Notes on bootstrapping the $O(N)$ vector models with a boundary

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Abstract: Abstract...

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1 Introduction

The nature of the boundary universality classes in the $O(N)$ models in $d = 3$ dimensions is an interesting problem which has not yet been settled in the literature of statistical mechanics. Recently, using innovative methods in the renormalization group, [**?**] was able to show that the models could afford a slew of interesting features as a function of N. Key results from [**?**] were that the special fixed point exists for $N = 2$ and also for $N \to 2^+$ but it separates the ordinary transition from an "extra-ordinary log" transition. Upon increasing N, eventually the special FP moves and fuses into the ordinary FP at some critical value N_c (not necessarily an integer). If $N_c > 3$, these phases could be realized in experiments. In the approach of [**?**], N_c depends on certain universal constants related to the normal fixed point (which exists for all N in $d = 3$). These constants are accessible by conformal bootstrap, and in these notes we outline our efforts in calculating them from known methods of the boundary bootstrap program.

1.1 Conformal bootstrap in the presence of a boundary

The idea of conformal bootstrap is quite old, dating back to Polyakov [**?**] but it has gained traction since 2010 with the advent of powerful methods which work generally and have also been matched by the computational power required to solve the bootstrap equations. The boundary bootstrap was described in the same spirit as this revival in [**?**] using some mathematical results from [**?**]. In this section, we recall important results from these works, especially about the boundary conformal blocks.

Consider a d-dimensional CFT with a codimension-1 boundary (say, at $x^d = 0$). In this case, the full conformal symmetry of the theory is broken down to a subgroup that leaves $x^d = 0$ invariant. Under this subgroup, one can construct the invariant cross ration ξ,

$$\xi = \frac{(\mathbf{x} - \mathbf{y})^2}{4x^d y^d} \tag{1.1}$$

The existence of an invariant quantity composed of two points $x = (\mathbf{x}, x^d)$ and $y = (\mathbf{y}, y^d)$ implies the existence of an arbitrary function of ξ in the two-point functions. Conformal blocks are decompositions of n-point functions in terms of the operator product expansion (OPE) of the CFT. For bulk CFTs, non-trivial conformal blocks occur only for 4-point functions. However, in the presence of a boundary, due to the coupling between the surface and the bulk operators, even 2-point functions could have non-trivial conformal block expansions.

Consider the (bulk) OPE involving two identical scalars

$$\mathcal{O} \times \mathcal{O} \sim \sum_k \mathcal{O}_k \tag{1.2}$$

We will assume that the one-point functions of operators are, in general, not zero. Instead, they are given by the identity contribution of the boundary OPE. This is because, the operators that live on the boundary, $\hat{\mathcal{O}}_l$ form a valid basis for representing the 2-point functions. Thus, we have

$$\mathcal{O}_k \sim \hat{\mathbb{1}} + \sum_l \hat{\mathcal{O}}_l \tag{1.3}$$

where $\hat{\mathcal{O}}_l$ belong to the $SO(d,1)$ CFT that exists at the boundary $x^d = 0$. As the one-point function of the identity in any CFT is non-zero,

$$\langle \mathcal{O}_k(x) \rangle = \frac{a_k}{(2x^d)^{\Delta_k}} \tag{1.4}$$

where a_k is an OPE coefficient between the bulk operator and the boundary identity.

Using these OPEs, we can find the behavior of 2-point functions of bulk operators in different limits. Consider the bulk limit, $\xi \to 0$ or $\mathbf{x} - \mathbf{y} \ll x^d, y^d$,

$$\begin{aligned} \langle \mathcal{O}(x)\mathcal{O}(y) \rangle &= \sum_k \frac{\lambda_k \left\langle \mathcal{O}_k\left(\frac{x+y}{2}\right)\right\rangle}{(\mathbf{x}-\mathbf{y})^{2(\Delta_{\mathcal{O}}-\Delta_k/2)}} \tilde{f}_{\text{bulk}}(\Delta_k, x, y) & (1.5) \\ &= \sum_k \frac{a_k \lambda_k}{(\mathbf{x}-\mathbf{y})^{2\Delta_{\mathcal{O}}}} \xi^{\Delta_k/2} \tilde{f}_{\text{bulk}}(\Delta_k, x, y) \equiv \sum_k a_k \lambda_k f_{\text{bulk}}(\Delta_k, \xi) & (1.6) \end{aligned}$$

where λ_k are bulk OPE coefficients. In the boundary limit ($\xi \to \infty$), each operator is expanded in the basis of boundary operators *first*, and then two-point functions are calculated in the boundary CFT. Here, we must use a different conformal block f_{bdy}:

