

AER210 Abridged

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Chapter 1

Review: Stuff to Have Memorized

1.1 Trig Functions and Derivatives

$$\frac{d}{dx} \sin(x) = \cos(x) \quad \frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x) \quad \frac{d}{dx} \sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \quad \frac{d}{dx} \cot(x) = -\csc^2(x)$$

1.2 Inverse Trig Derivatives

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

1.3 How to complete the Square

1. Put $ax^2 + bx$ in brackets and forcefully factor out the a
2. Add $(\frac{b}{2})^2$ to the inside of the brackets and subtract it from the outside (you got it)
3. Factor and be happy that you've completed the square;

1.4 Trig Angle Sums

1. $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

2. $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

3. $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

4. $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

1.5 Identities Assorted

$$\sin^2(x) = 1/2(1 - \cos(2x)) \quad (1.1)$$

$$\cos^2(x) = 1/2(1 + \cos(2x)) \quad (1.2)$$

$$\sin x \cos x = 1/2 \sin 2x \quad (1.3)$$

$$\sin A \cos B = 1/2[\sin(A - B) + \sin(A + B)] \quad (1.4)$$

$$\sin A \sin B = 1/2[\cos(A - B) - \cos(A + B)] \quad (1.5)$$

$$\cos A \cos B = 1/2[\cos(A - B) + \cos(A + B)] \quad (1.6)$$

$$\frac{d}{dx} \csc(x) = -\csc(x) \cot(x) \quad (1.7)$$

$$\cot^2(x) = \csc^2(x) - 1 \quad (1.8)$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x) \quad (1.9)$$

Chapter 2

In-Class Review

2.1 Vectors & Vector Functions

- Vector = magnitude + direction
- If the origin of the vector is the origin of the coordinate system, it's a position vector.
- Dot product: $\vec{a} \cdot \vec{b} = \vec{a}_1 \cdot \vec{b}_1 + \dots + \vec{a}_n \cdot \vec{b}_n$
- Cross product: $\vec{a} \times \vec{b} = \det(i, j, k; \vec{a}^T; \vec{b}^T)$
- Cross product is the area of the parallelogram traced out by the two vectors.
- Scalar triple product: $\vec{a} \cdot (\vec{b} \times \vec{c})$, produces a scalar, represents the volume of the parallelepiped formed by the three vectors.
- To get the derivative of a vector function, simply take the derivative of each of the internal functions and package them into a new vector function.

2.2 Arc Length

2.2.1 One-Variable Functions

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

2.2.2 Parametric Functions

Let $y(t)$ and $x(t)$ describe a parametric function in 2 dimensions. Then the arc length would be:

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

2.2.3 Vector Functions

Let $\vec{r}(t)$ describe a vector function that converts a scalar t into a vector. Then the arclength function would be:

$$L = \int_a^b |\vec{r}'(t)| dt$$

2.2.4 Reparameterizing with respect to Arc Length

What is this? Let there be a vector function $\vec{r}(t)$ and its corresponding arc length function $s(t)$. Since s is strictly increasing, we can safely **reparameterize** $\vec{r}(t)$ to be $\vec{r}(s(t)) \rightarrow \vec{r}(s)$.

Why would you want to do this? This type of reparameterization is useful because now we do not have to rely on any particular coordinate system.

Steps to Reparameterizing

1. Find $s(t) = \int_a^t |\vec{r}'(u)| du$.
2. Put s in terms of t .
3. Substitute the expression found in part 2 in the original $\vec{r}(t)$.

2.3 Partial Derivatives

2.3.1 Functions of Several Variables

A function of two variables transforms each pair of Reals (x, y) in a given set to a single real number. The given set is the domain, and the set of reals that the pair is transformed to is the range.

Level functions are functions that have $f(x, y) = k$ for given ranges of (x, y)

Functions of 3 or more variables are pretty easy to extrapolate from functions of two variables, tbh.

2.3.2 Limits and Continuity with Functions of Several Variables

Limits

Definition of limit with many variables:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ s.t. if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

How to find: Regard the non-mentioned variable in the notation as a constant and differentiate with respect to the mentioned variable.

2.3.3 Higher Partial Derivatives

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial z}{\partial y \partial x}$$

Clairaut's Theorem

If f_{xy} and f_{yx} are both **defined** and **continuous** on disk D then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

2.4 Gradient

Think of the gradient like an operator that applies to functions of many variables (functions of vectors). The ∇ just calculates the partial derivative of the function with respect to each of its input variables and puts it into a vector.

$$\nabla f(x, y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

Or, more generally for a function $f(\vec{x})$,

$$\nabla f(\vec{x}) = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$$

2.5 Chain Rule with Many Variables

Let there be a function $f(\vec{x})$. Let \vec{x} of length n be a function of \vec{t} of length m . We take the partial derivative of f with respect to t_i by the following:

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

Chapter 3

Multiple Integrals

Pretty much the same as regular integrals, you just do two. You can apply them to volumes under surfaces.

