Exercise solutions for

Categorical Realizability

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Note to the Reader

The hyperlinks in this document reference the URL MGS-categorical-realizability.pdf, which should be the PDF of the lecture notes, and in the same folder as this document.

Solutions to Chapter 2

Exercise 2.11. Each of the properties (i)–(iii) are proved by induction on the structure of the body term t.

(i) The claim holds when $t = x$, when $t = y$ is a variable distinct from x, and when $t = \mathrm{a} \in \mathcal{A}$.

Finally, by the induction hypothesis the set of variables in the term

$$\langle x \rangle.\,(t_1\, t_2) = \mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)$$

is exactly $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$, where $\mathcal{V}(t)$ denotes the set of variables in the term t. The result in this case now follows since

$$\mathcal{V}(t_1\, t_2) \setminus x = (\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x).$$

(ii) The term $\langle x \rangle.\, t$ is defined when t is a variable or an element of $\mathcal{A}$ since, by Definition 2.1, $\mathbf{K}\,\mathrm{a}$ and $\mathbf{S}\,\mathrm{a}\,\mathrm{b}$ are defined for any $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

Finally, if $\langle x \rangle.\, t_1$ and $\langle x \rangle.\, t_2$ are defined then any substitution into the variables of

$$\langle x \rangle.\,(t_1\, t_2) = \mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)$$

yields an element of $\mathcal{A}$ of the form $\mathbf{S}\ \mathrm{a}\ \mathrm{b}$ for some $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

(iii) By straightforward computation in the case where t is not an application. When $t = t_1\, t_2$,

$$\begin{aligned} &(\langle x \rangle.\,(t_1\, t_2))\ \mathrm{a} \\ =\ &\mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)\ \mathrm{a} \\ \simeq\ &((\langle x \rangle.\, t_1)\ \mathrm{a})\ ((\langle x \rangle.\, t_2)\ \mathrm{a}) \\ \simeq\ &(t_1[\mathrm{a}/x])\ (t_2[\mathrm{a}/x]) \qquad \text{(by the induction hypothesis)} \\ \simeq\ &(t_1\, t_2)[\mathrm{a}/x]. \end{aligned}$$

Exercise 2.14.

(i) $\mathbf{pair}\,\mathrm{a}\,\mathrm{b} = (\langle xyz \rangle.\, zxy)\,\mathrm{a}\,\mathrm{b} = \langle z \rangle.\, z\,\mathrm{a}\,\mathrm{b}$ is defined by Exercise **2.11**.

(ii) By computation, taking care to note throughout that all applications are defined in $\mathcal{A}$.

Exercise 2.15. A possible set of definitions is

$$\begin{aligned} \mathbf{iszero} &:= \mathbf{fst} \\ \mathbf{succ} &:= \mathbf{pair}\,\mathbf{false} \\ \mathbf{pred} &:= \langle n \rangle.\, \mathbf{if}\,(\mathbf{iszero}\, n)\ \overline{0}\ (\mathbf{snd}\, n) \end{aligned}$$

(check that these satisfy the required equations).

Exercise 2.16. From the specification of **primrec**, we would like our definition to satisfy the equation

$$\mathbf{primrec}\,\mathtt{a}\,\mathtt{f} \simeq \langle n\rangle.\,\big(\mathbf{if}\,(\mathbf{iszero}\,n)\;\mathtt{a}\;\big(\mathtt{f}\,(\mathbf{pred}\,n)\,(\mathbf{primrec}\,\mathtt{a}\,\mathtt{f}\,(\mathbf{pred}\,n))\big)\big)$$

for any $\mathtt{a}, \mathtt{f} \in \mathcal{A}$.

This suggests that the term $\mathbf{primrec}\,\mathtt{a}\,\mathtt{f}$ should be constructed as a fixed point of the abstraction

$$\langle r\rangle.\,\big(\langle n\rangle.\,\big(\mathbf{if}\,(\mathbf{iszero}\,n)\;\mathtt{a}\;\big(\mathtt{f}\,(\mathbf{pred}\,n)\,(r\,(\mathbf{pred}\,n))\big)\big)\big), \tag{1}$$

and so we might try to define

$$\mathbf{spec}' := \langle af\rangle.\,\langle rn\rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\,a\,\big(f\,(\mathbf{pred}\,n)\,(r\,(\mathbf{pred}\,n))\big)$$

and

$$\mathbf{primrec}' := \langle af\rangle.\,\mathrm{Z}\,(\mathbf{spec}'\,a\,f).$$

However, this definition does not satisfy the requirement that $\mathbf{primrec}'\,\mathtt{a}\,\mathtt{f}\,\overline{0}$ is always defined (expand the definition and check!).

Instead, we tweak the abstraction (1) whose fixed point we take, and define

$$\begin{aligned}\mathbf{spec} &:= \langle af\rangle.\,\langle rn\rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\,(\mathrm{K}\,a)\,(\mathrm{S}\,f\,r)(\mathbf{pred}\,n),\\ \mathbf{primrec} &:= \langle af\rangle.\,\mathrm{Z}\,(\mathbf{spec}\,a\,f).\end{aligned}$$

We can then check (do so!) that the required equations are satisfied.

