AER210 Abridged

Aman Bhargava

September 2019

Contents

Chapter 1

Review: Stuff to Have Memorized

1.1 Trig Functions and Derivatives

$\frac{d}{dx}sin(x) = cos(x)$ $\frac{d}{dx}csc(x) = -csc(x)cot(x)$

$\frac{d}{dx}cos(x) = -sin(x)$ $\frac{d}{dx}sec(x) = sec(x)tan(x)$

$\frac{d}{dx}tan(x) = sec^2(x)$ $\frac{d}{dx}cot(x) = -csc^2(x)$

1.2 Inverse Trig Derivatives

$\frac{d}{dx}sin^{-1}(x) = \frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}cos^{-1}(x) = \frac{-1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}tan^{-1}(x) = \frac{1}{1+x^2}$

1.3 How to complete the Square

1. Put $ax^2 + bx$ in brackets and forcefully factor out the a
2. Add $(\frac{b}{2})^2$ to the inside of the brackets and subtract it from the outside (you got it)
3. Factor and be happy that you've completed the square;

1.4 Trig Angle Sums

1. $sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$
2. $cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$
3. $sin(A-B) = sin(A)cos(B) - cos(A)sin(B)$
4. $cos(A-B) = cos(A)cos(B) + sin(A)sin(B)$

1.5 Identities Assorted

$$sin^2(x) = 1/2(1 - cos(2x)) \tag{1.1}$$
$$cos^2(x) = 1/2(1 + cos(2x)) \tag{1.2}$$
$$sinxcosx = 1/2sin2x \tag{1.3}$$

$$sinAcosB = 1/2[sin(A-B) + sin(A+B)] \tag{1.4}$$
$$sinAsinB = 1/2[cos(A-B) - cos(A+B)] \tag{1.5}$$
$$cosAcosB = 1/2[cos(A-B) + cos(A+B)] \tag{1.6}$$

$$\frac{d}{dx}csc(x) = -csc(x)cot(x) \tag{1.7}$$
$$cot^2(x) = csc^2(x) - 1 \tag{1.8}$$
$$\frac{d}{dx}cot(x) = -csc^2(x) \tag{1.9}$$

Chapter 2

In-Class Review

2.1 Vectors & Vector Functions

- Vector = magnitude + direction
- If the origin of the vector is the origin of the coordinate system, it's a position vector.
- Dot product: $\vec{a} \cdot \vec{b} = \vec{a}_1 \cdot \vec{b}_1 + ... + \vec{a}_n \cdot \vec{b}_n$
- Cross product: $\vec{a} \times \vec{b} = det(i, j, k; \vec{a}^T; \vec{b}^T)$
- Cross product is the area of the paralellogram traced out by the two vectors.
- Scalar triple product: $\vec{a} \cdot (\vec{b} \times \vec{c})$, produces a scalar, represents the volume of the parallelepiped formed by the three vectors.
- To get the derivative of a vector function, simply take the derivative of each of the internal functions and package them into a new vector function.

2.2 Arc Length

2.2.1 One-Variable Functions

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

2.2.2 Parametric Functions

Let $y(t)$ and $x(t)$ describe a parametric function in 2 dimensions. Then the arc length would be:

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dx$$

2.2.3 Vector Funtions

Let $\vec{r}(t)$ describe a vector function that converts a scalar t into a vector. Then the arclength function would be:

$$L = \int_a^b |\vec{r}(t)| dt$$

2.2.4 Reparamerizing with respect to Arc Length

What is this? Let there be a vector function $\vec{r}(t)$ and its correponding arc length function $s(t)$. Since s is strictly increasing, we can safely **reparameterize** $\vec{r}(t)$ to be $\vec{r}(s(t)) \rightarrow \vec{r}(s)$.

Why would you want to do this? This type of reparameterization is useful because now we do not have to rely on any particular coordinate system.

Steps to Reparameterizing

1. Find $s(t) = \int_a^b |\vec{r}(u)| du$.
2. Put s in terms of t.
3. Substitute the expression found in part 2 in the original $\vec{r}(t)$.