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \sum_k \frac{\mu^2_{\mathcal{O}k} \langle \hat{\mathcal{O}}_k(\mathbf{x}) \hat{\mathcal{O}}_k(\mathbf{y}) \rangle}{(4x^d y^d)^{(\Delta_{\mathcal{O}} - \Delta_{\hat{k}})}} \tilde{f}_{\text{bdy}}(\Delta_{\hat{k}}, x, y) \tag{1.7}$$

where $\mu_{\mathcal{O}k}$ is an OPE coefficient of **??** and $\Delta_{\hat{k}}$ is the corresponding scaling dimension of the boundary operator. Using

$$\langle \hat{\mathcal{O}}_k(\mathbf{x}) \hat{\mathcal{O}}_k(\mathbf{y}) \rangle = \frac{1}{(\mathbf{x}-\mathbf{y})^{2\Delta_{\hat{k}}}}, \tag{1.8}$$

the 2-point function takes the form

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \sum_k \frac{\mu^2_{\mathcal{O}k}}{(4x^d y^d)^{\Delta_{\mathcal{O}}}} \xi^{-\Delta_{\hat{k}}} \tilde{f}_{\text{bdy}}(\Delta_{\hat{k}}, x, y) \equiv \sum_k \mu^2_{\mathcal{O}k} f_{\text{bdy}}(\Delta_{\hat{k}}, \xi) \tag{1.9}$$

As explained in the Appendix A of [**?**], these leading behaviors are enough to derive the functions f_{bulk} and f_{bdy} from the Casimirs of $SO(d+1,1)$ and $SO(d,1)$ respectively.

2 Deriving the bootstrap equations

In this section, we write the relevant OPEs and derive the crossing equations therefrom. We will be working around the normal fixed point, so the boundary is forced to order along, say, the 1-axis. Thus, it is important to consider the longitudinal component separately from the transverse ones. Following [?], we call these σ and ϕ_i respectively, with i running from 2 to N. As is common in boundary bootstrap literature [?][?], we will denote operators living on the boundary with a hat $(\hat{\sigma}, \hat{\phi})$. We will be considering the following two-point function:

$$\langle\sigma(\mathbf{x}, x^d)\sigma(\mathbf{y}, y^d)\rangle + \left\langle \sum_{i=2}^{N} \phi_i(\mathbf{x}, x^d)\phi_i(\mathbf{y}, y^d) \right\rangle \tag{2.1}$$

for which we need to write down the OPEs of the fields in the bulk and boundary channels. Note that in each correlator in ??, the boundary identity is implied (e.g. $\langle\sigma\sigma\hat{\mathbb{1}}\rangle$). In the bulk channel, both σ and ϕ_i should have the same OPEs, as we expect $O(N)$ symmetry to be preserved in that limit. Thus, they are characterised by the same CFT data (scaling dimensions and OPE coefficients). The allowed terms in the OPE are dictated by the representation theory of $O(N)$ [?]:

$$\phi_i \times \phi_j \sim \sum_{S^+} \delta_{ij}\mathcal{O} + \sum_{T^+} \mathcal{O}_{(ij)} + \sum_{A^-} \mathcal{O}_{[ij]} \tag{2.2}$$

where S, T and A denote $O(N)$ singlets, symmetric traceless tensors and anti-symmetric tensors respectively. The effect of considering the two-point function ?? is to eliminate the last two terms in the bulk-channel OPE. Thus we only deal with $O(N)$ singlets in the bulk channel.

