Subobject Classifiers

Following

Sheaves in Geometry and Logic
Chapter I, Sections 3 & 4

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We begin with the ur-example. Recall that in Set, for any $A \subseteq S$ we have

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, hook, "0"] \\ S \arrow[r, "\chi_A"'] & \{0,1\} \end{tikzcd}$$

where

$$\chi_A(s) = \begin{cases} 0 \text{ if } s \in A \\ 1 \text{ otherwise.} \end{cases}$$

That is, the mono $A \overset{i}{\hookrightarrow} S$ is a pullback of the mono $* \overset{0}{\hookrightarrow} 2$ along the characteristic function χ_A. From this we can already extract the ideas for the following two definitions:

Definition 1. A *subobject* of X is an isomorphism class of monos into X.

To elaborate: for any category $\mathcal{C}$ and $X \in \mathcal{C}$, the monos into X form a full subcategory $\mathrm{Mono}_{\mathcal{C}}(X)$ of $\mathcal{C}/X$. This is a preorder, so “being isomorphic” is a property (not structure). Such an isomorphism class is a subobject.

Example 2. In Set, $\{0,1\} \subset \{0,1,2\}$, and the inclusion $i\colon \{0,1\} \hookrightarrow \{0,1,2\}$ witnesses $\{0,1\}$ as a subobject of $\{0,1,2\}$. But also the injection

$$f\colon \{2,3\} \rightarrowtail \{0,1,2\},$$

where $f(2) = 1$, $f(3) = 0$, represents (“is”) the same subobject.

Definition 3. A *subobject classifier* in a category $\mathcal{C}$ with terminal $\mathbf{1}$ consists of an object Ω and a mono $\top\colon \mathbf{1} \to \Omega$, such that any mono arises as the pullback of $\top$ along a unique arrow into Ω.

Explicitly: for any $X \in \mathcal{C}$ and mono $m\colon S \rightarrowtail X$, there’s a unique *characteristic* or *classifying map* $\chi_S\colon X \to \Omega$ making

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\top"] \\ X \arrow[r, dashed, "\chi_S"'] & \Omega \end{tikzcd}$$

a pullback.

Exercise 4. If $T \rightarrowtail X$ represents the same subobject as $S \rightarrowtail X$ then

$$\begin{tikzcd} T \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\top"] \\ X \arrow[r, dashed, "\chi_S"'] & \Omega \end{tikzcd}$$

is also a pullback square. That is, the characteristic map χ is the same for isomorphic monos. This justifies the name "subobject" classifier (as opposed to "mono" classifier).

For every preorder $\lesssim$ we get a partial order by quotienting by

$$\sim \, := \, (\lesssim \cap \gtrsim).$$

Applying this to $\mathrm{Mono}_{\mathcal{C}}(X)$ gives $\mathrm{Sub}_{\mathcal{C}}(X)$, the poset of subobjects of X. Forgetting the poset structure, for every object X we have a class $\mathrm{Sub}_{\mathcal{C}}(X)$ of subobjects of X. This is functorial: for $X \xrightarrow{f} Y$ we have the pullback map

$$f^*\colon \mathcal{C}/Y \to \mathcal{C}/X$$

which restricts to

$$f^*\colon \mathrm{Mono}_{\mathcal{C}}(Y) \to \mathrm{Mono}_{\mathcal{C}}(X)$$

(because pullbacks of monos are mono), and which further passes to the quotient

$$f^*\colon \mathrm{Sub}_{\mathcal{C}}(Y) \to \mathrm{Sub}_{\mathcal{C}}(X),$$

acting via $f^*[g] = [f^*g]$ (one can check that this is well defined). The upshot is that

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{op} \to \mathrm{Set}$$

is a (possibly large) presheaf on $\mathcal{C}$.

Definition 5. $\mathcal{C}$ is *well powered* if for all $X \in \mathcal{C}$, $\mathrm{Sub}_{\mathcal{C}}(X)$ is small.

What's the significance of all this to subobject classifiers? Well, the definition of a subobject classifier as

> "A mono $\mathbf{1} \xrightarrow{\top} \Omega$ (representing a subobject), such that every subobject (represented by) $S \xrightarrow{m} X$ arises uniquely as a pullback of $\top$"

amounts to saying that there is a bijection

$$\mathrm{Hom}(X, \Omega) \cong \mathrm{Sub}_{\mathcal{C}}(X).$$

Furthermore, by properties of pullback, this isomorphism ends up being natural in X. That is, the (target of the) subobject classifier Ω is a representing object for the subobject functor. In fact, the converse is also true.