3.1 Basic Meaning and Solving

3.2 Leibniz Integral Rule (Differentiability of Integral with Respect to Parameter)

The Leibniz integral rule simply lets you more easily take the derivative of the integral of a multivariable function where the variable you are integrating with respect to is not the same as the variable you are taking the derivative with respect to.

$$\frac{d}{dx} \int_a^b f(x, t) dt$$

3.2.1 Constant Bounds of Integration

When you are integrating from one constant to another $[a, b]$, the result is quite simple and elegant.

$$\frac{d}{dx} \int_a^b f(x, t) dt = \int_a^b \frac{\partial f}{\partial x} dt$$

3.2.2 Derivation

Let's write $\frac{d}{dx} \int_a^b f(x, t) dt$ in terms of the definition of the derivative:

$$\begin{aligned} &= \frac{\int_a^b f(x + \Delta x, t) dt - \int_a^b f(x, t) dt}{\Delta x} \\ &= \frac{\int_a^b f(x, t) dx + \int_a^b \frac{\partial f}{\partial x} \Delta x dt - \int_a^b f(x, t) dt}{\Delta x} \\ &= \int_a^b \frac{\partial f}{\partial x} dx \end{aligned}$$

3.2.3 Variable Bounds of Integration

Final result:

$$\frac{d}{dx} \int_{a(t)}^{b(t)} f(x, t) dt = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial x} dt - f(a, t) \frac{da}{dt} + f(b, t) \frac{db}{dt}$$

3.3 Polar Coordinates with Multiple Integrals

Recall $r^2 = x^2 + y^2$, $x = r \cos(\theta)$, $y = r \sin(\theta)$

A polar rectangle is of the form

$$R = (r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Basic form of double integral in polar coordinates:

$$\iint_R g dA = \int_{\alpha}^{\beta} \int_a^b g(r, \theta) dr d\theta$$

3.3.1 Change to Polar Coordinates in Double Integrals

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos(\theta), r \sin(\theta)) * r dr d\theta$$

Make sure not to forget the r in the integral!

3.3.2 Variable Bounds of Integration for r

$$D = (r, \theta) | \alpha \leq \theta \leq \beta, h(\theta) \leq r \leq g(\theta)$$

Then:

$$\iint_D f(r, \theta) = \int_{\alpha}^{\beta} \int_{h(\theta)}^{g(\theta)} f(r, \theta) * r \, dr \, d\theta$$

3.4 Applications of Multiple Integrals

The obvious application of multiple integrals is to compute volume. But, like with regular integrals, there are a lot more ways you can apply them (think back to centre of mass, arc length, etc).

3.4.1 Density and Mass

Lamina is a plan of mass and surface density. With double integrals, we can now calculate mass of a lamina with **variable** density!

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$
$$\rho(x, y) = \frac{\partial m}{\partial x \partial y}$$

Calculating Mass:

$$m = \iint_D \rho(x, y) dA$$

3.4.2 Moments of Intertia

Moment M_x is the distance of the centre of mass from the x -axis times the mass. Likewise with y .

Centre of Mass is at (\bar{x}, \bar{y}) where $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

If you're confused as to why the \bar{x} and M_y are related, just remember that the moment about the y -axis (M_y) is defined as the **distance from the y -axis to the centre of mass** times the mass of the lamina. That distance is the \bar{x} value!

3.4.3 Moments of Intertia

Recall that $I = mr^2$ for particles.

$$I_x = \iint_D y^2 \rho(x, y) dA$$

$$I_y = \iint_D x^2 \rho(x, y) dA$$

$$I_o = I_x + I_y = \iint_D (x^2 + y^2) \rho(x, y) dA$$

Makes sense because $x^2 + y^2$ is the distance of the chunk from the origin.

Radius of Gyration

About Arbitrary Axis: R is the radius of gyration if $mR^2 = I$. Basically, if mass were concentrated at R , then it would have the same moment of inertia as the original lamina.

About the Origin: (\bar{x}, \bar{y}) is the point of gyration about the origin.

3.4.4 Probability

The probability function of a random variable x is:

$$f(x) > 0 \forall x, \int_{-\infty}^{\infty} f(x) dx = 1$$

Probability of $a < x < b$: $\int_a^b f(x) dx$

2-D Probability Function: Chance of $(x, y) \in D$ is $\iint_D f(x, y) dA$.

