# Introduction to $\mathbb{Z}_2$ Gauge Theories

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Gauge symmetry has been crucial to our understanding of fundamental forces in the universe. The interest in *lattice* gauge theories first arose due to the problem of simulating the gauge bosons on a discrete spacetime lattice. However, it was soon realized that lattice gauge theories are interesting beasts of their own, and provide a window into new physics such as entanglement and topological order. In these notes, we discuss the simplest gauge theory on the lattice, one obtained by gauging the Abelian group $\mathbb{Z}_2$. In studying lattice gauge theory, we seek to understand the answer to the question

> *What are the physical implications of redundancy in our description of a statistical system?*

The notes are organized as follows. In Sec. 1, we review how one defines a (classical) gauge theory on a lattice. From thereon, we show how the $\mathbb{Z}_2$ gauge theory exhibits a phase transition with a non-local order parameter. The quantum version of the $\mathbb{Z}_2$ gauge theory is discussed in Sec. 2 of the notes. It has seen recent interest since Kitaev ([3]) presented a related model that could describe robust qubits for quantum computation. The quantum Ising gauge theory has since been a platform to study fractionalized excitations and quantum spin liquids, both of which are beyond the scope of these introductory notes. However, Sec 2.1 touches upon some of the obvious topological properties of the ground state of this model. Finally, in Sec. 3, we reproduce a duality first derived by Wegner which proves that the deconfinement phase transition is in the Ising universality class.

Due to the pedagogical nature of this article, we refrain from citing all the original works in the body, but most of the material is sourced from reviews [2], [4], and [5].

## 1 Classical model

To start, let us consider a simple classical system consisting of Ising spins $s_e = \pm 1$ at each edge $e$ on a three dimensional cubic lattice. Our quest is to write a model Hamiltonian that preserves a form of *local* $\mathbb{Z}_2$ symmetry. Remembering that the *global* $\mathbb{Z}_2$ symmetry corresponds to the action being invariant upon "flipping all the spins" $s_e \to -s_e$, we can define a *gauged* version of $\mathbb{Z}_2$ symmetry as invariance under flipping an arbitrary subset of spins. Specifically, we define the smallest unit $G_i$ of such a transformation on the lattice site $i$ as flipping all the spins around a vertex:

$$G_i : s_j \to -s_j \tag{1.1}$$

where $j$ runs over the edges connected to $i$ (see figure 1). The simplest non-trivial Hamiltonian that preserves this symmetry is

$$\mathcal{H} = -J \sum_{n} \prod_{i \,\in\, \square_n} s_i \tag{1.2}$$

which consists of four-spin products over spins in a *plaquette* (figure 2). Evidently, the operation $G_i$ always inverts two spins of every plaquette it touches and leaves each product invariant.