Exercise 2.17.

(i) $\Longrightarrow$ (ii): Assuming $\mathbf{true} \neq \mathbf{false}$, we show that $\overline{\mathtt{m}} \neq \overline{\mathtt{n}}$ for all $n \in \mathbb{N}$ and $m < n$, by case distinction on n and then induction on $m < n$. This is trivial for $n = 0$.

Assume that $n = n' + 1$. For $m = 0$,

$$\mathbf{iszero}\,\overline{\mathtt{m}} = \mathbf{true} \neq \mathbf{false} = \mathbf{iszero}\,\overline{\mathtt{n}}$$

and so $\overline{\mathtt{m}} \neq \overline{\mathtt{n}}$. If $m + 1 < n$ then $m < n'$ and

$$\mathbf{pred}\,\overline{\mathtt{m+1}} = \overline{\mathtt{m}} \neq \overline{\mathtt{n'}} = \mathbf{pred}\,\overline{\mathtt{n}}$$

by induction, so $\overline{\mathtt{m+1}} \neq \overline{\mathtt{n}}$.

(ii) $\Longrightarrow$ (iii): Immediate.

(iii) $\Longrightarrow$ (i): Because if $\mathbf{true} = \mathbf{false}$ then

$$\mathtt{a} = \mathbf{if}\,\mathbf{true}\,\mathtt{a}\,\mathtt{b} = \mathbf{if}\,\mathbf{false}\,\mathtt{a}\,\mathtt{b} = \mathtt{b}$$

for all $\mathtt{a}, \mathtt{b} \in \mathcal{A}$.

Solutions to Chapter 3

Exercise 3.20. The inclusion maps

$$\mathrm{inl}\colon X \to X+Y \quad \text{and} \quad \mathrm{inr}\colon Y \to X+Y$$

are given by the usual coproduct inclusions in Set, and tracked by **left** and **right** respectively.

To check that the definition gives a coproduct in $\mathsf{Asm}_{\mathcal{A}}$, it's enough to show that for any assembly Z and assembly maps $f\colon X \to Z$ and $g\colon Y \to Z$, the induced function $[f,g]\colon |X+Y| \to |Z|$ is tracked. That is, we need $\mathtt{t}_{[f,g]} \in \mathcal{A}$ such that for all $x \in X$, $y \in Y$ and realizers $\mathtt{a} \Vdash_X x$ and $\mathtt{b} \Vdash_Y y$,

$$\mathbf{pair}\,\mathbf{false}\,\mathtt{a} \Vdash_{X+Y} \mathrm{inl}(x) \implies \mathtt{t}_{[f,g]}(\mathbf{pair}\,\mathbf{false}\,\mathtt{a}) \Vdash_Z [f,g](\mathrm{inl}(x)) = f(x)$$

and

$$\mathbf{pair}\,\mathbf{true}\,\mathtt{b} \Vdash_{X+Y} \mathrm{inr}(y) \implies \mathtt{t}_{[f,g]}(\mathbf{pair}\,\mathbf{true}\,\mathtt{b}) \Vdash_Z [f,g](\mathrm{inr}(y)) = g(y).$$

Denoting the trackers of f and g by $\mathtt{t}_f$ and $\mathtt{t}_g$ respectively, we may define

$$\mathtt{t}_{[f,g]} := \langle w\rangle.\,\mathbf{if}\,(\mathbf{fst}\,w)\ \mathtt{t}_f\ \mathtt{t}_g\,(\mathbf{snd}\,w).$$

Exercise 3.21. The map out of the coproduct

$$\begin{aligned} \mathbf{1}+\mathbf{1} &\to \mathbf{2} \\ \mathrm{inl}(\star) &\mapsto 0 \\ \mathrm{inr}(\star) &\mapsto 1 \end{aligned}$$

(induced by the constant maps $\mathbf{1} \to \mathbf{2}$ at 0 and 1) is bijective on the carriers. Its inverse function is tracked by

$$\langle w\rangle.\,\mathbf{pair}\ w\ \mathtt{a}$$

for any $\mathtt{a} \in \mathcal{A}$, and so we get a pair of inverse assembly isomorphisms.

Exercise 3.23.

(i) The full details of this depend on the particular set theory, but the idea of the proof will hold for any "good" definition of Set (in particular, for ZFC) and is as follows.

The morphisms $z\colon 1 \to \mathbb{N}$ and $s\colon \mathbb{N} \to \mathbb{N}$ are given by the constant function at 0 and the successor function, respectively. Given any set X and functions $x\colon 1 \to X$ and $f\colon X \to X$, recursively define a sequence of functions

$$\left(r_n\colon \{0,\ldots,n\} \to X\right)_{n\in\mathbb{N}}$$

by

$$r_0(0) := x,$$
$$r_{n+1}(m) := \begin{cases} r_n(m) & \text{if } m \leq n \\ f(r_n(n)) & \text{if } m = n+1 \end{cases}$$

(where we have, as is customary, used the same name x to refer to the constant function $1 \xrightarrow{x} X$ and its value $x \in X$).

