2-Coherent Internal Models of Homotopical Type Theory

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Abstract

The program of *internal type theory* seeks to develop the categorical model theory of dependent type theory using the language of dependent type theory itself. In the present work we study internal homotopical type theory by relaxing the notion of a category with families (cwf) to that of a *wild*, or *precoherent higher* cwf, and determine coherence conditions that suffice to recover properties expected of models of dependent type theory. The result is a definition of a *split 2-coherent wild cwf*, which admits as instances both the syntax and the "standard model" given by a universe type. This allows us to give a straightforward internalization of the notion of a 2-coherent reflection of homotopical type theory in itself—namely as a 2-coherent wild cwf morphism from the syntax to the standard model. Our theory also easily specializes to give definitions of "low-dimensional" higher cwfs, and conjecturally includes the container higher model as a further instance.

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1 Introduction

1.1 Internal type theory

Given a sufficiently expressive logical system $\mathfrak{L}$, it is interesting and productive to ask

To what extent does $\mathfrak{L}$ internalize itself?

In more detail, one seeks to develop a suitable notion of *interpreting structure*, aka *model*, of $\mathfrak{L}$, and to study the theory of such models, entirely within the language and logic of $\mathfrak{L}$ itself.[1] The techniques and perspectives granted by the study of these *inner models* have historically been used to surprising effect, e.g. to prove ZF-relative consistency of the axiom of choice and the continuum hypothesis [20], or to show independence of the Whitehead problem—a statement in homological algebra about short exact sequences of abelian groups—from the traditionally accepted set theoretic foundations of mathematics [36].

The program of *internal type theory*, first articulated by Dybjer [17], seeks to develop the same paradigm in the setting of dependent type theory[2] by studying the categorical model theory of intensional Martin-Löf type theory (MLTT) using MLTT. Among the type theory community, this is also known as *internal model theory* of type theory.

[1] In this paper, "model" always refers to this logical notion and not to homotopy theoretic *model structures*.
[2] For conciseness, henceforth simply "type theory".

The foundational task of internal type theory is then to give a type theoretic definition of the notion of a categorical "model" of type theory, ensuring that it captures all the examples we care about. In early work, Buisse and Dybjer [12] formalize the type of *1-categories with families* (*1-cwfs*) [17, 21] in MLTT, and discuss approaches to constructing term models and the initial cwf. Later, Ahrens, Lumsdaine and Voevodsky [4] consider a number of other kinds of model in the same setting (on occasion, assuming univalence), and show the equivalence of 1-cwfs with (split) type categories [39, Definition 2.2.1][30, 31] (aka categories with attributes [29, §6][21]) and representable natural transformations (aka natural models [10]), via relative universes.

Two structures that we would particularly like to exhibit as internal models of a type theory are its *syntax* and, if present, its *universe type*. Altenkirch and Kaposi [6] show that if MLTT with uniqueness of identity proofs (UIP) is extended with *quotient inductive-inductive types* (*QIITs*) [5, 24] then the strongly typed term-cwf sketched by Buisse and Dybjer [12, §6.1] can be internally constructed, thus yielding an internal syntactic model. Even better, from this inductive construction it immediately follows that internal 1-cwfs satisfy the *initiality principle*, which posits that syntax should give rise to an initial object in an appropriate (higher) category of models. Assuming UIP also allows any universe type in MLTT to be equipped with a canonical 1-cwf structure, giving rise to an internal model known in the literature as the *standard model* [6, §4][26, Example 3].

Now, for any internal definition of model of MLTT which includes as instances both the syntax $\mathcal{S}$ and a universe $\mathcal{U}$, we may ask if the type of model morphisms from $\mathcal{S}$ to $\mathcal{U}$ is definable and inhabited. Elements of such a type may be viewed as *self-interpretations*, or *reflections* of MLTT in itself.[3] By the preceding discussion, in MLTT+$\mathcal{U}$+UIP+QIITs the type of 1-cwf morphisms from the syntax to the standard universe model is definable, and moreover inhabited by eliminating the syntax into the universe. We may colloquially summarize all of this by saying that any model of MLTT+$\mathcal{U}$+UIP+QIITs has an inner syntactic model $\mathcal{S}$ of its MLTT fragment, as well as an internalized reflection function $\mathcal{S} \to \mathcal{U}$.

1.2 Internal homotopical type theory

Can we tell a similar story about inner models of *homotopical* MLTT, i.e. without assuming UIP? Unfortunately, the account given in the previous section depends heavily on the uniqueness of identity proofs—both for the semantics of QIITs, but also for the interpretation of the universe as an internal model. In particular, it is well known that types with their higher identities possess ∞-*groupoidal* structure in the absence of UIP [28, 38, 25], which means that the canonical standard model on the universe $\mathcal{U}$ is no longer a 1-cwf since its substitutions $\mathcal{U}(A, B) :\equiv A \to B$ are not generally sets.

We must therefore return to the foundational task of internal type theory, and ask what a good notion of internal model of homotopical type theory might look like. This question is considered by Kraus [26] in the setting of a two-level type theory (2LTT) [8] that extends homotopical MLTT with an additional layer of *strict*, aka *non-fibrant*, types, including an equality type former satisfying UIP. With this extra structure it is now possible to define ∞-categories as semi-Segal types [13] having idempotent equivalences [26, §III]; to answer the question of models by defining ∞-*categories with families*; and to show that the syntax QIIT, the universe standard model, and any slice of an ∞-cwf are all ∞-cwfs.

[3] The self-interpretation—as opposed to just the internal *representation*—of type theory is a fascinating topic of study; Rendel, Ostermann and Hofer [32, §2] give an excellent account of the subtle distinctions involved, albeit in the context of typed lambda calculi.

However, models of 2LTT are somewhat strong extensions of models of MLTT, and it might therefore be argued that ∞-cwfs should not be considered to be inner models of homotopical MLTT in itself.[4] Indeed, Kraus conjectures [26, §VI] that plain homotopy type theory (HoTT) [37] does not internalize itself, but instead considers the definition of ∞-cwfs to be a step towards showing that 2LTT internalizes HoTT and also itself.

Another potential approach is to work in the axiomatic variation of *simplicial homotopy type theory* [33, 34] developed by Gratzer, Weinberger and Buchholtz [18], in which it is possible to give a straightforward definition of $(\infty, 1)$-categories as Segal or Rezk types. However, in this theory representable presheaves are defined using additional modalities [19], again making it a rather strong extension of plain HoTT.

1.3 Contributions

Thus, in the present work we stick with the original question and fix homotopical MLTT to be the theory of our outer models (i.e. the theory in which we perform our constructions). Since defining a type of ∞-categories and, a fortiori, ∞-categorical models in this theory remains an open problem, we take the notion of a *wild*, or *precoherent higher* category with families as our starting point for "internal model of homotopical MLTT" (Section 4). We then determine coherence conditions such that sufficiently coherent wild cwfs satisfy many good properties expected of any model of type theory (Section 5). In particular, we show that requiring *2-coherence* suffices to equip any wild cwf $\mathscr{C}$ with the expected cloven fibrational structure, which furthermore satisfies a coherent "splitting" property when the category of contexts of $\mathscr{C}$ is either set-level or univalent (Section 6). This generalizes the well known result that 1-cwfs are also full split comprehension categories [11, 3]. In order to do this we develop a theory of *wild categories* and of pullbacks therein (Sections 2 and 3).

The benefit of our approach is that we are able to capture, in a single internally definable notion, both set-level models such as the syntax as well as untruncated higher models such as the universe. We are then able to prove results simultaneously for all such models by stating them as generally as possible in terms of coherence, without resorting to truncation assumptions or restricting to some homotopy n-level. Our theory is still easily applicable to models with "low-dimensional" higher homotopy: in particular, the simple generalization of the theory of 1-cwfs to allow the type presheaf to be valued in 1-types is (almost trivially) an instance of our theory.

1.4 Assumptions and conventions

Our definitions and constructions are in the setting of homotopical MLTT—specifically, MLTT with Π, Σ and intensional identity types, without assuming uniqueness of identity proofs. We assume function extensionality as well as η for Π-types throughout. We do not globally assume univalence [37, Axiom 2.10.3], but instead take it to be a property that may or may not hold for a given universe type. Of the reader, we assume familiarity with the basics of homotopy type theory as well as the theory of 1-categories in univalent foundations, standard references for which are [37] and [2].

[4] Of course, even the account we gave of inner models of MLTT+UIP in the previous section does not, strictly speaking, provide models of MLTT+UIP in itself. One source of tension in inner model theory is to make the gap between the theory of the outer and inner models (the "host" theory and the "object" theory) as small as possible. How large this gap is allowed to be is typically a matter of some subjective judgment.

We frequently work with explicit transports over complicated path concatenations, and use the notation $a\downarrow^P_e$ to denote the transport of an element $a : P(x)$ along an equality $e : x = y$. We will also use, without explicit comment, the equation $a\downarrow^P_e\downarrow^P_{e'} = a\downarrow^P_{e\cdot e'}$ which holds by [37, Lemma 2.3.9].

2 Wild Categories

To internalize categorical models of homotopical type theory we first need a theory of categorical structures that captures both the 1-categorical syntax as well as the ∞-categorical universe. This is the theory of wild categories, whose relevant points we recall briefly here.

▶ **Definition 2.1** (Wild categories). *A* wild *or* precoherent higher category $\mathscr{C}$ *is a model of the generalized algebraic theory of precategories [2, Definition 3.1], dropping the requirement that morphisms form sets.*

As usual, we denote the objects of a wild category $\mathscr{C}$ by $\mathscr{C}_0$, and for every $x, y : \mathscr{C}_0$, the *type* of morphisms from x to y by $\mathscr{C}(x, y)$. We write $\diamond$ for the composition operation on compatible morphisms, and

$$(h \diamond g) \diamond f \overset{\alpha_{f,g,h}}{=\!=\!=} h \diamond g \diamond f, \qquad \mathrm{id}_y \diamond f \overset{\lambda_f}{=\!=} f \qquad \text{and} \qquad f \diamond \mathrm{id}_x \overset{\rho_f}{=\!=} f$$

for the associator, left unitor, and right unitor respectively. Crucially, no further coherence laws on hom-types are required in the definition of a wild category. We frequently leave the indices on the associator, unitors and identity morphisms implicit.

We call equalities between morphisms in a wild category *2-cells* (and higher equalities, *higher cells*). We stress that these cells are not explicitly axiomatized as part of the definition of a wild category, but instead arise out the ambient homotopical type theory.

▶ **Examples 2.2** (Wild categories). We are primarily interested in two particular kinds of wild category, which are in fact completely coherent. The first kind consists of the wild categories whose morphisms form sets: any precategory—and hence, set-level or univalent 1-category—is immediately a wild category. The second kind consists of the type theoretic (sub-) universes. Any universe type $\mathscr{U}$ gives rise to a wild category, also denoted $\mathscr{U}$, with objects $\mathscr{U}_0 :\equiv \mathscr{U}$, and whose hom-types $\mathscr{U}(A, B)$ are the function types $A \to B$. Composition is given by function composition and identity morphisms by identity functions. The associativity and unit laws hold definitionally, i.e. α, λ and ρ are families of trivial equations. This definition applies equally well to any reflective subuniverse [37, Definition 7.7.1].

A large number of elementary concepts from univalent 1-category theory [2] and bicategory theory [1] straightforwardly transfer into, and are subsumed by, the wild categorical setting. In particular, we take as given the definitions of terminal objects, as well as the type of sections $\mathrm{Sect}(f)$ and retractions $\mathrm{Retr}(f)$ of morphisms f in wild categories.

▶ **Definition 2.3** (Wild equivalences). *A morphism $f : \mathscr{C}(x, y)$ in a wild category $\mathscr{C}$ is a* wild equivalence *if it has both a section and a retraction (i.e. is biinvertible). The type of wild equivalences from x to y in $\mathscr{C}$ is denoted $x \simeq_{\mathscr{C}} y$. Its elements are also called* $\mathscr{C}$-equivalences *to avoid confusion with type theoretic equivalences.*

▶ **Definition 2.4** (Whiskering). *Given an equality $\gamma : g = g'$ of morphisms $g, g' : \mathscr{C}(x, y)$, for any morphism $f : \mathscr{C}(w, x)$ the* right whiskering $(\gamma * f)$ *of γ with f is the canonical equality*

$$\mathrm{ap}\,(\_ \diamond f)\,\gamma : g \diamond f = g' \diamond f,$$

and for any $h : \mathscr{C}(y, z)$ *the* left whiskering $(h * \gamma)$ *of* γ *with* h *is the equality*

$$\mathrm{ap}\,(h \diamond \_)\,\gamma : h \diamond g = h \diamond g'.$$

▶ **Definition 2.5** (Dependent identity morphisms). *If* $x, y : \mathscr{C}_0$ *are objects of a wild category such that* $e : x = y$*, there is a morphism*

$$\mathrm{idd}(e) :\equiv \mathrm{id}_x \downarrow_e^{\mathscr{C}(x, \_)} : \mathscr{C}(x, y).$$

In precategories, idd is essentially idtoiso [2, Lemma 3.4]. From it, we get a family of maps $\mathrm{idtoeqv}_{\mathscr{C}} : x = y \to x \simeq_{\mathscr{C}} y$ indexed over $x, y : \mathscr{C}_0$.

▶ **Definition 2.6** (Wild univalence). *A wild category* $\mathscr{C}$ *is* univalent *if* $\mathrm{idtoeqv}_{\mathscr{C}}$ *is a family of equivalences.*

▶ **Examples 2.7** (Univalent wild categories). Univalent 1-categories are univalent wild categories, and any univalent universe $\mathscr{U}$ is univalent as a wild category.

The following two coherence conditions are familiar from bicategory theory.

▶ **Definition 2.8** (Triangle coherators). *A wild category* $\mathscr{C}$ *has* triangle coherators *if for all* $\mathscr{C}$*-morphisms* $x \xrightarrow{f} y \xrightarrow{g} z$ *there is an equality* $\triangle_{f,g} : \alpha \cdot (g * \lambda) = \rho * f$ *filling the triangle*

$$\begin{tikzcd}
(g \diamond \mathrm{id}) \diamond f \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\rho * f"'] & & g \diamond \mathrm{id} \diamond f \arrow[dl, Rightarrow, "g * \lambda"] \\
& g \diamond f &
\end{tikzcd}$$

▶ **Definition 2.9** (Pentagonators). *A wild category* $\mathscr{C}$ *has* pentagon coherators for associators*, or* (α-) pentagonators*, if for all composable chains* $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$ *of* $\mathscr{C}$*-morphisms there is an equality* $\pentagon_{f,g,h,k} : (\alpha * f) \cdot \alpha \cdot (k * \alpha) = \alpha \cdot \alpha$ *filling the pentagon*

$$\begin{tikzcd}
& ((k \diamond h) \diamond g) \diamond f \arrow[dl, Rightarrow, "\alpha * f"'] \arrow[dr, Rightarrow, "\alpha"] & \\
(k \diamond h \diamond g) \diamond f \arrow[d, Rightarrow, "\alpha"'] & & (k \diamond h) \diamond g \diamond f \arrow[d, Rightarrow, "\alpha"] \\
k \diamond (h \diamond g) \diamond f \arrow[rr, Rightarrow, "k * \alpha"'] & & k \diamond h \diamond g \diamond f
\end{tikzcd}$$

▶ **Definition 2.10** (2-coherent wild categories). *A wild category is* 2-coherent *if it has triangle and pentagon coherators.*

▶ **Proposition 2.11.** *Any 2-coherent wild category is also a* prebicategory *in the sense of Ahrens et al. [1, Definition 2.1], by taking the type of 2-cells from* f *to* g *to be the equality type* $f = g$*.*

▶ **Examples 2.12** (2-coherent higher categories). Any precategory trivially satisfies all higher equalities between equalities of morphisms. Universe wild categories (Examples 2.2) have definitionally unital and associative composition of morphisms, and thus trivial triangle and pentagon coherators.