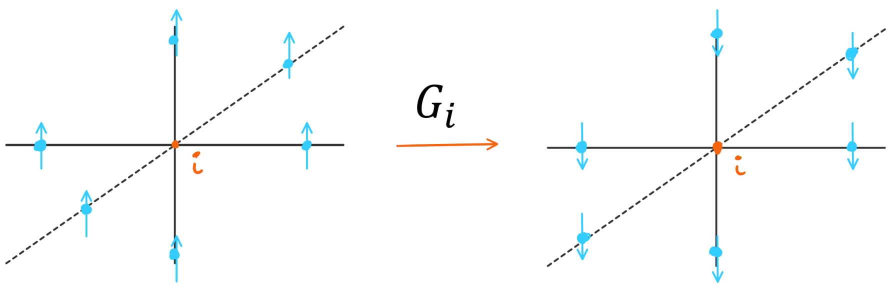


**Figure 1**: An example of a $\mathbb{Z}_2$ gauge transformation on a three dimensional lattice.

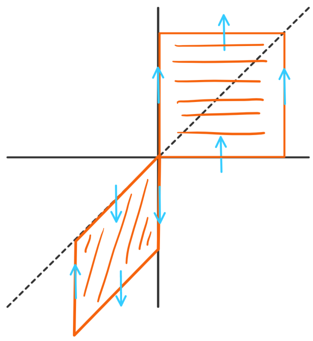

**Figure 2**: Plaquette terms which appear in the Hamiltonian, eq. 1.2.

This model, called the Ising gauge theory, was first studied by Wegner in 1971 ([1]). In this gauged theory, there is no phase transition because of spontaneous symmetry breaking of the $\mathbb{Z}_2$ symmetry as in the regular Ising model. This is because of Elitzur's theorem, which states that any expectation value, such as $\langle s \rangle$ which has variables that are not invariant under a local gauge transformation, must be zero. A qualitative argument for this is the following: due to the defining feature of gauge transformations, any two "gauge symmetry breaking" configurations of spins are connected by a *non-extensive* amount of energy even in the presence of a finite magnetic field. In this sense, it is like the one-dimensional Ising model, where domain walls proliferate at zero energy cost and order is destroyed at any non-zero temperature. In a more abstract sense, gauge symmetry fundamentally corresponds to some redundancy in our labeling of the configurations, and therefore the underlying spin variables are in some sense, *unphysical.* The fundamental building blocks of the Hamiltonian are gauge invariant quantities (the spin plaquettes), so the underlying spins are not observable.

### 1.1 An unusual phase transition

Once we have excluded a phase transition under a local order parameter, one wonders if this model has any kind of a non-trivial phase diagram. Wegner showed that it does have distinct high-temperature and low-temperature phases, but the order parameter describing the transformation must be non-local. Specifically, one considers the product of spins over a closed loop $\mathbf{C}$ in space, referred to in literature as a **Wegner-Wilson loop**

$$W_C = \left\langle \prod_{i \in \mathbf{C}} s_i \right\rangle .$$

It is explicitly gauge invariant for the same reason the Hamiltonian is gauge invariant. Now, as we will see, the phase transition concerns how $W_C$ grows with the size of the loop. At high temperatures it scales as the minimal surface area enclosed by the loop $A(\mathbf{C})$

$$\lim_{\beta \to 0} W_C \sim e^{-A(\mathbf{C})} \tag{1.3}$$

whereas at low temperatures $m_C$ scales like the perimeter of the loop $P(\mathbf{C})$

$$\lim_{\beta \to \infty} W_C \sim e^{-P(\mathbf{C})}. \tag{1.4}$$

In the quantum setting, the area-law phase is called 'confined' while the perimeter-law phase is called 'deconfined'.

#### High temperature expansion

To show the area law scaling in the $\beta \to 0$ limit, we can check the leading behavior of $W_C$. We know that

$$W_C = \left\langle \prod_{j \in \mathbf{C}} s_j \right\rangle = \frac{1}{Z} \sum_{\text{configs}} \left( \prod_{j \in \mathbf{C}} s_j \exp\left( \beta J \sum_n \prod_{i \,\in\, \square_n} s_i \right) \right) . \tag{1.5}$$

Let $\tilde{s}_n \equiv \prod_{i \,\in\, \square_n} s_i$ represent the product of the four spins in a plaquette $n$. In the rest of this section, we will set $J = 1$ for convenience. As $\tilde{s}_n = \pm 1$, we have the identity

$$\exp\left(\beta \tilde{s}_n\right) = \cosh\beta + \tilde{s}_n \sinh\beta = \left(1 + \tilde{s}_n \tanh\beta\right) \cosh\beta. \tag{1.6}$$

Thus,

$$W_C = \frac{\cosh\beta}{Z} \sum_{\text{configs}} \prod_{i \in C} s_i \prod_n (1 + \tilde{s}_n \tanh\beta) . \tag{1.7}$$

When $\beta \ll 1$, $\tanh\beta \ll 1$, which allows us to expand the numerator in powers of $\tanh\beta$ to get the leading order result. To performing the sum over spin configurations, we note that $\sum s_i = 0$ and $\sum s_i^2 = 2$, and thus the only terms which appear are of the kind where some power of $\tanh\beta$ is multiplied by even powers of all spins appearing in the product. For example,

$$\sum_{\text{configs}}\prod_{i\in C}s_i\prod_n(1+\tilde{s}_n\tanh\beta)\approx\sum_{\text{configs}}\prod_{i\in C}s_i+O(\tanh\beta),$$

but $\sum_{\text{configs}}\prod_{i\in C}s_i=0$ because all the spins on the loop only appear once in the sum. It follows that, the first non-zero term in the expansion corresponds to including all the plaquettes enclosed by the loop $\mathbf{C}$. The total number of these plaquettes, $N_C$ is proportional to the area enclosed by the loop $A(\mathbf{C})$ which will eventually give us the area law:

$$W_C\sim(\tanh\beta)^{A(\mathbf{C})}=e^{(\ln\tanh\beta)A(\mathbf{C})}. \tag{1.8}$$

The denominator $Z$ does not play any interesting role here because the leading order term is 1 and the next non-zero term simply contains *all* the plaquettes, $\sim(\tanh\beta)^{N_p}$.

