

Semantic Types

Joshua Chen

Computational Logic
Department of Computer Science
University of Innsbruck

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Types, generally

In this talk, we'll be concerned with logical and mathematical specification, formalization, and verification.

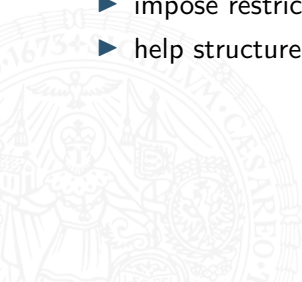


Types, generally

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Types...

- ▶ distinguish logical classes of objects,
- ▶ impose restrictions on formulas,
- ▶ help structure specifications and proofs.



Types in type theory

Dependent type theories used in major proof assistants are theoretically elegant, logically expressive, and functionally powerful.

Even “simple” type theory (HOL) has had a lot of success (and some advantages over DTT).



“Rigid” vs. “soft” types

Types as treated by type theory—“rigid” types:

- ▶ baked into the logical foundation,
- ▶ with accompanying syntactic features.



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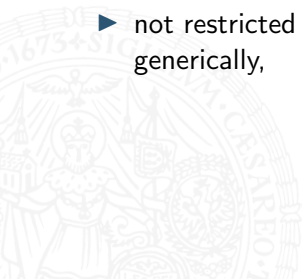
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“Soft” types:

- ▶ *not* part of the underlying logical formalism,
- ▶ not restricted to *any* particular logical system—can be used generically,
- ▶ correspond, essentially, to predicates of the underlying logic.

Types in set theory

Consider axiomatic set theory on top of first-order classical logic.

Only one logical type: *everything is a set*.



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Types in set theory

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Only one logical type: *everything is a set*—effectively untyped!

But in practice, single out various kinds of sets by particular properties. . .



Types in set theory

Type		Definition (first-order formula)
A set n is a natural number	iff	$n \in \omega$, where ω is the first nonzero limit ordinal.
A set x is an ordered triple	iff	x is of the form $(a, (b, c))$, where we use the Kuratowski definition of the ordered pair.
An ordered triple (G, \cdot, e) is a group	iff	$(\cdot: G \times G \rightarrow G) \wedge (\text{associativity of } \cdot) \wedge \dots$
A group (G, \cdot, e) is abelian	iff	$\forall g, h \in G (g \cdot h = h \cdot g)$.

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Here n , x , G , etc. are all sets, but we consider them instances of a particular *type* (of natural numbers, ordered triples, groups, ...) if they satisfy some defining property.

Semantic types

Basic idea

A **semantic type** T of a logical theory \mathcal{L} is given by a predicate ϕ_T in the language of \mathcal{L} . The **inhabitants of** T are the objects of \mathcal{L} that satisfy ϕ_T .

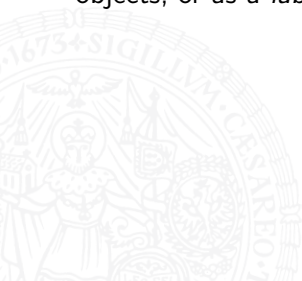


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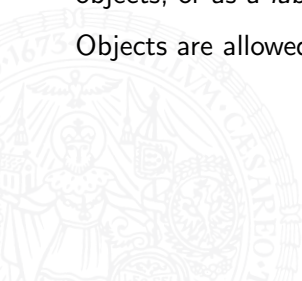
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Extensions to this basic idea are possible! (e.g. Mizar's soft type system.)

Syntactic vs. semantic types



Syntactic vs. semantic types

Types...

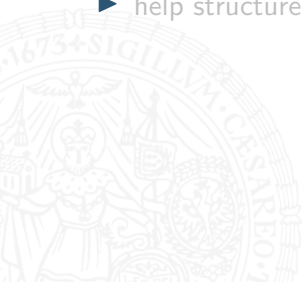
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Syntactic vs. semantic types

Type-theoretic (“syntactic”) types...

