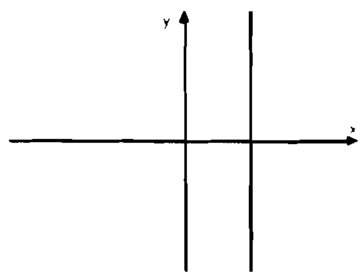
S_4. $y_4 = x_2$

S_5. "Purdue Women's Basketball Team Wins NCAA"

S_6.

```
R :=func(x);
      if is_number(x) then
      if x > 0 then return x**2+1;
      elseif x=0 then return 1;
      else return x+1;
      end;
      end;
   end;
```

S_7.



S_8. $\{[1,2x] : x \text{ in } \{1\ldots100\}\}$

S_9. $x^2 + 3x + 2 = 0$

S_{10}. $\{2^n > n^2 + 3n : n \text{ in } [1\ldots100]]$

S_{11}. A square in the plane centered at the origin is rotated clockwise by 90°.

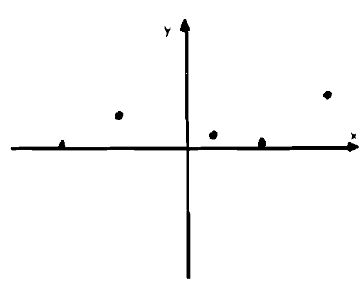
S_{12}. A record of all NCAA Division 1 men's basketball teams giving, for the 1987-88 season, each teams field goal shooting percentage at home and its field goal shooting percentage away.

S_{13}. $2x^3y - x \log y = 2$

S_{14}. $\{[x^2, x^2] : \text{x in } \{1\ldots100\}\}$

S_{15}. $y^4 = x^3$

S_{16}.



S_{17}. $\{[x,y]$: x,y in the set of all rational numbers$\}$

S_{18}.

$$x = t^3 + t$$
$$y = 1 - 3t + 2t^4$$

t is a real number

S_{19}. A swimmer starts from shore and swims to the other side of the lake.

S_{20}. The club members' dues status.

Name	Owed
Sue	\$17
John	6
Sam	27
Bill	0
Iris	6
Eve	12
Henry	14
Louis	6
Jane	12

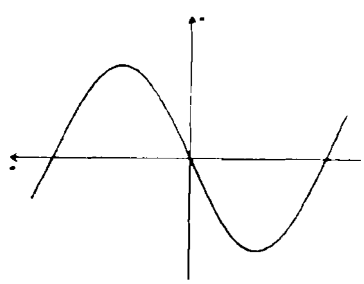
S_{21}. "ELPRHZAUPQDRMW"

S_{22}.

```
Q := Func(x)
    if is _integer (x)
        then return 23 + random(abs(x));
    end;
end;
```

S_{23}. [2n + random(9) : n in [1..50]]

S_{24}.



A conceptual analysis of the situations

This wide variation of situations was essential in this investigation because it would be very difficult to draw models of students' conception of function from their responses to just one or two types of situation. For example, most of the students in this study were able to describe S_1 by a function, but not as many agreed that S_{20} can be described in terms of a function. Moreover, the concept of function is very complex and consists of many notions, all depending, to some extent, on the student's prior experiences with specific situations that involve functions and the level to which these situations were abstracted in the sense of being used to construct a function schema. It is almost impossible to analyze and investigate the quality of these schemata from responses to a limited variety of situations. We found that a student can express a certain notion of function in response to one situation and ignore or not express it explicitly in another.

The notion of "uniqueness to the right" is an example. We have protocols of students who, in one situation will understand how this requirement is essential to thinking about a function as a process; in another situation, only slightly removed in time, will confuse it with the property of being one-to-one; and in still another situation will ignore it completely. This is part of a general phenomena, pointed out by Breidenbach et al., (in press), in which mathematical knowledge involves no more than a *tendency* to use a mathematical idea. That is, "a person's mathematical knowledge is her or his tendency to respond to certain kinds of perceived problem situations by constructing, reconstructing and organizing mental processes and objects to use in dealing with the situation" (Breidenbach et al., in press). Thus, in our attempt to draw models for students' function schemata we presented them with situations in a wide variety of contexts which potentially can evoke different versions of their function conception.

We will try to relate the eight categories of situations mentioned above with action and process conceptions of functions. It is important to realize that the nature of the conception is not contained in the situation, but rather in the subject's relationship to the situation. Therefore we will suggest, where appropriate for a category, both an action and a process conception.

