We want vac-vac so:

$$\langle 0|0\rangle = \int \mathcal{D}\phi \, exp\left(i\int d^4x\mathcal{L}_0\right) \tag{1}$$

Define a new object that will help find expectation values with the help of the current J

$$\langle 0|0\rangle_J = \int \mathcal{D}\phi \, exp\left(i\int d^4x\mathcal{L}_0 + J\phi\right) \tag{2}$$

The expectation value of ϕ is:

$$\langle 0|\,\phi\,|0\rangle = -i\delta_J \langle 0|0\rangle_J \Big|_{J=0} \tag{3}$$

operating once:

$$-i\delta_J \langle 0|0\rangle_J \Big|_{J=0} = -i\cdot i\phi \langle 0|0\rangle_J \Big|_{J=0} = \phi\cdot(1(\text{normalised}Z_0)) \tag{4}$$

ofc we're free to normalise shit as we want, so the obvious choice here is to normalise:

$$Z_0 = \langle 0|0\rangle_J \Big|_{J=0} = 1 \tag{5}$$

Now for the interaction one that will get a new term $\mathcal{L}_1$, bearing in mind we want vacuum again:

$$\langle 0|0\rangle_J = Z = Z_1 Z_0 = \int \mathcal{D}\phi \, exp\left(i\int d^4x\mathcal{L}_0 + \mathcal{L}_1 + J\phi\right) = \tag{6}$$

$$= \int \mathcal{D}\phi \, exp\left(i\int d^4x\mathcal{L}_1\right) exp\left(i\int d^4x\mathcal{L}_0 + J\phi\right) = \langle 0|0\rangle_J \int \mathcal{D}\phi \, exp\left(i\int d^4x\mathcal{L}_1\right) \tag{7}$$

and now somehow by magic i'm supposed to believe based on all the stupid shit above that:

$$\mathcal{L}_1(\phi) \to \mathcal{L}_1(-i\delta_J) \implies \phi^N \to (-i\delta_J)^N \tag{8}$$

but my problem here is that we still want $\langle 0|0\rangle_J$ and *not* $\langle 0|\,\phi\,|0\rangle$ which actually *would* allow me to do this fucking stupid move or at least partially up to the implication because for some reason it's assumed that $\langle 0|\,\phi\,|0\rangle = \phi$?? like i can *sorta* follow the logic, but can't do it explicitly for some reason, and absolutely no goddamn book in existence does this explicitly and i swear at this point i must be failing some absotulely basic highschool manoeuoevre

defining

$$\mathcal{L}_1 = \phi^3 \tag{9}$$

taylor expanding

$$\int \mathcal{D}\phi \, exp\left(\phi^3\right) \, exp(free) \propto \int \mathcal{D}\phi \sum \frac{(\phi^3)^n}{n!} \, exp(free) \tag{10}$$