3 Wild Pullbacks

The theory of pullbacks in wild categories jointly generalizes comma objects in $(2,1)$-categories (including 1-categorical pullbacks) as well as type theoretic homotopy pullbacks [9]. We will use it later to study context comprehension in wild cwfs (Section 6).

▶ **Definition 3.1** (Commuting squares). *In a wild category* $\mathscr{C}$*, a* commuting square *on a* cospan $\mathfrak{c} :\equiv A \xrightarrow{f} C \xleftarrow{g} B$ with source $X : \mathscr{C}_0$ *is an element of*

$$\mathrm{CommSq}_{\mathfrak{c}}(X) :\equiv \textstyle\sum (m_A : \mathscr{C}(X,\,A))\,(m_B : \mathscr{C}(X,\,B))\,(\gamma : f \diamond m_A = g \diamond m_B).$$

We call $\mathrm{CommSq}(\mathfrak{c}) :\equiv \sum (X : \mathscr{C}_0)\,\mathrm{CommSq}_{\mathfrak{c}}(X)$ *the type of* commuting squares on $\mathfrak{c}$*.*

▶ **Definition 3.2** (Precomposing squares with morphisms). *For any cospan* $\mathfrak{c} \equiv A \xrightarrow{f} C \xleftarrow{g} B$ *and* $X, Y : \mathscr{C}_0$ *in a wild category* $\mathscr{C}$*, there is a* precomposition *map*

$$\_ \mathbin{\square} \_ : \mathrm{CommSq}_{\mathfrak{c}}(Y) \to \mathscr{C}(X,\,Y) \to \mathrm{CommSq}_{\mathfrak{c}}(X)$$

defined by $(m_A, m_B, \gamma) \mathbin{\square} m \;:\equiv\; (m_A \diamond m,\; m_B \diamond m,\; \alpha^{-1} \cdot (\gamma * m) \cdot \alpha)$*.*

▶ **Definition 3.3** (Pullbacks and weak pullbacks). *Let*

$$\mathfrak{P} \;:\equiv\; \begin{tikzcd} P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\mathfrak{p}"] & C \end{tikzcd}$$

be a commuting square on $\mathfrak{c} :\equiv A \xrightarrow{f} C \xleftarrow{g} B$ *with source* P*, in a wild category* $\mathscr{C}$*. By specializing the precomposition map (Definition 3.2) at* $\mathfrak{P}$*, we obtain the family of maps*

$$\mathfrak{P} \mathbin{\square}\_\,\_ : \textstyle\prod (X : \mathscr{C}_0)\,\mathscr{C}(X,\,P) \to \mathrm{CommSq}_{\mathfrak{c}}(X).$$

We say that $\mathfrak{P}$ *is a* pullback of $\mathfrak{c}$ *if* $(\mathfrak{P} \mathbin{\square}\_\,\_)$ *is a family of equivalences, and a* weak pullback of $\mathfrak{c}$ *if* $(\mathfrak{P} \mathbin{\square}\_\,\_)$ *is a family of retractions, or split surjections.*

"Being a pullback" is a propositional predicate on $\mathrm{CommSq}_{\mathfrak{c}}(P)$. We thus have the subtypes $\mathrm{Pullback}_{\mathfrak{c}}(P)$ and $\mathrm{Pullback}(\mathfrak{c})$ of $\mathrm{CommSq}_{\mathfrak{c}}(P)$ and $\mathrm{CommSq}(\mathfrak{c})$, respectively.

In 2-coherent wild categories, we can prove the expected pullback pasting lemma.

▶ **Definition 3.4** (Horizontal pasting). *From a diagram of commuting squares* $\mathfrak{Q} :\equiv (i, f', \mathfrak{q})$ *and* $\mathfrak{P} :\equiv (j, g', \mathfrak{p})$ *on their respective cospans, we get the* horizontal pasting $\mathfrak{Q} \mid \mathfrak{P}$ *as depicted,*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "i"'] & B' \arrow[r, "g'"] \arrow[d, "j"] & C' \arrow[d, "k"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\mathfrak{q}"] & B \arrow[r, "g"'] \arrow[ur, phantom, "\mathfrak{p}"] & C \end{tikzcd} \quad\mapsto\quad \mathfrak{Q} \mid \mathfrak{P} \;:\equiv\; \begin{tikzcd} A' \arrow[r, "g' \diamond f'"] \arrow[d, "i"'] & C' \arrow[d, "k"] \\ A \arrow[r, "g \diamond f"'] \arrow[ur, phantom, "\mathfrak{q} \mid \mathfrak{p}"] & C \end{tikzcd},$$

where $\mathfrak{q} \mid \mathfrak{p} \;:\equiv\; \alpha \cdot (g * \mathfrak{q}) \cdot \alpha^{-1} \cdot (\mathfrak{p} * f') \cdot \alpha$*.*

▶ **Lemma 3.5** (Horizontal pullback pasting). *Suppose we have a diagram of commuting squares*

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "i"'] & B' \arrow[r, "g'"] \arrow[d, "j"] & C' \arrow[d, "k"] \\ A \arrow[r, "f"'] \arrow[ur, phantom, "\mathfrak{q}"] & B \arrow[r, "g"'] \arrow[ur, phantom, "\mathfrak{p}"] & C \end{tikzcd}$$

in a 2-coherent wild category $\mathscr{C}$*. Then if* $\mathfrak{P} :\equiv (j, g', \mathfrak{p})$ *is a pullback of* (g, k)*, the commuting square* $\mathfrak{Q} :\equiv (i, f', \mathfrak{q})$ *is a pullback of* (f, j) *if and only if* $\mathfrak{Q} \mid \mathfrak{P}$ *is a pullback of* $(g \diamond f, k)$*.*

Finally, we observe the following about the truncation level of the type of pullbacks.

▶ **Lemma 3.6.** *Suppose* $\mathfrak{c}$ *is a cospan in a wild category* $\mathscr{C}$*. If* $\mathscr{C}$ *is set-level, then* $\mathrm{Pullback}(\mathfrak{c})$ *is a set. If* $\mathscr{C}$ *is univalent and 2-coherent, then* $\mathrm{Pullback}(\mathfrak{c})$ *is a proposition.*

4 Wild Categories with Families

Now we develop precoherent higher internal models of homotopical MLTT, by taking Dybjer's generalized algebraic definition of a category with families [17] and allowing contexts to form wild categories. This notion has previously been briefly considered by Kraus [26, Definition 5]; we recall its definition here in order to explicitly record the necessary transports.

4.1 Typed term structures

▶ **Definition 4.1** (Typed term structures on wild categories). *Let* $\mathscr{U}$ *be a universe and* $\mathscr{C}$ *a wild category. A* typed term structure *on* $\mathscr{C}$ *(valued in* $\mathscr{U}$*) consists of the following data:*

- *A wild* $\mathscr{U}$*-valued presheaf of* $\mathscr{C}$-types *over* $\mathscr{C}$*, presented as a generalized algebraic theory by the components*[5]

$$\begin{aligned} \mathsf{Ty} &: \mathscr{C}_0 \to \mathscr{U} \\ \_[\_]_{\mathsf{T}} &: \mathsf{Ty}\,\Delta \to \mathscr{C}(\Gamma, \Delta) \to \mathsf{Ty}\,\Gamma \end{aligned}$$

  and equations[5] *expressing functoriality*

$$\begin{aligned} [\mathsf{id}]_{\mathsf{T}} &: A[\mathrm{id}_\Gamma]_{\mathsf{T}} = A && \textit{for all } A : \mathsf{Ty}\,\Gamma \\ [\diamond]_{\mathsf{T}} &: A[\tau \diamond \sigma]_{\mathsf{T}} = A[\tau]_{\mathsf{T}}[\sigma]_{\mathsf{T}} && \textit{for all } A : \mathsf{Ty}\,\mathrm{E},\ \sigma : \mathscr{C}(\Gamma, \Delta),\ \tau : \mathscr{C}(\Delta, \mathrm{E}). \end{aligned}$$

- *A wild* $\mathscr{U}$*-valued presheaf of* $\mathscr{C}$-terms *over the (wild) category of elements of the* $\mathscr{C}$*-type presheaf, presented*[5] *by*

$$\begin{aligned} \mathsf{Tm} &: (\Gamma : \mathscr{C}_0) \to \mathsf{Ty}\,\Gamma \to \mathscr{U} \\ \_[\_]_{\mathsf{t}} &: \mathsf{Tm}_\Delta\, A \to (\sigma : \mathscr{C}(\Gamma, \Delta)) \to \mathsf{Tm}_\Gamma\,(A[\sigma]_{\mathsf{T}}) \quad \textit{for all } A : \mathsf{Ty}\,\Delta \end{aligned}$$

  and

$$\begin{aligned} [\mathsf{id}]_{\mathsf{t}} &: a[\mathrm{id}_\Gamma]_{\mathsf{t}} = a{\downarrow}^{\mathsf{Tm}\,\Gamma}_{[\mathsf{id}]_{\mathsf{T}}^{-1}} && \textit{for all } A : \mathsf{Ty}\,\Gamma,\ a : \mathsf{Tm}_\Gamma\, A \\ [\diamond]_{\mathsf{t}} &: a[\tau \diamond \sigma]_{\mathsf{t}} = a[\tau]_{\mathsf{t}}[\sigma]_{\mathsf{t}}{\downarrow}^{\mathsf{Tm}\,\Gamma}_{[\diamond]_{\mathsf{T}}^{-1}} && \textit{for all } A : \mathsf{Ty}\,\mathrm{E},\ a : \mathsf{Tm}_{\mathrm{E}}\, A \\ & && \sigma : \mathscr{C}(\Gamma, \Delta),\ \tau : \mathscr{C}(\Delta, \mathrm{E}). \end{aligned}$$

The actions $\_[\_]_{\mathsf{T}}$ *and* $\_[\_]_{\mathsf{t}}$ *of the type and term presheaves on morphisms are called* substitution in types *and* substitution in terms*, respectively.*

We often denote a typed term structure on a wild category simply by the object parts of its component presheaves $(\mathsf{Ty}, \mathsf{Tm})$. We also frequently elide the first argument of Tm and write, for example, $\mathsf{Tm}\,A$ instead of $\mathsf{Tm}_\Gamma\,A$. Note also the following equivalences and equalities in typed term structures $(\mathsf{Ty}, \mathsf{Tm})$ on wild categories $\mathscr{C}$.

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[5] Implicitly quantifying over objects $\Gamma, \Delta, \mathrm{E} : \mathscr{C}_0$ as needed.

▶ **Proposition 4.2.** *For every $\Gamma : \mathscr{C}_0$ and $A : \mathsf{Ty}\,\Gamma$, the equation $[\mathsf{id}]_{\mathsf{t}}$ (Definition 4.1) implies that the function $\_[\mathrm{id}]_{\mathsf{t}} : \mathsf{Tm}\,A \to \mathsf{Tm}\,(A[\mathrm{id}]_{\mathsf{T}})$ sending $a \mapsto a[\mathrm{id}]_{\mathsf{t}}$ is equal to transport in Tm_{Γ} along ${[\mathsf{id}]_{\mathsf{T}}}^{-1}$, and is hence an equivalence.*

▶ **Definition 4.3.** *Assume $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Delta$, and an equality $e : \sigma = \tau$ of morphisms $\sigma, \tau : \mathscr{C}(\Gamma,\, \Delta)$. We write $[^{=}e]_{\mathsf{T}} :\equiv \mathrm{ap}\,(A[\_]_{\mathsf{T}})\,e$ for the induced equality $A[\sigma]_{\mathsf{T}} = A[\tau]_{\mathsf{T}}$.*

By [37, Lemma 2.2.2], $[^{=}\_]_{\mathsf{T}}$ respects trivial, composite and inverse equalities.

▶ **Proposition 4.4.** *For any $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Delta$, $a : \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}})$ and morphisms $\sigma, \tau : \mathscr{C}(\Gamma,\, \Delta)$ such that $e : \sigma = \tau$, the equality $a \downarrow_{e}^{\mathsf{Tm}\,(A[\_]_{\mathsf{T}})} = a \downarrow_{[^{=}e]_{\mathsf{T}}}^{\mathsf{Tm}}$ holds by [37, Lemma 2.3.10].*

▶ **Definition 4.5.** *Suppose that $A, A' : \mathsf{Ty}\,\Delta$ are $\mathscr{C}$-types such that $e : A = A'$. For any $\sigma : \mathscr{C}(\Gamma,\, \Delta)$, we write $e[\sigma]_{\mathsf{T}} :\equiv \mathrm{ap}\,(\_[\sigma]_{\mathsf{T}})\,e$ for the induced equality $A[\sigma]_{\mathsf{T}} = A'[\sigma]_{\mathsf{T}}$.*

▶ **Proposition 4.6** (Substitution in transported terms)**.** *If $e : A =_{\mathsf{Ty}\,\Delta} A'$, then for any $a : \mathsf{Tm}\,A$ and morphism $\sigma : \mathscr{C}(\Gamma,\, \Delta)$, the equality $(a \downarrow_{e}^{\mathsf{Tm}})[\sigma]_{\mathsf{t}} = a[\sigma]_{\mathsf{t}} \downarrow_{e[\sigma]_{\mathsf{T}}}^{\mathsf{Tm}}$ holds by induction on e.*

▶ **Proposition 4.7** ($[\diamond]_{\mathsf{T}}$ is a natural isomorphism)**.** *Suppose that $\sigma, \sigma' : \mathscr{C}(\Gamma,\, \Delta)$ are morphisms such that $e : \sigma = \sigma'$. By induction on e, the squares*

$$\begin{tikzcd} A[\tau \diamond \sigma]_{\mathsf{T}} \arrow[r, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] \arrow[d, Rightarrow, "{[^{=}\tau * e]_{\mathsf{T}}}"'] & A[\tau]_{\mathsf{T}}[\sigma]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}e]_{\mathsf{T}}}"] \\ A[\tau \diamond \sigma']_{\mathsf{T}} \arrow[r, Rightarrow, "{[\diamond]_{\mathsf{T}}}"'] & A[\tau]_{\mathsf{T}}[\sigma']_{\mathsf{T}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B[\sigma \diamond \varrho]_{\mathsf{T}} \arrow[r, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] \arrow[d, Rightarrow, "{[^{=}e * \varrho]_{\mathsf{T}}}"'] & B[\sigma]_{\mathsf{T}}[\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}e]_{\mathsf{T}}[\varrho]_{\mathsf{T}}}"] \\ B[\sigma' \diamond \varrho]_{\mathsf{T}} \arrow[r, Rightarrow, "{[\diamond]_{\mathsf{T}}}"'] & B[\sigma']_{\mathsf{T}}[\varrho]_{\mathsf{T}} \end{tikzcd}$$

canonically commute. Here, $A : \mathsf{Ty}\,\mathrm{E}$, $\tau : \mathscr{C}(\Delta,\, \mathrm{E})$, $B : \mathsf{Ty}\,\Delta$ and $\varrho : \mathscr{C}(\mathrm{B},\, \Gamma)$.

