Letter of Intent

Prospects of Formal Mathematics

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Joshua Chen
University of Nottingham

I am a fourth-year PhD student at the University of Nottingham, advised by Nicolai Kraus. My current research is on the logical and higher categorical aspects of dependent type theory, motivated in turn by the goal of applying the tools of homotopy type theory (HoTT) [Uni13] to the study of ∞-category theory. There is great scope for computer proof assistance in both the discovery and formalization of the mathematics involved, and this is true for the topics I hope to be able to work on with other participants while at the Hausdorff research trimester, described in more detail below.

Besides my current research, I have previously investigated the possibilities of integrating dependent typing (whether based on types [Che19; Che21] or sets [CKKK21]) into the Isabelle proof assistant, as optional aids in formalizations. I would be glad for the chance to interact again with experts in formal math and proof assistants from outside of the dependent type theory community, with the goal of mutually improving our theories and tools for the formalization of mathematics.

As a late-stage PhD student with substantial experience with proof assistants as well as the mathematics of type theory and ∞-category theory, I also greatly appreciate the opportunity to form research connections (as there was minimal opportunity to do so in the first year and a half of my PhD studies due to the Covid pandemic). I would be glad to share my experience, and would benefit from interacting with experts in category theory, type theory, and the various proof assistants that will be represented at the trimester program.

The following suggested research topics are based on my current research, but I am happy to collaborate with others on other projects in my areas of knowledge.

Understanding the "∞-category theoretic strength" of HoTT

Technical details. Univalent type theory[1] is a logic which can be interpreted in any $(\infty, 1)$-topos [Shu19] (up to equivalence of model structures). Constructions in this logic are automatically homotopically coherent, without the need to explicitly make them so, or to consider the presentation details of models of higher categories.

An initial hope was to be able to exploit this phenomenon to formally construct $(\infty, 1)$-categories in HoTT more simply than could be done in non-type-theoretic settings. This hope has not yet materialized, as all candidate definitions to date have been rendered type-incorrect by so-called *coherence problems* [Buc22]. In definitions of ∞-categories based on simplicial sets, this manifests as the seeming intractability of defining *semisimplicial types*, a type theoretic formulation of semisimplicial sets, i.e. presheaves on the direct part of the simplex category Δ [nLa23]. Frustratingly, neither has anyone been able to prove that constructing semisimplicial types in plain "book" HoTT is impossible, making this problem one of the longest standing open problems in homotopy type theory.

A major strand of my research is focused on investigating the obstructions to constructing semisimplicial types—and by extension $(\infty, 1)$-categories—in univalent type theory from a type theoretic perspective. We generalize the problem of constructing semisimplicial types to that of constructing Reedy fibrant diagrams [Rie14; RV14] over simple index categories [Mak95], in the category of contexts of a category with families [Dyb96] modeling Martin-Löf type theory. From this point of view, the coherence problem becomes the problem of internally defining a function that gives the matching objects of these Reedy fibrant diagrams.

Now, instead of categories with families (cwfs) in a typical set-based metatheory, we consider *wild cwfs in HoTT itself* [Kra21], which we call *internal wild cwfs* $\mathcal{C}$. In this homotopically-aware setting we can use the language of HoTT to discover necessary and sufficient coherence conditions that allow for the definition of the matching contexts, improving on previous work of Herbelin [Her15]. We can also cast the problem of defining the matching contexts of simple $\mathcal{I}$-diagrams into that of internally defining a (pseudo?-) functor from some category of cosieves in $\mathcal{I}$ to the category of contexts of $\mathcal{C}$. Since $\mathcal{C}$ is internal to HoTT, this amounts to defining some

[1] That is, Martin-Löf type theory with the univalence axiom, aka "book" HoTT [Uni13] without higher inductive types.

higher functor into a (possibly non-coherent) higher category of contexts. I would like to further investigate the details of this.

