Exercise solutions for

Categorical Realizability

Midlands Graduate School (MGS)

8–12 April 2024, Leicester, UK

Joshua Chen & Tom de Jong

School of Computer Science
University of Nottingham

April 2024

Solutions to Chapter 2

Exercise 2.11. Each of the properties (i)–(iii) are proved by induction on the structure of the body term t.

(i) The claim holds when $t = x$, when $t = y$ is a variable distinct from x, and when $t = \mathrm{a} \in \mathcal{A}$.

Finally, by the induction hypothesis the set of variables in the term

$$\langle x\rangle.\,(t_1\,t_2) = \mathbf{S}\,(\langle x\rangle.\,t_1)\,(\langle x\rangle.\,t_2)$$

is exactly $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$, where $\mathcal{V}(t)$ denotes the set of variables in the term t. The result in this case now follows since

$$\mathcal{V}(t_1\,t_2) \setminus x = (\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x).$$

(ii) The term $\langle x\rangle.\,t$ is defined when t is a variable or an element of $\mathcal{A}$ since, by Definition 2.1, $\mathbf{K}\,\mathrm{a}$ and $\mathbf{S}\,\mathrm{a}\,\mathrm{b}$ are defined for any $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

Finally, if $\langle x\rangle.\,t_1$ and $\langle x\rangle.\,t_2$ are defined then any substitution into the variables of

$$\langle x\rangle.\,(t_1\,t_2) = \mathbf{S}\,(\langle x\rangle.\,t_1)\,(\langle x\rangle.\,t_2)$$

yields an element of $\mathcal{A}$ of the form $\mathbf{S}\ \mathrm{a}\ \mathrm{b}$ for some $\mathrm{a}, \mathrm{b} \in \mathcal{A}$.

(iii) By straightforward computation in the case where t is not an application. When $t = t_1\,t_2$,

$$\begin{aligned}
&(\langle x\rangle.\,(t_1\,t_2))\,\mathrm{a} \\
=\ & \mathbf{S}\,(\langle x\rangle.\,t_1)\,(\langle x\rangle.\,t_2)\ \mathrm{a} \\
\simeq\ & ((\langle x\rangle.\,t_1)\,\mathrm{a})\,((\langle x\rangle.\,t_2)\,\mathrm{a}) \\
\simeq\ & (t_1[\mathrm{a}/x])\,(t_2[\mathrm{a}/x]) \qquad \text{(by the induction hypothesis)} \\
\simeq\ & (t_1\,t_2)[\mathrm{a}/x].
\end{aligned}$$

Exercise 2.14.

(i) $\mathbf{pair}\,\mathrm{a}\,\mathrm{b} = (\langle xyz\rangle.\,zxy)\,\mathrm{a}\,\mathrm{b} = \langle z\rangle.\,z\,\mathrm{a}\,\mathrm{b}$ is defined by Exercise **2.11**.

(ii) By computation, taking care to note throughout that all applications are defined in $\mathcal{A}$.

Exercise 2.15. A possible set of definitions is

$$\begin{aligned}
\mathbf{iszero} &:= \mathbf{fst} \\
\mathbf{succ} &:= \mathbf{pair}\,\mathbf{false} \\
\mathbf{pred} &:= \langle n\rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\ \overline{0}\,(\mathbf{snd}\,n)
\end{aligned}$$

(check that these satisfy the required equations).

Exercise 2.16. From the specification of **primrec**, we would like our definition to satisfy the equation

$$\mathbf{primrec}\,\mathtt{a}\,\mathtt{f} \simeq \langle n\rangle.\,\big(\mathbf{if}\,(\mathbf{iszero}\,n)\ \mathtt{a}\ \big(\mathtt{f}\,(\mathbf{pred}\,n)\,(\mathbf{primrec}\,\mathtt{a}\,\mathtt{f}\,(\mathbf{pred}\,n))\big)\big)$$

for any $\mathtt{a}, \mathtt{f} \in \mathcal{A}$.

This suggests that the term $\mathbf{primrec}\,\mathtt{a}\,\mathtt{f}$ should be constructed as a fixed point of the abstraction

$$\langle r\rangle.\,\big(\langle n\rangle.\,\big(\mathbf{if}\,(\mathbf{iszero}\,n)\ \mathtt{a}\ \big(\mathtt{f}\,(\mathbf{pred}\,n)\,(r\,(\mathbf{pred}\,n))\big)\big)\big), \tag{1}$$

and so we might try to define

$$\mathbf{spec}' := \langle af\rangle.\,\langle rn\rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\,a\,\big(f\,(\mathbf{pred}\,n)\,(r\,(\mathbf{pred}\,n))\big)$$

and

$$\mathbf{primrec}' := \langle af\rangle.\,\mathbf{Z}\,(\mathbf{spec}'\,af).$$

However, this definition does not satisfy the requirement that $\mathbf{primrec}'\,\mathtt{a}\,\mathtt{f}\,\overline{0}$ is always defined (expand the definition and check!).

Instead, we tweak the abstraction (1) whose fixed point we take, and define

$$\begin{aligned} \mathbf{spec} &:= \langle af\rangle.\,\langle rn\rangle.\,\mathbf{if}\,(\mathbf{iszero}\,n)\,(\mathbf{K}\,a)\,(\mathbf{S}\,f\,r)(\mathbf{pred}\,n), \\ \mathbf{primrec} &:= \langle af\rangle.\,\mathbf{Z}\,(\mathbf{spec}\,af). \end{aligned}$$

We can then check (do so!) that the required equations are satisfied.

Exercise 2.17.

(i) $\Longrightarrow$ (ii): Assuming $\mathbf{true} \neq \mathbf{false}$, we show that $\overline{\mathtt{m}} \neq \overline{\mathtt{n}}$ for all $n \in \mathbb{N}$ and $m < n$, by case distinction on n and then induction on $m < n$. This is trivial for $n = 0$.

Assume that $n = n' + 1$. For $m = 0$,

$$\mathbf{iszero}\,\overline{\mathtt{m}} = \mathbf{true} \neq \mathbf{false} = \mathbf{iszero}\,\overline{\mathtt{n}}$$

and so $\overline{\mathtt{m}} \neq \overline{\mathtt{n}}$. If $m + 1 < n$ then $m < n'$ and

$$\mathbf{pred}\,\overline{\mathtt{m}+1} = \overline{\mathtt{m}} \neq \overline{\mathtt{n}'} = \mathbf{pred}\,\overline{\mathtt{n}}$$

by induction, so $\overline{\mathtt{m}+1} \neq \overline{\mathtt{n}}$.

(ii) $\Longrightarrow$ (iii): Immediate.

(iii) $\Longrightarrow$ (i): Because if $\mathbf{true} = \mathbf{false}$ then

$$\mathtt{a} = \mathbf{if}\,\mathbf{true}\,\mathtt{a}\,\mathtt{b} = \mathbf{if}\,\mathbf{false}\,\mathtt{a}\,\mathtt{b} = \mathtt{b}$$

for all $\mathtt{a}, \mathtt{b} \in \mathcal{A}$.