ISETL Procedures (S_6, S_{22}). A computer procedure is a detailed, step-by-step description of the transformation that is applied to input data in order to obtain a result. By itself, the procedure seems most related to an action conception of function. A student who fails to understand how a procedure is an implementation of, or specifies, or constructs a function, may not have even an action conception of function and is probably exhibiting a prefunction response.

Thinking about and discussing a procedure in terms of the individual steps is what would be expected from an action response.

Because the action response is so natural for a procedure, one cannot say, without additional information, anything about the student's process conception. If he or she is able to think about the procedure independently of the steps, to discuss the input and its transformation in general terms, to combine the procedure with other procedures and even to reverse the action of the procedure, then one might consider that the student is displaying a process conception.

Finite sequences (S_2, S_{10}, S_{23}). By itself, a sequence of quantities does not represent a function until the subject adds something to the structure of the situation. It is necessary to think in terms of the first term, the second term, and so on. Then the subject can construct, in her or his mind, a process of accepting a positive integer, seeing it as an ordinal specification of one of the quantities in the sequence, and taking that quantity as the result of the process. When there is an indication that this is what the subject was doing, then one can consider that a process conception of function is being used.

If a sequence is presented as a tuple former with an expression, (as were the three in the 24 situations), then the notation strongly suggests some of the construction that is necessary. Therefore, much of the great success that the students in this study had with seeing a sequence as a function (see Breidenbach et al., in press) can only be ascribed to an action conception, with no information about whether they were capable of a process conception. The fact that one of the sequences had boolean values for output and another involved random numbers does provide us with a context for suggesting the possibility that those who were successful were capable of more than an action conception of function.

Strings (S_5, S_{20}). The case of character strings is, from the mathematical point of view, conceptually identical to sequences, (the position of a character is the input, the character is the output), the only point being that with a string, the output values are restricted to being characters. Psychologically, however, there are more serious differences. In particular, the strings in this study were only presented as "fait accompli," just sitting there after having been formed. Therefore, the only possibility is for the subject to construct the process of going from the index (domain) to the value (character) as we described for sequences. We can see that this presents an added difficulty for our subjects from the fact that their high degree of success with tuples (which were all sequences in which a defining expression was provided) was a bit reduced for the situations involving strings (no expression provided). For details see Breidenbach et al. (in press).

Graphs (S_3, S_7, S_{16}, S_{24}). A graph is another example in which it is hard to see an action conception because there is no description of a transformation. The subject must construct a process of locating a domain value on one axis and then moving in the direction of the other axis until the curve is met. The amount moved is the output of the function.

Some students appeared to recognize that a graph was a function only from memory. They were unable to deal with examples in which instead of starting at the horizontal axis, it was necessary to use the vertical axis for domain. They also did not see a set of individual points as a graph which represents a function (with a very small domain). We feel that none of these difficulties would persist if the subject could bring to bear on the situation a process conception of function. Therefore, the protocols that involved these examples had potential to provide indications of the subject's ability to use a process conception of function.

Set of ordered pairs (S_1, S_8, S_{14}, S_{17}). The idea of a set of ordered pairs presents a number of difficulties for students. In the first place, particularly when a process conception is just emerging, students can confuse the process of constructing the set of ordered pairs with the function itself. The latter can be constructed only after the set of ordered pairs is already there. At that point, the process must come almost entirely from the subject and consists of finding the domain value as a first member of some pair and taking as the result of applying the function, the quantity that appears as the second member. This is only possible, of course, if no quantity appears as the first member of two pairs, the so-called "condition of uniqueness to the right."

It is extremely common for subjects at all levels to have difficulty with this uniqueness condition and confuse it with the notion of one-to-one. It is our contention that only with the help of a strong process conception of function can such difficulties be avoided. The set of ordered pairs is a bellwether type of situation for detecting the presence and strength of a process conception of function.

In contrast with this point of view, some students take another tack. They consider that the set formation is a process that defines a function. For example, the set former, {[1,2x] : x in {1..100}}, is interpreted as assigning to each x the pair [1,2x] . In this way a function is constructed in a situation in which, according to traditional mathematics there is not a function. This point of view also differs from the traditional one in that, once the operation has been performed, the function is lost since there is no way to see how a pair comes from an x.

There were no students who took both points of view.

Tables (S_{20}). For our purposes, tables are very similar to sets of ordered pairs. The subject must construct the process as the act of going from a quantity in the first column of the table to the corresponding quantity in the second column. A student who rejects this and insists on looking for some rule that relates the first number to the second is probably displaying an action conception of function.

