# Subobject Classifiers

Following

*Sheaves in Geometry and Logic*
Chapter I, Sections 3 & 4

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We begin with the ur-example. Recall that in Set, for any $A \subseteq S$ we have

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, hook, "0"] \\ S \arrow[r, "\chi_A"'] & \{0,1\} \end{tikzcd}$$

where

$$\chi_A(s) = \begin{cases} 0 \text{ if } s \in A \\ 1 \text{ otherwise.} \end{cases}$$

That is, the mono $A \overset{i}{\hookrightarrow} S$ is a pullback of the mono $* \overset{0}{\hookrightarrow} 2$ along the characteristic function $\chi_A$. From this we can already extract the ideas for the following two definitions:

**Definition 1.** A *subobject* of $X$ is an isomorphism class of monos into $X$.

To elaborate: for any category $\mathcal{C}$ and $X \in \mathcal{C}$, the monos into $X$ form a full subcategory $\mathrm{Mono}_{\mathcal{C}}(X)$ of $\mathcal{C}/X$. This is a preorder, so "being isomorphic" is a property (not structure). Such an isomorphism class is a subobject.

*Example* 2. In Set, $\{0,1\} \subset \{0,1,2\}$, and the inclusion $i\colon \{0,1\} \hookrightarrow \{0,1,2\}$ witnesses $\{0,1\}$ as a subobject of $\{0,1,2\}$. But also the injection

$$f\colon \{2,3\} \rightarrowtail \{0,1,2\},$$

where $f(2) = 1$, $f(3) = 0$, represents ("is") the same subobject.

**Definition 3.** A *subobject classifier* in a category $\mathcal{C}$ with terminal $\mathbf{1}$ consists of an object $\Omega$ and a mono $\top\colon \mathbf{1} \to \Omega$, such that any mono arises as the pullback of $\top$ along a unique arrow into $\Omega$.

Explicitly: for any $X \in \mathcal{C}$ and mono $m\colon S \rightarrowtail X$, there's a unique *characteristic map* $\chi_S\colon X \to \Omega$ making

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\top"] \\ X \arrow[r, dashed, "\chi_S"'] & \Omega \end{tikzcd}$$

a pullback.

*Exercise* 4. If $T \rightarrowtail X$ represents the same subobject as $S \rightarrowtail X$ then

$$\begin{tikzcd} T \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, tail, "\top"] \\ X \arrow[r, dashed, "\chi_S"'] & \Omega \end{tikzcd}$$

is also a pullback square. That is, the characteristic map $\chi$ is the same for isomorphic monos. This justifies the name "subobject" classifier (as opposed to "mono" classifier).

For every preorder $\lesssim$ we get a partial order by quotienting by

$$\sim \;:=\; (\lesssim \cap \gtrsim).$$

Applying this to $\mathrm{Mono}_{\mathcal{C}}(X)$ gives $\mathrm{Sub}_{\mathcal{C}}(X)$, the class of subobjects of $X$ (in fact, this is a category, but we won't use this). So for every object $X$ we have a class $\mathrm{Sub}_{\mathcal{C}}(X)$ of subobjects of $X$. This is functorial: for $X \xrightarrow{f} Y$ we have the pullback map

$$f^* \colon \mathcal{C}/Y \to \mathcal{C}/X$$

which restricts to

$$f^* \colon \mathrm{Mono}_{\mathcal{C}}(Y) \to \mathrm{Mono}_{\mathcal{C}}(X)$$

(because pullbacks of monos are mono), and which further passes to the quotient

$$f^* \colon \mathrm{Sub}_{\mathcal{C}}(Y) \to \mathrm{Sub}_{\mathcal{C}}(X),$$

acting via $f^*[g] = [f^*g]$ (one can check that this is well defined). The upshot is that

$$\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{op} \to \mathrm{Set}$$

is a (possibly large) presheaf on $\mathcal{C}$.

**Definition 5.** $\mathcal{C}$ is *well powered* if for all $X \in \mathcal{C}$, $\mathrm{Sub}_{\mathcal{C}}(X)$ is small.

What's the significance of all this to subobject classifiers? Well, the definition of a subobject classifier as

> "A mono $\mathbf{1} \xrightarrow{\top} \Omega$ (representing a subobject), such that every subobject (represented by) $S \xrightarrow{m} X$ arises uniquely as a pullback of $\top$"

amounts to saying that

$$\hom(X, \Omega) \cong \mathrm{Sub}_{\mathcal{C}}(X)$$

as classes—and furthermore, by properties of pullback, this isomorphism ends up being natural in $X$. That is, the (target of the) subobject classifier $\Omega$ is a representing object for the subobject functor. The converse is also true.

**Proposition 6** (Proposition 1, SGL I§3)**.** *A locally small category $\mathcal{C}$ with terminal object and "enough" pullbacks has a subobject classifier if and only if*

$$\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{op} \to \mathrm{Set}$$

*is representable.*

*Proof.* In meeting. Maybe I'll type it up later. $\square$

**Corollary 6.1.** *A locally small category with subobject classifier is well powered.*[1]

**Corollary 6.2.** *Subobject classifiers are unique up to isomorphism.*

[1] i.e. each object has set-many subobjects, which I think means (later) that the power object is nice and can be defined.