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The Role of Isomorphisms in Mathematical Cognition

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Recognizing and exploiting structural relationships between situations differing in surface features is an inherent part of mathematical cognition. Laboratory-based experimental studies have shown that subjects generally show little awareness of such relationships when presented with isomorphic problems. However, these findings should be interpreted in the context of unmotivated participants performing abstract tasks over a short period with minimal opportunity for development of structural awareness. Another collection of studies has demonstrated that people often fail to apply mathematics principles to situations outside the classroom, to which they are at least potentially applicable; these findings reflect a major shortcoming in mathematics education. We recommend that awareness of structure, including specifically the recognition of isomorphisms, should be nurtured in children as part of the general development of expertise in constructing representational acts. A balanced view of the goals of mathematics education encompasses both the need to teach mathematics so that its applicability to many contexts is recognized, and a recognition of the importance and power of mathematics as desituated cognition.

... the idea came to me, apparently with nothing whatever in my previous thoughts having prepared me for it, that the transformations which I had used to define Fuchsian functions were identical with those of non-Euclidean geometry.

—Poincaré, 1913.

At this point Brandon's insight moved him to exhilaration. He pointed out that the group of four towers with exactly one red cube was like the four pizzas with one topping in his chart. He carefully moved each tower and placed it on top of the corresponding pizza code on the chart thereby validating the relationship he had organized. He then explained how the red cubes in each tower corresponded to the "0"s in his pizza chart and how the yellow cubes in each tower corresponded to the "1"s on his chart.

—Maher, Martino, & Alston, 1993.

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These are two examples of mathematicians—the first a great French genius, the second a fourth grade New Jersey student—experiencing insights about structural identity underlying what, on the surface, appear to be different situations.

The quotation from Poincaré is part of his account of his own mathematical discoveries, and the incident described is one of several where sudden insight into a mathematical problem, on which he had spent considerable time, came while he was otherwise occupied. Bell (1937, p. 529) commented that Poincaré's "comprehensive grasp of all the machinery of the theory of functions of a complex variable" was turned to magnificent use "in disclosing hitherto unsuspected connections between distant branches of mathematics, for example between ... groups and linear algebra."

The second quotation is from a case study which was part of an investigation of students working—at different times—on two tasks between which an isomorphism could be constructed. The "Towers" task required them to build as many different towers of four plastic cubes as possible, when the cubes were of two colours, and then to convince other students that the set was complete and without any duplicates. The "Pizza" task, on which the students worked some weeks later, involved finding the possible combinations of pizzas with some subset of four different toppings; again the students were expected to give convincing explanations that all the combinations had been found, without repetition. By a mapping such as the following (there is a degree of arbitrariness in the choice) an isomorphism can be established between the two situations:

Top cube:	red <--> no peppers	yellow <--> peppers
2nd cube:	red <--> no sausage	yellow <--> sausage
3rd cube:	red <--> no mushrooms	yellow <--> mushrooms
Bottom cube:	red <--> no pepperoni	yellow <--> pepperoni

This mapping establishes a 1-1 correspondence between the 16 possible towers and the 16 possible pizzas.

Brandon's insight into the isomorphism between the two problem situations was not the result of a sudden recognition, but rather the culmination of a lengthy process of construction, mediated by the notational system he had devised, by discussion with a partner, and by explanations of his thinking. All of this took place over an extended period, and within an instructional environment promoting thoughtfulness about mathematical situations, and challenging children to reorganize and extend their knowledge.

In striking contrast, there are many reports in the literature of students failing to appreciate links between structurally related mathematical problems. For example, Af Ekenstam and Greger (1983), interviewing 12-13-year-old students, presented some of them with these two problems in succession:

P1: A cheese weighs 5 kg. 1 kg costs 28 kr. Find out the price of the cheese. Which operation would you have to perform?

$$28/5 \quad 5 \times 28 \quad 5 + 28 \quad 28 + 28 + 28 + 28$$

P2: A piece of cheese weighs 0.923 kg. 1 kg costs 27.50 kr. Find out the price of the cheese. Which operation would you have to do?

$$27.50 + 0.923 \quad 27.50 \div 0.923 \quad 0.923 \times 27.50 \quad 27.50 - 0.923$$

Many gave the correct answer for the first but chose division for the second. There is strong indication from a number of studies that a plausible explanation is that these children realized that the answer would be numerically smaller than 27.50, and this realization interacted with the misconception that multiplication always makes bigger and division always makes smaller to produce the results noted. Even with probing, these children showed no awareness that the operation required to solve a problem of this type is invariant under changes in the numbers involved. Such a response has been termed "non-conservation of operations" (Greer, 1987; see also Greer, 1988, 1994; Harel, 1995).

Examples of inability to recognize the applicability of more advanced formal knowledge to specific situations are to be found in the experiments of Tversky and Kahnemann (1974). Take this question:

> **P3:** A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50% of all babies are boys. The exact percentage of baby boys, however, varies from day to day. Sometimes it may be higher than 50%, sometimes lower. For a period of one year, each hospital recorded the days on which more than 60% of the babies born were boys. Which hospital do you think recorded more such days—the larger hospital, the smaller hospital, or about the same?

In one study, of 37 university psychology students who had been taught about the effects of sample size on variability as applied to the binomial distribution, only 14 gave the correct response, namely the smaller hospital (Greer & Semrau, 1984). Such a finding is typical for this and similar problems.

In this paper, we briefly review related research from cognitive psychology on the generally very limited ability of people to make links between structurally related situations differing in surface features. Then we present an alternative perspective on the construction of isomorphisms as representational acts, consider the role that appreciation of structural relatedness plays in mathematical problem solving, and examine implications for the teaching of mathematics.

1. RESEARCH ON SOLVING ISOMORPHIC PROBLEMS

A considerable amount of laboratory-based research was carried out in the 1970s and 1980s which took the general form of asking participants to solve two or more isomorphic problems and probing for evidence of transfer of solutions (see various chapters in Goldin & McClintock, 1984).