Equations (S_4, S_9, S_{13}, S_{15}, S_{18}). An equation in two variables can be considered to be a function if one solves for one of the variables to obtain it as equal to an expression in the other variable, that is an explicit equation in which one variable is on one side alone and the other side is an expression in the other variable. An action conception of function in such a situation would involve explicitly making this calculation and insisting on having a specific expression. A student who can talk about such a situation without actually obtaining the expression is probably displaying at least the beginning of a process conception of function.

An interesting question arises if the equation cannot be explicitly solved. Also, suppose there is only one variable; there is a way of thinking about such a situation that leads to construction of a function in any of these cases. The process is to take a value of the variable(s), plug into the equation and return the value true or false depending on whether or not the equation is satisfied. Again it means that the process comes entirely from the subject.

This approach was mentioned in class, so it cannot be claimed that the students who used it in the interviews were inventing it. Rather, it is interesting to note that, with suggestion, some students were able to understand and use this approach while others were not. Again this is a good indication of the student's capability of using a process conception of function.

Statements (S_{11}, S_{19}, S_{12}). The statements were vague descriptions of physical or geometrical situations and constituted the most open-ended of all the situations which were presented to the students. Our interest here was to see what sort of function conception the student would construct, given a minimum of structure in the statement of the problem.

ANALYSIS OF INTERVIEWS

In this section we will present analyses of 4 interviews out of the 13 interviews taken at the end of the teaching experiment, whose findings exemplify the overall findings obtained from all the 13 interview analyses.

Analysis 1

The first question all the interviewees were asked was "what is a function, what does it mean to you?" Ay's response to this question was:

> Um, a function to me is where you have a set of numbers, your domain, that are numbers that you are going to put into some type of process, and then the process may or may not manipulate the numbers that you are inputting, and then give you some numbers out. And then those numbers that you, I don't know, you just get some numbers out.

It is not reasonable, from such a general answer to such a general question, to conclude anything definitive about Ay's function concept. We can only use such a statement as a first approximation, from which we can make guesses to be tested against her responses to more detailed questions.

Ay's reference to a vague and general set of numbers going in resulting in some numbers coming out does suggest a process-conception. Ay, however, strongly emphasizes numbers (repeated six times) in characterizing what a function is, which raises questions about the quality of her process-conception. In emphasizing numbers she might be thinking in terms of specific actions, which is an indication that her function-conception is based on an action-conception. Another possibility is that although she is not thinking in terms of specific actions, her function schema includes the quantity restriction; that is, inputs and/or outputs, for Ay, must be numbers. Finally, it may well be that the term numbers is a manifestation of the manipulation restriction in her function conception.

Indeed, Ay uses the term "manipulate" in her definition, by explicitly stating that "the process may or may not manipulate the numbers that you are inputting." We must find out what meaning Ay assigns to the word manipulate. If she uses this term in the sense of calculations according to formulas---which is consistent with an action-conception---then an interpretation of the fact that she is explicit about the non-requirement of the presence of manipulations is of special importance in characterizing her function conception.

It does seem from this first response that a reasonable working hypothesis is that Ay's function conception is somewhere between action and process. It is possibly in transition, hopefully from action to process. This is supported by her response to S_2 which Ay considered to be a function and described in terms of a function, saying:

> ... you're taking numbers between 1 and 100 and you're.... that's, um, your variable n and you're taking that number and putting it into the equation $2n + n^3$. And then after you have done the process of $2n + n^3$ for whatever value of n from 1 to 100 the number you get out. Then you get a number, a different number out.

This does not help us much in making the distinction. She does speak, in the second sentence, about a total process in general terms without needing to refer to specific values of the input variable n. On the other hand, in the first sentence, she considers a very explicit formula which certainly leads to a calculation suggesting an action. This formula was given as part of the problem, however, and there is nothing in what Ay says that tells us how important it was for her. What we would like to know is whether the process for this function is something that Ay has interiorized, or whether it depends, for her, entirely on the expression $2n + n^3$.

Her response to another situation (S_{17}) is a little more helpful. This is an example which mathematicians would consider to be a set of ordered pairs which is a relation but not a function because it does not satisfy the condition of "uniqueness to the right." Ay also felt in her original response that this does not represent a function and she confirmed that in the interview explaining,

> ... no because you're not really doing anything to the x and y that you're ... I mean you're just putting them into a set of tuples. You're not really manipulating the numbers in any sort of way. You're just taking two numbers and just setting them aside in a tuple.
> You're putting something in and I just, I just don't see it doing anything.

In this response, when trying to justify her position that this situation is not a function, the main reason that Ay appears to be giving is that there is no manipulation. This appears to contradict her earlier statement that "the process may or may not manipulate the numbers..." The important question for us is whether the phrase "doing anything" which she uses in her repetition a moment later refers to manipulation or something more general.