In the boundary channel, the longitudinal component σ must couple to the boundary identity, so that it acquires a non-zero expectation value. We also know that the next leading operator for the longitudinal component must be the displacement operator $\hat{D}$ with dimension $\Delta_{\hat{D}} = d = 3$ associated with the breaking of translation invariance along the d-axis. Thus, we have

$$\sigma \sim \hat{\mathbb{1}} + \hat{D} + \dots \tag{2.3a}$$

$$\phi_i \sim \hat{\phi}_i + \dots \tag{2.3b}$$

where $\hat{\phi}_i$ has scaling dimension $d-1$. Using these operator product expansions, the two-point function can be written in terms of the boundary conformal blocks ([?]). In the bulk channel, we have

$$\langle\sigma\sigma\rangle + \sum_{i=2}^{N}\langle\phi_i\phi_i\rangle = \frac{N}{(x-y)^{2\Delta_\phi}} + \sum_k N\lambda_{\phi\phi k} a_k \tilde{f}_{\text{bulk}}(\Delta_k, x, y), \tag{2.4}$$

where the sum runs over relevant $\mathcal{O}(N)$ singlet, Lorentz scalar operators. $\Delta_\phi = \frac{1+\eta}{2}$ is a known exponent whereas all OPE coefficients are to be treated as unknowns. In the boundary channel,

$$\langle\sigma\sigma\rangle+\sum_{i=2}^{N}\langle\phi_i\phi_i\rangle=\frac{a_\sigma^2}{(2x^d)^{\Delta_\phi}(2y^d)^{\Delta_\phi}}+\mu_{\sigma D}^2\tilde{f}_{\rm bdy}(d;\mathbf{x},\mathbf{y})$$
$$+(N-1)\mu_{\phi\phi}^2\tilde{f}_{\rm bdy}(d-1;\mathbf{x},\mathbf{y})+\ldots \quad (2.5)$$

The functions $\tilde{f}_{\rm bulk}$ and $\tilde{f}_{\rm bdy}$ can be expressed in terms of the invariant cross ratio $\xi(x,y)$ as hypergeometric functions upto an overall scale factor [?][?]. Following the discussion in Section ??,

$$f_{\rm bulk}(\Delta,\xi)=\xi^{\Delta/2}\,{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2};\Delta+1-\frac{d}{2},-\xi\right)\equiv(x-y)^{2\Delta_\phi}\tilde{f}_{\rm bulk}, \quad (2.6a)$$

$$f_{\rm bdy}(\Delta,\xi)=\xi^{-\Delta}\,{}_2F_1\left(\Delta,\Delta+1-\frac{d}{2};2\Delta+2-d;-\frac{1}{\xi}\right)\equiv(4x^dy^d)^{\Delta_\phi}\tilde{f}_{\rm bdy}. \quad (2.6b)$$

With these definitions, we can express crossing symmetry in the bulk and boundary channels as

$$N+\sum_k N\lambda_{\phi\phi k}a_k f_{\rm bulk}(\Delta_k,\xi)=\xi^{\Delta_\phi}\left(a_\sigma^2+(N-1)\mu_{\phi\phi}^2 f_{\rm bdy}(d-1,\xi)+\mu_{\sigma D}^2 f_{\rm bdy}(d,\xi)\right) \quad (2.7)$$

up to the order in the boundary operators that we have defined in the OPE. This crossing equation is amenable to be solved self-consistently for the unknown CFT data by Gliozzi's truncation method ([?]).

The gist of the truncation method is that certain OPE's are 'truncable', so that we can only keep the low-lying operators in the spectrum. Then, the crossing equation ?? can be expanded around some cross-ratio value, say $\xi=1$. This way, we obtain infinitely many homogeneous equations involving the derivatives of conformal blocks evaluated at $\xi=1$, of which we can keep the first M. The inhomogeneous equation becomes

$$-\left(\sum_{k=1}^{n_{\rm bulk}} Np_k f_{\rm bulk}(\Delta_k,1)\right)+a_\sigma^2+(N-1)\mu_{\phi\phi}^2 f_{\rm bdy}(d-1,1)+\mu_{\sigma D}^2 f_{\rm bdy}(d,1)=N, \quad (2.8)$$

where $p_k=\lambda_{\phi\phi k}a_k$. The homogeneous equations are,

$$-\left(\sum_{k=1}^{n_{\rm bulk}} Np_k\,\partial_m f_{\rm bulk}(\Delta_k,\xi)|_{\xi=1}\right)+(N-1)\mu_{\phi\phi}^2\,\partial_m(\xi^{\Delta_\phi}f_{\rm bdy}(d-1,\xi))\big|_{\xi=1}$$
$$+\mu_{\sigma D}^2\,\partial_m(\xi^{\Delta_\phi}f_{\rm bdy}(d,\xi))\big|_{\xi=1}+(\Delta_\phi)_m\,a_\sigma^2=0 \quad (2.9)$$

for $m=1,2,\ldots M$. This forms a homogeneous system of equations for the vector $(Np_k,(N-1)\mu_{\phi\phi}^2,\mu_{\sigma D}^2,a_\sigma^2)$ of dimension $L=n_{\rm bulk}+3$ that is overconstrained if we choose $M>L$. It has a non-trivial solution only if the minimum singular value of the matrix of derivatives is zero [?]. Once the unknown dimensions are found, we can solve the system of homogeneous equations along with the inhomogeneous equation to obtain the OPE coefficients.

