AER210 Abridged

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Contents

Chapter 1

Review: Stuff to Have Memorized

1.1 Trig Functions and Derivatives

$\frac{d}{dx}sin(x) = cos(x)$ $\quad$ $\frac{d}{dx}csc(x) = -csc(x)cot(x)$

$\frac{d}{dx}cos(x) = -sin(x)$ $\quad$ $\frac{d}{dx}sec(x) = sec(x)tan(x)$

$\frac{d}{dx}tan(x) = sec^2(x)$ $\quad$ $\frac{d}{dx}cot(x) = -csc^2(x)$

1.2 Inverse Trig Derivatives

$\frac{d}{dx}sin^{-1}(x) = \frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}cos^{-1}(x) = \frac{-1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}tan^{-1}(x) = \frac{1}{1+x^2}$

1.3 How to complete the Square

1. Put $ax^2 + bx$ in brackets and forcefully factor out the a
2. Add $(\frac{b}{2})^2$ to the inside of the brackets and subtract it from the outside (you got it)
3. Factor and be happy that you've completed the square;

1.4 Trig Angle Sums

1. $sin(A + B) = sin(A)cos(B) + cos(A)sin(B)$
2. $cos(A + B) = cos(A)cos(B) - sin(A)sin(B)$
3. $sin(A - B) = sin(A)cos(B) - cos(A)sin(B)$
4. $cos(A - B) = cos(A)cos(B) + sin(A)sin(B)$

1.5 Identities Assorted

$$sin^2(x) = 1/2(1 - cos(2x)) \tag{1.1}$$
$$cos^2(x) = 1/2(1 + cos(2x)) \tag{1.2}$$
$$sinxcosx = 1/2sin2x \tag{1.3}$$

$$sinAcosB = 1/2[sin(A - B) + sin(A + B)] \tag{1.4}$$
$$sinAsinB = 1/2[cos(A - B) - cos(A + B)] \tag{1.5}$$
$$cosAcosB = 1/2[cos(A - B) + cos(A + B)] \tag{1.6}$$

$$\frac{d}{dx}csc(x) = -csc(x)cot(x) \tag{1.7}$$
$$cot^2(x) = csc^2(x) - 1 \tag{1.8}$$
$$\frac{d}{dx}cot(x) = -csc^2(x) \tag{1.9}$$

Chapter 2

In-Class Review

2.1 Vectors & Vector Functions

- Vector = magnitude + direction
- If the origin of the vector is the origin of the coordinate system, it's a position vector.
- Dot product: $\vec{a} \cdot \vec{b} = \vec{a}_1 \cdot \vec{b}_1 + ... + \vec{a}_n \cdot \vec{b}_n$
- Cross product: $\vec{a} \times \vec{b} = det(i, j, k; \vec{a}^T; \vec{b}^T)$
- Cross product is the area of the paralellogram traced out by the two vectors.
- Scalar triple product: $\vec{a} \cdot (\vec{b} \times \vec{c})$, produces a scalar, represents the volume of the parallelepiped formed by the three vectors.
- To get the derivative of a vector function, simply take the derivative of each of the internal functions and package them into a new vector function.

2.2 Arc Length

2.2.1 One-Variable Functions

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

2.2.2 Parametric Functions

Let $y(t)$ and $x(t)$ describe a parametric function in 2 dimensions. Then the arc length would be:

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dx$$

2.2.3 Vector Funtions

Let $\vec{r}(t)$ describe a vector function that converts a scalar t into a vector. Then the arclength function would be:

$$L = \int_a^b |\vec{r}(t)| dt$$

2.2.4 Reparamerizing with respect to Arc Length

What is this? Let there be a vector function $\vec{r}(t)$ and its correponding arc length function $s(t)$. Since s is strictly increasing, we can safely **reparameterize** $\vec{r}(t)$ to be $\vec{r}(s(t)) \rightarrow \vec{r}(s)$.

Why would you want to do this? This type of reparameterization is useful because now we do not have to rely on any particular coordinate system.

Steps to Reparameterizing

1. Find $s(t) = \int_a^b |\vec{r}(u)| du$.
2. Put s in terms of t.
3. Substitute the expression found in part 2 in the original $\vec{r}(t)$.

