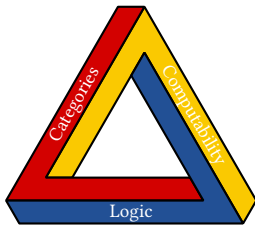


Exercise solutions for



# CATEGORICAL REALIZABILITY

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## Solutions to Chapter 2

**Exercise 2.11.** Each of the properties (i)–(iii) are proved by induction on the structure of the body term  $t$ .

- (i) The claim holds when  $t = x$ , when  $t = y$  is a variable distinct from  $x$ , and when  $t = a \in \mathcal{A}$ .

Finally, by the induction hypothesis the set of variables in the term

$$\langle x \rangle. (t_1 t_2) = S (\langle x \rangle. t_1) (\langle x \rangle. t_2)$$

is exactly  $(\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x)$ , where  $\mathcal{V}(t)$  denotes the set of variables in the term  $t$ . The result in this case now follows since

$$\mathcal{V}(t_1 t_2) \setminus x = (\mathcal{V}(t_1) \setminus x) \cup (\mathcal{V}(t_2) \setminus x).$$

- (ii) The term  $\langle x \rangle. t$  is defined when  $t$  is a variable or an element of  $\mathcal{A}$  since, by Definition 2.1,  $K a$  and  $S a b$  are defined for any  $a, b \in \mathcal{A}$ .

Finally, if  $\langle x \rangle. t_1$  and  $\langle x \rangle. t_2$  are defined then any substitution into the variables of

$$\langle x \rangle. (t_1 t_2) = S (\langle x \rangle. t_1) (\langle x \rangle. t_2)$$

yields an element of  $\mathcal{A}$  of the form  $S a b$  for some  $a, b \in \mathcal{A}$ .

- (iii) By straightforward computation in the case where  $t$  is not an application. When  $t = t_1 t_2$ ,

$$\begin{aligned} & (\langle x \rangle. (t_1 t_2)) a \\ &= S (\langle x \rangle. t_1) (\langle x \rangle. t_2) a \\ &\simeq ((\langle x \rangle. t_1) a) ((\langle x \rangle. t_2) a) \\ &\simeq (t_1[a/x]) (t_2[a/x]) \quad (\text{by the induction hypothesis}) \\ &\simeq (t_1 t_2)[a/x]. \end{aligned}$$

**Exercise 2.14.**

- (i)  $\text{pair } a b = (\langle xyz \rangle. zxy) a b = \langle z \rangle. z a b$  is defined by Exercise 2.11.  
(ii) By computation, taking care to note throughout that all applications are defined in  $\mathcal{A}$ .

**Exercise 2.15.** A possible set of definitions is

$$\begin{aligned} \text{iszero} &:= \text{fst} \\ \text{succ} &:= \text{pair false} \\ \text{pred} &:= \langle n \rangle. \text{if } (\text{iszero } n) \ \bar{0} \ (\text{snd } n) \end{aligned}$$

(check that these satisfy the required equations).

**Exercise 2.16.** From the specification of **primrec**, we would like our definition to satisfy the equation

$$\mathbf{primrec} \ a \ f \simeq \langle n \rangle. \left( \text{if } (\text{iszero } n) \ a \ (f \ (\text{pred } n) \ (\mathbf{primrec} \ a \ f \ (\text{pred } n))) \right)$$

for any  $a, f \in \mathcal{A}$ .

This suggests that the term  $\mathbf{primrec} \ a \ f$  should be constructed as a fixed point of the abstraction

$$\langle r \rangle. \left( \langle n \rangle. \left( \text{if } (\text{iszero } n) \ a \ (f \ (\text{pred } n) \ (r \ (\text{pred } n))) \right) \right), \quad (1)$$

and so we might try to define

$$\text{spec}' := \langle af \rangle. \langle rn \rangle. \text{if } (\text{iszero } n) \ a \ (f \ (\text{pred } n) \ (r \ (\text{pred } n)))$$

and

$$\mathbf{primrec}' := \langle af \rangle. Z \ (\text{spec}' \ af).$$

However, this definition does not satisfy the requirement that  $\mathbf{primrec}' \ a \ f \ \bar{0}$  is always defined (expand the definition and check!).

Instead, we tweak the abstraction (1) whose fixed point we take, and define

$$\begin{aligned} \text{spec} &:= \langle af \rangle. \langle rn \rangle. \text{if } (\text{iszero } n) \ (K \ a) \ (S \ f \ r) \ (\text{pred } n), \\ \mathbf{primrec} &:= \langle af \rangle. Z \ (\text{spec} \ af). \end{aligned}$$

We can then check (do so!) that the required equations are satisfied.

**Exercise 2.17.**

(i)  $\implies$  (ii): Assuming **true**  $\neq$  **false**, we show that  $\bar{m} \neq \bar{n}$  for all  $n \in \mathbb{N}$  and  $m < n$ , by case distinction on  $n$  and then induction on  $m < n$ . This is trivial for  $n = 0$ .

Assume that  $n = n' + 1$ . For  $m = 0$ ,

$$\text{iszero } \bar{m} = \text{true} \neq \text{false} = \text{iszero } \bar{n}$$

and so  $\bar{m} \neq \bar{n}$ . If  $m + 1 < n$  then  $m < n'$  and

$$\text{pred } \overline{m+1} = \bar{m} \neq \bar{n}' = \text{pred } \bar{n}$$

by induction, so  $\overline{m+1} \neq \bar{n}$ .

(ii)  $\implies$  (iii): Immediate.

(iii)  $\implies$  (i): Because if **true** = **false** then

$$a = \text{if } \text{true} \ a \ b = \text{if } \text{false} \ a \ b = b$$

for all  $a, b \in \mathcal{A}$ .