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Semantic types...

- ▶ distinguish logical classes of objects,
- ▶ impose **semantic** restrictions on formulas (ill-“typed” sentences are always false/logically irrelevant/undefined),
- ▶ help structure specifications and proofs.

Semantic types in set theory

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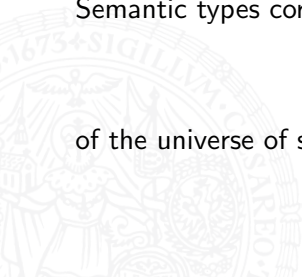
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In set theory,

Semantic types correspond to subclasses

$$T = \{x \mid \phi_T(x)\}$$

of the universe of sets.



Semantic types in set theory

Type construction	Formulation as semantic types
Subtypes $A < B$	$A < B \iff \forall x (\phi_A(x) \longrightarrow \phi_B(x)).$
Intersection types $A \cap B$	$A \cap B = \{x \mid \phi_A(x) \wedge \phi_B(x)\}.$
Dependent types $P(x_1 : X_1, \dots, x_n : X_n)$	$P = \{x \mid \phi_P(x; x_1, \dots, x_n)\},$ where ϕ_P has parameters $x_i \in X_i.$
W-types (well-founded inductive types) T	$T = \bigcup \{F(z) \mid z \in Z\},$ where Z is a well-founded set and $F : Z \rightarrow V$ is defined by well-founded recursion.



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Example

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$$Gp = \{x \mid \exists G, \cdot, e (x = (G, \cdot, e) \wedge \cdot : G \times G \rightarrow G \wedge \dots)\}.$$

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Only need the following version of the recursion theorem on $(\mathbb{N}, <)$:
For any function $G: V \rightarrow V$ there is a unique function $F: \mathbb{N} \rightarrow V$
such that

$$\begin{aligned} F(0) &= G(0), \\ F(n+1) &= G(F(n)). \end{aligned}$$



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$$\begin{aligned} F(0) &= \{0\}, \\ F(1) &= \{(n, 0) \mid n \in \mathbb{N}\}, \\ F(2) &= \{(m, (n, 0)) \mid m, n \in \mathbb{N}\}, \text{ etc.} \end{aligned}$$

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Then $\text{Nat List} = \bigcup \{F(n) \mid n \in \mathbb{N}\}.$

Semantic types in Mizar

Mizar: proof assistant based on axiomatic set theory on top of first-order logic.



Semantic types in Mizar

Mizar: proof assistant based on axiomatic set theory on top of first-order logic.

Semantic types (“modes” and “attributes”) defined via predicates.



Semantic types in Mizar

The type of thin subsets of a space X with respect to a measure M on X :