2.3 Partial Derivatives

2.3.1 Functions of Several Variables

A function of two variables transforms each pair of Reals (x, y) in a given set to a single real number. The given set is the domain, and the set of reals that the pair is transformed to is the range.

Level functions are functions that have $f(x, y) = k$ for given ranges of (x, y)

Functions of 3 or more variables are pretty easy to extrapolate from functions of two variables, tbh.

2.3.2 Limits and Continuity with Functions of Several Variables

Limits

Definition of limit with many variables:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ s.t. if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

How to find: Regard the non-mentioned variable in the notation as a constant and differentiate with respect to the mentioned variable.

2.3.3 Higher Partial Derivatives

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial z}{\partial y \partial x}$$

Clairaut's Theorem

If f_{xy} and f_{yx} are both **defined** and **continuous** on disk D then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

2.4 Gradient

Think of the gradient like an operator that applies to functions of many variables (functions of vectors). The ∇ just calculates the partial derivatative of the function with respect to each of its input variables and puts it into a vector.

$$\nabla f(x, y) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$$

Or, more generally for a function $f(\vec{x})$,

$$\nabla f(\vec{x}) = [\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}]$$

2.5 Chain Rule with Many Variables

Let there be a function $f(\vec{x})$. Let $\vec{x}$ of length n be a function of $\vec{t}$ of length m. We take the partial derivative of f with respect to t_i by the following:

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + ... + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Chapter 3

Multiple Integrals

Pretty much the same as regular integrals, you just do two. You can apply them to volumes under surfaces.

3.1 Basic Meaning and Solving

3.2 Leibniz Integral Rule (Differentiability of Integral with Respect to Parameter)

The Leibniz integral rule simply lets you more easily take the derivative of the integral of a multivariable function where the variable you are integrating with respect to is not the same as the variable you are taking the derivative with respect to.

$$\frac{d}{dx}\int_a^b f(x,t)dt$$

3.2.1 Constant Bounds of Integration

When you are integrating from one constant to another $[a, b]$, the result is quite simple and elegant.

$$\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial f}{\partial x}dt$$

3.2.2 Derivation

Let's write $\frac{d}{dx}\int_a^b f(x,t)dt$ in terms of the definition of the derivative:

$$\begin{aligned}
&= \frac{\int_a^b f(x+\Delta x,t)dt - \int_a^b f(x+\Delta x,t)dt}{\Delta x} \\
&= \frac{\int_a^b f(x,t)dx + \int_a^b \frac{\partial f}{\partial x} dt \Delta t dx - \int_a^b f(x,t)dt}{\Delta x} \\
&= \int_a^b \frac{\partial f}{\partial t} dx
\end{aligned}$$

3.2.3 Variable Bounds of Integration

Final result:

$$\frac{d}{dx}\int_{a(t)}^{b^t} f(x,t)dt = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial x} dt - f(a,t)\frac{da}{dt} + f(b,t)\frac{db}{dt}$$

3.3 Polar Coordinates with Multiple Integrals

Recall $r^2 = x^2 + y^2$, $x = rcos(\theta)$, $y = rsin(\theta)$

A polar rectangle is of the form

$$R = (r,\theta)|a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Basic form of double integral in polar coordinates:

$$\iint_R gdA = \int_\alpha^\beta \int_a^b g(r,\theta)\, dr\, d\theta$$

3.3.1 Change to Polar Coordinates in Double Integrals

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(rcos(\theta), rsin(\theta)) * r\, dr\, d\theta$$

Make sure not to forget the r in the integral!

3.3.2 Variable Bounds of Integration for r

$$D = (r, \theta) | \alpha \leq \theta \leq \beta, h(\theta) \leq r \leq g(\theta)$$

Then:

$$\iint_D f(r, \theta) = \int_\alpha^\beta \int_{h(\theta)}^{h(\theta)} f(r, \theta) * r \, dr \, d\theta$$

3.4 Applications of Multiple Integrals

The obvious application of multiple integrals is to compute volume. But, like with regular integrals, there are a lot more ways you can apply them (think back to centre of mass, arc length, etc).