The following two definitions are analogous to the conditions for a pseudofunctor between weak $(2,1)$-categories.[6] In the case that $\mathscr{C}$ is a 2-coherent wild category, they improve the wild presheaf Ty of a typed term structure on $\mathscr{C}$ to what might be called a *wild weak (2, 1)-presheaf.*

▶ **Definition 4.8** (Type triangulators)**.** *A typed term structure on $\mathscr{C}$ is said to have* type triangulators *if for all morphisms $\sigma : \mathscr{C}(\Gamma,\, \Delta)$ and $\mathscr{C}$-types $A : \mathsf{Ty}\,\Delta$ the following triangles commute:*

$$\begin{tikzcd} A[\mathrm{id} \diamond \sigma]_{\mathsf{T}} \arrow[rr, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] \arrow[dr, Rightarrow, "{[^{=}\lambda]_{\mathsf{T}}}"'] & & A[\mathrm{id}]_{\mathsf{T}}[\sigma]_{\mathsf{T}} \arrow[dl, Rightarrow, "{[\mathsf{id}]_{\mathsf{T}}[\sigma]_{\mathsf{T}}}"] \\ & A[\sigma]_{\mathsf{T}} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A[\sigma \diamond \mathrm{id}]_{\mathsf{T}} \arrow[rr, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] \arrow[dr, Rightarrow, "{[^{=}\rho]_{\mathsf{T}}}"'] & & A[\sigma]_{\mathsf{T}}[\mathrm{id}]_{\mathsf{T}} \arrow[dl, Rightarrow, "{[\mathsf{id}]_{\mathsf{T}}}"] \\ & A[\sigma]_{\mathsf{T}} & \end{tikzcd}$$

▶ **Definition 4.9** (Type pentagonators)**.** *A typed term structure on $\mathscr{C}$ has* type pentagonators *if for all sequences of morphisms*

$$\Gamma \xrightarrow{\varrho} \Delta \xrightarrow{\sigma} \mathrm{E} \xrightarrow{\tau} \mathrm{Z}$$

---

[6] See e.g. [23, §4.1].

and $\mathscr{C}$*-types* $A : \mathsf{Ty}\,\mathrm{Z}$*, the following pentagon commutes:*

$$\begin{tikzcd}
 & & A[\tau \diamond \sigma \diamond \varrho]_{\mathsf{T}} \arrow[dll, Rightarrow, "{[^{=}\alpha^{-1}]_{\mathsf{T}}}"'] \arrow[drr, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] & & \\
A[(\tau \diamond \sigma) \diamond \varrho]_{\mathsf{T}} \arrow[dr, Rightarrow, "{[\diamond]_{\mathsf{T}}}"'] & & & & A[\tau]_{\mathsf{T}}[\sigma \diamond \varrho]_{\mathsf{T}} \arrow[dl, Rightarrow, "{[\diamond]_{\mathsf{T}}}"] \\
 & A[\tau \diamond \sigma]_{\mathsf{T}}[\varrho]_{\mathsf{T}} \arrow[rr, Rightarrow, "{[\diamond]_{\mathsf{T}}[\varrho]_{\mathsf{T}}}"'] & & A[\tau]_{\mathsf{T}}[\sigma]_{\mathsf{T}}[\varrho]_{\mathsf{T}} &
\end{tikzcd}$$

4.2 Context extension structures and wild cwfs

▶ **Definition 4.10** (Context extension structures)**.** *Assume a typed term structure* $(\mathsf{Ty}, \mathsf{Tm})$ *on a wild category* $\mathscr{C}$*. A* context extension structure *on* $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$ *is given*[7] *by the following components*

$$\begin{aligned}
_ . _ \;&:\; (\Gamma : \mathscr{C}_0) \to \mathsf{Ty}\,\Gamma \to \mathscr{C}_0 \\
\mathsf{p} \;&:\; (A : \mathsf{Ty}\,\Gamma) \to \mathscr{C}(\Gamma.A,\, \Gamma) \\
\mathsf{q} \;&:\; (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Tm}_{\,\Gamma.A}\,(A[\mathsf{p}_A]_{\mathsf{T}}) \\
_ \,, _ \;&:\; (\sigma : \mathscr{C}(\Gamma,\, \Delta)) \to \mathsf{Tm}_{\,\Gamma}\,(A[\sigma]_{\mathsf{T}}) \to \mathscr{C}(\Gamma,\, \Delta.A) \quad \textit{for all} \;\; A : \mathsf{Ty}\,\Delta
\end{aligned}$$

and equations (note Definition 4.3)

$$\begin{array}{lcll}
\mathsf{p}\beta & : & \mathsf{p}_A \diamond (\sigma\,, a) = \sigma & \textit{and} \\
\mathsf{q}\beta & : & \mathsf{q}_A[\sigma\,, a]_{\mathsf{t}} = a\downarrow^{\mathsf{Tm}}_{[^{=}\mathsf{p}\beta]_{\mathsf{T}}{}^{-1}\cdot[\diamond]_{\mathsf{T}}} & \textit{for all} \;\; \sigma : \mathscr{C}(\Gamma,\, \Delta), \\
 & & & \qquad\;\; A : \mathsf{Ty}\,\Delta,\;\; a : \mathsf{Tm}_{\,\Gamma}\,(A[\sigma]_{\mathsf{T}}) \\
,\eta & : & (\mathsf{p}_A\,, \mathsf{q}_A) = \mathrm{id}_{\Gamma.A} & \textit{for all} \;\; A : \mathsf{Ty}\,\Gamma \\
,\diamond & : & (\tau\,, a) \diamond \sigma = (\tau \diamond \sigma\,, a[\sigma]_{\mathsf{t}}\downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}}) & \textit{for all} \;\; \sigma : \mathscr{C}(\Gamma,\, \Delta),\;\; \tau : \mathscr{C}(\Delta,\, \mathrm{E}), \\
 & & & \qquad\;\; A : \mathsf{Ty}\,\mathrm{E},\;\; a : \mathsf{Tm}_{\,\Delta}\,(A[\tau]_{\mathsf{T}}).
\end{array}$$

We call p *the* display map*, and* q *the* generic term *of the context extension structure.*

We sometimes elide the argument $A : \mathsf{Ty}\,\Gamma$ to the display map p and the generic term q of a context extension structure. When we need to be concise, we denote the display map $\Gamma.A \xrightarrow{\mathsf{p}_A} \Gamma$ by $\Gamma.A \twoheadrightarrow \Gamma$.

▶ **Definition 4.11** (Cwf structures on wild categories)**.** *If* $\mathscr{C}$ *is a wild category, a* cwf structure *on* $\mathscr{C}$ *consists of:*

- *a terminal object* $\blacklozenge : \mathscr{C}_0$*,*
- *a typed term structure* $(\mathsf{Ty}, \mathsf{Tm})$ *on* $\mathscr{C}$*, and*
- *a context extension structure on* $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$*,*

which model the structural rules of a Martin-Löf type theory over $\mathscr{C}$*.*

▶ **Definition 4.12** (Wild categories with families)**.** *A* wild category with families (wild cwf) *is a wild category* $\mathscr{C}$ *together with a cwf structure on* $\mathscr{C}$*. In this case, we call* $\mathscr{C}$ *the* category of contexts *of the wild cwf, its objects* contexts*, and its morphisms* substitutions*.*

---

[7] Again, implicitly generalizing over $\Gamma, \Delta, \mathrm{E} : \mathscr{C}_0$ as needed.

We usually denote a wild cwf by its category of contexts. Every 1-cwf is immediately a wild cwf. Any universe type is also a wild cwf:

▶ **Example 4.13** (Universe cwfs). If a universe wild category $\mathcal{U}$ (Examples 2.2) has Σ-types that satisfy the η-rule, then it supports a canonical wild cwf structure given as follows.

- The terminal context $\blacklozenge$ is the unit type $\mathbb{1} : \mathcal{U}$.
- The typed term structure is as follows. $\mathcal{U}$-types in context Γ are Γ-indexed type families

  $$\mathsf{Ty} : \mathcal{U} \to \mathcal{U}^+, \;\; \mathsf{Ty}\,\Delta :\equiv \Delta \to \mathcal{U},$$

  while $\mathcal{U}$-terms $a : \mathsf{Tm}_\Delta\, A$ are sections of $A : \mathsf{Ty}\,\Delta$, i.e. $\mathsf{Tm}_\Delta\, A :\equiv \Pi\, \Delta\, A$.
  Substitution of $\sigma : \mathcal{U}(\Gamma,\, \Delta)$ in $\mathcal{U}$-types $A : \mathsf{Ty}\,\Delta$ and $\mathcal{U}$-terms a is given by precomposition

  $$A[\sigma]_{\mathsf{T}} :\equiv A \circ \sigma \quad \text{and} \quad a[\sigma]_{\mathsf{t}} :\equiv a \circ \sigma.$$

  This action is definitionally functorial—that is, $[\mathsf{id}]_{\mathsf{T}}$, $[\diamond]_{\mathsf{T}}$, $[\mathsf{id}]_{\mathsf{t}}$ and $[\diamond]_{\mathsf{t}}$ are all families of trivial identities.
- The context extension structure is given by dependent pairing. The extended context $\Delta.\,A$ is $\Sigma\,\Delta\,A$, and p and q are the functions fst and snd respectively. For $\sigma : \mathcal{U}(\Gamma,\, \Delta)$ and $t : \mathsf{Tm}_\Gamma\, (A \circ \sigma)$, the extended substitution $(\sigma\,,t) : \mathcal{U}(\Gamma,\, \Sigma\,\Delta\,A)$ is given by

  $$(\sigma\,, t)(\gamma) :\equiv (\sigma(\gamma), t(\gamma)).$$

  Again, the equations for context extension structures hold definitionally. In particular, the η-rule for Σ-types is used for $\,,\eta$.

Variations of this canonical universe cwf structure appear throughout the literature as the "standard model".

▶ **Examples 4.14** (Subuniverse cwfs and the 1-cwf of sets). The construction of the typed term and context extension structures of Example 4.13 works equally well for any *sub*universe wild category (Examples 2.2) that has a terminal object and is closed under Σ-types with η. In particular, the 1-cwf of sets $\mathrm{Set}_{\mathcal{U}}$ is a subuniverse cwf of $\mathcal{U}$.

▶ **Definition 4.15** (Univalent wild cwfs). *A wild cwf $\mathscr{C}$ is called* univalent *if its category of contexts is univalent.*

▶ **Example 4.16.** Any subuniverse of a univalent universe $\mathcal{U}$ yields a univalent cwf. In particular, $\mathrm{Set}_{\mathcal{U}}$ is a univalent cwf if $\mathcal{U}$ is univalent.

4.3 Structural properties of wild cwfs

The following structural properties, familiar for 1-cwfs, also hold for arbitrary wild cwfs $\mathscr{C}$.

▶ **Lemma 4.17** (Substitutions into extended contexts are pairs). *Let $\Gamma, \Delta : \mathscr{C}_0$ be contexts, and $A : \mathsf{Ty}\,\Delta$ a $\mathscr{C}$-type. There is an equivalence*

$$\begin{tikzcd} \mathscr{C}(\Gamma,\, \Delta.\,A) \arrow[r, bend left, "\sigma \,\mapsto\, (\mathsf{p}_A \diamond \sigma\,,\; \mathsf{q}_A[\sigma]_{\mathsf{t}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}-1})"] \arrow[r, phantom, "\simeq"] & \sum\,(\sigma : \mathscr{C}(\Gamma,\, \Delta))\,,\; \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}}), \arrow[l, bend left, "(\sigma\,,a) \,\mapsfrom\, (\sigma, a)"] \end{tikzcd}$$

where the reverse function sends a pair (σ, a) to the extended substitution $(\sigma\,, a)$ given by the context extension structure (Definition 4.10).

▶ **Corollary 4.18** (Equality of substitutions into extended contexts). *If σ and τ are substitutions from Γ to $\Delta.A$, then by Lemma 4.17, the fact that equivalences induce equivalent identity types [37, Theorem 2.11.1], and Proposition 4.4, an equality $\sigma = \tau$ is equivalent to a pair of equalities $e : \mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau$ and $\mathsf{q}[\sigma]_{\mathsf{t}} \downarrow_{[\diamond]_{\mathsf{T}}{}^{-1} \cdot [=e]_{\mathsf{T}} \cdot [\diamond]_{\mathsf{T}}} = \mathsf{q}[\tau]_{\mathsf{t}}$.*

An alternative but equivalent formulation is the following—for substitutions of the form $(\sigma, a), (\tau, b) : \mathscr{C}(\Gamma,\, \Delta.A)$,

$$\big((\sigma, a) =_{\mathscr{C}(\Gamma,\, \Delta.\, A)} (\tau, b)\big) \;\simeq\; \big(\textstyle\sum (e : \sigma = \tau)\,,\; a \downarrow^{\mathsf{Tm}}_{[=e]_{\mathsf{T}}} = b\big).$$

▶ **Lemma 4.19** (Terms are sections of display maps). *For all contexts $\Gamma : \mathscr{C}_0$ and $\mathscr{C}$-types $A : \mathsf{Ty}\,\Gamma$, there is an equivalence $\mathsf{Tm}\,A \simeq \mathrm{Sect}(\mathsf{p}_A)$ whose forward map sends the $\mathscr{C}$-term a to the section $(\mathrm{id}\,, a[\mathrm{id}]_{\mathsf{t}})$ of p_A, witnessed by $\mathsf{p}\beta$.*

Analogues of Lemmas 4.17 and 4.19 were already observed for set-level 1-cwfs by Dybjer [17] and by Hofmann [21] in a traditional 1-categorical setting. In that setting, these properties essentially follow from the fact that context extension structures on $(\mathscr{C}, \mathsf{Ty}, \mathsf{Tm})$ are choices of representing objects for particular presheaves on slices of $\mathscr{C}$.

It may seem slightly surprising that the fully coherent homotopical versions of the same properties hold for arbitrary, even noncoherent, wild cwfs. On the other hand, given that a context extension structure for 1-cwfs essentially spells out the universal property of representability of a certain presheaf (and, relatedly, that the equivalent natural models [10] have a relatively simple axiomatization in terms of representability of pullbacks of presheaves), it is perhaps to be expected that certain consequences would carry over immediately to higher generalizations even without the explicit imposition of additional coherence conditions.

5 2-Coherence for Context Extension

We now construct and analyze in more detail a characterization of the equality of substitutions into extended contexts.