Lave (1988) presented an ethnographic analysis of four typical studies of this genre, namely those of Reed, Ernst, and Banerji (1974), Hayes and Simon (1977), Gick and Holyoak (1980), and Gentner and Gentner (1983). From a perspective of what is now called situated cognition, Lave presented a strong critique of the "culture" of such experiments and of the epistemological assumptions underlying them. One of Lave's points was that these experiments show very little evidence of transfer or awareness of structural relatedness. Likewise, Detterman, concluded that:

> the amazing thing about all these studies is not that they don't produce transfer. The surprise is the extent of similarity it is possible to have between two problems without subjects realizing that the two situations are identical and require the same solution. Evidently the only way to get subjects to see the similarity is to tell them or to point it out in some not-so-subtle way. (DeHerman, 1993, p. 13).

Arguably, this is overstated. Cases of successful transfer have been reported in various studies using tasks based on mathematical groups (e.g. Halford, 1975; Somerville & Wellman, 1978). Dienes and Jeeves (1965) reported that children related the structure of the 2-group (in which there are two elements, which combine thus: $(x, x) \dashrightarrow x$, $(x, y) \dashrightarrow y$, $(y, x) \dashrightarrow y$, $(y, y) \dashrightarrow x$) to various familiar situations, such as magnetic attractions between poles and the rules for multiplying positive and negative numbers. Some of their examples show how isomorphism is related to metaphor (Presmeg, this volume), as in one child's explanation that the element which acts as identity operator (x in the above characterisation, which combined with x or y yields the same) is the Conservative, who wants everything to stay the same, whereas the other element (y, which in combination with x or y yields the other) is the Communist, who wants everything to change.

An experiment reported in Jeeves and Greer (1983, Chapter 5) showed a developmental trend whereby structurally mediated transfer between successive tasks based on the 2-group differing only in the symbols used appears to emerge at about age 11.

Lave (1988, p. 23) commented that "learning transfer is assumed to be the central mechanism for bringing school-taught knowledge to bear in life after school." She suggested the metaphor of school knowledge as a set of general-purpose tools. Quite early, the limitations of this view began to be exposed, in particular through a number of studies showing that, when abstract and "realistic" problems sharing the same structure are presented to subjects: (a) the realistic variant is generally much easier, and (b) subjects show little transfer between the variants and little awareness of the structural relationship.

A typical example is to be found within the extensive research on Wason's "selection task." It has frequently been reported that there is very little transfer between abstract and contextualized versions of the task and that very few participants show any awareness of the relationship between the tasks. This was the case for the study reported by Johnson-Laird, Legrenzi and Legrenzi (1972) using the versions shown in Figure 1.

Such findings, along with other evidence, were interpreted by Wason (1983) in terms of schema theory as the "realism effect." This theoretical explanation suggests that specific experience is stored in terms of organized structures, that elicitation of these structures allows the information in them to be manipulated, and that people are usually only capable of logical inference within such content-specific schemas. Evidence in support of this, in relation to the letter-sorting task, is that experience of a rule about differential postage rates for sealed and unsealed letters appears to be important; subjects in the USA, where such a rule does not apply, and younger British subjects, who hadn't experienced it, performed much worse on this variant of the selection task than older British subjects who had (Wason, 1983). Elshout (1992) suggested that even more crucial is that the task such as the letter-sorting variant make good sense, whereas the formal variants are "senseless."

Similar views were expressed by Johnson-Laird (1983) in rejecting what he called "the doctrine of mental logic." In relation to the selection task, he stated:

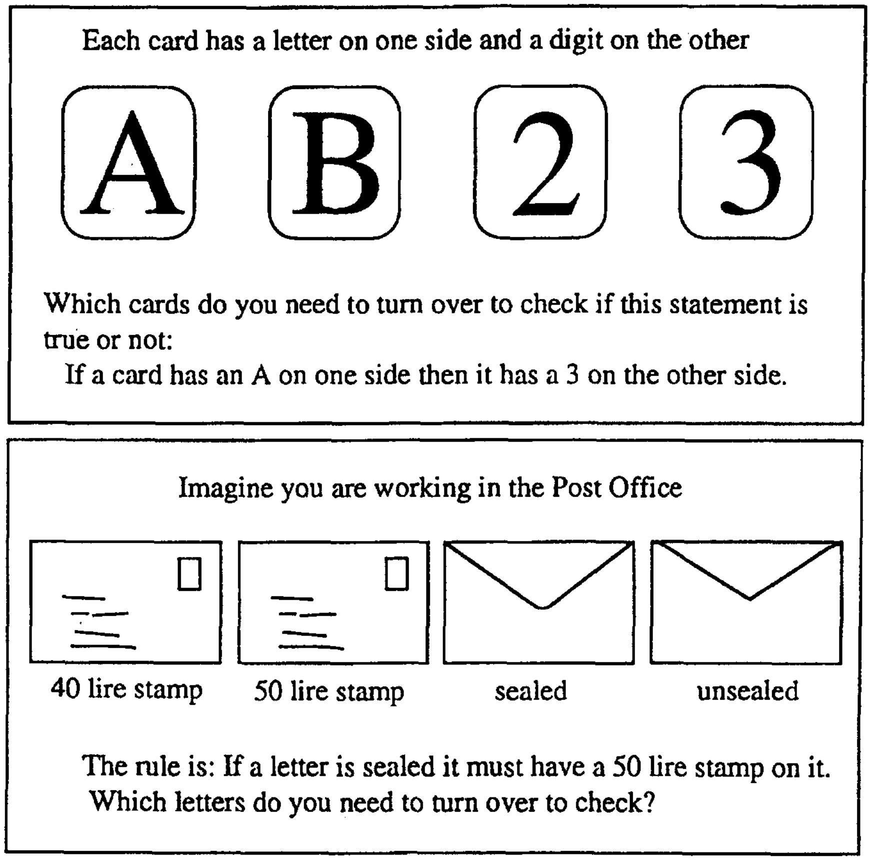

FIGURE 1. Abstract and contextualized versions of the "selection task" (after Johnson-Laird, Legrenzi, & Legrenzi, 1972).

