SHM path integral

Start with a hammy:

$$H\,(P,Q) = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2} \tag{1}$$

In path integrals, operators are functions so $P \to p$ and $Q \to q$. Here we're interested in gound state to ground state, because of reasons. Using:

$$\langle 0|0\rangle = \langle 0|q_n\rangle \langle q_{n-1}|q_{n-2}\rangle \dots \langle q_1|0\rangle \tag{2}$$

$$\Downarrow$$

$$\langle 0|0\rangle = \int \mathcal{D}p\mathcal{D}q\, exp\left[i\int_{-\infty}^{\infty} dt\,(p\dot{q} - (1-i\epsilon)H + fq))\right] \tag{3}$$

Where H is Weyl-ordered (average of normal and anti-normal ordering).
Applying $(1-i\epsilon)$ on H will pick out the ground states in $\pm\infty$ time, leads to the following transforms:

$$\frac{1}{2}m\omega^2 q \to \frac{1}{2}(1-i\epsilon)m\omega^2 \tag{4}$$

$$and \tag{5}$$

$$\frac{1}{2m}p^2 \to \frac{1}{2(1-i\epsilon)m}p^2 = \frac{(1-i\epsilon)(1+i\epsilon)}{2m(1+i\epsilon)}p^2 = \frac{1+i\epsilon-i\epsilon+\mathcal{O}(2)}{2m(1+i\epsilon)}p^2 \tag{6}$$

$$\Rightarrow \frac{1}{2m}p^2 \to \frac{1}{2(1+i\epsilon)m}p^2 \tag{7}$$

subbing back into 3:

$$\langle 0|0\rangle = \int \mathcal{D}p\mathcal{D}q\, exp\left[i\int_{-\infty}^{\infty} dt\left(p\dot{q} - \frac{p^2}{(1+i\epsilon)2m} - \frac{1-i\epsilon}{2}m\omega^2 q^2 + fq\right)\right] \tag{8}$$

now the sweet insides can be integrated out over $\mathcal{D}p$ to turn it into a laggy, using $\partial_p\mathcal{H} = \dot{q}$:

$$\langle 0|0\rangle = \int \mathcal{D}q\, exp\left[i\int_{-\infty}^{\infty} dt\left(\frac{1}{2}(1+i\epsilon)m\dot{q}^2 - \frac{1}{2}(1-i\epsilon)m\omega^2 q^2 + fq\right)\right] \tag{9}$$

Next up, perform a fourier transform to get this shit into functions of energies and shit. Use these variables, they're good, trust me, you are me after all:

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{\tau}e^{-iEt}\tilde{q}(E) \tag{10}$$

$$\dot{q}(t) = \int_{-\infty}^{\infty} -\frac{dE}{\tau}\,iE\,e^{-iEt}\tilde{q}(E) \tag{11}$$

$$\tilde{q}(E) = \int_{-\infty}^{\infty} dt\,e^{iEt}q(t) \tag{12}$$

Now take all that rubbish and shove it into the terms in 9
do not forget that there's squared variables so we'll have to integrate over two different variables, thus E and E', and t and t'

$$\langle 0|0\rangle_f = \int \mathcal{D}q\, exp\left\{\frac{i}{2}\int_{-\infty}^{\infty} \frac{dE}{\tau}\frac{dE'}{\tau}\,e^{-i(E+E')t}\left[\Big(-(1+i\epsilon)EE' - (1-i\epsilon)\omega^2\Big)\tilde{q}(E)\tilde{q}(E') + \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)\right]\right\} \tag{13}$$

Now it looks like a fucking goddamn mess, but we can integrate over E' using a neat delta function:

$$\tau\delta(a-b) = \int dx\,e^{i(a-b)x} \implies \frac{1}{2}\int \frac{dE\,dE'}{\tau^2}\,\delta(E+E')[\dots] \tag{14}$$

Reslutting in:

$$\langle 0|0\rangle_f = \int \mathcal{D}q\, exp\left\{\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{\tau}\left[\underbrace{\Big((1+i\epsilon)E^2-(1-i\epsilon)\omega^2\Big)}_{\downarrow}\tilde{q}(E)\tilde{q}(-E)+\tilde{f}(E)\tilde{q}(-E)+\tilde{f}(-E)\tilde{q}(E)\right]\right\} \tag{15}$$

$$E^2-\omega^2+i(E^2+\omega^2)\epsilon \implies E^2-\omega^2-i\epsilon \tag{16}$$