First, consider the case where σ and τ are substitutions from Γ to *arbitrary* contexts Δ, with $A : \mathsf{Ty}\,\Delta$, $a : \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}})$ and $b : \mathsf{Tm}\,(A[\tau]_{\mathsf{T}})$. From the backward equivalence of Lemma 4.17, we obtain

$$\big((\sigma, a) =_{\Sigma\,(\mathscr{C}(\Gamma,\, \Delta.\, A))\,(\mathsf{Tm}\,(A[\_]_{\mathsf{T}}))} (\tau, b)\big) \xrightarrow[\mathrm{ap}\,(\_\,,\,\_)]{\sim} \big((\sigma\,, a) =_{\mathscr{C}(\Gamma,\, \Delta.\, A)} (\tau\,, b)\big).$$

Precomposing this with

$$\begin{aligned}\big(\textstyle\sum (e : \sigma = \tau)\,,\; a \downarrow^{\mathsf{Tm}}_{[=e]_{\mathsf{T}}} = b\big) &\xrightarrow[\mathrm{id} \times \varphi]{\sim} \textstyle\sum (e : \sigma = \tau)\,,\; a \downarrow^{\mathsf{Tm}\,(A[\_]_{\mathsf{T}})}_{e} = b \\ &\xrightarrow[\mathrm{pair}^{=}]{\sim} \big((\sigma, a) =_{\Sigma\,(\mathscr{C}(\Gamma,\, \Delta.\, A))\,(\mathsf{Tm}\,(A[\_]_{\mathsf{T}}))} (\tau, b)\big),\end{aligned}$$

where

$$\varphi : \textstyle\prod (e : \sigma = \tau)\; \big(a \downarrow^{\mathsf{Tm}}_{[=e]_{\mathsf{T}}} = b \xrightarrow{\sim} a \downarrow^{\mathsf{Tm}\,(A[\_]_{\mathsf{T}})}_{e} = b\big)$$

is the family of equivalences induced by Proposition 4.4 and $\mathrm{pair}^{=}$ is the standard characterization of the equality of Σ-types, we get an equivalence

$$\begin{aligned}&\mathrm{sub}_0^{=} : \big(\textstyle\sum (e : \sigma = \tau)\,,\; a \downarrow^{\mathsf{Tm}}_{[=e]_{\mathsf{T}}} = b\big) \xrightarrow{\sim} (\sigma, a) =_{\mathscr{C}(\Gamma,\, \Delta.\, A)} (\tau, b)\\ &\mathrm{sub}_0^{=}\; (e, e') :\equiv \mathrm{ap}\,(\_\,,\_)\,(\mathrm{pair}^{=}\,(e, \varphi_e e')).\end{aligned}$$

▶ **Lemma 5.1** (pβ is a natural isomorphism)**.** *By definition,* $\mathrm{sub}_0^{=}\,(\mathrm{refl}, \mathrm{refl}) \equiv \mathrm{refl}$. *Hence for all substitutions* $\sigma, \tau : \mathscr{C}(\Gamma,\, \Delta.\, A)$*,* $\mathscr{C}$*-terms* $a : \mathsf{Tm}\,(A[\sigma]_\mathsf{T})$ *and* $b : \mathsf{Tm}\,(A[\tau]_\mathsf{T})$*, and equalities* $e : \sigma = \tau$ *and* $e' : a_{\downarrow\,[=e]_\mathsf{T}} = b$*, we have that the square*

$$\begin{tikzcd}
\mathsf{p}_A \diamond (\sigma\,, a) \arrow[r, Rightarrow, "\mathsf{p}\beta"] \arrow[d, Rightarrow, "\mathsf{p}_A * \mathrm{sub}_0^{=}\,(e, e')"'] & \sigma \arrow[d, Rightarrow, "e"] \\
\mathsf{p}_A \diamond (\tau\,, b) \arrow[r, Rightarrow, "\mathsf{p}\beta"'] & \tau
\end{tikzcd}$$

canonically commutes by induction on e *and* e'*. Equivalently,* $\mathsf{p}_A * \mathrm{sub}_0^{=}\,(e, e') = \mathsf{p}\beta \cdot e \cdot \mathsf{p}\beta^{-1}$.

▶ **Definition 5.2** (η-equality of substitutions)**.** *For all* $A : \mathsf{Ty}\,\Delta$ *and* $\sigma : \mathscr{C}(\Gamma,\, \Delta.\, A)$*, denote by*

$$\eta^{\mathrm{sub}}_\sigma : (\mathsf{p}_A \diamond \sigma\,, \mathsf{q}_A[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}}) = \sigma$$
$$\eta^{\mathrm{sub}}_\sigma :\equiv\ ,\diamond^{-1} \cdot (\,, \eta * \sigma) \cdot \lambda$$

the equality in the first part of the proof of Lemma 4.17. This is an η*-rule for substitutions into extended contexts.*

▶ **Definition 5.3** (Equality of substitutions into extended contexts, revisited)**.** *Suppose that* $\sigma, \tau : \mathscr{C}(\Gamma,\, \Delta.\, A)$ *are substitutions into an extended context. We define an equivalence*

$$\mathrm{sub}^{=}_{\sigma,\tau} : \left(\textstyle\sum\,(e : \mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau)\,,\ \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}\cdot[=e]_\mathsf{T}\cdot[\diamond]_\mathsf{T}} = \mathsf{q}[\tau]_\mathsf{t}\right) \xrightarrow{\sim} \sigma = \tau$$

as follows: if $e : \mathsf{p}_A \diamond \sigma = \mathsf{p}_A \diamond \tau$ *and* $e' : \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}\cdot[=e]_\mathsf{T}\cdot[\diamond]_\mathsf{T}} = \mathsf{q}[\tau]_\mathsf{t}$*, then take* $\mathrm{sub}^{=}_{\sigma,\tau}\,(e, e')$ *to be the composite*

$$\sigma \ \overset{\eta^{\mathrm{sub}\,-1}_\sigma}{=\!=\!=\!=} \ (\mathsf{p} \diamond \sigma\,, \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}}) \ \overset{\mathrm{sub}_0^{=}\,(e, e'')}{=\!=\!=\!=\!=\!=} \ (\mathsf{p} \diamond \tau\,, \mathsf{q}[\tau]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}}) \ \overset{\eta^{\mathrm{sub}}_\tau}{=\!=\!=} \ \tau,$$

where $e'' : \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}\,\downarrow\,[=e]_\mathsf{T}} = \mathsf{q}[\tau]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}}$ *is canonically constructed from* e'*. This definition yields an equivalence, being essentially the composition of* $\mathrm{sub}_0^{=}$ *with the equivalence given by path-composing with* $\eta^{\mathrm{sub}\,-1}_\sigma$ *and* η^{sub}_τ*.*

Now, it is natural to ask if a β-rule holds for the first argument of $\mathrm{sub}^{=}_{\sigma,\tau}$, i.e. if, for all σ and τ, the composition

$$\left(\textstyle\sum\,(\mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau)\,,\ \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}\cdot[=\_]_\mathsf{T}\cdot[\diamond]_\mathsf{T}} = \mathsf{q}[\tau]_\mathsf{t}\right) \xrightarrow{\mathrm{sub}^{=}_{\sigma,\tau}} \sigma = \tau \xrightarrow{\mathsf{p}\,*\,\_} \mathsf{p} \diamond \sigma = \mathsf{p} \diamond \tau$$

is equal to the first projection. By Lemma 5.1, we calculate that

$$\mathsf{p} * \mathrm{sub}^{=}_{\sigma,\tau}\,(e, e') = (\mathsf{p} * \eta^{\mathrm{sub}\,-1}_\sigma) \cdot \mathsf{p}\beta \cdot e \cdot \mathsf{p}\beta^{-1} \cdot (\mathsf{p} * \eta^{\mathrm{sub}}_\tau)$$

for all e and e'. The desire to have this expression be equal to e motivates the next definition, whence Proposition 5.5 immediately follows.

▶ **Definition 5.4** (Coherators for η^{sub})**.** *We say that a wild cwf* $\mathscr{C}$ has coherators for η^{sub} *if for all* $\Gamma, \Delta : \mathscr{C}_0$*,* $A : \mathsf{Ty}\,\Delta$ *and* $\sigma : \mathscr{C}(\Gamma,\, \Delta.\, A)$*, we have that* $\mathsf{p}_A * \eta^{\mathrm{sub}}_\sigma = \mathsf{p}\beta$ *as 2-cells of type* $\mathsf{p}_A \diamond (\mathsf{p}_A \diamond \sigma\,, \mathsf{q}_A[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}}) = \mathsf{p}_A \diamond \sigma$*.*

▶ **Proposition 5.5** (β-reduction for $\mathrm{sub}^{=}_{\sigma,\tau}$)**.** *Suppose* $\sigma, \tau : \mathscr{C}(\Gamma,\, \Delta.\, A)$ *are equal substitutions, witnessed by* $e : \mathsf{p}_A \diamond \sigma = \mathsf{p}_A \diamond \tau$ *and* $e' : \mathsf{q}[\sigma]_{\mathsf{t}\,\downarrow\,[\diamond]_\mathsf{T}^{-1}\cdot[=e]_\mathsf{T}\cdot[\diamond]_\mathsf{T}} = \mathsf{q}[\tau]_\mathsf{t}$*. If* $\mathscr{C}$ *has coherators for* η^{sub}*, then* $\mathsf{p}_A * \mathrm{sub}^{=}_{\sigma,\tau}\,(e, e') = e$*.*

In fact, a wild cwf $\mathscr{C}$ has coherators for $\mathsf{\eta}^{\mathrm{sub}}$ if its category of contexts has triangle coherators (Definition 2.8), and it further satisfies the following coherence condition:

▶ **Definition 5.6** (Coherators for context extension)**.** *A wild cwf* $\mathscr{C}$ has coherators for context extension *if, for all* $\Gamma, \Delta : \mathscr{C}_0$*,* $A : \mathsf{Ty}\,\Delta$ *and* $\sigma : \mathscr{C}(\Gamma, \Delta.A)$*, the following diagrams of equalities commute:*

$$\begin{tikzcd}[column sep=small] & \mathsf{p}_A \diamond (\mathsf{p}_A\,,\mathsf{q}_A) \arrow[dl, Rightarrow, "\mathsf{p}_A{*}\,,\,\mathsf{\eta}"'] \arrow[dr, Rightarrow, "\mathsf{p\beta}"] & \\ \mathsf{p}_A \diamond \mathrm{id} \arrow[rr, Rightarrow, "\rho"'] & & \mathsf{p}_A \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathsf{p}_A \diamond (\mathsf{p}_A\,,\mathsf{q}_A) \diamond \sigma \arrow[r, Rightarrow, "\alpha^{-1}"] \arrow[d, Rightarrow, "\mathsf{p}_A{*}\,,\,\diamond"'] & (\mathsf{p}_A \diamond (\mathsf{p}_A\,,\mathsf{q}_A)) \diamond \sigma \arrow[d, Rightarrow, "\mathsf{p\beta}{*}\sigma"] \\ \mathsf{p}_A \diamond (\mathsf{p}_A \diamond \sigma\,,\mathsf{q}_A[\sigma]_{\mathsf{t}} \downarrow [\diamond]_{\mathsf{T}}{}^{-1}) \arrow[r, Rightarrow, "\mathsf{p\beta}"'] & \mathsf{p}_A \diamond \sigma \end{tikzcd}$$

▶ **Lemma 5.7.** *If a wild cwf $\mathscr{C}$ has triangle coherators as well as coherators for context extension, then it has coherators for $\mathsf{\eta}^{\mathrm{sub}}$. Thus, the conclusion of Proposition 5.5 also holds if $\mathscr{C}$ has coherators for context extension.*

Presumably, these coherence conditions for context extension structures would also arise out of the universal properties of sufficiently coherent representable presheaves à la a formulation via wild natural models. In any case, we can now define:

▶ **Definition 5.8** (Structurally 2-coherent wild cwfs)**.** *We say that a wild cwf $\mathscr{C}$ is* (structurally) 2-coherent *if $\mathscr{C}$ has*

- *a 2-coherent wild category of contexts (Definition 2.10),*
- *type triangulators (Definition 4.8) and type pentagonators (Definition 4.9), and*
- *coherators for context extension (Definition 5.6).*

▶ **Examples 5.9** (2-coherent internal cwfs)**.** Any set-level 1-cwf is immediately 2-coherent, and it is straightforward to check that the universe cwfs have type triangulators, type pentagonators and coherators for context extension.

▶ **Conjecture 5.10** (The container model)**.** *We also expect that the higher* container model *of Altenkirch and Kaposi [7] is 2-coherent, to be shown by forthcoming work of Damato and Altenkirch [16].*

6 Context Comprehension in 2-Coherent Wild Cwfs

A central feature of the categorical semantics of dependent type theory is that types form a cartesian fibration over contexts that is preserved by context extension. This is summed up in the notion of a *comprehension category* [22], and it is widely known that 1-cwfs are equivalent to full split comprehension categories [11, 3]. In this section we show that analogous statements hold for 2-coherent wild cwfs.

6.1 The universal property of context extension

▶ **Lemma 6.1** (2-coherent substitution in types is weak pullback)**.** *Suppose that $\mathscr{C}$ is a 2-coherent wild cwf, $\sigma : \mathscr{C}(\Gamma, \Delta)$ is a substitution, and $A : \mathsf{Ty}\,\Delta$ is a $\mathscr{C}$-type. Then there is a substitution*

$$\sigma \cdot A :\equiv (\sigma \diamond \mathsf{p}_{A[\sigma]_{\mathsf{T}}}\,, \mathsf{q}_{A[\sigma]_{\mathsf{T}}} \downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}})$$

from $\Gamma.\, A[\sigma]_\mathsf{T}$ *to* $\Delta.\, A$*, such that the square*

$$\mathfrak{P}_{\sigma,\, A} \;:\equiv\; \begin{tikzcd} \Gamma.\, A[\sigma]_\mathsf{T} \arrow[r, "\sigma\,\cdot\,A"] \arrow[d, "\mathsf{p}"'] & \Delta.\, A \arrow[d, "\mathsf{p}"] \\ \Gamma \arrow[r, "\sigma"'] \arrow[ur, phantom, "\mathsf{p}\beta^{-1}"] & \Delta \end{tikzcd}$$

is a weak pullback in $\mathscr{C}$*. That is, for any* $\mathrm{B} : \mathscr{C}_0$ *and commuting square* $\mathfrak{S} :\equiv (\tau, \varrho, \gamma)$ *with source* B*, the fiber* $(\mathfrak{P}_{\sigma,\, A} \,\square\, \_\_)^{-1}(\mathfrak{S})$ *is pointed: there is a mediating substitution* $\mu_{\sigma,A,\mathfrak{S}} : \mathscr{C}(\mathrm{B},\, \Gamma.\, A[\sigma]_\mathsf{T})$ *such that* $\mathfrak{P}_{\sigma,\, A} \,\square\, \mu_{\sigma,A,\mathfrak{S}} = \mathfrak{S}$*.*

The proof of Lemma 6.1 gives a family $\mu_{\sigma,A}$ of sections to the precomposition map $(\mathfrak{P}_{\sigma,\, A} \,\square\_\_\_)$. We can show that $\mu_{\sigma,A}$ is also a family of retractions to $(\mathfrak{P}_{\sigma,\, A} \,\square\_\_\_)$, and thus:

▶ **Theorem 6.2** (2-coherent substitution in types is pullback)**.** *The weak pullbacks* $\mathfrak{P}_{\sigma,\, A}$ *of Lemma 6.1 are pullbacks.*

6.2 Cleavings of 2-coherent wild cwfs

▶ **Definition 6.3** (Cleavings)**.** *Let* $\mathscr{C}$ *be a 2-coherent wild cwf. A* cleaving *of* $\mathscr{C}$ *is an assignment*

$$\mathfrak{cl} : \textstyle\prod\, (\Gamma, \Delta : \mathscr{C}_0)\, (\sigma : \mathscr{C}(\Gamma,\, \Delta))\, (A : \mathsf{Ty}\, \Delta)\ \mathrm{Pullback}_{(\sigma, \mathsf{p}_A)}(\Gamma.\, A[\sigma]_\mathsf{T})$$

of pullbacks

$$\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A) \;:\equiv\; \begin{tikzcd} \Gamma.\, A[\sigma]_\mathsf{T} \arrow[r, "\ell_{\sigma,A}"] \arrow[d, two heads] & \Delta.\, A \arrow[d, two heads] \\ \Gamma \arrow[r, "\sigma"'] \arrow[ur, phantom, "\mathfrak{p}_{\sigma,A}"] & \Delta \end{tikzcd}$$

to each cospan in $\mathscr{C}$ *of the form* $\Gamma \xrightarrow{\sigma} \Delta \xleftarrow{\mathsf{p}_A} \Delta.\, A$*. We call the component* $\ell_{\sigma,A}$ *of the pullback* $\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A)$ *the* chosen lift of σ at A*.*

The upshot of Theorem 6.2 is then that every 2-coherent wild cwf has a cleaving $\mathfrak{cl}_{\Gamma,\Delta}(\sigma, A) :\equiv \mathfrak{P}_{\sigma,\, A}$ which we call the *type substitution cleaving*. In particular, $\sigma\,\cdot\,^A$ is the chosen lift of a substitution $\sigma : \mathscr{C}(\Gamma,\, \Delta)$ at $A : \mathsf{Ty}\, \Delta$. With the type substitution cleaving, the chosen lift of the identity is the canonical dependent identity.