> The subjects in the card-turning task do, indeed, search for counterexamples, but their search is only comprehensive with realistic materials that relate to an existing mental model. Their conspicuous failure with abstract materials is difficult to reconcile with syntactic principles of inference that are independent of content. (p. 34)

In a paper noting parallels between aspects of performance on mathematical problems and on Wason's selection task and its variants, Bell (1984) pointed out that:

> Mathematics teaching is generally based on the assumption that a principle or method may be learned in the abstract, or in one context, and then applied without essential change to other contexts.
>
> ... There is now an emerging realisation that differences between contexts (and also between the types of numbers involved) affect the way in which the problem is approached to such an extent that structural similarities visible to the mathematician's eye in the initial problems do not exist in the problem as seen and solved by the pupil.

The emerging awareness of context-dependency has been consolidated by more recent studies showing lack of awareness in people in general of relationships between formal school-based mathematical knowledge and situations outside of school to which that knowledge could be applied (Lave, 1988; Nunes, Schliemann, & Carraher, 1993).

In a provocative chapter, Elshout (1992, p. 5) suggested that:

> Educational philosophy seems to be locked into a pendular motion, in some periods favoring rational formal schooling as its ideal, then swinging to the position that the best of learning is to be found in everyday life.

Is it possible to take a balanced position? Margaret Donaldson, who was one of the first to write about situated cognition (she called it "embedded"), commented:

> In order to handle the world with maximum competence it is necessary to consider the structure of things. It is necessary to become skilled in manipulating systems and in abstracting forms and patterns. This is a truth which, as a species, we have slowly come to know. If we were ever to renounce the activity, there would be a hefty price to pay (Donaldson, 1988, p. 82).

Likewise, Hatano (1992) argued that "mathematics is so powerful as an intellectual tool mainly because it enables us, through formalization, to detect structural commonalities in apparently different domains, and thus to apply general problem solving algorithms to them more or less consciously." He further suggested that:

> ... mathematical knowledge that experts (experienced users with mathematical understanding as well as mathematicians) possess is no [longer] associated tightly with the situation in which it was originally acquired.
>
> ... Their knowledge may be "desituated" in the sense of being useful even outside of the situations experienced, because experts can adapt known procedures or even invent new ones based on their understanding of a set of conceptual entities. Thus how a new mathematical concept is introduced in instruction is critical, but even more important is to try to develop students' conceptual understanding, unless the goal of instruction is to teach routine procedures that are useful only for limited problems.

Another of the main criticisms made by Lave (1988, p. 34) related to what she called the "culture of transfer experiments." In these experiments, subjects work individually on prefabricated puzzles or problems, within "a tight time-frame of an hour or less of unfamiliar activity" (p. 43), and where it is questionable "what motivates people to recognize and undertake to solve problems when not required to do so" (p. 42). The contrast with the study in which the case study of Brandon was set (see quotation at start of paper) is striking. This study was spread over several weeks, and took place within an instructional environment emphasizing social interaction and the formulation and discussion of representations; within this environment, the making of connections is a recognized activity.

In summary, it is clear from the experimental research reviewed that:

1. Insight into structural relationships, and transfer between isomorphic problems, rarely occur spontaneously in laboratory-based studies.

2. There is often little understanding of the relationship between formal school knowledge and everyday situations to which it is, at least potentially, applicable.

In response to the first point, we may argue that the conditions of laboratory experimentation (as discussed earlier) are not conducive to the construction of high-level representational acts, on account of such factors as the questionable motivation of the subjects, the use of semantically lean puzzle-like tasks, and, typically, a short time span.

With respect to the second conclusion, we would argue that mathematics can and should be taught with emphasis on its applicability within a wide variety of contexts. Goldin and Kaput (1996, p. 425) point out that "one meaning of powerful is that a system of representation has a wide and varied domain of applicability." Freudenthal (1991, p. 123) identified as one of the "big strategies" in the development of a mathematical disposition "identifying the mathematical structure within a context, if any is allowed, and barring mathematics where it does not apply" (the latter part is also very important).

It must also be recognized that people operating effectively, and more or less routinely, within very specific domains generally do not need to consider the relationship to domain-independent formulations or tasks within other domains. We have argued that one goal of mathematics education (at least for some students) is an appreciation of the power of general unifying representations (Goldin & Kaput, 1996). It is characteristic of mathematicians, as illustrated by the quotation from Poincaré to search out structural relationships underlying situations with very different surface characteristics. Various suggestions have been made as to how such skills could be taught (Luger, 1984). From a formal mathematical perspective, Dienes (1971) tackled this educational objective head-on with a six-stage model leading all the way from "play" with structured manipulatives, through comparison of isomorphic situations, to the development of a unifying representation (usually graphical), and culminating in an axiom system. Goldin has proposed a somewhat similar theory involving: (a) an inventive-semiotic stage, (b) a period of structural development, and (c) an autonomous stage (Goldin & Kaput, 1996, p. 424).

More recently, the ongoing classroom-based work of the Rutgers team (Maher et al., 1993) has shown how an instructional environment can be created within which children develop a disposition to look for, represent, and exploit structure. The example of Brandon with which this paper begins being a case in point.

2. ISOMORPHISM WITHIN REPRESENTATIONAL ACTS

The dominant view on isomorphism, in the body of research and theoretical tradition within cognitive psychology critiqued by Lave (1988), is ontological. That is, subjects' conceptions of similarities between situations are analyzed relative to normative criteria with respect to a predetermined mapping between the situations, built into the experimental set-up by the researcher.