2.3 Partial Derivatives

2.3.1 Functions of Several Variables

A function of two variables transforms each pair of Reals (x, y) in a given set to a single real number. The given set is the domain, and the set of reals that the pair is transformed to is the range.

Level functions are functions that have $f(x,y) = k$ for given ranges of (x,y)

Functions of 3 or more variables are pretty easy to extrapolate from functions of two variables, tbh.

2.3.2 Limits and Continuity with Functions of Several Variables

Limits

Definition of limit with many variables:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ s.t. if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$

How to find: Regard the non-mentioned variable in the notation as a constant and differentiate with respect to the mentioned variable.

2.3.3 Higher Partial Derivatives

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial z}{\partial y \partial x}$$

Clairaut's Theorem

If f_{xy} and f_{yx} are both **defined** and **continuous** on disk D then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

2.4 Gradient

Think of the gradient like an operator that applies to functions of many variables (functions of vectors). The ∇ just calculates the partial derivatative of the function with respect to each of its input variables and puts it into a vector.

$$\nabla f(x,y) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$$

Or, more generally for a function $f(\vec{x})$,

$$\nabla f(\vec{x}) = [\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}]$$

2.5 Chain Rule with Many Variables

Let there be a function $f(\vec{x})$. Let $\vec{x}$ of length n be a function of $\vec{t}$ of length m. We take the partial derivative of f with respect to t_i by the following:

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + ... + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Chapter 3

Multiple Integrals

Pretty much the same as regular integrals, you just do two. You can apply them to volumes under surfaces.

3.1 Basic Meaning and Solving

3.2 Leibniz Integral Rule (Differentiability of Integral with Respect to Parameter)

The Leibniz integral rule simply lets you more easily take the derivative of the integral of a multivariable function where the variable you are integrating with respect to is not the same as the variable you are taking the derivative with respect to.

$$\frac{d}{dx}\int_a^b f(x,t)dt$$

3.2.1 Constant Bounds of Integration

When you are integrating from one constant to another $[a, b]$, the result is quite simple and elegant.

$$\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial f}{\partial x}dt$$

3.2.2 Derivation

Let's write $\frac{d}{dx}\int_a^b f(x,t)dt$ in terms of the definition of the derivative:

$$
\begin{aligned}
&= \frac{\int_a^b f(x+\Delta x,t)dt - \int_a^b f(x+\Delta x,t)dt}{\Delta x} \\
&= \frac{\int_a^b f(x,t)dx + \int_a^b \frac{\partial f}{\partial x}dt\Delta t dx - \int_a^b f(x,t)dt}{\Delta x} \\
&= \int_a^b \frac{\partial f}{\partial t}dx
\end{aligned}
$$

3.2.3 Variable Bounds of Integration

Final result:

$$\frac{d}{dx}\int_{a(t)}^{b^t} f(x,t)dt = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial x}dt - f(a,t)\frac{da}{dt} + f(b,t)\frac{db}{dt}$$

3.3 Polar Coordinates with Multiple Integrals

Recall $r^2 = x^2 + y^2$, $x = rcos(\theta)$, $y = rsin(\theta)$

A polar rectangle is of the form

$$R = (r,\theta)|a \leq r \leq b, \alpha \leq \theta \leq \beta$$

Basic form of double integral in polar coordinates:

$$\iint_R gdA = \int_\alpha^\beta \int_a^b g(r,\theta)\, dr\, d\theta$$

3.3.1 Change to Polar Coordinates in Double Integrals

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(rcos(\theta), rsin(\theta)) * r\, dr\, d\theta$$

Make sure not to forget the r in the integral!

3.3.2 Variable Bounds of Integration for r

$$D = (r, \theta) | \alpha \leq \theta \leq \beta, h(\theta) \leq r \leq g(\theta)$$

Then:

$$\iint_D f(r, \theta) = \int_\alpha^\beta \int_{h(\theta)}^{h(\theta)} f(r, \theta) * r \, dr \, d\theta$$

3.4 Applications of Multiple Integrals