MAT292 Abridged

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0.1 Introduction

The textbook and lectures for this course offer a great comprehensive guide for the methods of solving ODE's. The goal here is to give a very concise overview of the things you need to know (NTK) to answer exam questions. Unlike some of our other courses, you don't need to be very intimately familiar with the derivations of everything in order to solve the problems (though it certainly doesn't hurt). Think of this as a really good cheat sheet.

Chapter 1

Qualitative Things and Definitions

1.1 Definitions

1. **Differential Equation:** Any equation that contains a differential of dependent variable(s) with respect to any independent variable(s)

2. **Order:** The order of the highest derivative present.

3. **Autonomous:** When the definition of the $\frac{dy}{dt}$ doesn't contain t

4. **ODE and PDE:** Ordinary derivatives or partial derivatives.

5. **Linear Differential Equations:** nth order Linear ODE is of the form:
$$\sum a_i(t)y^{(i)} = 0$$

6. **Homogenous:** if the 0th element of the above sum has $a_0(t) = 0$ for all t.

1.2 Qualitative Analytic Methods to Know

1. Phase lines

2. Slope fields

1.3 Types of Equilibrium

1. Asymptotic stable equilibrium

2. Unstable equilibrium

3. Semistable equilibirium

Chapter 2

1st Order ODE's

2.1 Separable 1st Order ODE's

If you can write the ODE as:

$$\frac{dy}{dx} = p(x)q(y)$$

Then you can put $p(x)$ with dx on one side and $q(y)$ with dy on the other and integrate them both so solve the ODE.

2.2 Method of Integrating Factors

This is used to solve ODE's that can be put into the form

$$\frac{dy}{dt} + p(t) * y = g(t)$$

The chain rule can be written as: $\int (f'(x)g(x) + f(x)g'(x))dx = f(x)g(x)$

We can use an **integrating factor** equivalent to $e^{\int p(t)dt}$ to multiply both sides and arrive at a form that can be integrated with ease using the reverse chain rule.

2.3 Exact Equations

If the equation is of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

and

$$M_y(x, y) = N_x(x, y)$$

then $\exists$ a function f satistfying

$$f_x(x, y) = M(x, y); f_y(x, y) = N(x, y)$$

The solution: $f(x, y) = C$ where C is an arbitrary constant.

2.4 Modeling with First Order Equations

These are some vague tips on how to solve these types of problems from textbook section 2.3

- To **create** the equation, state physical principles
- To **solve**, solve the equation and/or find out as much as you can about the nature of the solution.
- Try **comparing** the solution/equation to the physical phenomenon to 'check' your work.

2.5 Non-Linear vs. Linear DE's

Theorem on Uniqueness of 1st Order Solutions

$$y' + p(t)y = g(t)$$

There exists a unique solution $y = \Phi(t)$ for each starting point (y_0, t_0) if p, g are continuous on the given interval.

2.6 Population Dynamcis with Autonomous Equations

Autonomous: $\frac{dy}{dt} = f(y)$

2.6.1 Simple Exponential

$\frac{dy}{dt} = ry$ Problem: doesn't take into account the upper bound for population/sustainability.

2.6.2 Logistic equation

$$\frac{dy}{dt} = (r - ay)y$$

Equivalent form:

$$\frac{dy}{dt} = r(1 - \frac{y}{k})y$$

f is the *intrinsic growth rate*.

Chapter 3

Systems of Two 1st Order DE's

3.1 Set Up

Your first goal is to get the system in the form

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{K}\boldsymbol{u} + \boldsymbol{b}$$

Where $\boldsymbol{K}$ is a 2 by 2 matrix, $\boldsymbol{u}$ is your vector of values you want to predict, and $\boldsymbol{b}$ is a 2-long vector of constants.

More generally, the equation is of the type

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(t)x + \boldsymbol{g}(t)$$

Called a **first order linear system of two dimensions**. If $\boldsymbol{g}(t) = \boldsymbol{0}\forall t$ then it is called **homogenous**, else **non-homogenous**. We let x be composed of values

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

3.2 Existence and Uniqueness of Solutions

Theorem: $\exists$ unique solution to

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(t)x + \boldsymbol{g}(t)$$

so long as the functions $\boldsymbol{P}(t)$ exist and are continuous on the interval I in equestion.