**Low temperature expansion**

To obtain the low $T$ limit of the correlator $W_C$, we consider a spin-flip expansion. Let us utilise the gauge freedom to fix the $T=0$ configuration to be $s_i=+1$ for all spins. We choose this configuration to represent all configurations that can be reached by local gauge transformations. A gauge-invariant quantity such as $W_C$ should be oblivious to this choice. Each spin flip *frustrates* $4=2(d-1)$ plaquettes that the spin is part of, therefore it costs an energy of $8\beta$ to perform. Thus,

$$\sum_{\text{configs}}\left(\prod_{j\in\mathbf{C}}s_j\exp\left(\beta\sum_n\tilde{s}_n\right)\right)\approx e^{\beta N_p}\left(1+(N-2P(\mathbf{C}))\exp\left(-8\beta\right)+\ldots\right) \tag{1.9}$$

where $N_p$ is the number of plaquettes in the system and $N$ is the total number of links. The $(N-2P(\mathbf{C}))$ comes from the Wegner-Wilson loop operator taking a value of $+1$ when the spin flip occurs not on the loop and $-1$ when it does. In the same vein, we have for the partition function

$$Z=\sum_{\text{configs}}\exp\left(\beta\sum_n\tilde{s}_n\right)\approx e^{\beta N_p}\left(1+N\exp\left(-8\beta\right)+\ldots\right). \tag{1.10}$$

To a first approximation, we can consider all spin flips to be independent of each other. Thus, performing the sum in eq. 1.9 for $m$ spin flips yields

$$\sum_{\text{configs}}\left(\prod_{j\in\mathbf{C}}s_j\exp\left(\beta\sum_n\tilde{s}_n\right)\right)\approx\ e^{\beta N_p}\left(\sum_m\frac{(N-2P)^m}{m!}\exp\left(-8m\beta\right)\right). \tag{1.11}$$

Of course, this can be turned into an exponential:

$$W_C\approx\frac{\exp\left((N-2P)\exp\left(-8\beta\right)\right)}{\exp\left(N\exp\left(-8\beta\right)\right)}=\exp\left(-2e^{-8\beta}P(\mathbf{C})\right). \tag{1.12}$$

Thus, at low temperatures, $W_C$ scales exponentially with the perimeter of the loop.

As the Ising lattice gauge theory shows distinct phases, there must be a phase transition between them. The nature of this phase transition remains mysterious so far, and novel in the sense that it is driven by a fundamentally non-local order parameter. Moreover, it lacks a simple spontaneous symmetry breaking picture. However one can think of the perimeter law phase as "ordered" because the order parameter falls off much slower compared to the area law phase. In the next section, we will define and study the quantum version of the Ising gauge theory. Later, in section 3 we will use this quantum model to show that this phase transition actually falls in the $d = 3$ Ising universality class.

## 2 Quantum model

Every classical theory at finite temperature is dual to (lies in the same universality class as) some quantum theory at $T = 0$ in one lower dimension. In order to understand the non-local properties of the lattice gauge theory discussed above, it is useful to study the quantum model obtained by placing the three dimensional model on a space-time lattice and taking the continuum limit in the time direction. This procedure gives us a Hamiltonian of the form

$$H = -K \sum_{n} \prod_{i \,\in\, \Box_n} \sigma_i^z - g \sum_i \sigma_i^x \tag{2.1}$$

where $\sigma^x, \sigma^z$ are Pauli matrices acting on static spin-1/2 degrees of freedom sitting on the edges of a square lattice. The action of the local $\mathbb{Z}_2$ symmetry is realised by the operator

$$G_i = \prod_{i \,\in\, +} \sigma_i^x. \tag{2.2}$$

In the $\sigma^z$ basis, $G_i$ is the spin-flip operator acting on all edges emanating from a vertex $i$, analogous to what we had earlier in the classical model. Moreover, we have

$$[H, G_i] = 0 \tag{2.3}$$

for all vertices $i$ on the lattice. Thus, $H$ and $G_i$ can be simultaneously diagonalised and the Hilbert space is partitioned in two corresponding to the two eigenvalues of $G_i$, $\pm 1$. We will work with the $G_i = +1$ sector of the theory, referred to in literature as the even $\mathbb{Z}_2$ Gauge theory.

The phase transition is more apparent in the quantum theory because the two terms in $H$ do not commute. Thus, it is not unusual that the ground state is qualitatively different at $g/K \to 0$ compared to $g/K \to \infty$. Before studying the phase transition itself, let us look at some interesting aspects of the ground state in the two phases.