3 Results for low integer N

In this section, we present our bootstrap results for integer values of N, up to $N = 4$. For $N = 1$, the analysis in [?] carries over, and our results corroborate excellently with the values in their work. The only difference in our methods is that we minimize the minimum singular value of the derivative matrix whereas [?] find the dimensions by requiring that all minors of order L vanish. In the bulk channel, we kept four scalar operators apart from the identity. One of these is the energy (related to ν by $\Delta_\epsilon = 3 - 1/\nu$) and the next scalar operator is related to the exponent ω [?]. Bootstrap dictates the other two dimensions. Explicitly, we have the bulk OPE

$$\phi_i \times \phi_i \sim \mathbb{1} + \epsilon + \epsilon' + \epsilon'' + \epsilon''' \tag{3.1}$$

with

$$\Delta_\phi = \frac{1+\eta}{2}, \qquad \Delta_\epsilon = 3 - \frac{1}{\nu}, \qquad \Delta_{\epsilon'} = 3 + \omega. \tag{3.2}$$

We use the critical exponents gathered in [?] and for $N = 4$, we take the results from [?]. For $N = 1$, we found a consistent solution at

$$\Delta_{\epsilon''} = 7.34(84), \qquad \Delta_{\epsilon'''} = 13.4(28) \tag{3.3}$$

with the OPE coefficients

$$\begin{aligned} &a_\epsilon \lambda_{\phi\phi\epsilon} = 6.909, && a_{\epsilon'} \lambda_{\phi\phi\epsilon} = 2.262, && a_{\epsilon''} \lambda_{\phi\phi\epsilon''} = 0.186, \\ &a_{\epsilon'''} \lambda_{\phi\phi\epsilon'''} = 0.003924, && a_\sigma^2 = 6.7537, && \mu_{\sigma D}^2 = 0.06285. \end{aligned}$$

Repeating the process for $N = 2$, we found the minimum singular value at

$$\Delta_{\epsilon''} = 6.984, \qquad \Delta_{\epsilon'''} = 12.4285 \tag{3.4}$$

with the OPE coefficients

$$\begin{aligned} &a_\epsilon \lambda_{\phi\phi\epsilon} = 3.99, && a_{\epsilon'} \lambda_{\phi\phi\epsilon} = 1.359, && a_{\epsilon''} \lambda_{\phi\phi\epsilon''} = 0.143, \\ &a_{\epsilon'''} \lambda_{\phi\phi\epsilon'''} = 0.004, && a_\sigma^2 = 8.568, && \mu_{\sigma D}^2 = 0.07(26), \\ &\mu_{\phi\phi}^2 = 0.23(94). \end{aligned}$$

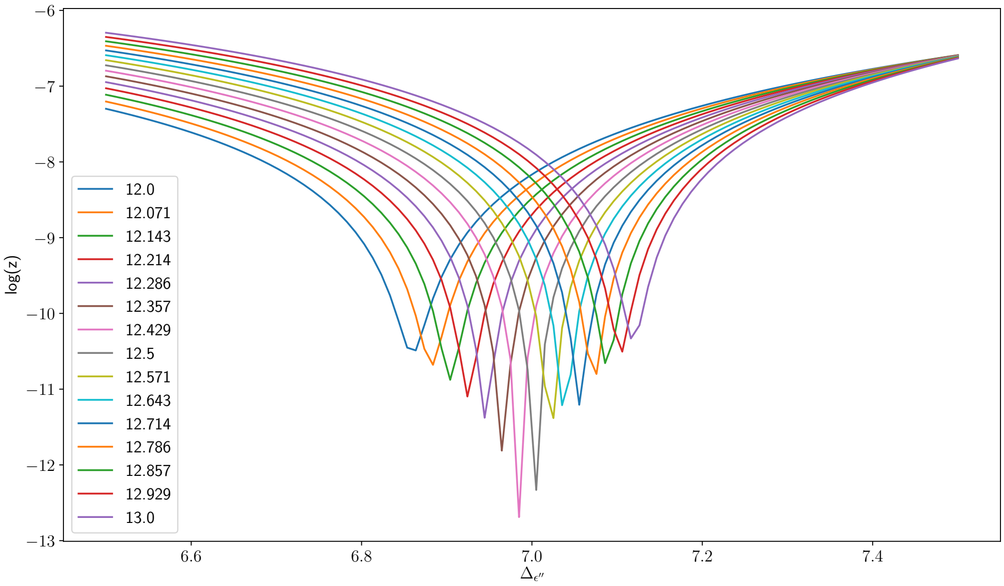
The search for the solution is plotted in Figure ??. Similarly, for $N = 3$, we found a consistent solution at