3.4.1 Density and Mass

Lamina is a plan of mass and surface density. With double integrals, we can now calculate mass of a lamina with **variable** density!

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

$$\rho(x, y) = \frac{\partial m}{dxdy}$$

Calculating Mass:

$$m = \iint_D \rho(x, y) dA$$

3.4.2 Moments of Intertia

Moment M_x is the distance of the centre of mass from the x-axis times the mass. Likewise with y.

Centre of Mass is at $(\bar{x}, \bar{y})$ where $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

If you're confused as to why the $\bar{x}$ and M_y are related, just remember that the moment about the y-axis (M_y) is defined as the **distance from the y-axis to the centre of mass** times the mass of the lamina. That distance is the $\bar{x}$ value!

3.4.3 Moments of Intertia

Recall that $I = mr^2$ for particles.

$$I_x = \iint_D y^2 \rho(x,y) dA$$
$$I_y = \iint_D x^2 \rho(x,y) dA$$
$$I_o = I_x + I_y = \iint_D (x^2 + y^2) \rho(x,y) dA$$

Makes sense because $x^2 + y^2$ is the distance of the chunk from the origin.

Radius of Gyration

About Arbitrary Axis: R is the radius of gyration if $mR^2 = I$. Basically, if mass were concentrated at R, then it would have the same moment of inertia as the original lamina.

About the Origin: $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point of gyration about the origin.

3.4.4 Probability

The probability function of a random variable x is:

$$f(x) > 0 \,\forall\, x, \int_{-\infty}^{\infty} f(x) dx = 1$$

Probability of $a < x < b$: $\int_a^b f(x) dx$

2-D Probability Function: Chance of $(x, y) \in D$ is $\iint_D f(x,y) dA$.
Note that $\iint_{\mathbb{R}^2} f(x,y) dA = 1$

Expected Values

For single variable x with probability density function f, the expected value is the average value of x across many trials.

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

Makes sense because you're just summing the products of the chance of a particular x and the value of that x for all x.

Multi-Variable Functions:

$$\mu_1 = X_{mean} = \iint_{\mathbb{R}^2} xf(x,y)dA$$

$$\mu_2 = Y_{mean} = \iint_{\mathbb{R}^2} yf(x,y)dA$$

Normal Distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

3.5 Surface Area

How it's derived: If you divide the surface in to an m by n grid, the surface area of each piece is the area of that parallelogram. That can be found by the cross product of the two vectors that define two sides of the parallelogram. Those vectors are just the width and length of the piece times the partial derivatives along that particular axis.

$$A(S) = \iint_D \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} dA$$

$$A(S) = \iint_D \sqrt{(\nabla f(x,y))^2 + 1} dA$$

The $+1$ term is there because it's $\frac{\partial f}{\partial f}$.

3.6 Triple Integrals

3.6.1 Type 1

Lies between continuous functions of $f(x,y)$

$$E = (x,y,z)|(x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)$$

Then

$$\iiint_E f(x,y,z)dV = \iint_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dA$$

3.6.2 Type 2

Lies between continuous functions of $f(y,z)$

$$E = (x,y,z)|(y,z) \in D, u_1(y,z) \leq x \leq u_2(y,z)$$

Then

$$\iiint_E f(x,y,z)dV = \iint_D [\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx]dA$$

3.6.3 Type 3

Lies between continuous functions of $f(x,z)$

$$E = (x,y,z)|(x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)$$

Then

$$\iiint_E f(x,y,z)dV = \iint_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy]dA$$

3.6.4 Applications of Triple Integrals

Obvious

$$V = \iiint_E dV$$

$$m = \iiint_E \rho(x,y,z)dV$$

Moments and CoM

REMEMBER: Moments are about a plane (e.g. (x,y) plane) while moments of INERTIA are about an axis (think about spinning vs. balancing - you balance along a plane, spin around an axis).

$$M_{yz} = \iiint_E x\rho(x,y,z)dV$$

etc.

$$I_x = \iiint_E (y^2+z^2)\rho(x,y,z)dV$$

3.7 Cylindrical Coordinates

These work pretty much how you would think they'd work based on your knowledge of polar coordiantes in 2-D. Just stick a conventional z-axis on the polar coordinate system and you have cylindrical coordinates.

$$x = r\cos\theta; \quad y = r\sin\theta; \quad z = z$$

$$\iiint_E f(x,y,z)dV = \iiint rf(r\cos(\theta), r\sin(\theta), z)dz\,dr\,d\theta$$

Don't forget that little r coefficient in there!