Note that $\iint_{\mathbb{R}^2} f(x, y) dA = 1$

Expected Values

For single variable x with probability density function f , the expected value is the average value of x across many trials.

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

Makes sense because you're just summing the products of the chance of a particular x and the value of that x for all x .

Multi-Variable Functions:

$$\mu_1 = X_{mean} = \iint_{\mathbb{R}^2} x f(x, y) dA$$

$$\mu_2 = Y_{mean} = \iint_{\mathbb{R}^2} y f(x, y) dA$$

Normal Distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

3.5 Surface Area

How it's derived: If you divide the surface into an m by n grid, the surface area of each piece is the area of that parallelogram. That can be found by the cross product of the two vectors that define two sides of the parallelogram. Those vectors are just the width and length of the piece times the partial derivatives along that particular axis.

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} dA$$

$$A(S) = \iint_D \sqrt{(\nabla f(x, y))^2 + 1} dA$$

The +1 term is there because it's $\frac{\partial f}{\partial f}$.

3.6 Triple Integrals

3.6.1 Type 1

Lies between continuous functions of $f(x, y)$

$$E = (x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)$$

Then

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA$$

3.6.2 Type 2

Lies between continuous functions of $f(y, z)$

$$E = (x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)$$

Then

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right] dA$$

3.6.3 Type 3

Lies between continuous functions of $f(x, z)$

$$E = (x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)$$

Then

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right] dA$$

3.6.4 Applications of Triple Integrals

Obvious

$$V = \iiint_E dV$$
$$m = \iiint_E \rho(x, y, z) dV$$

Moments and CoM

REMEMBER: Moments are about a plane (e.g. (x, y) plane) while moments of INERTIA are about an axis (think about spinning vs. balancing - you balance along a plane, spin around an axis).

$$M_{yz} = \iiint_E x \rho(x, y, z) dV$$

etc.

$$I_x = \iiint_E (y^2 + z^2) \rho(x, y, z) dV$$

Chapter 4

Vector Calculus

4.1 Review: Vector Fields, Gradient Fields

4.2 Line Integrals

From Stewart 16.2: It's like a regular integral, but we're integrating over a curve, not a range of values.

Definition: Let f be defined along smooth curve C . Then the line integral of f along C is

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Visually it's just like taking the single integral along the path $\langle x(t), y(t) \rangle$ as opposed to a straight line along the x or y axis.

4.2.1 Line Integrals with respect to x and y

You can also take the integral with respect to differences in x and y instead of with respect to s (the arc length).

$$\begin{aligned}\int_C f(x, y) dx &= \int_a^b f(x(t), y(t)) x'(t) dt \\ \int_C f(x, y) dy &= \int_a^b f(x(t), y(t)) y'(t) dt\end{aligned}$$

A shorthand for adding these together is as follows:

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$$

4.2.2 Line Integrals in Space

This section pertains to 3D space. Let C be a smooth space curve given by $\langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$

Now let f be a function of three variables. The line integral of f along C is then

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Which is equivalent to:

$$\int_a^b f(r(t)) |r'(t)| dt$$

where $r(t)$ is the vector function $r(t) = \langle x(t), y(t), z(t) \rangle$

4.2.3 Line Integrals of Vector Fields

Recall that the work done by force $f(x)$ on a particle from $a \rightarrow b$ along the x -axis is:

$$W = \int_a^b f(x) dx$$

Also recall that $W = F \cdot D$

Now we think of $F(x, y, z) = \langle p(x), q(y), r(z) \rangle$ as a force field in 3-space. How do we compute the work done by the force field on a moving particle?

$$W = \int_a^b F(r(t)) \cdot r'(t) dt = \int_C F \cdot T ds$$

Remembering that $F(r(t)) = F(x(t), y(t), z(t))$, so

$$\int_C F \cdot dr = \int_C P dx + Q dy + R dz$$

4.2.4 Useful Tips

The vector function that connects points r_0, r_1 is:

$$r(t) = (1 - t)r_0 + tr_1 \quad 0 \leq t \leq 1$$

4.3 Fundamental Theorem of Line Integrals

From 16.3 of Stewart.

Recall: the fundamental theorem of calculus states $\int_a^b F'(x)dx = F(b) - F(a)$ (a.k.a. the net change theorem).

Now let us generalize this to line integrals!

Fundamental Theorem of Line Integrals Let C be a smooth curve given by $r(t)$ and f is a differentiable function of 2-3 variables with a continuous gradient on C .

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$