When she was asked to compare S_{17} to S_1 which she correctly described by a function, she said:

> ... your x value in the first part of the tuple [in S_1], but then you're manipulating that x value to put it, to get a number for the second number in your tuple.

And when she was asked:

> Okay, so what, the difference, what distinguishes this [S_1] yes from the previous [S_2] one was... ?

she said:

> This one [S_1] manipulated the x.

One possible explanation of these responses is that, for Ay, manipulation with an explicit formula is a very important tool in deciding whether or not something is a function. She is happy to use this tool when it is available and, in a difficult situation, relies on its absence to justify her conclusion that a situation does not represent a function. At other times, perhaps when the question is not so troublesome, Ay does appear to recognize that the presence of manipulation is not absolutely necessary and that a more general notion is also a part, possibly just emerging, of her conception of function. For example, she further distinguishes [S_2] from [S_{10}] by saying (in reference to [S_{10}]):

> You have numbers being inputted, and a process being done. And then you're coming out with an answer. In this case being true or false.

One indication that Ay may not be totally bound to manipulation of numbers can be found in her response to S_{10} which, again correctly, she sees as a function. She is not bothered at all by the fact that, although numbers go in, boolean values of true or false come out. She describes this in quite general terms.

Ay's position that functions must involve manipulations of the input according to a formula is also somewhat ambiguous in her discussion of S_{13}. To describe it she first used a function that involves manipulation, saying:

> ... like for the other one, [Situations] 4 and 15, whatever value you put in for x you could manipulate the rest of the equation to find out what y was.

An important fact is that in this example, it is not really possible to actually perform the manipulations. This is a typical situation for an implicitly defined function. When the interviewer points this out by asking:

> ... What if you couldn't manipulate to find out for y? What if I give you an x value and there was no way you could manipulate this?

she abandoned her first function which she used to describe the given situation for another function in which she could manipulate the inputs, saying:

> If, if you couldn't manipulate ... Maybe, maybe you could get two, have two input values, one for x and one also for y and then manipulate them. Then your output would be true or false that the equation is ...

In responding to the interviewer's question:

> And the manipulation would be what?

Ay said:

> What this equation says, once you plug those numbers in and you solve for the left hand side of this equation and then you come up with your output of being true or false if that equation holds true.

This manipulation is not so clearly suggested in the given formula, at least according to how such an expression is dealt with normally in school and college mathematics, but must be constructed by Ay herself. This particular construction, if not invented on the spot is certainly very new to her and was mentioned only briefly in class.

This last is a fairly decent general description that might be given by someone with a process conception of function. Certainly, Ay's function schema includes a restriction on the nature of the process that relates the input to the output. However, the fact that she was able to suggest almost instantly a new interpretation that would satisfy her function conception is an indication of flexibility, and it is evidence that her repertoire of possible interpretations for a given situation is not limited to one.

The manipulation restriction did not prevent Ay from giving sophisticated descriptions of situations in terms of functions. Consider her response to S_5:

> I looked at it as saying that in order for, there's a lot of things that need to be put into the idea that the Purdue Women's Basketball team wins the NCAA. There's a lot of, I just took it as a statement of the things they have to do. They have to, like their input to getting to this goal would be to win all these games in order to get a bid. And then once they got their bid they had to go and, through the tournament and succeed in their games in their tournaments. And deciding who's going to play and who's not going to play. And substituting and scoring. And all those kind of things are taken, are the inputs. They are manipulated around to achieve the winnings. And then the winnings is going to get them to the, this output of winning the tournament.

This might be interpreted as saying there are many variables that determine winning or losing a game. Combinations of the values of these variables make up a domain of a function whose range is the set of values "winning" and "losing." Ay seems to consider all the combinations whose values are "winning." This interpretation is consistent with her response to the following question asked by the interviewer:

> ... Let me ask you this. If I now change the statement slightly and wrote, "In 1989 Purdue's Women Basketball Team wins NCAA" something that we already know the outcome to. ... And if I just make that statement, is it still a yes answer and is it still yes for the same reason?

Ay responded to this question, saying:

> Ay: Um, I think no because it's, ... this, when you're saying in 1989 Purdue women's basketball team wins NCAA, it's done. I don't know how to say this. ... It's just this, the other way, Purdue's women basketball team wins NCAA. It's kind of to me just a general statement. And you have to look at it and well, how did they derive to get to that point. But when you're saying in 1989 they already won it, and you know it, then it just kind of happened and you don't, there's nothing ...