3.8 Spherical Coordinates

Now we define points in 3-D space by the the following values:

1. p: the distance away from the origin (scalar)
2. θ: angle from 0 of the point when it's projected onto the x-y plane.
3. ψ: The angle from the z-axis.

So yeah, a bit of a pain in the ass to visualize and draw. At least it's useful when they give you literal spheres...

Anyway, here are the relationships between those newfangled varibles and our trusty old x, y, z system:

1. $x = p\sin\phi\cos\theta$
2. $y = p\sin\phi\sin\theta$
3. $z = p\cos\phi$
4. $p = \sqrt{x^2 + y^2 + z^2}$

Finally, here's the conversion formula for triple integrals:

$$\iiint_E f(x,y,z)dV = \iiint f(p\sin(\phi)\cos(\theta), p\sin(\phi)\sin(\theta), p\cos(\theta))p^2\sin(\phi)dp\,d\theta\,d\phi$$

3.9 Taylor Series with Two Variables

...

3.10 Jacobians

Remember that time when $\int_a^b f(x)dx = \int_c^d f(g(u))\frac{dx}{du}du$? Well imagine if we could do that with multiple integrals. How cool would that be?

Well guess what, we've been doing that for a while now! Whenever we transfer to cylindrical and spherical coordinates we're basically just applying that same rule. The only new thing we have to learn is the equivalent of that $\frac{dx}{du}$ bit when you have multiple independent variables.

3.10.1 How it Works

Let $(u, v) = (G(x, y), H(x, y))$ be our transformation we want to use to make the integral easier. And, let's say we want to change the followinig:

$$\iint_V f(x, y)dx\,dy \to \iint_{V_2} f(u, v)du\,dv$$

All we gotta do is apply the same rule that we're used to using for u-substitutions, but the $\frac{dx}{du}$ term becomes:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

You can kind of see where that Jacob guy was coming from. It sort of makes sense that $\Delta A \approx |\frac{\partial(x,y)}{\partial(u,v)}|du\,dv$.

3.10.2 How to Change your Variable with Jacobians

$$\iint_R f(x, y)dA = \iint_S f(x(u, v), y(u, v))|\frac{\partial(x, y)}{\partial(u, v)}|du\,dv$$

This applies to n-dimensional integrals. You can double check that the transformations to cylindrical and spherical coordinates work out if you want!

Chapter 4

Vector Calculus

4.1 Review: Vector Fields, Gradient Fields

4.1.1 Vector Fields

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is F that assigns each $(x, y, z) \in \mathbb{R}^3$ to a 3D vector $F(x, y, z)$.

4.1.2 Gradient Fields

The gradient operator is represented by ∇ and can be viewed as the operator

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \ldots \end{bmatrix}$$

A scalar field transforms each point $(x, y, ...) \in \mathbb{R}^n$ into a real number in $\mathbb{R}$ (e.g. $f(x, y, ...) \to \mathbb{R}$). A gradient field is a field of n-dimensional vectors that is made by taking the gradient of f at each point $(x, y, z, ...)$ in a given region $\nabla f(x, y, z, ...)$.

These differ from regular vector fields because **not every vector field can be made from the gradient of a scalar field**. Think about that, it's really cool!

4.2 Line Integrals

From Stewart 16.2: It's like a regular integral, but we're integrating over a curve, not a range of values.

Definition: Let f be defined along smooth curve C. Then the line integral of f along C is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

Visually it's just like taking the single integral along the path $< x(t), y(t) >$ as opposed to a straight line along the x or y axis. More generally:

$$\int_C f(\vec{r}(t))ds = \int_C f(\vec{r}(t))|r'(t)|dt$$

4.2.1 Line Integrals with respect to x and y

You can also take the integral with respect to differences in x and y instead of with respect to s (the arc length).

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

Physically, that means that the 'significance' of an infinitessimal part of the line is decided by the x or y-component of the piece of the line.