$$\Delta_{\epsilon''} = 6.927, \qquad \Delta_{\epsilon'''} = 12.4285 \tag{3.5}$$

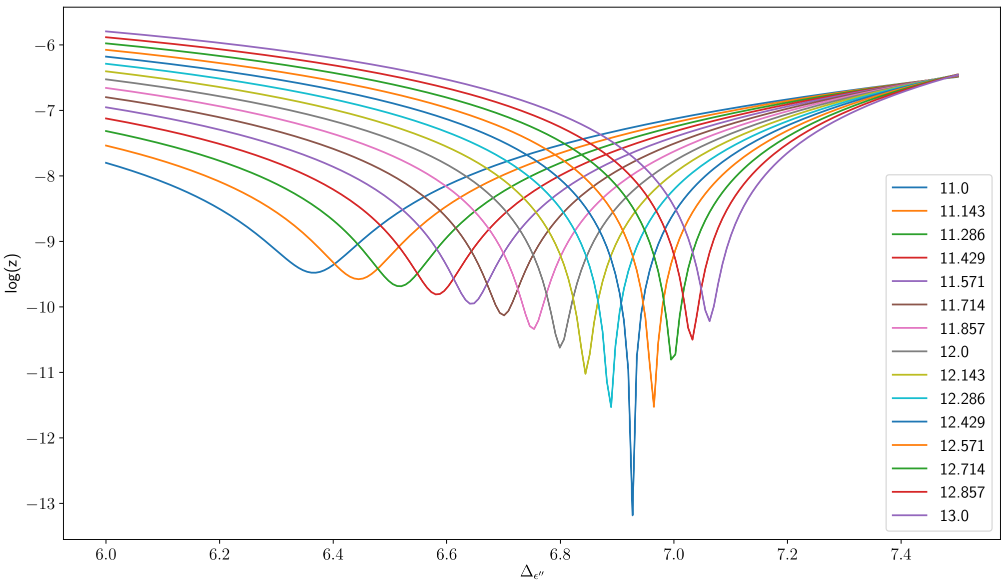
with the OPE coefficients

$$\begin{aligned} &a_\epsilon \lambda_{\phi\phi\epsilon} = 2.938, && a_{\epsilon'} \lambda_{\phi\phi\epsilon} = 1.026, && a_{\epsilon''} \lambda_{\phi\phi\epsilon''} = 0.118, \\ &a_{\epsilon'''} \lambda_{\phi\phi\epsilon'''} = 0.004, && a_\sigma^2 = 10.075, && \mu_{\sigma D}^2 = 0.0743, \\ &\mu_{\phi\phi}^2 = 0.2539. \end{aligned}$$

Notice that the dimensions $\Delta_{\epsilon''}$ and $\Delta_{\epsilon'''}$ are essentially the same as those for $N = 2$. However, the other CFT data is different. These are two different CFTs, but they have the same operators in their low-lying spectrum which is curious (maybe an artefact of the coarse grid?).



(a) $N = 2$



(b) $N = 3$

Figure 1: Log of minimum singular value of the derivative matrix for different values of $\Delta_{\epsilon'''}$ for $N = 2$ and $N = 3$. A consistent solution is found when the minimum singular value is zero.