We speculate that in this response Ay says that if the statement is reformulated in the past tense, as opposed to ongoing present tense, then it indicates one particular evaluation of the function described earlier at a point in its domain, not the function itself.

In summary, we can say that the responses from this student describe a process conception of function that is far from monolithic. Her response to the question, What is a function? is fairly reasonable. She is quite ambiguous on the manipulation restriction, which seems to be definitely present on some occasions but in other situations it may be that she is not bound by it. An important issue which we cannot entirely resolve is how autonomous this student can be in constructing a process herself. She seems able to do so when some form of algorithm is present, but seems hesitant to make such a construction entirely on her own. Perhaps she requires some sort of permission . She is also ambiguous on the quantity restriction. She insists on using the term number repeatedly but is not bothered by a situation in which the output of a function is a boolean value. Finally, Ay offers an intriguing analysis of the situation with the basketball team. It is possible to interpret her response as suggesting that she can introduce a vague form of function evaluation to distinguish between an ongoing situation involving several variables with sets of potential values for them and a fait accompli in which specific values are known.

Analysis 2

Compared to Ay, Dn displays a much clearer process conception of function. When Dn was asked to explain what the term "function" meant for her, she answered:

> Dn: I look at function as a process where you put something in, go through some sort of

process, and you get an output.

What did Dn mean by the term "process?" In her response to S_2, she uses it to refer to manipulation in a formula:

Dn: It's going to look at all the integers 1 to 100. It's going that expression $2^n + 3^n$ at 1, then at 2, at 3, you know, for your values at n and if this were just a tuple you could call up the number that is in the position, the first position of your tuple ... Like if this was your tuple and you called up a 1, you would get that expression evaluated at 1.
I: And the process in this one?
Dn: The process is evaluating $2^n + 3^n$.

In this example, we have a finite sequence with the general term given by a formula. It is absolutely natural for Dn to use that formula. In several other situations, however, she shows that she does not need the formula and is able to construct a process herself even when it is not explicitly present. Consider, for example, the discussion she had with the interviewer about S_1.

I: ...with your definition of function, if I say, does this, the way it is, represent a function to you, what would you answer now be?
Dn: Yes, this is a function.
I: Because?
Dn: Because it is a set of tuples. Your first ... This is a set of tuples. Your first element of your tuple is going to be an integer, and if it's the set of all integers you're not going to have any repeats in there.

That last comment is an explicit statement of the requirement that no value appears more than once as a first component of a tuple or pair.

I: Okay. So go back to your definition for me. So how does that satisfy your thought of what it needs to be?
Dn: Okay, when I look at these I think of putting them into the computer. Okay, I'm going to put this set into the computer.
Then like if I were going to call up like t(1), okay, I'm going to get out 3, is going to be my answer. I input the first element of the tuple, output comes the second, the second element of the tuple.

Here she has referred to the process, and, on request, she explains in the next few statements the nature of that process.

I: ...What about that manipulating, that process you were talking about?
Dn: The process is you input the first element of tuple. It looks at its tuples and finds one that had its first element in it, and outputs the second element.

The point here is that the process which Dn describes is not at all explicit in the information she is given in the presentation of the situation. It is something that she has to construct herself, a structure that she imposes on the situation in order to make it a function.

We are not suggesting here that Dn has invented this idea herself. It is, no doubt, something she learned in the course. What she has learned is to deal with certain kinds of situations by constructing a function, in this case by constructing the process of going from the first component to the second component of an ordered pair.

This positive estimate of Dn's process conception must be tempered by the confusion she displayed in dealing with Situations S_{15} and S_{13}. Here we have equations in two variables and Dn first thinks about plugging in a single value for one variable and trying to solve for the other variable, which is characteristic of an action approach. Notice also that she confuses the two variables considerably, for example when she says

> Dn: Oh! You could input a value for y, take the fourth root and get your x.
>
> ......
>
> Dn: I could could represent something, take the fourth root of that and I could find a corresponding number that has a cube root which is equal to that.

This shows considerable confusion in dealing with the equation $y^4 = x^3$. In dealing with S_{13} she quite explicitly insists in the following sequence that it be possible to find an expression for one variable in terms of the other or she will not call it a function. The interviewer has just asked her what she would do if it were not possible to solve for y in terms of x.

> Dn: Yea, because if my x should have been in my domain but there's no answer for it, then it's not a function.
> I: Okay, so if you can't solve for y it wouldn't be a function but as long as you can, it is a function?
> Dn: Yea, if I'm using x as my domain.
> I: Uh-huh.
> Dn: Yea.
> I: Well, what if it were y is your domain but you couldn't solve for x?
> Dn: Then no that wouldn't be a function.