A shorthand for adding these together is as follows:

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dxQ(x,y)dy$$

4.2.2 Path Independence

It is not necessarily the case that

$$\int_C f(x,y)dx = -\int_{-C} f(x,y)dx$$

However, when we take the line integral with respect to s (path length), equality does hold!

$$\int_C f(x,y)ds = -\int_{-C} f(x,y)ds$$

4.2.3 Line Integrals in Space

This section pertains to 3D space. Let C be a smooth space curve given by $< x(t), y(t), z(t) >$ for $a \leq t \leq b$

Now let f be a function of three variables. The line integral of f along C is then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$$

Which is equivalent to:

$$\int_a^b f(r(t))|r'(t)|dt$$

where $r(t)$ is the vector function $r(t) =< x(t), y(t), z(t) >$

4.2.4 Line Integrals of Vector Fields

Recall that the work done by force $f(x)$ on a particle from $a \to b$ along the x-axis is:

$$W = \int_a^b f(x)dx$$

Also recall that $W = F \cdot D$

Now we think of $F(x,y,z) =< p(x), q(y), r(z) >$ as a force field in 3-space. How do we compute the work done by the force field on a moving particle?

$$W = \int_a^b F(r(t)) \cdot r'(t)dt = \int_C F \cdot T ds$$

Remembering that $F(r(t)) = F(x(t), y(t), z(t)) = [P, Q, R]$, so

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$$

4.2.5 Useful Tips

The vector function that connects points r_0, r_1 is:

$$r(t) = (1-t)r_0 + tr_1 \;\; 0 \leq t \leq 1$$

4.3 Fundamental Theorem of Line Integrals

From 16.3 of Stewart.

Recall: the fundamental theorem of calculus states $\int_a^b F'(x)dx = F(b) - F(a)$ (a.k.a. the net change theorem).

Now let us generalize this to line integrals!

Fundamental Theorem of Line Integrals Let C be a smooth curve given by $r(t)$ and f is a differentiable function of 2-3 variables with a continuous gradient on C.

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

4.3.1 Path Independence

Here are some fun theorems to understand and have in the back of your mind:

1. **THM:** $\int_C \vec{F} \cdot d\vec{r}$ is path independent iff $\int_C \vec{F} \cdot d\vec{r} = 0 \, \forall$ closed paths
2. **THM:** A vector field is conservative iff it is path independent. $\exists$ a function f such that $\nabla f = F$ iff conservative/path independent.
3. **THM:** if $\vec{F}(x,y) = [P(x,y)\hat{i}, Q(x,y)\hat{j}]$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

assuming P, Q have cts first derivatives and the region is *simply connected*.

Some useful definitions:

1. A **simple curve** doesn't intersect itself between the ends.
2. A **simply connected region** has every simple, closed curve enclose only points in that region D.

4.4 Green's Theorem

Provides a relationship between the line integral of a **simple closed curve** C and the **double integral of region D that is bounded by C.**

Note on orientation: We assume a **SINGLE, COUNTERCLOCKWISE** traversal of C. The region is 'to the left' of $r(t)$ as it traverses C.

$$\int_C P\,dx + Q\,dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$$

It's kind of the counterpart to $\int_a^b F'(x)dx = F(b) - F(a)$.

4.5 Curl and Divergence

Curl is a measure of the rotation about a point in a vector field.

$$curlF = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Divergence is a measure of the amount of *stuff* going in (or out of) a point.

$$divF = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

4.5.1 Properties of Divergence and Curl

1. If F has continuous second order derivatives, then $divcurlF = 0$.

4.5.2 Vector form of Green's Theorem

$$\int_C F \cdot dr = \iint_D (curl\,F) \cdot kdA$$

In other words, you can infer the *curliness* of a surface in a vector field based on a line interal of the edge.

4.6 Divergence Theorem

E is simple solid region with boundary surface S with outward orientation. F is a vector field in that area with continuous partial derivatives. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\,\vec{F}dV$$

4.7 Stoke's Theorem

If S is an oriented piecewise-smooth surface with piecewise smooth closed boundary curve C inside of vector field F, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S curl\, F \cdot d\vec{S}$$