By contrast, we view isomorphisms as components of mental representations constructed by individuals in the course of assimilating and dealing with given situations. In this regard, we propose three basic models of isomorphism and its construction (see Figure 2).

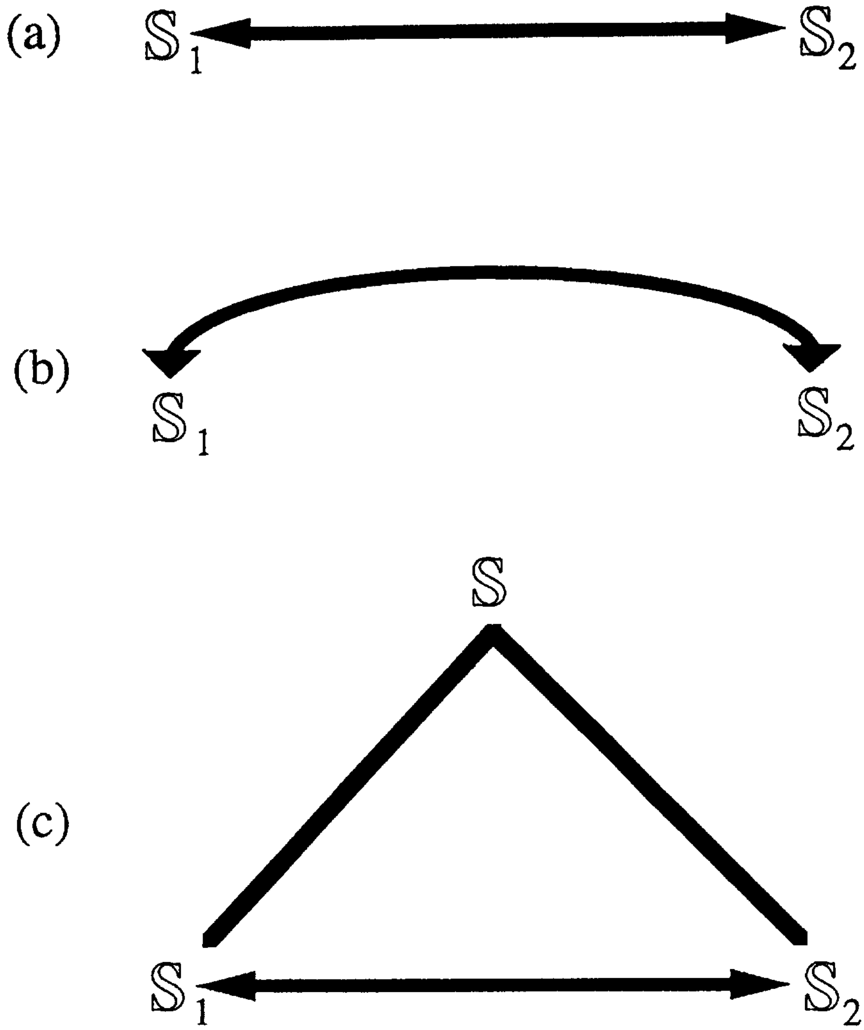

FIGURE 2. Three models for the construction of an isomorphism as a representational act.

MODEL 1: Surface-Level Isomorphism

A subject may read a problem such as:

P1. The weight of 3.62 liters of a liquid is 0.27 lb. How many liters of the liquid weigh 1 lb?

and, in the process of assimilating the problem situation, he or she may map it onto an easier problem such as:

P2. The weight of 6 liters of a liquid is 2 lb. How many liters of the liquid weigh 1 lb?

The mapping between Problems 1 and 2 is a conceptual act of forming a representation for Problem 1. The quality of this mapping is determined by the kind of reasoning employed by the subject in forming the representation. If the subject is an operation conserver, then she realizes that the appropriate operation is independent of the numbers (Greer, 1987). On the other hand, she may be unmindfully following the "conservation formula" (Harel, 1995), which may be formulated as follows:

S1: When you encounter a word problem with "nasty" numbers:

a. replace the "nasty" numbers with "friendly" numbers;
b. solve the problem with the "friendly" numbers;
c. transform back your solution to the problem with the "nasty" numbers.

Using this formula, the subject would construct a problem with a whole number divisor, such as Problem 2. Some children do not accept the validity of this method since they argue that the appropriate solution changes when the numbers change (Greer, 1987).

Schematically, Figure 2(a) depicts the formation of a mapping between two problem situations S_1 and S_2, that is based on surface considerations—such as those employed by an operation nonconserver in forming the mapping between Problems 1 and 2. The use of surface-level mappings of this sort is, unfortunately, quite pervasive among students in all levels. Consider, for example the following linear algebra problem:

P3. Determine whether the three polynomials:

$$p_1(x) = 2x^2 + 3x + 4, \quad p_2(x) = -2x^2 + 4x + 1, \quad p_3(x) = 5x^2 + 8x - 11$$

are linearly independent.

It is quite common to see students represent each polynomial by its coefficients as a triple and replace the given problem by:

P4. Determine whether the three vectors:

$$v_1(x) = (2, 3, 4), \quad v_2(x) = (-2, 4, 1), \quad v_3(x) = (5, 8, -11)$$

are linearly independent.

From a mathematical point of view, this method is correct, for $P_3[x]$ (the set of all polynomials of degree less than 3) is indeed isomorphic to $\mathbf{R}^3$, which guarantees that a set of vectors in $P_3[x]$ is linearly independent if and only if the corresponding set of vectors in $\mathbf{R}^3$ is linearly independent. However, students may, or may be taught to, employ this method before they have established the isomorphism, or even learned what isomorphism means.

It must be pointed out that the source of students' behavior in forming a surface-level mapping between problem situations is faulty instruction. Both cases described here—the use of the "conservation formula," and the translation of polynomials to n-tuples—are taught to students explicitly. We return to the instructional aspects of isomorphism formation in mathematics in the next section.