definition

```
let X be set;  
let S be SigmaField of X;  
let M be sigma_Measure of S;  
mode thin of M -> Subset of X means  
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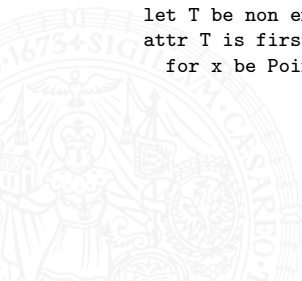
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► **Mizar “modes”**



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- Mizar “modes” are hierarchical **dependent types**

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- ▶ Mizar “modes” are hierarchical dependent types with implicit arguments.
- ▶ Mizar “attributes” modify specific types, extending the basic semantic type system.

Working with semantic types

Semantic type automation

As for type theoretic systems, so for semantic type systems.



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- Implicit argument inference



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- ▶ Implicit argument inference
- ▶ Type inference—possibility of multiple types means having to disambiguate in specific instances.

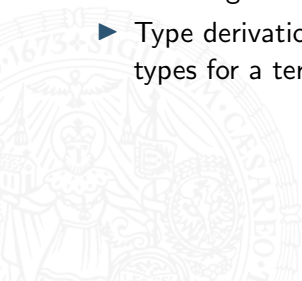


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- ▶ Type derivation—may need to widen, narrow, or derive other types for a term.

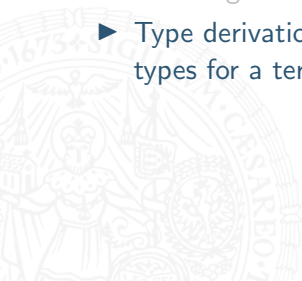


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Type derivation

Sometimes easy:

x is infinite set $\implies x$ is nonempty set.



Type derivation

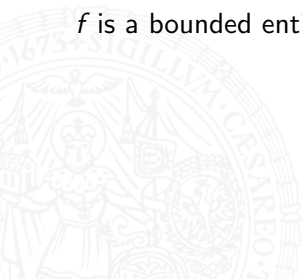
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G is a finite integral domain $\implies G$ is a field. (Wedderburn)

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Type derivation is needed everywhere in proof automation (when applying functions, instantiating premises of theorems, etc).

Type derivation, generally

Assume an ambient logical system \mathcal{L} with implication \Rightarrow .



Type derivation, generally

Assume an ambient logical system \mathcal{L} with implication \implies .

Definition

A **type rule** of a semantic type system in \mathcal{L} is a theorem of the form

$$\psi_1 \implies \cdots \implies \psi_n \implies t : T,$$

where the ψ_i are formulas, t is a term, and T is a semantic type.



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General type derivation task

Given collections \mathcal{S} of terms, \mathcal{T} of valid typing judgments, and \mathcal{R} of type rules, return all typing judgments for terms in \mathcal{S} that are derivable from \mathcal{T} using the rules in \mathcal{R} .

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To be clear, we are not deriving new type *rules*, but using existing rules to derive new typing *judgments*.

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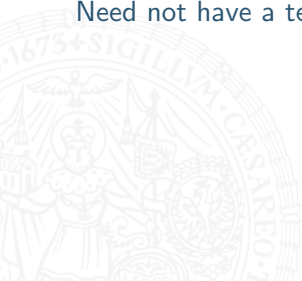


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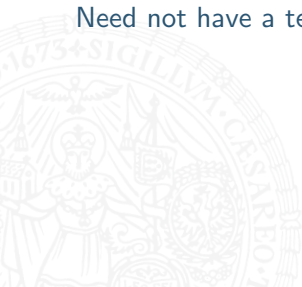


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Need not have a terminating algorithm in general.

Restricted versions of the problem do.



Type derivation in Mizar

Type rules (“cluster registrations”) in Mizar have the abstract form

$$\begin{aligned} & \llbracket x_1 : A_1, x_2 : A_2(x_1), \dots, x_k : A_k(x_1, \dots, x_{k-1}), \\ & \quad t_1(x_1, \dots, x_k) : B_1(x_1, \dots, x_k), \dots, \\ & \quad t_n(x_1, \dots, x_k) : B_n(x_1, \dots, x_k) \rrbracket \\ & \implies t(x_1, \dots, x_k) : B(x_1, \dots, x_k) \end{aligned}$$



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  cluster X \setminus Y -> open Subset of T;
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Abstract form:

$$\begin{aligned} & \llbracket T : \text{TopSpace}, X : \text{open Subset of } T, Y : \text{open Subset of } T \rrbracket \\ & \implies X \cup Y : \text{open Subset of } T \end{aligned}$$

Restricted type derivation

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Restricted type derivation task

Given *finite* collections \mathcal{S} of *closed* terms, \mathcal{T} of valid typing judgments for terms in \mathcal{S} , and \mathcal{R} of type rules of the form $(*)$, return all typing judgments for terms in \mathcal{S} that are derivable from \mathcal{T} using the rules in \mathcal{R} .

The type derivation algorithm

Algorithm: Type derivation

Input: Finite sets of closed terms \mathcal{S} , valid typings \mathcal{T} for \mathcal{S} , and type rules \mathcal{R} of the form $(*)$.

Output: A set \mathcal{T}' of typing judgments for terms in \mathcal{S} .

