Exercise solutions for

Categorical Realizability

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Chapter 2

Exercise 2.11.

Each of the properties (i)–(iii) are proved by induction on the structure of the body term t.

(i) The claim holds when $t = x$, when $t = y$ is a variable distinct from x, and when $t = \mathrm{a} \in \mathcal{A}$.

Finally, by the induction hypothesis the set of variables in the term

$$\langle x \rangle.\,(t_1\, t_2) = \mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)$$

is exactly $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$, where $\mathcal{V}(t)$ denotes the set of variables in the term t. The result in this case now follows since

$$\mathcal{V}(t_1\, t_2) \setminus x = (\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x).$$

(ii) The term $\langle x \rangle.\, t$ is defined when t is a variable or an element of $\mathcal{A}$ since, by Definition 2.1, $\mathbf{K}\,\mathrm{a}$ and $\mathbf{S}\,\mathrm{a}\,\mathrm{b}$ are defined for any $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

Finally, if $\langle x \rangle.\, t_1$ and $\langle x \rangle.\, t_2$ are defined then any substitution into the variables of

$$\langle x \rangle.\,(t_1\, t_2) = \mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)$$

yields an element of $\mathcal{A}$ of the form $\mathbf{S}\ \mathrm{a}\ \mathrm{b}$ for some $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

(iii) By straightforward computation in the case where t is not an application. When $t = t_1\, t_2$,

$$\begin{aligned}
&(\langle x \rangle.\,(t_1\, t_2))\ \mathrm{a} \\
=\ & \mathbf{S}\,(\langle x \rangle.\, t_1)\,(\langle x \rangle.\, t_2)\ \mathrm{a} \\
\simeq\ & ((\langle x \rangle.\, t_1)\ \mathrm{a})\,((\langle x \rangle.\, t_2)\ \mathrm{a}) \\
\simeq\ & (t_1[\mathrm{a}/x])\,(t_2[\mathrm{a}/x]) && \text{(by the induction hypothesis)} \\
\simeq\ & (t_1\, t_2)[\mathrm{a}/x].
\end{aligned}$$

Exercise 2.14.

(i) $\mathbf{pair}\,\mathrm{a}\,\mathrm{b} = (\langle xyz \rangle.\, zxy)\,\mathrm{a}\,\mathrm{b} = \langle z \rangle.\, z\,\mathrm{a}\,\mathrm{b}$ is defined by Exercise 2.11.

(ii) By computation, taking care to note throughout that all applications are defined in $\mathcal{A}$.

Exercise 2.15.

A possible set of definitions is

$$\begin{aligned}
\mathbf{iszero} &\coloneqq \mathbf{fst} \\
\mathbf{succ} &\coloneqq \mathbf{pair}\,\mathbf{false} \\
\mathbf{pred} &\coloneqq \langle n \rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\ \overline{0}\ (\mathbf{snd}\,n)
\end{aligned}$$

(check that these satisfy the required equations!)