This exchange requires an interpretation. One possibility is that by "can't solve for..." Dn means that one does not succeed in finding an expression. But in light of her other responses which tend to contradict this explanation, she might be referring to the fact that there does not exist any quantity.

In summary, this student definitely displays the ability to make an autonomous construction of a process in order to obtain a function. This strong process conception appears to help her avoid the "uniqueness to the right" confusion. On the other hand, this ability is not completely solid for her, but can fall apart in certain situations. An important issue to resolve in order to have a better understanding of the nature of her process conception of function would be her interpretation of the phrase "Cannot solve for ..." Does she take this to mean that it is not possible to find it explicitly or that it does not exist? Does she understand the difference?

Analysis 3

Dv's response to "what is your definition of function?" was:

> Um, let's see. Um, a function, a function, um, is a, well, what we've said, I think, is a process in the different ways you can use, um, different, um, I don't know how to say it. Well, we've studied different ISETL representations of functions, um, taking a number, going through a process and getting a result, or it doesn't have to be a number. It can go through a string, find a position, um, ... I don't know. I can't really put an exact, formal definition to it.

This answer, which is a mixture of a contextual definition and general definition, does not include explicit restrictions on the type of inputs and outputs, or on what the process should be. It is general only in the sense that it includes several examples, but it does not really specify the underlying structure in these examples. To digress, we might suggest that for Dv, his definition of function is an action not yet interiorized to be a process. We will see that there are a number of different notions regarding function that appear to be competing in Dv's mind. The first competing notion concerns the manipulation restriction. Consider the following discussion of S_3.

> Dv: Number 3 is a graph of a, of a line, or a function of a line. I thought it was a function because you can use a, there's, there's probably some formula for that line and for each point on the line. Okay, I have, You can use the formula for a line y=mx+b for each of the points on the line. If you have a value for y and you know the slope and you know the, or you, and you can find the slope and you can find the b, you can get an x. If you um, if you um, have a, let's say that this formula is f(x) equals this equation of this line, then you have this x you plotted in there, plotted, take the, take the function at that number and you'll get the, like say that's the y and you'll get the x value for it.

The interaction goes on and Dv sticks to this insistence on a formula of some kind that can be manipulated. Despite this, he is also able to give a purely graphical description of the process on request.

> I: Now, suppose you didn't have that line formula. Suppose all you had was that graph. How would you get the y value if you took...?
> Dv: Well you could, you could just take an x on here and...
> I: Uh-huh, on here...
> Dv: x, x on the x, um, bar...
> I: Okay. Thank you, for the tape.
> Dv: and draw a straight line up to where that line would cross the curved line on the graph and then take another line over to the y bar to find the value.

What we don't know here is the extent to which Dv is able to synthesize these two ways, formula and graph, of explaining the function process. At this point they may be two competing points of view for him. His responses to S_6 is an unambiguous description of a process.

Dv: If it is a number then if it's greater than zero then you return x^2+1, else if it equals zero then you return 1, and else return $x+1$, Um, well I, um, I think now I would put yes it's a function [referring to the "no" answer he gave to this situation in a written test he was given during the course]. It takes a number x, and if it's greater than zero it equals x^2+1. If it's equal to zero it returns 1 and if its's less than zero, um, it's going to add 1 to it.

Notice that, instead of concentrating on the algebraic expressions in the definition of this procedure, he emphasizes the transformational aspect of a single, coherent operation, with several alternatives steps. He gives a good process description of a function for S_{11} as well. Dv's response to S_{20} includes even stronger indications that his concept of function as a process is not restricted by the view that the process has to involve manipulations:

... The tables's like tuple. You have the name as the first component and the amount he owes as the second component. So that's what it seems to me. You, you're, you put in the first name and you get out what's owed. I mean it's a tuple of values.

This analysis shows that Dv's function schema involves competing notions regarding the manipulation restriction: in a few situations he seems to be bound by this restriction but in most cases he shows a high degree of freedom from it.

Despite the fact that Dv's schema of function is somehow free from the manipulation restriction, there are indications that he is bound by the quantity restriction which means that in his view the inputs and outputs must be quantities (as opposed to propositions, for example). This is a rather interesting observation because one might argue that the quantity restriction is entailed from the manipulation restriction: Students who insist that a function must involve a formula or (algebraic) expression are likely to expect the inputs and outputs to be algebraically manipulable objects. In the range of past experiences of these students, these objects are numbers, or quantities. In Dv's case, however, while the "manipulation restriction" is absent, the quantity restriction is present. It could be that we have a suggestion here of a sequence in the development of the process conception. For Dv, the manipulation restriction is overcome before the quantity restriction.