3.2.1 Linear Autonomous Systems

If the right side doesn't depend on t, it's autonomous. In this case, the autonomous version looks (familiarly) like:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}x + \boldsymbol{b}$$

Equilibrium points arise when $\boldsymbol{A}x = -\boldsymbol{b}$

3.3 Solving

3.3.1 General Solution

We start with $y' = \boldsymbol{A}y + b$

- Find eigen values λ s.t. $det(A - I\lambda) = 0$
- Find eigen vectors v s.t. $(A - I\lambda)v = 0$
- Enter and simplify $y(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$

Converting to homogenous equation: Let y_{eq} be the equilibrium value of y that can be found when $y' = 0 = Ay + b$.

$$y_{eq} + \bar{y} = y$$

and y_{eq} is the solution to $\bar{y}' = A\bar{y}$.

3.3.2 Special Case 1: Repeated Eigen Value

Start in the same fashion as above. You will easily be able to find the eigen value and at least one eigen vector. Then, the path diverges:

Case 1: Another can easily be found - Now you find your v_2 and proceed.

Case 2: Another cannot easily be found - You must use the following formula to find your second vector if this is the case:

$$(A - I\lambda)v_2 = v_1$$

This is known as the "general" eigen vector.

Final Form

Your final form for this case is going to be rather different than the others:

$$x = C_1 e^{\lambda_1 t} t v_1 + C_2 e^{\lambda_2 t} v_2$$

3.3.3 Special Case 2: Two Complex Eigen Values

Chapter 4

Numerical Methods

$$\frac{dy}{dt} = f(t, y)$$

4.1 Euler's Method

We start with a first order ODE. Let us define a fixed step Δt.

$$y_{n+1} = y_n + \Delta t(f(t_n, y_n))$$

Error $= |y(t_n) - y_n| \approx \Delta t$

4.1.1 Basic Idea: Integrate The ODE

$$\int_{t_n}^{t_n+1} \frac{dy}{dt} dt = \int_{t_n}^{t_n+1} f(t, y(t))dt$$

Euler's method makes the following approximation.

$$\int_{t_n}^{t_n+1} f(t, y(t))dt \approx \Delta t f(t_n, y_n)$$

But we can do better.

Mean Value Theorem for Integrals

If y is continuous on $[a, b]$ then $\exists c \in (a, b)$ so that

$$\frac{1}{b-a} \int_a^b g(t)dt = g(c)$$

Euler's method would just assume that $g(c)$ is at the far left hand side of the Riemann sum, so we can improve upon this! If we can guess c more accurately, our final answer will be a lot better.

Since c is more likely to be inside the interval $[t_n, t_{n+1}$, we could try the following estimations to improve upon Euler's method. We will now try **sampling**.

4.2 Improved Euler Method

Let $g(t) = f(t, y(t))$. We literally use that riemann sum trapezoidal rule for this approximation.

$$y_{n+1} - y_n \approx \frac{\Delta t}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$

Where we make the approximation for y_{n+1} as

$$y_{n+1} \approx y_n + \Delta t f(t_{n+1}, y_n)$$

Steps:

- Evaluate $K_1 = f(t_n, y_n)$
- Predict $u_{n+1} = y_n + \Delta t K_1$
- Evaluate $K_2 = f(t_{n+1}, u_{n+1})$
- Update $u_{n+1} = u_n + \Delta t \frac{k_1+k_2}{2}$

This method is consider **second order**, so

$$|y(t_n) - y_n| \approx C(\Delta t)^2$$

(global error).

The expense of a numerical method is roughly the **number of function calls to** $f()$. Therefore, improved Euler's method comes at the cost of one more function evaluation of $f()$.

4.3 Runge Kutta Method

Modern workhorse of solving ODE's. It's 4th order, so requires 4 function f calls.

Steps

- $k_1 = f(t_n, y_n)$
- $u_n = y_n + \frac{\Delta t}{2} k_1$ (half step)
- $k_2 = f(t_n + \frac{\Delta t}{2}, u_n$
- $v_n = y_n + \frac{\Delta t}{2} k_2$
- $k_3 = f(t_n + \frac{\Delta t}{2}, v_n$
- $w_n = y_n + \Delta t k_3$
- $k_{-1} = f(t_{n+1}, w_n$
- $y_{n+1} = y_n + \Delta t(\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$

4.4 Above and Beyond 4th Order

If we can just increase our accuracy by adding more functional evaluations, then why can't we just keep on adding function evaluations and increasing the order?

Order	1	2	3	4	5	6	7	8	9	10
Min Function Evaluations	1	2	3	4	**6**	7	**9**	11	14	?

Answer: Past a 4 evaluations, it's not really worth while.