▶ **Lemma 6.4.** *The type substitution cleaving of any 2-coherent wild cwf satisfies*

$$\mathrm{id}\,\cdot\,^A = \mathrm{idd}(\mathrm{ap}\,(\Gamma.\_\_)\,[\mathsf{id}]_\mathsf{T})$$

for any $\Gamma : \mathscr{C}_0$ *and* $A : \mathsf{Ty}\, \Gamma$*.*

▶ **Definition 6.5** (Split cleavings of 2-coherent wild cwfs)**.** *A cleaving* $\mathfrak{cl}$ *of a 2-coherent wild cwf* $\mathscr{C}$ *is called* split *if for all substitutions*

$$\mathrm{B} \xrightarrow{\tau} \Gamma \xrightarrow{\sigma} \Delta$$

and $\mathscr{C}$*-types* $A : \mathsf{Ty}\, \Delta$*, the equality type*

$$\big(\mathrm{B}.\, A[\sigma \diamond \tau]_\mathsf{T},\ \mathfrak{cl}_{\mathrm{B},\Delta}(\sigma \diamond \tau, A)\big) = \big(\mathrm{B}.\, A[\sigma]_\mathsf{T}[\tau]_\mathsf{T},\ \mathfrak{cl}_{\mathrm{B},\Gamma}(\tau, A[\sigma]_\mathsf{T}) \mid \mathfrak{cl}_{\Gamma,\Delta}(\sigma, A)\big)$$

of pullbacks on $\mathrm{B} \xrightarrow{\sigma\diamond\tau} \Delta \xleftarrow{\mathsf{p}_A} \Delta.\, A$ *is contractible.*

A split cleaving of a 2-coherent wild cwf, in our sense, is a coherent higher version of a splitting of a full comprehension category. As is to be expected, set-level internal cwfs have split type substitution cleavings. We can show that the type substitution cleavings of *univalent* 2-coherent wild cwfs are also always split.

▶ **Lemma 6.6.** *Suppose $\mathscr{C}$ is a 2-coherent wild cwf. For all substitutions* $\mathrm{B} \xrightarrow{\tau} \Gamma \xrightarrow{\sigma} \Delta$ *and $\mathscr{C}$-types* $A : \mathsf{Ty}\,\Delta$,

$$(\mathrm{B}.\, A[\sigma \diamond \tau]_{\mathsf{T}},\ \mathfrak{P}_{\sigma\diamond\tau,\, A}) = (\mathrm{B}.\, A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}},\ \mathfrak{P}_{\tau,\, A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,\, A}).$$

Since the type of pullbacks on a given cospan is a set in set-level categories and a proposition in univalent 2-coherent wild categories (Lemma 3.6), we get as consequences:

▶ **Theorem 6.7** (Split comprehension for set-level and univalent cwfs)**.** *The type substitution cleaving $\mathfrak{P}_{\sigma,\, A}$ of any set-level or univalent 2-coherent wild cwf $\mathscr{C}$ is split.*

▶ **Definition 6.8.** *We call a 2-coherent wild cwf* split *if its type substitution cleaving is split.*

▶ **Corollary 6.9.** *The syntax cwf QIIT of Altenkirch and Kaposi [6] as well as any univalent universe cwf $\mathscr{U}$ is split 2-coherent.*

7 Discussion

Working internally to homotopical MLTT we have given a unified account, via 2-coherent wild cwfs, of the cloven fibrational structure of internal models of homotopical dependent type theory, in such a way so as to include both set-level models (such as the syntax) and canonical higher models (given by universe types).

While the "splitness" of a cleaving of a 2-coherent wild cwf (Definition 6.5) is propositional and ensures that composition of lifts is coherently unique, the same cannot currently be said for lifts of identity substitutions. Even so, the type of morphisms of split 2-coherent wild cwfs is internally definable, which opens the possibility of internally speaking of "transfers" of constructions that respect cloven fibrations of internal models. Indeed, this is a large part of the motivation of the present work: the theory developed here is intended to provide the correct formal setting in which to investigate constructions of semisimplicial and other Reedy fibrant inverse diagrams [27] in internal models of homotopical type theory [15, 14].

Separately from questions of infinite higher coherent constructions, we hope that our theory can still be useful by immediately specializing to yield notions of 1-truncated "2-cwfs". For instance, by modifying the definition of a 2-coherent wild cwf to additionally require that (1) the category of contexts $\mathscr{C}$ is a precategory, (2) the presheaf of $\mathscr{C}$-types is valued in 1-types, and (3) the presheaf of $\mathscr{C}$-terms is set-valued, we obtain a simple higher generalization of the notion of a 1-cwf, which conjecturally includes the container higher model of type theory [7] as an instance. Then via Rezk completion [37, §9.9] and our results, any instance of such a higher cwf should be equivalent to a split one. A less naïve approach would be to use univalent bicategory theory [1], noting Proposition 2.11, to develop a full theory of 2-cwfs.

As a final remark, we have developed wild categories with families for their anticipated applicability to specific further internal constructions, but we also expect the study of wild *natural models* [10] to prove complementarily fruitful.

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A Wild Categories

▶ **Proposition A.1** (Properties of whiskering)**.** *By induction, the following equations hold for right whiskering:*

$$\begin{aligned} &\mathrm{refl} * f \equiv \mathrm{refl}, && \gamma * \mathrm{id} = \rho \cdot \gamma \cdot \rho^{-1}, \\ &(\gamma * f)^{-1} = \gamma^{-1} * f, && (\gamma \cdot \delta) * f = (\gamma * f) \cdot (\delta * f), \end{aligned}$$

as well as the analogous equations for left whiskering. We also have the following associativity laws expressing "naturality" of α*,*

$$\begin{aligned} &g * (f * \gamma) = \alpha^{-1} \cdot ((g \diamond f) * \gamma) \cdot \alpha, \\ &(\gamma * g) * f = \alpha \cdot (\gamma * (g \diamond f)) \cdot \alpha^{-1}, \\ &(g * \gamma) * f = \alpha \cdot (g * (\gamma * f)) \cdot \alpha^{-1}. \end{aligned}$$

Many coherences involving λ, ρ and α that hold in all bicategories also hold in 2-coherent wild categories—namely, those that do not rely on uniqueness of equality of 2-cells. In particular,

▶ **Proposition A.2.** *In a 2-coherent wild category* $\mathscr{C}$*, there are witnesses (not necessarily unique) that*

- $\lambda_{\mathrm{id}_x} = \rho_{\mathrm{id}_x}$ *for all* $x : \mathscr{C}_0$*, and*
- *the diagrams of equalities*

$$\begin{tikzcd} (\mathrm{id} \diamond g) \diamond f \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\lambda * f"'] & & \mathrm{id} \diamond g \diamond f \arrow[dl, Rightarrow, "\lambda"] \\ & g \diamond f & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} (g \diamond f) \diamond \mathrm{id} \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\rho"'] & & g \diamond f \diamond \mathrm{id} \arrow[dl, Rightarrow, "g * \rho"] \\ & g \diamond f & \end{tikzcd}$$

commute for all $f : \mathscr{C}(x, y)$ *and* $g : \mathscr{C}(y, z)$*.*

We refer to [23, Propositions 2.2.4 and 2.2.6] for proofs of these facts.

We will also use the following coherence.

▶ **Proposition A.3.** *Suppose that* $\gamma : g = \mathrm{id} \diamond f$ *in a 2-coherent wild category. Then*

$$\begin{tikzcd} \mathrm{id} \diamond g \arrow[r, Rightarrow, "\mathrm{id} * \gamma"] \arrow[d, Rightarrow, "\lambda"'] & \mathrm{id} \diamond \mathrm{id} \diamond f \arrow[d, Rightarrow, "\mathrm{id} * \lambda"] \\ g \arrow[r, Rightarrow, "\gamma"'] & \mathrm{id} \diamond f \end{tikzcd}$$

commutes, i.e. $\lambda_g \cdot \gamma =_{(\mathrm{id} \diamond g = \mathrm{id} \diamond f)} (\mathrm{id} * \gamma) \cdot (\mathrm{id} * \lambda_f)$*.*

Proof. We equivalently prove that $\mathrm{id} * \gamma = \lambda_g \cdot \gamma \cdot (\mathrm{id} * \lambda_f)^{-1}$. Since $\mathrm{id} * \gamma = \lambda_g \cdot \gamma \cdot \lambda^{-1}_{\mathrm{id} \diamond f}$ by properties of whiskering (Proposition A.1), the result follows if $\mathrm{id} * \lambda_f = \lambda_{\mathrm{id} \diamond f}$. Figure 1 shows how to construct such an equality: its interior is divided into two triangles and a bigon, which commute by the triangle coherator $\triangle_{f,\,\mathrm{id}}$ and the equalities in Proposition A.2. ◀

B Pullbacks in 2-Coherent Wild Categories

In this section we assume a wild category $\mathscr{C}$.

Figure 1

$$\begin{tikzcd}[column sep=large]
\mathrm{id} \diamond \mathrm{id} \diamond f \arrow[rr, bend left=40, Rightarrow, "\mathrm{id} * \lambda_f"] \arrow[rr, bend right=40, Rightarrow, "\lambda_{\mathrm{id} \diamond f}"'] & (\mathrm{id} \diamond \mathrm{id}) \diamond f \arrow[l, Rightarrow, "\alpha"'] \arrow[r, bend left, Rightarrow, "\rho_{\mathrm{id}} * f"] \arrow[r, bend right, Rightarrow, "\lambda_{\mathrm{id}} * f"'] & \mathrm{id} \diamond f
\end{tikzcd}$$

B.1 Commuting squares

▶ **Proposition B.1** (Equality of $\mathrm{CommSq}_{\mathfrak{c}}(X)$). *Let* $\mathfrak{c} :\equiv A \xrightarrow{f} C \xleftarrow{g} B$ *be a cospan,* $X : \mathscr{C}_0$ *an object, and* $\mathfrak{S} :\equiv (m_A, m_B, \gamma)$, $\mathfrak{S}' :\equiv ({m_A}', {m_B}', \gamma')$ *be elements of* $\mathrm{CommSq}_{\mathfrak{c}}(X)$ *in a wild category* $\mathscr{C}$*. Then the equality type* $\mathfrak{S} = \mathfrak{S}'$ *is equivalent to*

$$\sum (e_A : m_A = {m_A}')\,(e_B : m_B = {m_B}'),\ \gamma = (f * e_A) \cdot \gamma' \cdot (g * e_B)^{-1}.$$

Proof. By a routine application of the fundamental theorem of identity types [35, Theorem 11.2.2], the algebra of Σ-types, and Proposition A.1. ◀

▶ **Definition B.2** (Transpose). *For any cospan* $A \xrightarrow{f} C \xleftarrow{g} B$ *and* $X : \mathscr{C}_0$*, the* transpose *map*

$$\_^{\mathrm{T}} : \mathrm{CommSq}_{(f,g)}(X) \to \mathrm{CommSq}_{(g,f)}(X)$$

is given by $(m_A, m_B, \gamma)^{\mathrm{T}} :\equiv (m_B, m_A, \gamma^{-1})$.

▶ **Proposition B.3** (Transpose is an equivalence). *Transpose is involutive:* $(\mathfrak{S}^{\mathrm{T}})^{\mathrm{T}} = \mathfrak{S}$ *for all* $\mathfrak{S}$*, and so in particular* $\_^{\mathrm{T}}$ *is an equivalence.*

In Definition 3.4 we defined horizontal pasting of commuting squares. We also have:

▶ **Definition B.4** (Vertical pasting). *From a diagram*

$$\begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathfrak{q}"] & A \arrow[d, "f"] \\
B' \arrow[r, "j"] \arrow[d, "g'"'] \arrow[dr, phantom, "\mathfrak{p}"] & B \arrow[d, "g"] \\
C' \arrow[r, "k"'] & C
\end{tikzcd}$$

of commuting squares $\mathfrak{Q} :\equiv (f', i, \mathfrak{q})$ *and* $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$ *we get the* vertical pasting

$$\frac{\mathfrak{Q}}{\mathfrak{P}} :\equiv \begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "g' \diamond f'"'] \arrow[dr, phantom, "\frac{\mathfrak{q}}{\mathfrak{p}}"] & A \arrow[d, "g \diamond f"] \\
C' \arrow[r, "k"'] & C
\end{tikzcd}$$

where $\frac{\mathfrak{q}}{\mathfrak{p}} :\equiv \alpha^{-1} \cdot (\mathfrak{p} * f') \cdot \alpha \cdot (g * \mathfrak{q}) \cdot \alpha^{-1}$.

A straightforward calculation shows that $\frac{\mathfrak{Q}}{\mathfrak{P}} = (\mathfrak{Q}^{\mathrm{T}} \mid \mathfrak{P}^{\mathrm{T}})^{\mathrm{T}}$, and so one could simply vertically paste squares by horizontally pasting their transposes. However, this equality is only propositional, and it turns out to be more convenient for later proofs to use the canonical form of the vertical pasting as we have defined it.

▶ **Definition B.5** (Vertical pasting map). *For any $A : \mathscr{C}_0$, $f : \mathscr{C}(A, B)$ and $X : \mathscr{C}_0$, the vertical pasting with*

$$\mathfrak{P} \;:\equiv\; \begin{tikzcd} B' \arrow[r, "j"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ C' \arrow[r, "k"'] \arrow[ur, phantom, "\Nearrow\mathfrak{p}"] & C \end{tikzcd}$$

yields a map $\mathrm{CommSq}_{(j,f)}(X) \to \mathrm{CommSq}_{(k,g\diamond f)}(X)$. *That is, we have the family*

$$\frac{-}{\mathfrak{P}} : \prod\,(A : \mathscr{C}_0)\,(f : \mathscr{C}(A, B))\,(X : \mathscr{C}_0)\ \mathrm{CommSq}_{(j,f)}(X) \to \mathrm{CommSq}_{(k,g\diamond f)}(X).$$

▶ **Lemma B.6** (Right action of morphisms on commuting squares). *If $\mathfrak{S} : \mathrm{CommSq}_{\mathfrak{c}}(X)$ is a commuting square in a 2-coherent wild category then $\mathfrak{S} \mathbin{\square} \mathrm{id}_X = \mathfrak{S}$, and for all $f : \mathscr{C}(X, Y)$ and $g : \mathscr{C}(Y, Z)$, $\mathfrak{S} \mathbin{\square} (g \diamond f) = \mathfrak{S} \mathbin{\square} g \mathbin{\square} f$.*

Proof. By calculation, properties of whiskering (Proposition A.1), and coherence—namely, the right identity triangle coherence (Proposition A.2) for the first claim, and the pentagon coherence for the second. ◀

▶ **Corollary B.7.** *By induction on e and Lemma B.6,*

$$\mathfrak{S}\downarrow_e^{\mathrm{CommSq}_{\mathfrak{c}}(\_)} =_{\mathrm{CommSq}_{\mathfrak{c}}(X')} \mathfrak{S} \mathbin{\square} \mathrm{idd}(e^{-1})$$

for every $\mathfrak{S} : \mathrm{CommSq}_{\mathfrak{c}}(X)$ and $e : X = X'$.