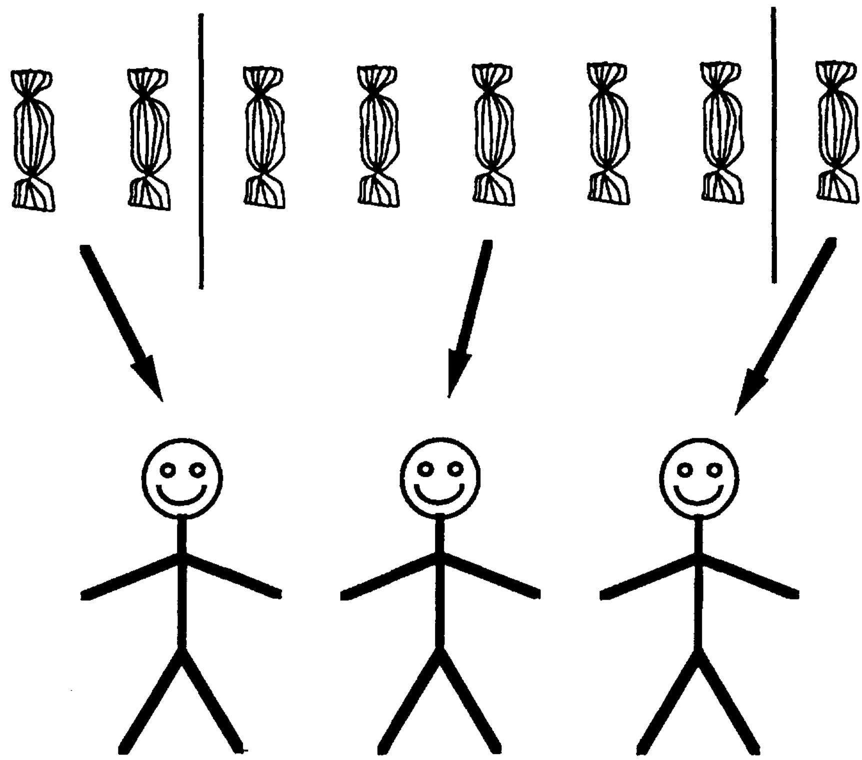

FIGURE 3. Subject's drawing to show the relationship between the number of partitions and the number of ways of placing 2 sticks in 7 positions.

Model 2: Deep Isomorphism

Under a different kind of reasoning, the two cases just discussed can exemplify a high quality of mapping formation. If the subject is an operation conserver, or the student understands the application of the isomorphism between $P_3[x]$ and $\mathbf{R}^3$ to linear independency, then the formations of the mappings between Problem 1 and 2 and Problems 3 and 4, respectively, are cases of deep isomorphism (schematically illustrated in Figure 2b).

A deep isomorphism is not necessarily formed through advanced (formal) content, i.e. theorems, as in the case of Problems 3 and 4, for example. As an example, consider the solution to the following problem formed by a high-school student in an algebra class:

P5. In how many ways can you distribute 8 candies among 3 children?

The student first drew eight candies. Then after long thought, he said that the given problem is "the same" as asking:

P6. In how many combinations can you position two sticks in the spaces between them?

and answered: $C_7^2 = 7!/(5!\cdot 2!)$. He drew a figure to illustrate his solution (Figure 3). This student did not appeal to a formal theorem to map the given problem onto another problem, which he was able to solve.

MODEL 3: Mediated Isomorphism

In contrast to Models 1 and 2, Model 3 involves a third situation, S, as a mediator in constructing an isomorphism between two situations, S_1 and S_2. The subject sees two specific situations, S_1 and S_2, as special cases of S, whereby she establishes an isomorphism between S_1 and S_2 (Figure 2c).

Consider the following example, inspired by Vinner (1991). Students A and B both know that:

S_1. Two perpendicular lines, x and y, represent a coordinate system of the plane.

Student A's understanding of this idea is:

S. Since (a) any point A in the plane can be determined uniquely by two points, X on x and Y on y, and (b) the points on a line are in one-to-one correspondence with the real numbers, the points in the plane are in a one-to-one correspondence with the ordered pairs of the real numbers.

Student B's understanding of S_1 is:

S′. The ordered pair of a point Z on the plane represents the distances of that point from the lines x and y. To find these distances, one should draw from the point Z two lines perpendicular to the lines x and y at points X and Y, respectively. This results in the rectangle OYZX, where O is the intersecting point of x and y. Then the desired distances of ZX and ZY can be obtained by measuring XO and YO, respectively.

The mathematics teacher of these two students said one day that:

S_2. Any intersecting lines, not necessarily perpendicular, can be a coordinate system for the plane.

This example illustrates how a subject can succeed, or fail, depending on her or his existing scheme, to construct an isomorphism. For Student A, the teacher's statement is just a generalization—the range of application of her/his coordinate system scheme was merely extended, and includes no other *instances* of coordinate system (see Harel & Tall, 1991). That is, Student A was able to form a mapping between S_1 and S_2 via S because S subsumes both S_1 and S_2 (Figure 4). Arrows 1, 2 and 3 represent a possible sequence of the chronological events in constructing the isomorphism between S_1 and S_2 by Student A—first abstracting S_1 into S, then projecting S onto S_2, whereby he or she realizes that S_1 and S_2 are isomorphic instances of S.