Finally, for $N = 4$, we have

$$\Delta_{\epsilon''} = 6.897, \qquad \Delta_{\epsilon'''} = 12.464 \tag{3.6}$$

with the OPE coefficients

$$\begin{aligned} &a_\epsilon \lambda_{\phi\phi\epsilon} = 2.49, && a_{\epsilon'} \lambda_{\phi\phi\epsilon} = 0.883, && a_{\epsilon''} \lambda_{\phi\phi\epsilon''} = 0.107, \\ &a_{\epsilon'''} \lambda_{\phi\phi\epsilon'''} = 0.003, && a_\sigma^2 = 11.8823, && \mu_{\sigma D}^2 = 0.07643, \\ &\mu_{\phi\phi}^2 = 0.25696. \end{aligned}$$

3.1 Comparison to large-N results

We compare our values of a_σ and $\mu_{\phi\phi}$ to the large-N results from [?]. The results are presented in Figure ??.

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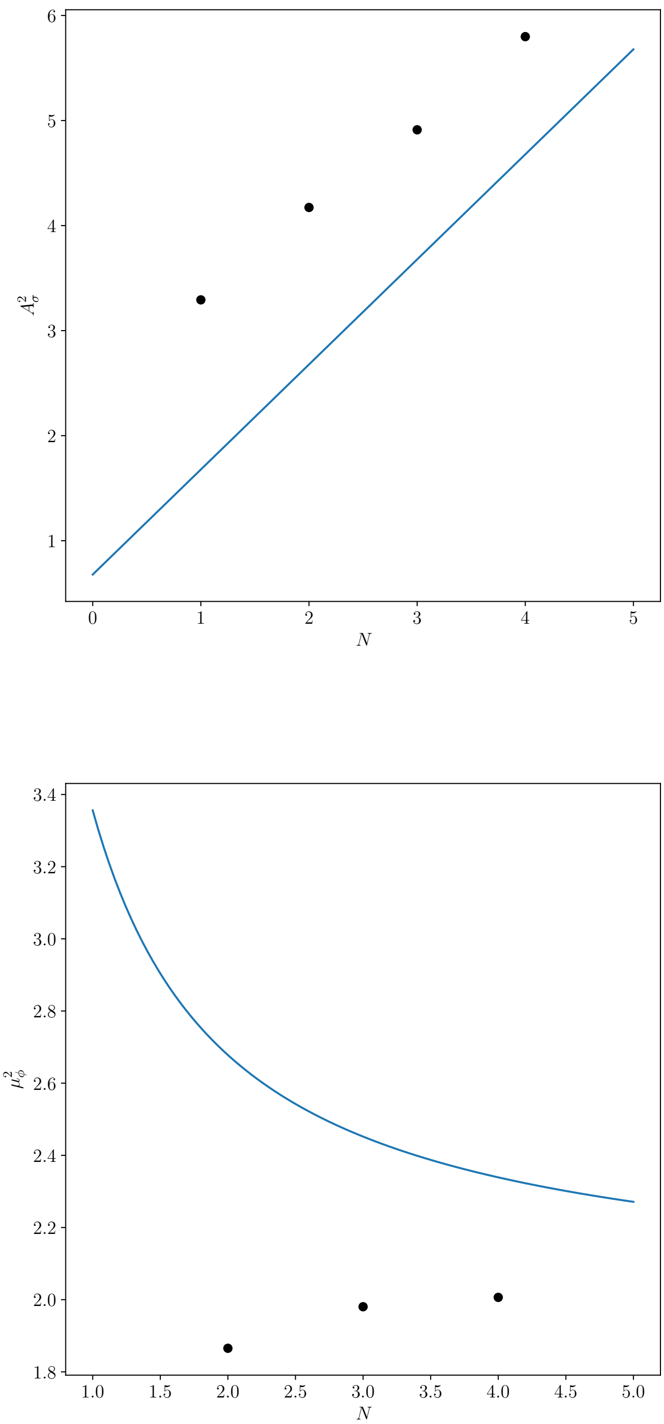

Figure 2: Bootstrap vs large-N: The plots compare results from [?] and our work. Dots represent bootstrap results. The OPE coefficients A_σ and μ_ϕ calculated in [?] differs from our a_σ and $\mu_{\phi\phi}$ by a factor of 2^{Δ_ϕ} and $2^{\Delta_\phi-(d-1)}$ respectively; we plot here the rescaled values and NOT our bare results presented in Section **??**