By induction, $r_{n+1} \upharpoonright \{0, \ldots, n\} = r_n$ for all $n \in \mathbb{N}$, and the universal morphism $r \colon \mathbb{N} \to X$ is the union of this sequence of functions. In particular, we have that

$$r(n+1) = r_{n+1}(n+1) = f(r_n(n)) = f(r(n))$$

for all $n \in \mathbb{N}$.

(ii) The zero and successor functions of $|\mathbf{N}| = \mathbb{N}$ are tracked by $\mathbf{K}\,\overline{0}$ and **succ** respectively. We claim that these are also the morphisms making $\mathbf{N}$ an nno in $\mathsf{Asm}_{\mathcal{A}}$.

To show this, it's enough to show that given any assembly X and maps $x \colon \mathbf{1} \to X$ and $f \colon X \to X$ such that $\mathtt{a} \Vdash_X x$ and $\mathtt{t}_f$ tracks f, the function $r \colon |\mathbf{N}| \to |X|$ defined in part (i) is tracked. And indeed it is: show, by induction, that

$$\mathbf{primrec}\ \mathtt{a}\ (\mathbf{K}\ \mathtt{t}_f)$$

tracks r.

Exercise 3.26. Any constant function $x \colon |\mathbf{1}| \to |X|$ for any assembly X is tracked (by $\mathbf{K}\ \mathtt{a}$ for any realizer $\mathtt{a} \Vdash_X x$), so

$$\mathsf{Asm}_{\mathcal{A}}(\mathbf{1}, X) \cong \mathsf{Set}(|\mathbf{1}|, |X|) \cong |X|.$$

Naturality of this bijection is immediate because the action of any assembly map is just the action of its underlying function.

Exercise 3.28. For any assembly X and set Y, we have that any function $|X| \to |\nabla(Y)|$ is tracked by $\mathbf{I}$, so

$$\mathsf{Set}(|X|, Y) = \mathsf{Set}(|X|, |\nabla(Y)|) \cong \mathsf{Asm}_{\mathcal{A}}(X, \nabla(Y)).$$

Again, naturality in the arguments holds straightforwardly (check!).

Exercise 3.30. If there is a nonconstant map $\nabla\{0,1\} \to \mathbf{2}$ with tracker $\mathtt{t}$, then for any $\mathtt{a} \in \mathcal{A}$ we have that $\mathtt{a} \Vdash_{\nabla\{0,1\}} 0$ and $\mathtt{a} \Vdash_{\nabla\{0,1\}} 1$, and hence that $\mathtt{t}\ \mathtt{a} \Vdash_{\mathbf{2}} 0$ and $\mathtt{t}\ \mathtt{a} \Vdash_{\mathbf{2}} 1$. But the realizers of 0 and 1 in $\mathbf{2}$ are **true** and **false** respectively, and so

$$\mathbf{true} = \mathtt{t}\ \mathtt{a} = \mathbf{false}.$$

By Exercise 2.17, this means that $\mathcal{A}$ is trivial.

Exercise 3.31. In particular, a right adjoint $R \colon \mathsf{Asm}_{\mathcal{A}} \to \mathsf{Set}$ would satisfy

$$\mathsf{Asm}_{\mathcal{A}}(\nabla\{0,1\}, \mathbf{2}) \cong \mathsf{Set}(\{0,1\}, R(\mathbf{2})).$$

By Exercise 3.30 and the fact that constant functions are always tracked, the left hand side has size 2 when $\mathcal{A}$ is nontrivial. On the other hand, the right hand set has size $|R(\mathbf{2})|^2$, which cannot be the case.

Exercise 3.32.

(i) We show that any two distinct assembly maps $f, g \colon X \to \mathbf{2}$ must have distinct trackers. To this end assume that $f(x) = 0$ and $g(x) = 1$ for some $x \in X$ with realizer $\mathtt{a} \Vdash_X x$. If $\mathtt{t} \in \mathcal{A}$ tracks both f and g, then $\mathtt{t}\, a \Vdash_{\mathbf{2}} f(x)$ and also $\mathtt{t}\, a \Vdash_{\mathbf{2}} g(x)$. Thus **true** = **false** and $\mathcal{A}$ is trivial, which we assumed was not the case. Hence f and g have distinct trackers, and there can be at most $|\mathcal{A}|$-many.

(ii) Because otherwise, by the universal property of coproducts there would be exactly $2^{|\mathcal{A}|} > |\mathcal{A}|$ maps from $\coprod_{\mathtt{a} \in \mathcal{A}} \mathbf{1}$ to $\mathbf{2}$.

(iii) Left adjoints preserve colimits, in particular coproducts. However, the $\mathcal{A}$-indexed coproduct of terminal objects exists in Set but not in $\mathsf{Asm}_{\mathcal{A}}$.