In addition, it is expected that the construction of Reedy fibrant diagrams in an internal cwf $\mathcal{C}$ is possible if we assume that $\mathcal{C}$ is set-based. Since the syntax is set-based, our construction would give a "two level" type theoretic definition of semisimplicial types, in a HoTT type that encodes the syntactic cwf. Now, if we could define the canonical interpretation function of the syntactic cwf back into HoTT, we would then have true semisimplicial types in HoTT. In this way we transform the problem of defining semisimplicial types to the (*a priori*, harder) problem of interpreting HoTT in itself[2], giving an interesting lower bound on the difficulty of a fundamental logical question (*"Can this logic internalize itself?"*) in a type theoretic setting.

Relevance. A brief note outlining (an early version of) these ideas was presented at the 2021 TYPES conference [CK21]. The ideas have since been developed further, and an Agda formalization [CK23] and paper on the construction is in progress. I would like to be able to talk about this project with experts in higher category theory and models of type theory who will be present.

This project involves the formalization of a lot of metatheory of type theory. A novel part of this is that we work without the assumption of uniqueness of identity proofs, thus paving the way toward a fuller understanding of the self-interpretation of homotopy type theory. In addition, one may view the formal diagram construction as the beginning of a type theoretic version of Shulman's inverse diagram model of HoTT [Shu15], an important result in the semantics of HoTT and a potential candidate for formalization.

Summary of working goals:

- Investigate obstructions to defining $(\infty, 1)$-categories in univalent type theory.
- Formalize metatheory of type theory, in particular higher-dimensional models of type theory in type theory, and Shulman's inverse diagram model.

∞-category theory: theory, formalization and proof assistants

Technical details. While the ontological status of ∞-categories in univalent type theory is not yet known, in the interim a number of extensions of HoTT that *do* allow for their construction have been proposed. Two such theories are the "simplicial" HoTT of Riehl and Shulman [RS23] and the two-level type theory of Kraus et al [ACKS23]. Simplicial HoTT (sHoTT) is directly geared towards being a setting for *synthetic* (i.e. axiomatic in an appropriate logic) ∞-category theory, using intuitions from the model of ∞-categories as (complete) Segal spaces [Rez01]. An amount of theory has already been developed in this setting [Mar22; BW23; KRW23], a number of results of which have been formalized in the recently developed rzk proof assistant [Kud23].

On the other hand, two-level type theory (2LTT) is geared more towards being a general setting for type theoretic higher mathematics, in which one can use strict equality to define constructions that one may later prove (co-)fibrant, i.e. equivalent to fully coherent structures. One benefit of this viewpoint is, for instance, the ability to talk about "definability" of higher structures type theoretically: one formulates a concept in 2LTT using strict equality as an *a priori* outer type A, and may then say the concept is definable in HoTT if A is fibrant [ACKS23].

Kraus conjectures[3] that one may model many extensions of HoTT inside 2LTT, and thereby provide a *shared standard* and *generous arena* [Mad19] in which to situate the various specific type theories that have been developed for higher category theory. If this were to be the case, one might imagine a proof assistant, based on 2LTT, which one could use to translate results between the various more specific type theories designed for ∞-category theory, analogously to how systems like dedukti are used to translate statements and proofs between various proof assistants.

Relevance. A possible project would be to work out a special case of the aforementioned conjecture of Kraus by modeling simplicial HoTT in 2LTT. One immediate benefit of such an encoding would be the possibility of using Agda (with its `--two-level` flag) to work in sHoTT. I would also be happy to contribute to formalization projects of ∞-category theory, whether in sHoTT/rzk or some other proof assistant. As a longer-term research project, I would like the chance to speak with people who have experience implementing dependently typed proof assistants, with a view toward further developing an implementation of 2LTT, whether in Agda or from scratch.

[2] To my knowledge, this idea first appears publicly in a blog post of Shulman [Shu14].

[3] Ongoing work.

Summary of working goals:

- Embed simplicial HoTT in two-level type theory in Agda, and investigate its suitability for formalizing ∞-category theory.
- Contribute to the formalization of ∞-category theory in rzk, Agda, or some other suitable proof assistant.

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