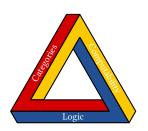
Exercise solutions for



CATEGORICAL REALIZABILITY

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Chapter 2

Exercise 2.11.

Each of the properties (i)–(iii) are proved by induction on the structure of the body term t.

(i) The claim holds when t = x, when t = y is a variable distinct from x, and when $t = a \in A$.

Finally, by the induction hypothesis the set of variables in the term

$$\langle x \rangle. (t_1 t_2) = S(\langle x \rangle. t_1)(\langle x \rangle. t_2)$$

is exactly $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$, where $\mathcal{V}(t)$ denotes the set of variables in the term t. The result in this case now follows since

$$V(t_1 t_2) \setminus x = (V(t_1) \setminus x) \cup (V(t_2) \setminus x).$$

(ii) The term $\langle x \rangle$. t is defined when t is a variable or an element of \mathcal{A} since, by Definition 2.1, K a and S a b are defined for any $a, b \in \mathcal{A}$.

Finally, if $\langle x \rangle$. t_1 and $\langle x \rangle$. t_2 are defined then any substitution into the variables of

$$\langle x \rangle$$
. $(t_1 t_2) = S(\langle x \rangle. t_1)(\langle x \rangle. t_2)$

yields an element of A of the form S a b for some $a, b \in A$.

(iii) By straightforward computation in the case where t is not an application. When $t = t_1 t_2$,

$$(\langle x \rangle, (t_1 t_2)) a$$

$$= S(\langle x \rangle, t_1) (\langle x \rangle, t_2) a$$

$$\simeq ((\langle x \rangle, t_1) a) ((\langle x \rangle, t_2) a)$$

$$\simeq (t_1[a/x]) (t_2[a/x])$$
 (by the induction hypothesis)
$$\simeq (t_1 t_2)[a/x].$$

Exercise 2.14.

- (i) **pair** a b = $(\langle xyz \rangle, zxy)$ a b = $\langle z \rangle, z$ a b is defined by Exercise 2.11.
- (ii) By computation, taking care to note throughout that all applications are defined in A.

Exercise 2.15.

A possible set of definitions is

iszero := fst
succ := pair false
pred :=
$$\langle n \rangle$$
. if (iszero n) $\overline{0}$ (snd n)

(check that these satisfy the required equations!)