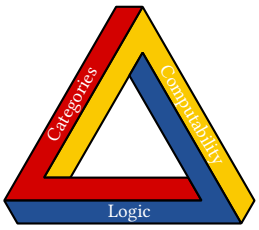
Exercise solutions for



Categorical Realizability

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Chapter 2

Exercise 2.11.

Each of the properties (i)–(iii) are proved by induction on the structure of the body term t.

(i) The claim holds when $t = x$, when $t = y$ is a variable distinct from x, and when $t = \mathtt{a} \in \mathcal{A}$.

Finally, by the induction hypothesis the set of variables in the term

$$\langle x \rangle.\, (t_1\ t_2) = \mathbf{S}\ (\langle x \rangle.\, t_1)\ (\langle x \rangle.\, t_2)$$

is exactly $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$, where $\mathcal{V}(t)$ denotes the set of variables in the term t. The result in this case now follows since

$$\mathcal{V}(t_1\ t_2) \setminus x = (\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x).$$

(ii) The term $\langle x \rangle.\, t$ is defined when t is a variable or an element of $\mathcal{A}$ since, by Definition 2.1, $\mathbf{K}\,\mathtt{a}$ and $\mathbf{S}\,\mathtt{a}\,\mathtt{b}$ are defined for any $\mathtt{a}, \mathtt{b} \in \mathcal{A}$.

Finally, if $\langle x \rangle.\, t_1$ and $\langle x \rangle.\, t_2$ are defined then any substitution into the variables of

$$\langle x \rangle.\, (t_1\ t_2) = \mathbf{S}\ (\langle x \rangle.\, t_1)\ (\langle x \rangle.\, t_2)$$

yields an element of $\mathcal{A}$ of the form $\mathbf{S}\ \mathtt{a}\ \mathtt{b}$ for some $\mathtt{a}, \mathtt{b} \in \mathcal{A}$.

(iii) By straightforward computation in the case where t is not an application. When $t = t_1\ t_2$,

$$\begin{aligned}
&(\langle x \rangle.\, (t_1\ t_2))\ \mathtt{a} \\
&= \mathbf{S}\ (\langle x \rangle.\, t_1)\ (\langle x \rangle.\, t_2)\ \mathtt{a} \\
&\simeq ((\langle x \rangle.\, t_1)\ \mathtt{a})\ ((\langle x \rangle.\, t_2)\ \mathtt{a}) \\
&\simeq (t_1[\mathtt{a}/x])\ (t_2[\mathtt{a}/x]) \qquad \text{(by the induction hypothesis)} \\
&\simeq (t_1\ t_2)[\mathtt{a}/x].
\end{aligned}$$

Exercise 2.14.

(i) $\mathbf{pair}\ \mathtt{a}\ \mathtt{b} = (\langle xyz \rangle.\, zxy)\ \mathtt{a}\ \mathtt{b} = \langle z \rangle.\, z\ \mathtt{a}\ \mathtt{b}$ is defined by Exercise 2.11.

(ii) By computation, taking care to note throughout that all applications are defined in $\mathcal{A}$.

Exercise 2.15.

A possible set of definitions is

$$\begin{aligned}
\mathbf{iszero} &:= \mathbf{fst} \\
\mathbf{succ} &:= \mathbf{pair}\ \mathbf{false} \\
\mathbf{pred} &:= \langle n \rangle.\, \mathbf{if}\ (\mathbf{iszero}\ n)\ \overline{0}\ (\mathbf{snd}\ n)
\end{aligned}$$

(check that these satisfy the required equations!)