Now as a magic trick we do a little quasigauge shift. We gunna introduce x as a shift of q and the inverse of 16. The benefit of this is that since it's a linear shift in q/x only the measure won't change:

$$\tilde{x}(E) = \tilde{q}(E) + \frac{\tilde{f}(E)}{E^2-\omega^2+i\epsilon} \tag{17}$$

$$\mathcal{D}q = \mathcal{D}x \tag{18}$$

Substituting back into 15 and splitting the path integral into a phase dependent on x and independent of x:

$$\langle 0|0\rangle_f = \underbrace{exp\left[\frac{i}{2}\int\frac{dE}{\tau}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right]}_{\text{basically } \textit{the} \text{ interesting part - the propagator comes from here}} \cdot \underbrace{\int\mathcal{D}x\, exp\left[\frac{i}{2}\int\frac{dE}{\tau}\tilde{x}(E)(E^2-\omega^2+i\epsilon)\tilde{x}(-E)\right]}_{\text{when } f=0 \implies \langle 0|0\rangle_f=1 \text{ which is the ground state}} \tag{19}$$

$$\tag{20}$$

As noted by Srednicki - a system in a ground state will remain in a ground state (aka itial and final states are ground $\langle 0|0\rangle$) unless acted on by an external force ($f \neq 0$) - which in this case it is, resulting in there being a propagator - a source term/current term (very much related to the J term in maxwell's eqns - in this case just the driving fq term.
To actually proceed solving this we'll idntify the partts of the first term that aren't $\tilde{f}(\textit{something})$ dependent, act with that on a phasor and call it a green function. this will let us do a fourier/delta transform thing.

$$\langle 0|0\rangle_f = exp\left[\frac{i}{2}\int\underbrace{\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}}_{\text{green fx}}\tilde{f}(E)\tilde{f}(-E)\right] \tag{21}$$

$$G(t-t') = \int\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}\, e^{-iE(t-t')} \tag{22}$$

Where the green function is a solution to the wave eqn

$$(\partial_t^2+\omega^2)G(t-t') = \delta(t-t') \tag{23}$$

in other words it's a well dressed delta function. or in other other words, a green function is the inverse of a differential operator. or in other other words it's a convolution function. there may be some other other other words, but idk, it should be pretty obv from its application how it works innit.
Most importantly however, a green function is what's more commonly known as the propagator.
In any case, let's actually test it out by chucking 22 into 23 explicitly:

$$(\partial_t^2+\omega^2)\int\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}\, e^{-iE(t-t')} = \int\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}\,\left(\partial_t^2 e^{-iE(t-t')}+\omega^2 e^{-iE(t-t')}\right) = \tag{24}$$

$$= \int\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}\,\left((-iE)(-iE)e^{-iE(t-t')}+\omega^2 e^{-iE(t-t')}\right) = \int\frac{dE}{\tau(-E^2+\omega^2-i\epsilon)}\,\left(-E^2+\omega^2\right)e^{-iE(t-t')} = \tag{25}$$

$$= \int\frac{dE}{\tau}\frac{\left(-E^2+\omega^2\right)}{(-E^2+\omega^2-i\epsilon)}\, e^{-iE(t-t')} \overset{\epsilon\to 0}{=} \int\frac{dE}{\tau}\frac{\left(-E^2+\omega^2\right)}{(-E^2+\omega^2)}\, e^{-iE(t-t')} = \int\frac{dE}{\tau}\, e^{-iE(t-t')} = \delta(t-t') \tag{26}$$

Beautiful, it fits. But back to business, but let's actually evaluate the propagator, we're gonna use funky ass Cauchy residue theorem or whatever the hell its properly called:

$$\oint_\gamma f(z)dz = \tau i\sum_{a_i\in\gamma} Res_{z=a_i} f(z) \tag{27}$$

Aka sum all the poles a_i within the γ contour. The residue is then just the value of the function without the particular pole. So first cleverly split the bottom term into the roots of a completed square (factoring out a minus overall to have nicer plusminuses to work with):

$$\int\frac{dE\; e^{-iE(t-t')}}{\tau(-E^2+\omega^2-i\epsilon)} = -\int\frac{dE\; e^{-iE(t-t')}}{\tau\left(E+\sqrt{\omega^2+i\epsilon}\right)\left(E+\sqrt{\omega^2-i\epsilon}\right)} = \tag{28}$$