▶ **Corollary B.8** (Equality of $\mathrm{CommSq}(\mathfrak{c})$). *Suppose that $(X, \mathfrak{S})$ and $(X', \mathfrak{S}')$ are two commuting squares on a cospan $\mathfrak{c} :\equiv (f, g)$. By equality of Σ-types and Corollary B.7, the equality $(X, \mathfrak{S}) =_{\mathrm{CommSq}(\mathfrak{c})} (X', \mathfrak{S}')$ is equivalent to*

$$\sum\,(e : X = X')\,,\ \mathfrak{S} = \mathfrak{S}' \mathbin{\square} \mathrm{idd}(e).$$

B.2 Pullbacks

▶ **Corollary B.9** (Universal property of (weak) pullbacks). *By Proposition B.1, for each $X : \mathscr{C}_0$ and commuting square $\mathfrak{S} :\equiv (m_A, m_B, \gamma)$ on $\mathfrak{c}$ with source X, the fiber of $(\mathfrak{P} \mathbin{\square}_X \_)$ at $\mathfrak{S}$ is equivalent to*

$$\begin{aligned} &\sum\,(m : \mathscr{C}(X, P))\,(e_A : \pi_A \diamond m = m_A)\,(e_B : \pi_B \diamond m = m_B)\,, \\ &\quad \alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha = (f * e_A) \cdot \gamma \cdot (g * e_B)^{-1}. \end{aligned}$$

Thus $\mathfrak{P}$ is a pullback (respectively, a weak pullback) when this type is contractible (respectively, pointed) for every X and $\mathfrak{S}$.

▶ **Proposition B.10** (Pullbacks are closed under transpose). *$\mathfrak{P}^{\mathrm{T}}$ is a (weak) pullback if $\mathfrak{P}$ is.*

Proof. By a straightforward calculation, $(\mathfrak{P}^{\mathrm{T}} \mathbin{\square}_X \_) = (\_^{\mathrm{T}}) \circ (\mathfrak{P} \mathbin{\square}_X \_)$ for all $X : \mathscr{C}_0$. Since $\_^{\mathrm{T}}$ is an equivalence (Proposition B.3), $(\mathfrak{P}^{\mathrm{T}} \mathbin{\square}_X \_)$ is an equivalence (respectively, a retraction) when $(\mathfrak{P} \mathbin{\square}_X \_)$ is. ◀

The following lemma is inspired by the proof of [9, Proposition 4.1.11] and used in the proof of the pullback pasting lemma (Lemma B.12).

▶ **Lemma B.11** (Pasting maps of (weak) pullbacks)**.** *A commuting square $\mathfrak{P}$ in a 2-coherent wild category $\mathscr{C}$ is a pullback (respectively, a weak pullback) if and only if the vertical pasting map $\bar{\bar{\mathfrak{P}}}$ (Definition B.5) is a family of equivalences (respectively, retractions).*

Proof. Let $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$ be a commuting square on (k, g) as in Definition B.5. For any $A : \mathscr{C}_0$, $f : \mathscr{C}(A, B)$ and $X : \mathscr{C}_0$, the fiber of $\bar{\bar{\mathfrak{P}}}_{A,f,X}$ at

$$\mathfrak{X} \;:\equiv\; \begin{tikzcd} X \arrow[r, "m_A"] \arrow[d, "m_{C'}"'] \arrow[dr, phantom, "\xi"] & A \arrow[d, "g \diamond f"] \\ C' \arrow[r, "k"'] & C \end{tikzcd}$$

is equivalent to the Σ-type

$$\begin{aligned} &\textstyle\sum (m : \mathscr{C}(X, B'))\,(i : \mathscr{C}(X, A))\,(\gamma : j \diamond m = f \diamond i)\,, \\ &\quad (e_{C'} : g' \diamond m = m_{C'}) \times (e_A : i = m_A) \\ &\quad \times\ (\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha \cdot (g * \gamma) \cdot \alpha^{-1} = (k * e_{C'}) \cdot \xi \cdot (g \diamond f * e_A)^{-1}), \end{aligned}$$

by Proposition B.1. Contracting the singleton formed by the components i and e_A, this is equivalent to

$$\begin{aligned} &\textstyle\sum (m : \mathscr{C}(X, B'))(e_{C'} : g' \diamond m = m_{C'})(e_B : j \diamond m = f \diamond m_A)\,, \\ &\quad \alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha = (k * e_{C'}) \cdot (\xi \cdot \alpha) \cdot (g * e_B)^{-1}. \end{aligned}$$

But this type is also the fiber of the precomposition map $(\mathfrak{P} \,\square_X\, \_)$ at the commuting square

$$\begin{tikzcd} X \arrow[r, "f \diamond m_A"] \arrow[d, "m_{C'}"'] \arrow[dr, phantom, "\xi \cdot \alpha"] & B \arrow[d, "g"] \\ C' \arrow[r, "k"'] & C \end{tikzcd}$$

obtained by "reparenthesizing" the diagram $\mathfrak{X}$. Thus if $\mathfrak{P}$ is a pullback (respectively, a weak pullback) then by its universal property (Corollary B.9) the fiber $\left(\bar{\bar{\mathfrak{P}}}_{A,f,X}\right)^{-1}(\mathfrak{X})$ is contractible (respectively, pointed).

Conversely, for any $X : \mathscr{C}_0$, we claim that the map

$$\begin{aligned} &\varphi : \mathscr{C}(X, B') \to \mathrm{CommSq}_{(j, \mathrm{id}_B)}(X) \\ &\varphi(m) :\equiv (m, j \diamond m, {\lambda_{j \diamond m}}^{-1}) \end{aligned}$$

is an equivalence, and that the diagram of types and functions

$$\begin{tikzcd} & \mathrm{CommSq}_{(j,\mathrm{id}_B)}(X) \arrow[d, "\bar{\bar{\mathfrak{P}}}_{B,\mathrm{id}_B,X}"] \\ \mathscr{C}(X, B') \arrow[ur, "\varphi", "\sim"'] \arrow[dr, "\mathfrak{P} \square_X \_"'] & \mathrm{CommSq}_{(k, g \diamond \mathrm{id}_B)}(X) \arrow[d, "\psi", "\sim"'] \\ & \mathrm{CommSq}_{(k,g)}(X) \end{tikzcd}$$

commutes, where ψ is the equivalence $(m_{C'}, m_B, \gamma) \mapsto (m_{C'}, m_B, \gamma \cdot (\rho * m_B))$. That is, $(\mathfrak{P} \square_X \_)$ is the pre- and post-composition of $\bar{\bar{\mathfrak{P}}}_{B,\mathrm{id}_B,X}$ by equivalences. Thus, if $\bar{\bar{\mathfrak{P}}}$ is a family of equivalences then so is $(\mathfrak{P} \square_{\_} \_)$, and if $\bar{\bar{\mathfrak{P}}}$ is a family of retractions then so is $(\mathfrak{P} \square_{\_} \_)$.

Now, the map φ is clearly a section of $\mathrm{fst} : \mathrm{CommSq}_{(j,\mathrm{id})}(X) \to \mathscr{C}(X,\, B')$. We show that it's also a retraction of fst, i.e. that

$$\varphi(m_{B'}) \equiv (m_{B'}, j \diamond m_{B'}, \lambda^{-1}) = (m_{B'}, m_B, \gamma)$$

for all $m_{B'} : \mathscr{C}(X,\, B')$, $m_B : \mathscr{C}(X,\, B)$ and $\gamma : j \diamond m_{B'} = \mathrm{id} \diamond m_B$. Taking the straightforward equalities $\mathrm{refl} : m_{B'} = m_{B'}$ and $\gamma \cdot \lambda : j \diamond m_{B'} = m_B$ of the morphism parts of the commuting squares, we lastly need the equality of commutativity witnesses

$$\lambda^{-1} = (j * \mathrm{refl}) \cdot \gamma \cdot (\mathrm{id} * (\gamma \cdot \lambda)^{-1}),$$

or equivalently, that $\lambda \cdot \gamma = (\mathrm{id} * \gamma) \cdot (\mathrm{id} * \lambda)$. This holds by Proposition A.3.

Finally, given $m : \mathscr{C}(X,\, B')$ we calculate that

$$\mathfrak{S} :\equiv \big(\psi \circ \bar{\bar{\mathfrak{P}}}_{B,\mathrm{id}_B,X} \circ \varphi\big)(m) \quad \text{and} \quad \mathfrak{S}' :\equiv \mathfrak{P} \square_X m$$

are commuting squares of type $\mathrm{CommSq}_{(k,g)(X)}$ with the same morphism components $g' \diamond m$ and $j \diamond m$. The commutativity witness of $\mathfrak{S}$ is $\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha \cdot (g * \lambda^{-1}) \cdot \alpha^{-1} \cdot (\rho * (j \diamond m))$, while that of $\mathfrak{S}'$ is $\alpha^{-1} \cdot (\mathfrak{p} * m) \cdot \alpha$, and these are equal since $(g * \lambda^{-1}) \cdot \alpha^{-1} \cdot (\rho * (j \diamond m)) = \mathrm{refl}$ by the triangle coherator. ◀

▶ **Lemma B.12** (Vertical pullback pasting)**.** *Suppose we have a diagram of commuting squares*

$$\begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "j"] \arrow[d, "g'"'] \arrow[ur, phantom, "\Nearrow \mathfrak{q}"] & B \arrow[d, "g"] \\
C' \arrow[r, "k"'] \arrow[ur, phantom, "\Nearrow \mathfrak{p}"] & C
\end{tikzcd}$$

in a 2-coherent wild category $\mathscr{C}$. Then if $\mathfrak{P} :\equiv (g', j, \mathfrak{p})$ is a pullback of (k, g), the commuting square $\mathfrak{Q} :\equiv (f', i, \mathfrak{q})$ is a pullback of (j, f) if and only if the vertical pasting $\frac{\mathfrak{Q}}{\mathfrak{P}}$ is a pullback of $(k, g \diamond f)$.

Proof. We claim that for any $X : \mathscr{C}_0$, the triangle

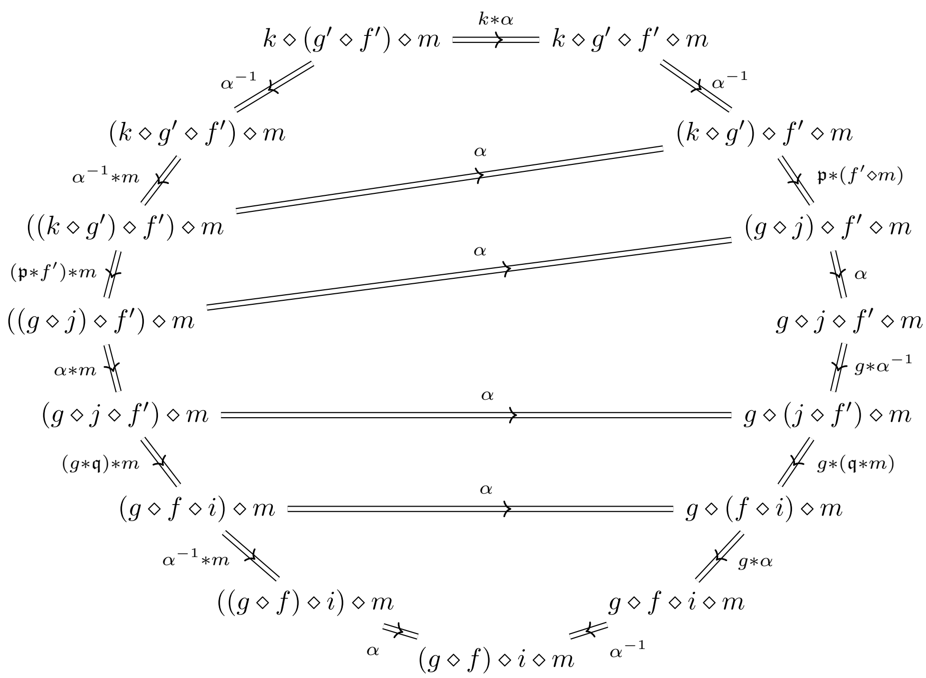
$$\begin{tikzcd}
 & \mathrm{CommSq}_{(j,f)}(X) \arrow[dd, "\bar{\bar{\mathfrak{P}}}_{A,f,X}"] \\
\mathscr{C}(X,\, A') \arrow[ur, "\mathfrak{Q} \square_X _"] \arrow[dr, "\frac{\mathfrak{Q}}{\mathfrak{P}} \square_X _"'] & \\
 & \mathrm{CommSq}_{(k,g \diamond f)}(X)
\end{tikzcd}$$

commutes. Then since $\mathfrak{P}$ is a pullback, the map $\bar{\bar{\mathfrak{P}}}_{A,f,X}$ is an equivalence (Lemma B.11), and it follows that $(\mathfrak{Q} \square \_)$ is a family of equivalences if and only if $(\frac{\mathfrak{Q}}{\mathfrak{P}} \square \_)$ is.

What remains is to construct a homotopy $\big(\frac{\mathfrak{Q}}{\mathfrak{P}} \square_X \_\big) = \big(\bar{\bar{\mathfrak{P}}}_{A,f,X}\big) \circ (\mathfrak{Q} \square_X \_)$ for any X, i.e. a witness that, for any $m : \mathscr{C}(X,\, A')$, the commuting squares

$$\frac{\mathfrak{Q}}{\mathfrak{P}} \square_X m \;\equiv\; \Big((g' \diamond f') \diamond m,\; i \diamond m,\; \alpha^{-1} \cdot \big(\tfrac{\mathfrak{q}}{\mathfrak{p}} * m\big) \cdot \alpha\Big)$$

Figure 2 Construction of $\alpha^{-1} \cdot \left(\frac{\mathfrak{q}}{\mathfrak{p}} * m\right) \cdot \alpha = (k * \alpha) \cdot \frac{\alpha^{-1} \cdot (\mathfrak{q} * m) \cdot \alpha}{\mathfrak{p}}$.



and

$$\frac{\mathfrak{Q} \square_X m}{\mathfrak{P}} \equiv \left(g' \diamond f' \diamond m,\ i \diamond m,\ \frac{\alpha^{-1} \cdot (\mathfrak{q} * m) \cdot \alpha}{\mathfrak{p}}\right)$$

are equal. By Proposition B.1 together with the canonical equalities $\alpha : (g' \diamond f') \diamond m = g' \diamond f' \diamond m$ and $\mathrm{refl} : i \diamond m = i \diamond m$, it's enough to show that

$$\alpha^{-1} \cdot \left(\frac{\mathfrak{q}}{\mathfrak{p}} * m\right) \cdot \alpha = (k * \alpha) \cdot \frac{\alpha^{-1} \cdot (\mathfrak{q} * m) \cdot \alpha}{\mathfrak{p}}.$$

With a little path algebra (noting Proposition A.1) this amounts to showing commutativity of the outer boundary of Figure 2. By inserting associators α as shown in the interior of the diagram, we decompose the outer shape into a pasting of three commuting pentagons (by the pentagonators) and two commuting squares (by Proposition A.1). Thus the entire diagram commutes. ◀

Proof of Lemma 3.5. Since the transpose of a pullback is a pullback (Proposition B.10), we deduce the more familiar horizontal version of the pullback pasting lemma from Lemma B.12 by taking transposes as appropriate. ◀