```
1 Set  $i := 0$ ,  $T_i := \mathcal{T}$ , and  $T_{i+1} := \{\}$ 
2 foreach term  $s$  in  $\mathcal{S}$  do
3   foreach rule  $r$  in  $\mathcal{R}$  do
4     if  $s$  unifies with  $t(x_1, \dots, x_k)$  via  $\sigma$  then
5       Let  $n$  be the number of premises of  $r\sigma$ 
6       foreach  $n$ -tuple  $tys$  of typings (with repetition) in  $T_i$  do
7         Try to discharge the premises of  $r\sigma$  with the entries of  $tys$  in
          order, instantiating the variables  $x_i$ 
8         if all premises are discharged to yield a typing  $ty$  then
9            $T_{i+1} := T_{i+1} \cup \{ty\}$ 
10        end
11      end
12    end
13  if  $T_i \neq T_{i+1}$  then set  $i := i + 1$  and go to Line 2
14  else return  $T_i$ 
```

The type derivation algorithm

Proposition

The type derivation algorithm is correct and terminates.



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- ▶ Correctness is an invariant of the sets T_i —the typings in $T_0 = \mathcal{T}$ are valid by definition, and every typing in T_{i+1} is the result of discharging a type rule with valid typings (by the induction hypothesis), and is thus valid.



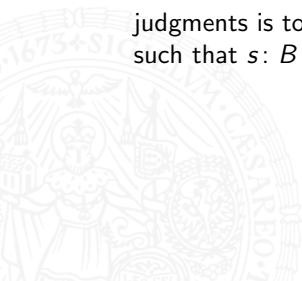
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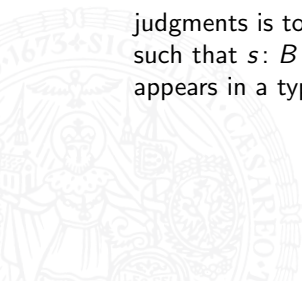
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- ▶ But there are only finitely many choices for $B(x_1, \dots, x_k)$ since \mathcal{R} is finite, and there are also finitely many choices for the s_j since, by induction, only terms from \mathcal{S} appear in typing judgments of the T_i .

□

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Recasts the difficulty of deriving “enough” type information as the problem of choosing a good initial set of terms \mathcal{S} .



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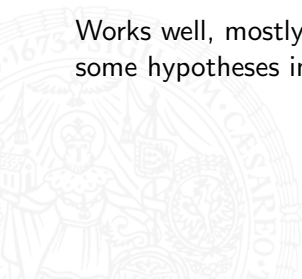


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Works well, mostly. In rare cases need to explicitly instantiate some hypotheses in order to add specific terms to \mathcal{S} .



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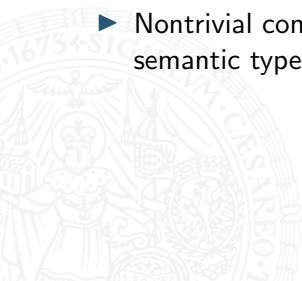
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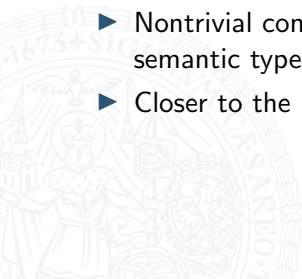
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- ▶ May motivate greater adoption of proof assistants?

Semantic types in type theory...?

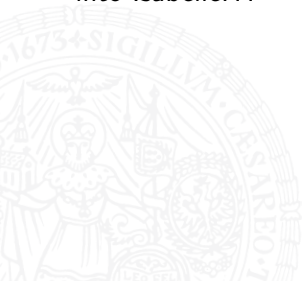
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Current discussion about integrating support for semantic types into Isabelle...



Thanks!