Dv's responses to S_9 and S_{13} are those which indicate the "quantity restriction" in his process conception of function.

I: What about number 9?
Dv: $x^2+3x+2=0$. Um, I, well, ..., you can solve for x but you can't, ... I don't think it's a function because you're, you can't ... It equals zero. You can't, it doesn't ask to put in anything to receive an answer to put an integer through that equation to get an answer. I mean it's going, unless it's one that has to be zero. But I don't think it's a function. It's just an equation.
I: So you'll put in an x and it either satisfies that equation or not. Right?
Dv: Well, yea. Well, it could return ... Your equation could be true or, I mean you could

return true or false.
I: If you thought about it that way what would it be?
Dv: If, uh, ... Yea, for x, if you put in x with this equation and it does equal zero then you could do a function that returns true. And if it doesn't equal zero it returns false. But you don't really have those variables in number 9.

Again, we see here that two ideas are present and it takes only a slight suggestion from the interviewer to bring out the more powerful, but less familiar notion. A few minutes later, when Dv was asked what he thinks about S_{13}, he responded:

> Dv: Well, it looks similar to number 9 in that it's an equation with an answer. You would either need an x and y value to plug in to see if that equation equals two and you could return true or false. But I would answer no now because it's an equation. You don't have any specifics on what it's going to return. What you're, what the process ... If this is process, what do you want it to return? Do you want it to return true or false or do you want it to return just values it is equal to?
> I: So for you, unless the person who presents it puts in something specific about wanting it to return true or false, you know, you say no, that's not good enough. That needs to be specific there.
> Dv: Yea, I think it needs to be specified.

We see in this last exchange, a possible emergence of an issue of authority. Who has the right to specify the domain and range, to construct the function. Dv's quantity restriction, displayed in his obvious discomfort at any suggestion of the values of a function being quantities other than numbers, is in direct contradiction to his general definition of function in which he says

> ...it [the result] doesn't have to be a number.

The question that this raises has to do with the severity of this quantity restriction for Dv. It appears to be that, although he is willing to accept it, he simply does not like it, may not feel that he has the authority to do it.. In other words, the generality of the nature of the objects that a function can produce seems to be part of Dv's *concept image* (see Vinner, 1983) in that, however reluctantly, he uses it to accept that an equation in two variables can be a function even though it produces boolean values for output. He seems quite hesitant, however, about including this generality in his *concept definition* . This is very different from what usually happens.

The authority issue arises again in Dv's discussion of S_{19}. He again expresses reluctance to make up input and output quantities or construct a process. But when the interviewer says,

> I: But if you were the one describing it, if you were the one making up the function, could you think of something you would put in and something you would get out?

Dv acts as if he were given permission to decide these things and constructs a reasonable function process to describe what the swimmer is doing.

Dv: Okay, if you knew the distance and then you could enter in that x could be like the length of his stroke and you put that in and you know the distance, you could divide the distance by that length of the stroke and get the numbers of strokes to get to the other side.

In doing this quite unusual (for Dv) operation, he reverts back to a formula - but at least it came from him!

We can best summarize this student's responses by pointing out the presence of competing notions about functions with respect to both the quantity restriction and the manipulation restriction. He avoids the former in his statement about what a function is and seems able to work with domains and ranges consisting of other kinds of values, but he appears to be quite uncomfortable about it. Similarly, with the manipulation restriction, he is sometimes free of it, but on other occasions appears to prefer dealing with specific manipulations. When prompted, he seems able to drop this restriction easily. Perhaps we see here an item in the student's *zone of proximal knowledge* in the sense of Vygotsky (1978). Another possibility is that this student, although capable of making autonomous constructions, may need permission to do so. It could be that he is concerned about who has the authority to specify domains and ranges or construct processes.

Analysis 4

Re is an example of a student who has begun her transition from an action conception toward a process conception of function, and who has abandoned previous restrictions on what a function is. In her response to S_{16}, Re explicitly admitted that on her original written response given during the course prior to this interview, she thought that a graph that represents a function must be continuous. It seems that the instructional treatment she received in this course helped to free her from this restriction. The course, however, seems to have been less successful with the other restriction she possesses, the manipulation (a pattern, in her words) restriction.

Re: I think it is [a function] because, um, well, the reason I thought it wasn't was because it wasn't continuous. I guess I was thinking it wasn't going on forever. But you definitely have an x and a y value, x and a f(x) value. I mean, I don't really know what ... There's really no pattern. I don't see any pattern but you definitely have an input and for that specific x value you have a y ....
I: Uh-huh.
Re: ... but I don't see what the, if there is a pattern or anything, or what the process would be. But ....