For Student B, on the other hand, a major reconstruction is needed in order to understand the teacher's latter statement, S_2. S′, Student B's existing scheme for coordinate systems, is insufficient to interpret S_1 and S_2 as instances of one structure, the structure

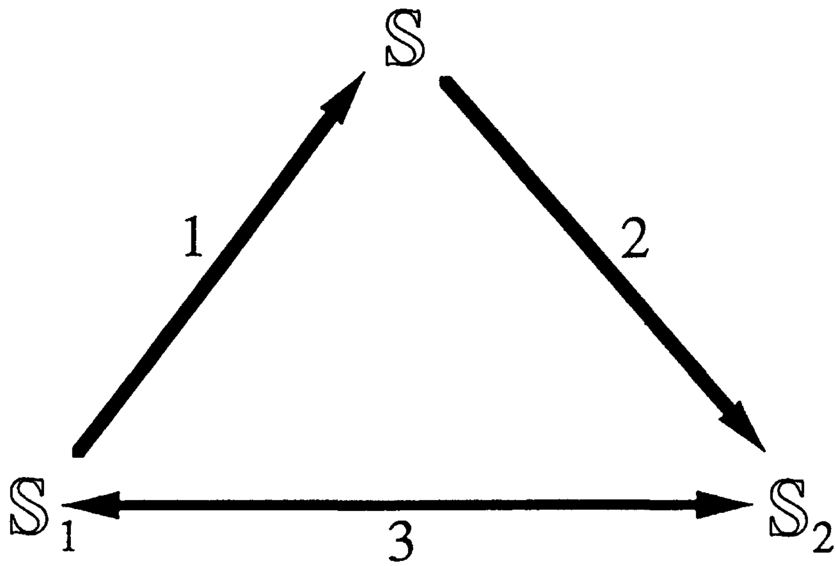

FIGURE 4. Postulated sequence by which Subject A constructed an isomorphism between S_1 and S_2.

induced by S. As a result, Student B would be unable to form an isomorphism between S_1 and S_2 without considerable conceptual restructuring.

3. EDUCATIONAL IMPLICATIONS

3.1. Linkage to Existing Knowledge

It is common for mathematicians to recognize a problem they are working on as having the same structure as some already formulated mathematical knowledge. This has already been illustrated in the quotation from Poincaré with which we began this paper; indeed, Poincaré once described mathematics as the art of giving the same name to different things.

Another example has been related by Martin Gardner (1966). Members of the Trinity Mathematical Society at Cambridge in 1936-38 investigated whether a square with integral sides could be dissected into squares, all of a different size, with integral sides. At one point, when a number of rectangles which could be so dissected had been found, further progress was made possible by the realization of an isomorphism between these dissections and electrical networks satisfying a set of equations (Kirchoff's Laws). They were thus able to exploit the extensive theory already existing for those equations.

3.2. Using an Isomorphic Problem as Solution Aid

"Think of a simpler problem" was one of Polya's heuristics for problem-solving. A specific version of this heuristic is "think of a problem which you know to be isomor-

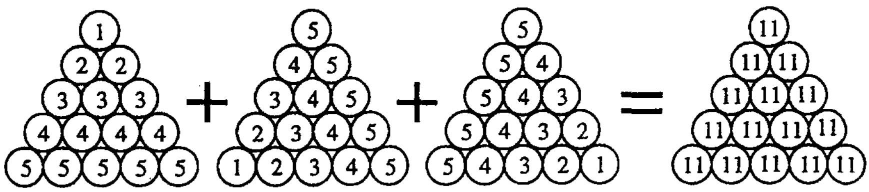

FIGURE 5. Use of spatial configurations to demonstrate the formula for the sum of the first five squares (for the general formula, which can similarly be shown schematically, the number of circles is $n\ (n + 1)/2$ and the total in each circle on the right is $2n + 1$).

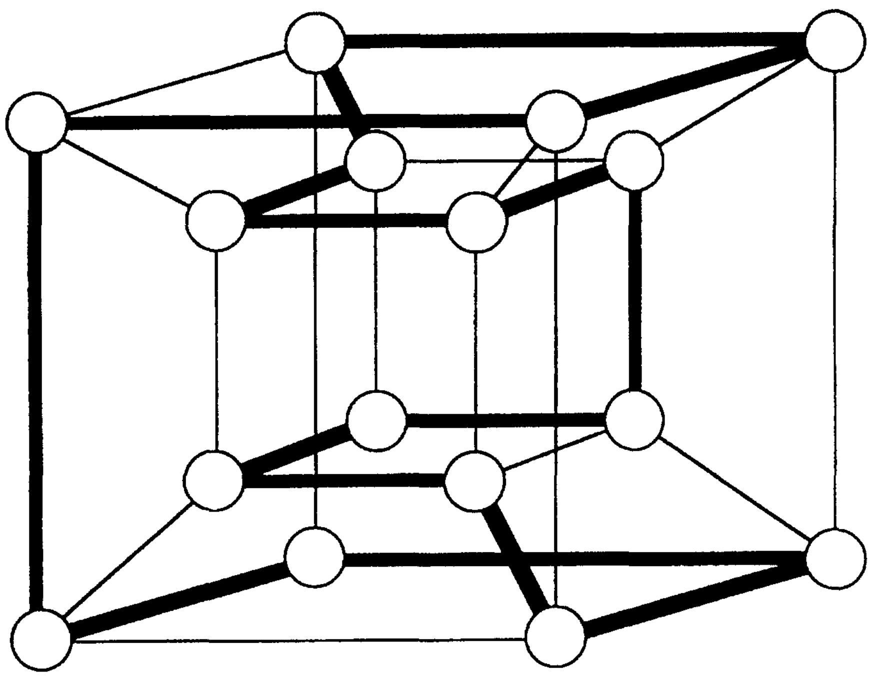

FIGURE 6. Route round a hypercube as a representation for certain problems (see text for details). The vertices of the hypercube may be labelled with coordinates (0, 0, 0, 0), (0, 0, 0, 1) etc.

phic and which is easier to solve." An example discussed above is the "conservation formula" for deciding on what operation to use for a word problem with "nasty numbers." Geometric problems may be converted to algebraic problems for easier solution, or vice versa, and numerical problems may be illuminated by spatial configurations (Figure 5).