C Further Proofs

Proof of Lemma 4.17. For one composition, it's enough to show that for all $\sigma : \mathscr{C}(\Gamma, \Delta. A)$,

$$(\mathsf{p}_A \diamond \sigma\,, \mathsf{q}_A[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}^{-1}}) \overset{,\,\diamond^{-1}}{=\!=} (\mathsf{p}_A\,, \mathsf{q}_A) \diamond \sigma \overset{,\,\eta*\sigma}{=\!=} \mathrm{id} \diamond \sigma \overset{\lambda}{=\!=} \sigma.$$

Figure 3 Coherators for $\mathsf{\eta}^{\mathrm{sub}}$ from coherators for context extension.

$$\begin{tikzcd}
\mathsf{p}_A \diamond (\mathsf{p}_A \,, \mathsf{q}_A) \diamond \sigma \arrow[rrr, Rightarrow, "\mathsf{p}{*}(\,,\mathsf{\eta}{*}\sigma)"] \arrow[dr, Rightarrow, "\alpha^{-1}"] \arrow[dd, Rightarrow, "\mathsf{p}{*}\,,\,\diamond"'] & & & \mathsf{p}_A \diamond \mathrm{id} \diamond \sigma \arrow[dd, Rightarrow, "\mathsf{p}{*}\lambda"] \\
 & (\mathsf{p}_A \diamond (\mathsf{p}_A \,, \mathsf{q}_A)) \diamond \sigma \arrow[r, Rightarrow, "(\mathsf{p}{*}\,,\,\mathsf{\eta}){*}\sigma"] \arrow[ddrr, Rightarrow, bend right, "\mathsf{p\beta}{*}\sigma"] \arrow[dl, phantom, "\circlearrowright"] \arrow[drr, phantom, "\circlearrowright"] & (\mathsf{p}_A \diamond \mathrm{id}) \diamond \sigma \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\rho{*}\sigma"] & \\
\mathsf{p}_A \diamond (\mathsf{p}_A \diamond \sigma \,, \mathsf{q}_A[\sigma]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}) \arrow[rrr, Rightarrow, "\mathsf{p\beta}"'] & & & \mathsf{p}_A \diamond \sigma
\end{tikzcd}$$

For the other, we show the equality of pairs $(\mathsf{p}_A\diamond(\sigma \,, a), \mathsf{q}_A[\sigma \,, a]_{\mathsf{t}\downarrow[\diamond]_{\mathsf{T}}{}^{-1}}) = (\sigma, a)$. Equality of the first components is given by $\mathsf{p\beta}$ (Definition 4.10), and for the second components we have that

$$\begin{aligned}
\mathsf{q}_A[\sigma \,, a]_{\mathsf{t}\downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}}\downarrow^{\mathsf{Tm}\,(A[_]_{\mathsf{T}})}_{\mathsf{p\beta}}} &= \mathsf{q}_A[\sigma \,, a]_{\mathsf{t}\downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\mathsf{p\beta}]_{\mathsf{T}}}} && \text{(by Proposition 4.4)} \\
&= a && \text{(by } \mathsf{p\beta} \text{ and properties of transport).}
\end{aligned}$$ ◀

Proof of Lemma 4.19. We have that

$$\begin{aligned}
& \mathrm{Sect}(\mathsf{p}_A) \\
\equiv\; & \textstyle\sum (\sigma : \mathscr{C}(\Gamma, \Gamma . A)) \,, \; \mathsf{p}_A \diamond \sigma = \mathrm{id} \\
\simeq\; & \textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \mathsf{Tm}\,(A[\mathrm{id}]_{\mathsf{T}})) \,, \; \mathsf{p}_A \diamond (\mathrm{fst}\, u \,, \mathrm{snd}\, u) = \mathrm{id} && \text{(by Lemma 4.17)} \\
\simeq\; & \textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \mathsf{Tm}\,(A[\mathrm{id}]_{\mathsf{T}})) \,, \; \mathrm{fst}\, u = \mathrm{id} && \text{(by } \mathsf{p\beta} \text{ (Definition 4.10))} \\
\simeq\; & \textstyle\sum (u : \Sigma\,(\sigma : \mathscr{C}(\Gamma, \Gamma)), \sigma = \mathrm{id}) \,, \; \mathsf{Tm}\,(A[\sigma]_{\mathsf{T}}) && \text{(assoc. of } \Sigma \text{ and comm. of } \times) \\
\simeq\; & \mathsf{Tm}\,(A[\mathrm{id}]_{\mathsf{T}}) && \text{(contractibility of singletons)} \\
\simeq\; & \mathsf{Tm}\, A && \text{(by the inverse of the equivalence } _[\mathrm{id}]_{\mathsf{t}} \text{ (Proposition 4.2)).}
\end{aligned}$$

Tracing the composition of this chain of equivalences, we compute that its inverse is equal to the map $a \mapsto ((\mathrm{id} \,, a[\mathrm{id}]_{\mathsf{t}}), \mathsf{p\beta})$. ◀

Proof of Lemma 5.7. Having coherators for $\mathsf{\eta}^{\mathrm{sub}}$ is equivalent to having the outer boundary of Figure 3 commute for all $\Gamma, \Delta : \mathscr{C}_0$, $A : \mathsf{Ty}\,\Delta$ and $\sigma : \mathscr{C}(\Gamma, \Delta . A)$. We show that this holds by pasting together the regions shown in the interior of the diagram, where the topmost interior square is filled by associativity of whiskering (Proposition A.1), the rightmost triangle by the triangle coherator, and the regions marked $\circlearrowright$ using the coherators for context extension. ◀

Proof of Lemma 6.1. We claim that a mediating substitution is given by

$$\mu_{\sigma,A,\mathfrak{S}} :\equiv (\tau \,, \mathsf{q}_A[\varrho]_{\mathsf{t}\downarrow^{\mathsf{Tm}}_{[\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\gamma]_{\mathsf{T}}{}^{-1}\cdot[\diamond]_{\mathsf{T}}}}),$$

where $\mathsf{q}_A[\varrho]_{\mathsf{t}} : \mathsf{Tm}\, A[\mathsf{p}_A]_{\mathsf{T}}[\varrho]_{\mathsf{T}}$ is transported in the family $\mathsf{Tm}_{\,\mathrm{B}}$ along

$$A[\mathsf{p}]_{\mathsf{T}}[\varrho]_{\mathsf{T}} \overset{[\diamond]_{\mathsf{T}}{}^{-1}}{=\!=\!=} A[\mathsf{p} \diamond \varrho]_{\mathsf{T}} \overset{[^{=}\gamma]_{\mathsf{T}}{}^{-1}}{=\!=\!=} A[\sigma \diamond \tau]_{\mathsf{T}} \overset{[\diamond]_{\mathsf{T}}}{=\!=} A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}.$$

For brevity, denote $\mu_{\sigma,A,\mathfrak{S}}$ by μ. By Proposition B.1, constructing $\theta : \mathfrak{P}_{\sigma,\,A} \mathbin{\square} \mu = \mathfrak{S}$ is equivalent to constructing witnesses $\delta : \mathsf{p}_{A[\sigma]_\mathsf{T}} \diamond \mu = \tau$ and $\epsilon : \sigma \cdot^A \diamond \mu = \varrho$ such that

$$\alpha^{-1} \cdot (\mathsf{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \delta) \cdot \gamma \cdot (\mathsf{p}_A * \epsilon)^{-1}.$$

Let $\delta{:}{\equiv}\mathsf{p}\beta$. Using the equivalence $\mathrm{sub}^{=}_{\_,\_}$ (Definition 5.3), we define $\epsilon{:}{\equiv}\mathrm{sub}^{=}_{(\sigma \cdot^A \diamond \mu),\,\varrho}(\epsilon_0, \epsilon_1)$, where $\epsilon_0 : \mathsf{p}_A \diamond \sigma \cdot^A \diamond \mu = \mathsf{p}_A \diamond \varrho$ and $\epsilon_1 : \mathsf{q}_A[\sigma \cdot^A \diamond \mu]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathsf{q}_A[\varrho]_\mathsf{t}$ are the 2-cells[8] constructed as follows.

First, let ϵ_0 be the concatenation of equalities

$$\mathsf{p}_A \diamond \sigma \cdot^A \diamond \mu \xlongequal{\alpha^{-1} \cdot (\mathsf{p}\beta * \mu) \cdot \alpha} \sigma \diamond \mathsf{p}_{A[\sigma]_\mathsf{T}} \diamond \mu \xlongequal{\sigma * \mathsf{p}\beta} \sigma \diamond \tau \xlongequal{\gamma} \mathsf{p}_A \diamond \varrho.$$

Now calculate that

$$\begin{aligned}
\mathsf{q}_A[\sigma \cdot^A \diamond \mu]_\mathsf{t} &= \mathsf{q}_A[\sigma \cdot^A]_\mathsf{t}[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}{}^{-1}} && \text{(by } [\diamond]_\mathsf{t}) \\
&\equiv \mathsf{q}_A[\sigma \diamond \mathsf{p}_{A[\sigma]_\mathsf{T}}\,, \mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow_{[\diamond]_\mathsf{T}{}^{-1}}]_\mathsf{t}[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}{}^{-1}} \\
&= (\mathsf{q}_{A[\sigma]_\mathsf{T}} \downarrow_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T}})[\mu]_\mathsf{t} \downarrow_{[\diamond]_\mathsf{T}{}^{-1}} && \text{(by } \mathsf{q}\beta) \\
&= \mathsf{q}_{A[\sigma]_\mathsf{T}}[\mu]_\mathsf{t} \downarrow_{([\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T})[\mu]_\mathsf{T} \cdot [\diamond]_\mathsf{T}{}^{-1}} && \text{(by Proposition 4.6)} \\
&= \mathsf{q}_A[\varrho]_\mathsf{t} \downarrow_e && \text{(by } \mathsf{q}\beta)
\end{aligned}$$

where the transports are all in Tm_B, and e is the composition

$$e :\equiv [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\gamma]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T} \cdot ([\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\mathsf{p}\beta]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T})[\mu]_\mathsf{T} \cdot [\diamond]_\mathsf{T}{}^{-1}.$$

So to construct ϵ_1, we may as well show that $\mathsf{q}_A[\varrho]_\mathsf{t} \downarrow_{e \cdot [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathsf{q}_A[\varrho]_\mathsf{t}$. We do this by showing that the equality $\widetilde{e} :\equiv e \cdot [\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}$ is in fact equal to the trivial identity. Some path algebra shows that $\widetilde{e}$ is equal to the outer boundary of Figure 4. This boundary commutes, since we can fill the interior of the diagram with the commuting regions:

- (1), which commutes straightforwardly,
- (2) and (4), which commute by Proposition 4.7, and
- (3) and (5), which are filled by the type pentagonators.

This shows that $\widetilde{e} = \mathrm{refl}$, which completes the proof ϵ_1 that

$$\mathsf{q}_A[\sigma \cdot^A \diamond \mu]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\epsilon_0]_\mathsf{T} \cdot [\diamond]_\mathsf{T}} = \mathsf{q}_A[\varrho]_\mathsf{t} \downarrow_{\widetilde{e}} = \mathsf{q}_A[\varrho]_\mathsf{t},$$

and thus also the proof $\epsilon :\equiv \mathrm{sub}^{=}_{(\sigma \cdot^A \diamond \mu),\,\varrho}(\epsilon_0, \epsilon_1)$ that $\sigma \cdot^A \diamond \mu = \varrho$.

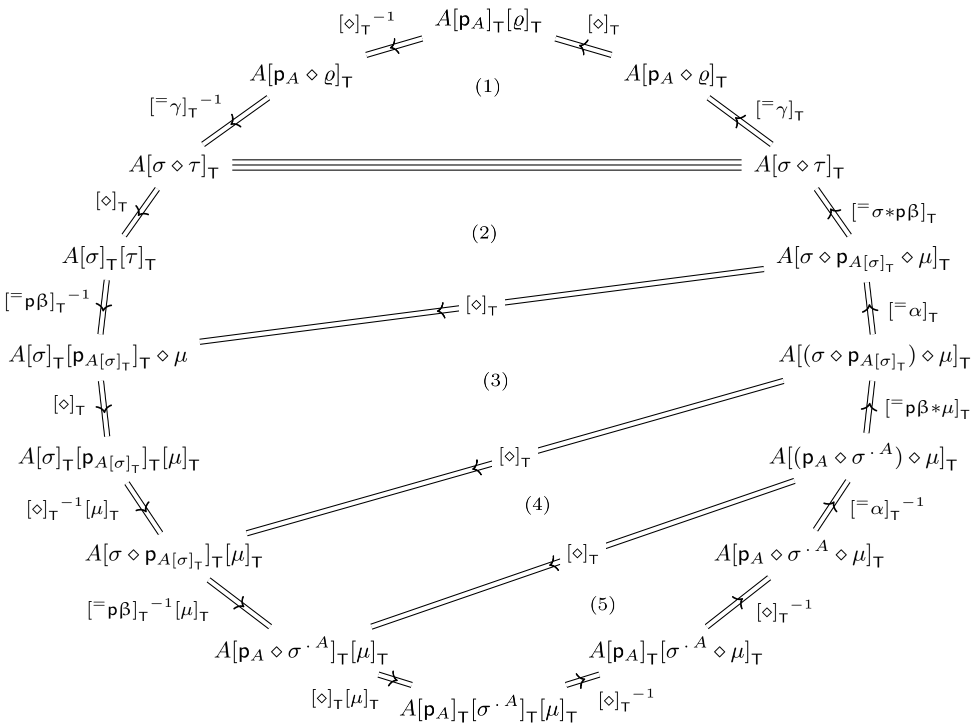
Finally, what remains is to show that $\alpha^{-1} \cdot (\mathsf{p}\beta^{-1} * \mu) \cdot \alpha = (\sigma * \delta) \cdot \gamma \cdot (\mathsf{p}_A * \epsilon)^{-1}$. But by Lemma 5.7 we have that $(\mathsf{p}_A * \epsilon)^{-1} = \epsilon_0^{-1}$ on the right hand side, and the equality then follows by calculation. ◀

Proof of Theorem 6.2. By Lemma 6.1 we have that, for any $\mathrm{B} : \mathscr{C}_0$, the map

$$\begin{aligned}
&\mu_{\sigma,A} : \mathrm{CommSq}_{(\sigma,\mathsf{p}_A)}(\mathrm{B}) \to \mathscr{C}(\mathrm{B}, \Gamma.\, A[\sigma]_\mathsf{T}) \\
&\mu_{\sigma,A}((\tau, \varrho, \gamma)) :\equiv (\tau\,, \mathsf{q}_A[\varrho]_\mathsf{t} \downarrow^{\mathsf{Tm}}_{[\diamond]_\mathsf{T}{}^{-1} \cdot [^{=}\gamma]_\mathsf{T}{}^{-1} \cdot [\diamond]_\mathsf{T}})
\end{aligned}$$

[8] Since $\mathscr{C}$-terms correspond to display maps in $\mathscr{C}$ (Lemma 4.19), we are justified in also calling ϵ_1 a 2-cell.

Figure 4 $e \cdot [\diamond]_{\mathsf{T}}{}^{-1} \cdot [{}^{=}\epsilon_0]_{\mathsf{T}} \cdot [\diamond]_{\mathsf{T}} = \mathrm{refl}$.



is a section of the precomposition map $(\mathfrak{P}_{\sigma,A} \,\square_{\mathrm{B}}\, \_\_)$. We show that it's a retraction of the same, and therefore that $(\mathfrak{P}_{\sigma,A} \,\square_{\_}\, \_\_)$ is a family of equivalences.