Although Re seems to be bothered by the fact that a pattern is absent from the situation, she seems to be confident that it represents a function. Additional evidence for the progress she made toward the process conception of function, can be found in her response to S_{13}:

Re: Oh! I forgot how to do logs. That's why I didn't do this one . I would think it would be a function though because it's going to give you just a true/false answer for whatever input you put in.
I: Okay. Uh-huh. So what would, what would you be inputting in this case?
Re: I guess just numbers. All numbers. All real numbers.
I: And then where would the numbers go?
Re: Ho! Um, would you have to have inputs for both x and y? It's been a long time since I've had logs.
I: Okay. Well, you said this would give you a true or false.
Re: Yea. There would have to be specific values of x's and y's in order to equal 2. So it would test all of the values and trues or falses as output. But I can't think of how that process works right now, logarithms and stuff.

Notice that although Re did not remember log, that is she did not have an explicit expression to use as a process, she still described the situation in terms of function.

Her ability to construct a process seems to depend on the situation. When asked about S_{21}, for example, she says,

Re: Its just like a, that was just a bunch of letters. And I wouldn't think this would be a function because there's nothing going on there. Its just a, you know, where did this come from? You don't know. That's about like the first one.
I: Is there something I could do to make this a function?
Re: I don't know what you could do. Maybe something with consonants or vowels or something. I don't know.

Her need for an explicit algorithm in a process also occurs in her discussion of the table, S_{20}. In S_{21} she simply thought that no process existed. In S_{20}, she seems to think there is a process if she could only figure it out. She is not really trying to construct a process here, but rather an action. When she does not succeed, she rejects the idea that this is a function. With prompting from the interviewer at the end, she appears to accept reluctantly a vague process without an explicit algorithm. There does not seem much reason to hope that this last is a very stable part of Re's knowledge structure at this point.

Re: I had no idea what this was doing. I didn't see... I was trying to find some relationship between, I mean how do you know? I don't see how you know any, with what you are given, the amount that they owe. I don't see any correlation between that and the name. How do you know Bill doesn't owe anything? If there was more information... I just don't see any process going on. It's just a bunch of names and a bunch of values. But I don't see how, why Henry is 14 and Sue 17. What's the difference?
I: Uh-huh.
Re: So that's why I put no for that one.
I: All right. Is there something we could do then to make that a function? What would I have to do?
Re: You'd have to put some more conditions in there stating, um, I don't know what all this represents, like T-shirts were $10 or... I don't know. Or Sue bought a T-shirt. I don't know. Maybe set a value to different objects purchased. I don't know what to do. If it was dues, how many meetings you went to correlates to... I don't know.

I: Okay, suppose I gave you all that information, and you were satisfied that it was a function, what kind of input values would you have? Describe, you know, as far as the input and output.
Re: Well, if you dealt with how many meetings they went to, you know, you'd have like the input the number of meetings and then you'd have all these conditions in here, blah, blah, blah, and then the output would be how much they owed or something like that.

In summary, we can see this student's conception of function in transition. She seems to have succeeded in overcoming a continuity restriction and is quite aware of that. She still has trouble, however, with a manipulation restriction on occasion. She is able to make autonomous constructions of processes in some situation but not in others. We can see her trying to construct an action which must be done before it can be interiorized into a process. For this student the distinction between being unable to construct an action and determining that one does not exist becomes explicit.

SUMMARY

As we have indicated earlier, these findings address the question of how far beyond an action conception and how much into process conception was each of the students at the end of the teaching experiment, in terms of several factors: the restrictions possessed by the students on what a function is --- the *manipulation restriction* , the *quantity restriction*, and the *continuity of a graph restriction* --- and the severity of these restrictions; the ability to construct a process when none is explicit in the situation, and students' autonomy in such a construction; and confusion between the 1-1 property and the uniqueness to the right condition. We have presented analyses of 4 interviews out of the 13 interviews taken at the end of the teaching experiment, whose findings exemplify the overall findings obtained from all the 13 interview analyses.

These findings can be summarized as follows. First, the question of whether or not the manipulation can actually be performed is a crucial factor for some students in determining whether a given situation can be described by a function. Second, students' autonomy in constructing a process depends on whether some form of algorithm is present and whether the student feels that he or she has the permission to do so. Despite this, students may be able to offer an intriguing analysis in which they invent a process of function to describe some of the given situations. Third, a strong process-conception appears to help students to avoid the "uniqueness to the right" confusion. Finally, most students have overcome the continuity restriction, and some are even aware of the fact they they had possessed this restriction.