The isomorphic problem evoked in these circumstances could be viewed as a representation of the original problem. In fact, a clear distinction cannot be drawn between thinking of an isomorphic problem and simply thinking of a good representation, as illustrated by the following example:

E1: Given 16 logic blocks differing in size (large, small), thickness (thin, thick), shape (square, triangle) and colour (red, blue), can you arrange them in a circle so that each neighbouring pair differ by only one attribute?

This becomes easy to solve if the two values for each attribute are coded as 0 and 1 (note that this is similar to the coding used by Brandon for the tower and pizza problems), then the 16 blocks are interpreted as the vertices of a hypercube (Figure 6). Since vertices joined by edges correspond to blocks differing in only one attribute, the problem becomes one of finding a closed route around the hypercube as shown. (Incidentally, the same representation can be used to solve the problem of the re-entrant knight's tour on a 4 × 4 torus (Stewart, 1992, p. 107).

Spatial representations are particularly powerful for solving problems of many types, including problems in logic (Carroll, 1958) and probability (Ichikawa, 1989). Goldin and Kaput (1996) suggest that reasons for the power of external spatial representation include the fact that it affords random access and supports imagery. We may also suggest that it is evolutionarily developed (Rav, 1993) and grounded in bodily experience (Johnson, 1987). Certainly, spatial representations allow many immediate affordances—order, relative area, continuous path, and so on, and permit easy differentiation of salience.

3.3. The Problem of Inert Knowledge

Whitehead (1929) referred to the educational problem of inert knowledge, namely knowledge that is available and can be recalled on request, but is not spontaneously recognized as applicable in situations where it is relevant.

Students very frequently fail to recognize applications of a mathematical structure which is supposed to be within their repertoire.

Many examples have been reported relating to probability and statistics, such as the following, from Kahnemann and Tversky (1982, p. 495). They presented the following question to many squash players:

E2: As you know, a game of squash can be played either to 9 or to 15 points. Holding all other rules of the game constant, if A is a better player than B, which scoring system will give A a better chance of winning?

All those asked had some knowledge of statistics, but most said the scoring system should not make any difference. They were asked to consider the argument that an atypical outcome is less likely to occur in a large sample than in a small one; with very few exceptions, the respondents accepted the force of this argument as applying to the squash question. Kahnemann and Tversky commented that:

E3: Evidently, our informants had some appreciation of the effect on sample size on sampling errors, but they failed to code the length of a squash game as an instance of sampling size.

3.4. Isomorphisms as Teaching Aids

The approach of replacing a perceived-as-difficult problem by a perceived-as-easy, yet isomorphic, problem is quite pervasive in mathematical activities at all levels, including mathematical research. However, teachers should remember that conceptual isomorphism is not a universal phenomenon—two situations may be thought of as obviously isomorphic by one subject but conceived as unrelated by another. For example, studies by Silver (1979) and others have shown that students strong in mathematics tend to categorize problems on the basis of their underlying structure, whereas students weak in mathematics do so on the basis of surface characteristics.

Here we present two cases to demonstrate the pedagogical and cognitive implications of the use of isomorphism in the teaching of mathematics. The first is from Harel (1995). An interview was conducted by Harel (Guershon) simultaneously with two children, a 13-year-old girl (Tami), and an 8 year-old boy, (Dan).

Guershon: One pound of candy cost \$7. How much would 3 pounds of candy cost?
Tami: Three times seven, 21.
Dan: I agree, three times seven.
Guershon: What if I changed the 3 into 0.31? What if the problem were: One pound of candy cost \$7. How much would 0.31 of a pound cost?
Tami: The same. It is the same problem, you have just changed the number. 0.31 times 7.
Dan: No way! It isn't the same. Can't be (angrily). It isn't times. How did you (speaking to the interviewer) agree with her?
Guershon: I didn't agree with her, I'm just listening to both of you. How would *you* solve the problem?
Dan: (After a short pause) You take 1 and you divide by 0.31. You take that number, whatever that number is, and you divide 7 by that number.

It took the interviewer a long moment to realize, not before applying some algebraic manipulations, that Dan's solution was correct. Dan's solution was to first perform the operation $1 \div 0.31$, then take the result of this operation and divide 7 by it, namely: $7 \div (1 \div 0.31)$. Indeed, $7 \div (1/0.31) = 7 \text{ x } (0.31/1) = 7 \times 0.31$. Dan saw the two problems as different and therefore rejected the idea that the solution of the first problem can be conserved as a solution to the second. Despite this he was able to offer a correct and, more importantly, a meaningful solution. An instructional approach that imposes an isomorphism interpretation between the two problems asked by the interviewer would harm children like Dan, who is still in the process of moving from an operation non-conservation state to an operation conservation stage (see, Harel, 1995, for more details).

The second case we use here to demonstrate the need for pedagogical sensitivity in introducing isomorphic-to-us problems is from the domain of geometry. A student had difficulty solving the following problem:

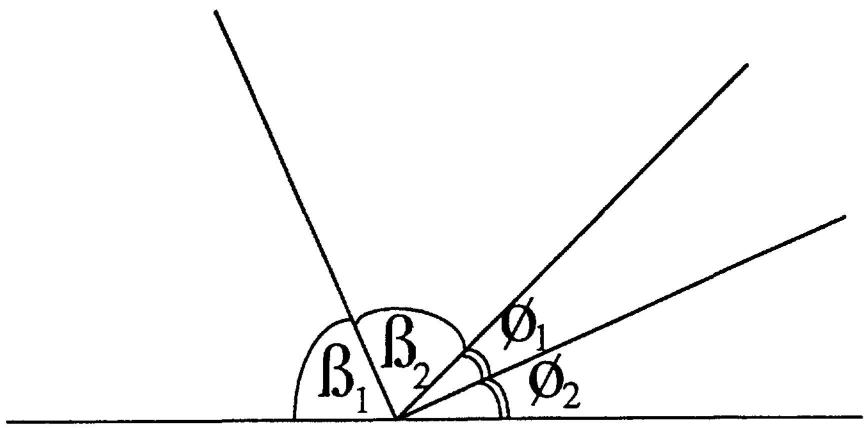

FIGURE 7. If $\beta_1 = \beta_2$ and $\phi_1 = \phi_2$ what is the value of $\beta_2 + \phi_1$?

