SHM path integral

Start with a hammy:

$$H\left(P,Q\right)=\frac{P^2}{2m}+\frac{m\omega^2Q^2}{2} \tag{1}$$

In path integrals, operators are functions so $P\rightarrow p$ and $Q\rightarrow q$. Here we're interested in gound state to ground state, because of reasons. Using:

$$\langle 0|0\rangle = \langle 0|q_n\rangle\,\langle q_{n-1}|q_{n-2}\rangle\dots\langle q_1|0\rangle \tag{2}$$

$$\Downarrow$$

$$\langle 0|0\rangle = \int \mathcal{D}p\mathcal{D}q\,exp\left[i\int_{-\infty}^{\infty}dt\left(p\dot{q}-(1-i\epsilon)H+fq\right)\right] \tag{3}$$

Where H is Weyl-ordered (average of normal and anti-normal ordering).
Applying $(1-i\epsilon)$ on H will pick out the ground states in $\pm\infty$ time, leads to the following transforms:

$$\frac{1}{2}m\omega^2 q\rightarrow\frac{1}{2}(1-i\epsilon)m\omega^2 \tag{4}$$

$$and \tag{5}$$

$$\frac{1}{2m}p^2\rightarrow\frac{1}{2(1-i\epsilon)m}p^2=\frac{(1-i\epsilon)(1+i\epsilon)}{2m(1+i\epsilon)}p^2=\frac{1+i\epsilon-i\epsilon+\mathcal{O}(2)}{2m(1+i\epsilon)}p^2 \tag{6}$$

$$\Rightarrow\frac{1}{2m}p^2\rightarrow\frac{1}{2(1+i\epsilon)m}p^2 \tag{7}$$

subbing back into 3:

$$\langle 0|0\rangle = \int \mathcal{D}p\mathcal{D}q\,exp\left[i\int_{-\infty}^{\infty}dt\left(p\dot{q}-\frac{p^2}{(1+i\epsilon)2m}-\frac{1-i\epsilon}{2}m\omega^2q^2+fq\right)\right] \tag{8}$$

now the sweet insides can be integrated out over $\mathcal{D}p$ to turn it into a laggy, using $\partial_p\mathcal{H}=\dot{q}$:

$$\langle 0|0\rangle = \int \mathcal{D}q\,exp\left[i\int_{-\infty}^{\infty}dt\left(\frac{1}{2}(1+i\epsilon)m\dot{q}^2-\frac{1}{2}(1-i\epsilon)m\omega^2q^2+fq\right)\right] \tag{9}$$

Next up, perform a fourier transform to get this shit into functions of energies and shit. Use these variables, they're good, trust me, you are me after all:

$$q(t)=\int_{-\infty}^{\infty}\frac{dE}{\tau}e^{-iEt}\tilde{q}(E) \tag{10}$$

$$\dot{q}(t)=\int_{-\infty}^{\infty}-\frac{dE}{\tau}\,iE\,e^{-iEt}\tilde{q}(E) \tag{11}$$

$$\tilde{q}(E)=\int_{-\infty}^{\infty}dt\,e^{iEt}q(t) \tag{12}$$

Now take all that rubbish and shove it into the terms in 9
do not forget that there's squared variables so we'll have to integrate over two different variables, thus E and E', and t and t'

$$\langle 0|0\rangle_f=\int \mathcal{D}q\,exp\left\{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{\tau}\frac{dE'}{\tau}\,e^{-i(E+E')t}\left[\Big(-(1+i\epsilon)EE'-(1-i\epsilon)\omega^2\Big)\tilde{q}(E)\tilde{q}(E')+\tilde{f}(E)\tilde{q}(E')+\tilde{f}(E')\tilde{q}(E)\right]\right\} \tag{13}$$

Now it looks like a fucking goddamn mess, but we can integrate over E' using a neat delta function:

$$\tau\delta(a-b)=\int dx\,e^{i(a-b)x}\implies\frac{1}{2}\int\frac{dE\,dE'}{\tau^2}\,\delta(E+E')[\dots] \tag{14}$$

Reslutting in:

$$\langle 0|0\rangle_f = \int \mathcal{D}q\, exp\left\{\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{\tau}\left[\underbrace{\Big((1+i\epsilon)E^2-(1-i\epsilon)\omega^2\Big)}_{\downarrow}\tilde{q}(E)\tilde{q}(-E)+\tilde{f}(E)\tilde{q}(-E)+\tilde{f}(-E)\tilde{q}(E)\right]\right\} \tag{15}$$

$$E^2-\omega^2+i(E^2+\omega^2)\epsilon \implies E^2-\omega^2-i\epsilon \tag{16}$$

Now as a magic trick we do a little quasigauge shift. We gunna introduce x as a shift of q and the inverse of 16. The benefit of this is that since it's a linear shift in q/x only the measure won't change:

$$\tilde{x}(E) = \tilde{q}(E) + \frac{\tilde{f}(E)}{E^2-\omega^2+i\epsilon} \tag{17}$$

$$\mathcal{D}q = \mathcal{D}x \tag{18}$$

Substituting back into 15 and splitting the path integral into a phase dependent on x and independent of x:

$$\langle 0|0\rangle_f = exp\left[\frac{i}{2}\int\frac{dE}{\tau}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right]\cdot\underbrace{\int\mathcal{D}x\, exp\left[\frac{i}{2}\int\frac{dE}{\tau}\tilde{x}(E)(E^2-\omega^2+i\epsilon)\tilde{x}(-E)\right]}_{\text{when} f=0 \implies \langle 0|0\rangle_f = 1 \text{which is the ground state}} \tag{19}$$