Proposition 6 (Proposition 1, SGL I§3)**.** *A locally small category $\mathcal{C}$ with terminal object and "enough" pullbacks has a subobject classifier if and only if*

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{op} \to \mathrm{Set}$$

is representable.

Proof. ($\Rightarrow$) Let $\mathbf{1} \xrightarrow{\top} \Omega$ be a subobject classifier for $\mathcal{C}$. We claim that Ω is a representing object for $\mathrm{Sub}_{\mathcal{C}}$. Firstly, for any $X \in \mathcal{C}$ the map

$$\begin{aligned} \varphi\colon \mathrm{Hom}(X,\Omega) &\to \mathrm{Sub}_{\mathcal{C}}(X) \\ f &\mapsto [f^*\top] \end{aligned}$$

that sends f to the subobject represented by $f^*\top$ is a bijection, since, by the universal property of Ω, f is the unique preimage of $[f^*\top]$ under φ.

One also easily checks that φ is also natural: for any $g\colon X \to Y$ in $\mathcal{C}$, the commutativity of

$$\begin{tikzcd} \mathrm{Hom}(Y,\Omega) \arrow[r, "\varphi"] \arrow[d, "-\circ g"'] & \mathrm{Sub}_{\mathcal{C}}(Y) \arrow[d, "g^*"] \\ \mathrm{Hom}(X,\Omega) \arrow[r, "\varphi"'] & \mathrm{Sub}_{\mathcal{C}}(X) \end{tikzcd}$$

boils down to the fact that for any k, $g^*(f^*k)$ is isomorphic to $(f \circ g)^*k$ in $\mathcal{C}/X$.

($\Leftarrow$) Conversely, assume that Ω is a representing object for $\mathrm{Sub}_{\mathcal{C}}$ via a natural isomorphism

$$\psi\colon \mathrm{Sub}_{\mathcal{C}} \xrightarrow{\cong} \mathrm{Hom}(-,\Omega).$$

Define our candidate "true" subobject, represented by some mono

$$\top\colon \Omega' \rightarrowtail \Omega,$$

to be the correspondent of id_Ω under ψ, i.e. $[\top] \coloneqq \psi^{-1}(\mathrm{id}_\Omega)$.

Now let $[m] \in \mathrm{Sub}_{\mathcal{C}}(X)$ be a subobject of X. Take its candidate characteristic map $\chi \in \mathrm{Hom}(X,\Omega)$ to be its correspondent under ψ, i.e.

$$\chi \coloneqq \psi([m]).$$

Naturality of ψ means that

$$\mathrm{id}_\Omega \circ \chi = \psi([\top]) \circ \chi = \psi([\chi^*\top]),$$

so that m and $\chi^*\top$ represent the same subobject and are thus isomorphic, and so χ is an arrow along which m is a pullback. Furthermore χ is the unique such arrow since ψ is an isomorphism.

Finally it remains to show that Ω' is terminal. One easily checks that for any $X \in \mathcal{C}$ and $f, g\colon X \to \Omega'$, the two squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & \Omega' \arrow[d, tail, "\top"] \\ X \arrow[r, "\top\circ f"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] & \Omega' \arrow[d, tail, "\top"] \\ X \arrow[r, "\top\circ g"'] & \Omega \end{tikzcd}$$

are pullbacks, so that $\top \circ f = \top \circ g$ by uniqueness of the bottom arrows, and hence $f = g$ by monocity of $\top$. $\square$

Corollary 6.1. *A locally small category with subobject classifier is well powered.*[1]

Corollary 6.2. *Subobject classifiers are unique up to isomorphism.*

[1] i.e. each object has set-many subobjects, which I think means (later) that the power object is nice and can be defined.

Addendum

In the meeting Elisabeth pointed out that $\mathrm{Sub}_{\mathrm{Set}}$ is the contravariant powerset functor, whose action on maps is to take $f\colon X \to Y$ to the map $\mathcal{P}(Y) \to \mathcal{P}(X)$ that sends $A \subseteq Y$ to the preimage $f^{-1}(A) \subseteq X$.

There is also the *covariant* powerset functor that sends $f\colon X \to Y$ to the *direct image* map

$$\begin{aligned} \mathcal{P}(X) &\to \mathcal{P}(Y) \\ A &\mapsto f(A). \end{aligned}$$

It should be possible to generalize this to arbitrary toposes using the (regular) epi-mono factorization. We'll possibly discuss this in the next meeting.