P7. Consider the figure (Figure 7). In this figure, it is given that

$\beta_1 = \beta_2$ and $\phi_1 = \phi_2$. Find the measure of $\beta_2 + \phi_1$

In an attempt to help the student with this problem, the instructor (Harel) introduced the following problem which, to him, was obviously isomorphic to the previous one.

P8. You and your sister had 180 dollars altogether. Your sister gave me half of what she had and you gave me half of what you had. How much money do you have left between you?

While the student solved Problem 8 easily, he saw no connection between the two problems. It was only *after* he had constructed a solution for Problem 7 that he made the connection to Problem 8, realizing that the two problems possess essentially the same structure.

Traditionally, isomorphism has been used to describe similarities between new ideas to be learned and familiar ones that are outside the content area of immediate interest (Reigeluth & Faith, 1983). Teachers use isomorphism (or more general forms of structural relationship, such as analogy and metaphor) to motivate and introduce new ideas by pointing out to the students the correspondence, as it is viewed by the teacher, between the idea to-be-learned and a concept that is already possessed by the student. The use of water flow as a conceptual model to teach electricity typifies this instructional approach (Gentner & Gentner, 1984). Recently, some mathematics educators have raised doubts about the pedagogical benefit of this approach. These doubts are grounded in both experimental findings and philosophical analyses of learning.

3.5. The Case of Manipulatives

Hart (1993) has summarized evidence that children fail to form bridges between manipulatives and the formalizations they are intended to help develop; in the words of one child "sums is sums and bricks is bricks" (p. 25).

Cobb, Yackel, and Wood (1992) argued that the mapping between two domains (for example, in the approach to teaching the standard subtraction algorithm that emphasized the analogy between symbols used in standard written notation and Dienes blocks designed to highlight the quantitative meaning of place value) can be spelled out by the instructor because he or she has already contructed a relatively sophisticated conception of place value numerations. In contrast to the discussion of learning that characterizes students as actively trying to make sense of their worlds, the transition to instructional considerations brings with it an almost exclusive emphasis on the instructor's relatively sophisticated understanding of the domains (see also Gravemeijer, 1991).

3.6. The Use of Analogies

An important implication from the argument of Cobb, Yackel, and Wood is that analogy should be used to reinforce already existing conceptions rather than building new conceptions. Thus, students who have constructed the ideas of place value and the subtraction algorithm for multi-digit numbers, can strengthen their understanding of these ideas by interpreting Dienes blocks in terms of their conceptions. This interpretation is essentially a formation of a mapping between two representational systems (Goldin & Kaput, 1996)—the world of symbols used in standard written notation, S_1, and the Dienes blocks, S_2—via their conception, S (see Model 3, Figure 2c).

In an analysis of linear algebra textbooks, Harel (1987) found that analogy is used as one of four strategies to motivate students and to integrate new mathematical ideas with material previously studied. Two kinds of analogies were found to be used by textbooks, namely analogies to non-mathematical content (e.g., the analogy between economics problems and problems about matrices) and analogies to mathematical content (e.g., defining self-adjoint operator by analogy to the characterization of real numbers in the field of complex numbers).

Analogies relating a mathematical content situation to a non-mathematical content situation have many nonanalogous aspects. For a student to distinguish between the analogous aspects and the nonanalogous aspects, he or she must first abstract the two situations and only then be able to distinguish between features that are relevant or irrelevant to the analogy. This, obviously, raises doubts about the anticipated motivational effect on students.

In some cases, indeed, the use of analogy may produce conceptual confusion. For example, in some textbooks in linear algebra the definition of inner product is motivated by the idea of computing the total price of items in quantities $x_1, x_2, \ldots, x_n$ whose unit prices are $p_1, p_2, \ldots, p_n$, respectively, by the operation:

$$(x_1, x_2, \ldots, x_n)^* (p_1, p_2, \ldots, p_n) = \Sigma\, x_i \cdot p_i$$

The student may erroneously think that this operation is an inner product. In fact, it is not an inner product but a bilinear form on two spaces, the space of items and the space of prices, which may be regarded as the dual space. Accordingly, the use of analogies needs to be handled with circumspection.

4. CONCLUSIONS

Much of the flavor of recent work in mathematics education may be summarized by the statement that mathematics is a human activity embedded in social, cultural, and historical contexts. As Elshout (1992) has commented, the pendulum of educational philosophy has swung towards everyday learning. For balance, it must be remembered that intellectual activity—including the activity of mathematical abstraction—is also part of being human (Greer, 1996). We agree with Thompson (1993, p. 283) when he stated that:

> I consider it imperative that the mathematics education community regain the sense that mathematics is a deep and abstract intellectual achievement.

Likewise, we agree with Goldin and Kaput (1996, p. 426) that a central goal of mathematics education is to increase the power of students' representations. An aspect of such a goal not addressed in this paper, but of increasing importance, is the extension of dynamic, interactive and recording representational media (Goldin & Kaput, 1996, p. 411) made possible by computers.

The detection and exploitation of structural relationships, and the very idea of isomorphism, is an essential component of mathematics as "a deep and abstract intellectual achievement" which can be developed in children. We have outlined a reconceptualization locating this form of thinking within a framework of representational acts. From this perspective, both laboratory-based research using artificial problems, and research demonstrating the weak linkage between formal mathematics and cognition situated within highly circumscribed contexts, are of limited relevance. A more appropriate form of investigation is research situated within an instructional environment such as that within which Brandon developed his insights.

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