That is, for $m : \mathscr{C}(\mathrm{B}, \Gamma.\,A[\sigma]_{\mathsf{T}})$, we want the equality of substitutions $\mu_{\sigma,A}(\mathfrak{P}_{\sigma,A} \,\square\, m) = m$. By Corollary 4.18 and a calculation similar to the one in the proof of Lemma 6.1 we have that $\mu_{\sigma,A}(\mathfrak{P}_{\sigma,A} \,\square\, m) = (\mathsf{p} \diamond m\,, \mathsf{q}_{A[\sigma]_{\mathsf{T}}}[m]_{\mathsf{t}\downarrow e})$, where

$$e :\equiv ([\diamond]_{\mathsf{T}}{}^{-1} \cdot [{}^{=}\mathsf{p}\beta]_{\mathsf{T}}{}^{-1} \cdot [\diamond]_{\mathsf{T}})[m]_{\mathsf{T}} \cdot [\diamond]_{\mathsf{T}}{}^{-1} \cdot [\diamond]_{\mathsf{T}}{}^{-1} \cdot [{}^{=}\alpha^{-1} \cdot (\mathsf{p}\beta * m) \cdot \alpha]_{\mathsf{T}} \cdot [\diamond]_{\mathsf{T}}.$$

On the other hand, $m = (\mathsf{p} \diamond m\,, \mathsf{q}_{A[\sigma]_{\mathsf{T}}}[m]_{\mathsf{t}\downarrow [\diamond]_{\mathsf{T}}{}^{-1}})$ by Lemma 4.17, and thus by Corollary 4.18 again it's enough to show that $e = [\diamond]_{\mathsf{T}}{}^{-1}$.

By path algebra this amounts to showing the commutativity of a diagram of equalities that looks like the one formed by regions (3), (4) and (5) of Figure 4, but where we replace μ with m. Commutativity of this diagram then follows as in the proof of Lemma 6.1, i.e. by Proposition 4.7 and type pentagonators. ◀

▶ **Proposition C.1.** *Let $e : A = A'$ be an equality of $\mathscr{C}$-types $A, A' : \mathsf{Ty}\,\Gamma$. Then*

$$\mathrm{idd}(\mathrm{ap}\,(\Gamma.\_\_)\,e) = (\mathsf{p}_A\,, \mathsf{q}_{A\downarrow^{\mathsf{Tm}}_{e[\mathsf{p}_A]_{\mathsf{T}}}}).$$

Proof. By induction on e it's enough to show that $\mathrm{idd}(\mathrm{refl}_{\Gamma.\,A}) = (\mathsf{p}_A\,, \mathsf{q}_A)$, which holds by $,\eta$ (Definition 4.10). ◀

Proof of Lemma 6.4. By Proposition C.1 it's enough to show that

$$\mathrm{id}^{\cdot A} = (\mathsf{p}_{A[\mathrm{id}]_{\mathsf{T}}}\,, \mathsf{q}_{A[\mathrm{id}]_{\mathsf{T}}\downarrow [\mathrm{id}]_{\mathsf{T}}[\mathsf{p}_{A[\mathrm{id}]_{\mathsf{T}}}]_{\mathsf{T}}}),$$

which holds by Corollary 4.18 and the left type triangulator. ◀

Proof of Lemma 6.6. By Corollary B.8 it's enough to give an equality

$$e : \mathrm{B}.\, A[\sigma \diamond \tau]_{\mathsf{T}} = \mathrm{B}.\, A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}$$

such that $\mathfrak{P}_{\sigma\diamond\tau,\, A} = (\mathfrak{P}_{\tau,\, A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,\, A})\square\mathrm{idd}(e)$. Take $e {:\equiv} \mathrm{ap}\,(\mathrm{B}.\_\_)\,[\diamond]_{\mathsf{T}}$, then by Proposition C.1 we may as well show that $\mathfrak{P}_{\sigma\diamond\tau,\, A}$ is equal to $(\mathfrak{P}_{\tau,\, A[\sigma]_{\mathsf{T}}} \mid \mathfrak{P}_{\sigma,\, A})\,\square\,(\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}})$, or, equivalently, give three equalities

$$\begin{aligned} &\delta : \mathsf{p}_{A[\sigma\diamond\tau]_{\mathsf{T}}} = \mathsf{p}_{A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}} \diamond (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}}), \\ &\epsilon : (\sigma \diamond \tau)^{\cdot\, A} = (\sigma^{\cdot\, A} \diamond \tau^{\cdot\, A[\sigma]_{\mathsf{T}}}) \diamond (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}}) \\ \text{and}\quad &\eta : \mathfrak{p}_{\sigma\diamond\tau,A} = \big((\sigma\diamond\tau) * \delta\big)\cdot \alpha^{-1} \cdot \big((\mathfrak{p}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{p}_{\sigma,A}) * (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}})\big)\cdot\alpha\cdot(\mathsf{p}_A * \epsilon^{-1}). \end{aligned}$$

Take $\delta :\equiv \mathsf{p}\beta^{-1}$. We will define $\epsilon :\equiv \mathrm{sub}^{=}(\epsilon_0, \epsilon_1)$, where $\mathrm{sub}^{=}$ is the equivalence defined at Definition 5.3, and where we seek equalities

$$\begin{aligned} &\epsilon_0 : \mathsf{p} \diamond (\sigma\diamond\tau)^{\cdot\, A} = \mathsf{p} \diamond (\sigma^{\cdot\, A} \diamond \tau^{\cdot\, A[\sigma]_{\mathsf{T}}}) \diamond (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}}) \\ \text{and}\quad &\epsilon_1 : \mathsf{q}[(\sigma\diamond\tau)^{\cdot\, A}]_{\mathsf{t}\,\downarrow\,[\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\epsilon_0]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}} = \mathsf{q}[(\sigma^{\cdot\, A} \diamond \tau^{\cdot\, A[\sigma]_{\mathsf{T}}}) \diamond (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}})]_{\mathsf{t}}. \end{aligned}$$

Now, from Proposition 5.5 we have that $\mathsf{p}_A * \epsilon^{-1} = (\mathsf{p}_A * \epsilon)^{-1} = \epsilon_0^{-1}$, and by rearranging the type of η we may take

$$\epsilon_0 :\equiv \mathfrak{p}_{\sigma\diamond\tau,A}{}^{-1} \cdot \big((\sigma\diamond\tau) * \delta\big)\cdot\alpha^{-1}\cdot\big((\mathfrak{p}_{\tau,A[\sigma]_{\mathsf{T}}} \mid \mathfrak{p}_{\sigma,A}) * (\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}})\big)\cdot\alpha.$$

What remains, then, is to construct ϵ_1. By applying $\mathsf{q}\beta$ and $[\diamond]_{\mathsf{t}}$ to reduce the generic terms on the left and right, we calculate that its type is equivalent to

$$\mathsf{q}_{A[\sigma\diamond\tau]_{\mathsf{t}}\,\downarrow\, e_1}^{\mathsf{Tm}\,\mathrm{B}} = \mathsf{q}_{A[\sigma\diamond\tau]_{\mathsf{t}}\,\downarrow\, e_2}^{\mathsf{Tm}\,\mathrm{B}},$$

where the left and right hand sides are transported, respectively, over equalities

$$\begin{aligned} &e_1 :\equiv [\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}\cdot[^{=}\epsilon_0]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}} \\ \text{and}\quad &e_2 :\equiv [\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}\cdot[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}\cdot\widetilde{e}\,[\mathsf{p}\,,\mathsf{q}_{\downarrow\,[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}{}^{-1} \end{aligned}$$

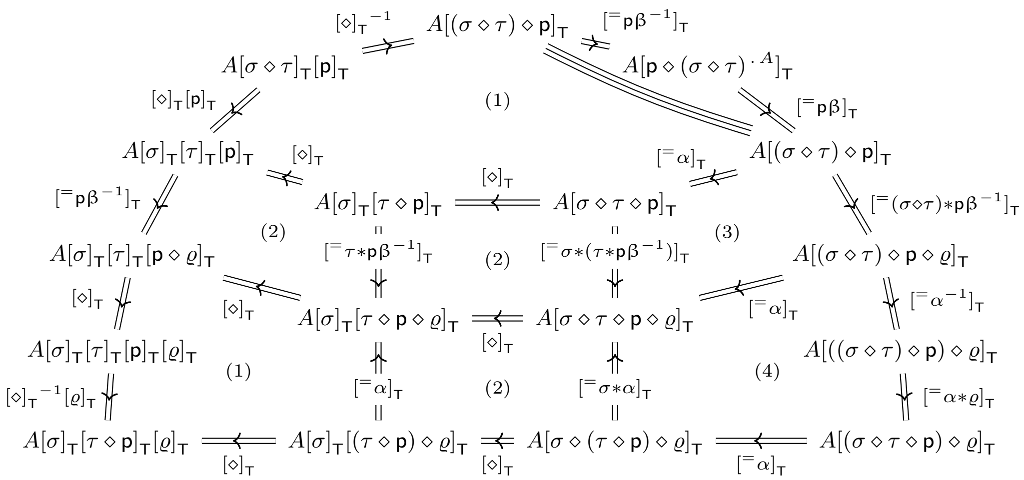
and where $\widetilde{e} :\equiv [\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}\cdot\big([\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}\big)[\tau^{\cdot\, A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}}{}^{-1}$. It's now enough to show that $e_1 = e_2$. This amounts to giving a filling of Figure 5, which we divide into three regions filled with coherence cells as shown in Figure 6 and Figure 7. ◀

Figure 5 The pasting proof of $e_1 = e_2$ splits into three regions, which are filled with the cells shown in Figure 6 and Figure 7. We abbreviate the substitution $(\mathsf{p}\,,\mathsf{q}_{\downarrow[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}})$ by ϱ and the substitution $([\diamond]_{\mathsf{T}}{}^{-1}\cdot[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}\cdot[\diamond]_{\mathsf{T}})[\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}[\varrho]_{\mathsf{T}}$ by ξ.

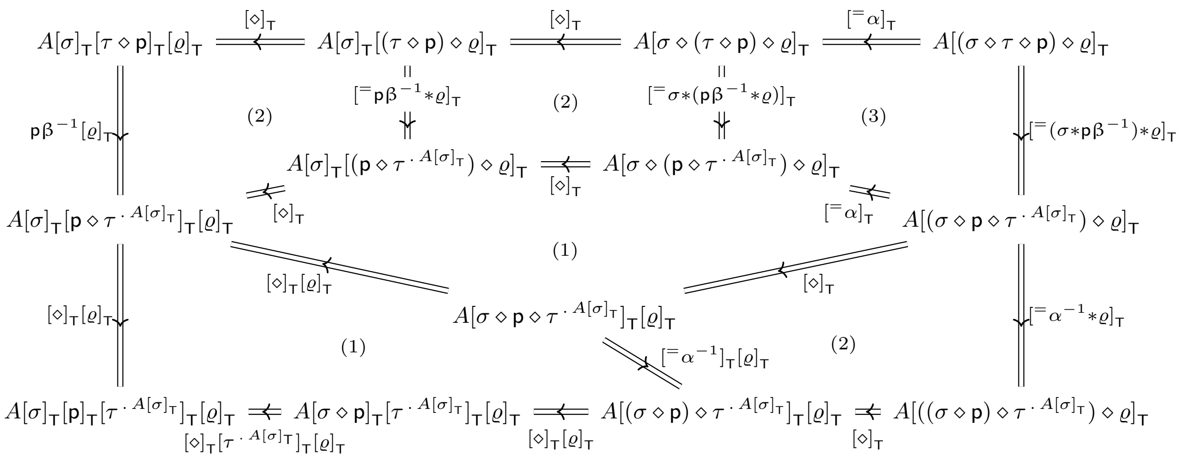
$$
\begin{tikzcd}
 & A[(\sigma\diamond\tau)\diamond\mathsf{p}]_{\mathsf{T}} \arrow[dr, Rightarrow, "{[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}}"] & \\
A[\sigma\diamond\tau]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}} \arrow[ur, Rightarrow, "{[\diamond]_{\mathsf{T}}{}^{-1}}"] \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}}"'] & & A[\mathsf{p}\diamond(\sigma\diamond\tau)^{\,\cdot\,A}]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\mathsf{p}\beta]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}}"'] & & A[(\sigma\diamond\tau)\diamond\mathsf{p}]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}(\sigma\diamond\tau)*\mathsf{p}\beta^{-1}]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}[\mathsf{p}\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}}"'] & \text{(I)} & A[(\sigma\diamond\tau)\diamond\mathsf{p}\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\alpha^{-1}]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\tau]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}{}^{-1}[\varrho]}"'] & & A[((\sigma\diamond\tau)\diamond\mathsf{p})\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\alpha*\varrho]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\tau\diamond\mathsf{p}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "{[^{=}\mathsf{p}\beta^{-1}]_{\mathsf{T}}[\varrho]_{\mathsf{T}}}"'] & & A[(\sigma\diamond\tau\diamond\mathsf{p})\diamond\varrho]_{\mathsf{T}} \arrow[ll, Rightarrow] \arrow[d, Rightarrow, "{[^{=}(\sigma*\mathsf{p}\beta^{-1})*\varrho]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\mathsf{p}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}[\varrho]_{\mathsf{T}}}"'] & \text{(II)} & A[(\sigma\diamond\mathsf{p}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\alpha^{-1}*\varrho]_{\mathsf{T}}}"] \\
A[\sigma]_{\mathsf{T}}[\mathsf{p}]_{\mathsf{T}}[\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "\xi"'] & & A[((\sigma\diamond\mathsf{p})\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} \arrow[ll, Rightarrow] \arrow[d, Rightarrow, "{[^{=}(\mathsf{p}\beta^{-1}*\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})*\varrho]_{\mathsf{T}}}"] \\
A[\mathsf{p}]_{\mathsf{T}}[\sigma^{\,\cdot\,A}]_{\mathsf{T}}[\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}{}^{-1}[\varrho]_{\mathsf{T}}}"'] & \text{(III)} & A[((\mathsf{p}\diamond\sigma^{\,\cdot\,A})\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\alpha*\varrho]_{\mathsf{T}}}"] \\
A[\mathsf{p}]_{\mathsf{T}}[\sigma^{\,\cdot\,A}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}}]_{\mathsf{T}}[\varrho] \arrow[d, Rightarrow, "{[\diamond]_{\mathsf{T}}{}^{-1}}"'] & & A[(\mathsf{p}\diamond\sigma^{\,\cdot\,A}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} \arrow[d, Rightarrow, "{[^{=}\alpha]_{\mathsf{T}}}"] \\
A[\mathsf{p}]_{\mathsf{T}}[(\sigma^{\,\cdot\,A}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} & & A[\mathsf{p}\diamond(\sigma^{\,\cdot\,A}\diamond\tau^{\,\cdot\,A[\sigma]_{\mathsf{T}}})\diamond\varrho]_{\mathsf{T}} \arrow[ll, Rightarrow, "{[\diamond]_{\mathsf{T}}}"']
\end{tikzcd}
$$

Figure 6 Filling regions (I) and (II) of Figure 5 with coherence cells. Regions marked (1) are filled using type pentagonators (Definition 4.9); those marked (2), by naturality of $[\diamond]_{\mathsf{T}}$ (Proposition 4.7); (3), by associativity of whiskering (Proposition A.1); and (4), by the pentagon associator of the category of contexts (Definition 4.9).

(a) Region (I)



(b) Region (II)



■ **Figure 7** Filling region (III) of Figure 5 with coherence cells. Regions marked (1) are filled using type pentagonators (Definition 4.9); regions marked (2), by naturality of $[\diamond]_\mathsf{T}$ (Proposition 4.7).