### 2.1 Toric code and Topological order in the $g = 0$ limit

When $g = 0$, the Hamiltonian only consists of the $\sigma^z$ plaquette terms, which can be minimized, for example, by a state where all spins are in $|\uparrow\rangle^z$. This state is not an eigenstate of $G_i$, so we utilise the facts that all $G_i$'s commute with $H$ and $G_i^2 = 1$ to write the ground state as

$$|0\rangle = \prod_i (1 + G_i) |\Uparrow\rangle \tag{2.4}$$

where $|\Uparrow\rangle = \bigotimes_n |\uparrow\rangle_n^z$. There is a nice visual interpretation of this state. Consider the action of one $G_i$ on $|\Uparrow\rangle$, which flips four spins. Let this state be visually represented by a loop surrounding the vertex $i$ on the *dual* square lattice (see figure 3). In the diagram, we represent every $|\downarrow\rangle$ spin by drawing a line perpendicular to the edge in the dual lattice and the $|\uparrow\rangle$ spins contribute no lines.

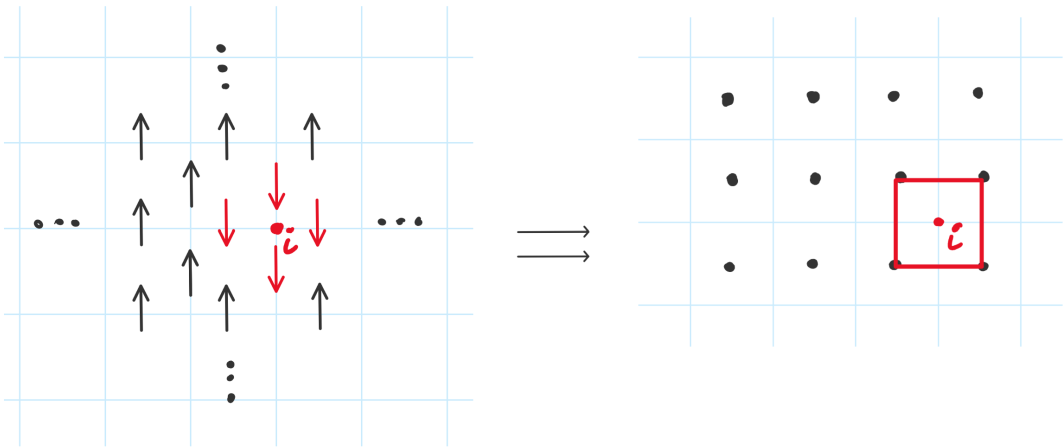


**Figure 3**: Diagrammatic representation of $G_i |\Uparrow\rangle$ on the regular lattice (left) by a loop in the dual lattice (right).

The many body ground state $|0\rangle$ is thus a linear combination of all the states represented by closed loops on the dual lattice. It is also the ground state of the related *toric code* Hamiltonian $H_{tc}$ introduced by Kitaev ([3])

$$H_{tc} = -\sum \prod_{i \,\in\, \square} \sigma_i^z - \sum \prod_{i \,\in\, +} \sigma_i^x \tag{2.5}$$

which is simply the gauge theory at $g = 0$ with the symmetry terms $G_i$ included in the Hamiltonian. Both the toric code and the gauge theory have a curious property: they are sensitive to the *topology* of the manifold they are defined on. Specifically, imposing periodic boundary conditions on the lattice (equivalent to defining the model on a torus) makes the ground state 4-fold degenerate. Let us see how this degeneracy arises and how it is *topologically protected*, making the toric code a candidate for fault-tolerant quantum computation.

The ground state is not unique if the lattice is defined on the torus, which is best understood from the diagrammatic representation. The degeneracy arises because on the torus, not all loops are *contractible*. There are two types of non-contractible loops (see figure 4), and although these loops satisfy both the symmetry condition and mimimize the energy, they are not reachable from $|0\rangle$ by applying $G_i$'s. Let $V_x$ and $V_y$ be the operators that create the non-contractible loops of spin flips. These operators commute with $H$ and $H_{tc}$ but they cannot be constructed out of $G_i$'s. Hence we have four ground states all with the same energy $-K$ (figure 5):

$$|0\rangle \qquad V_x\,|0\rangle \qquad V_y\,|0\rangle \qquad V_xV_y\,|0\rangle \tag{2.6}$$

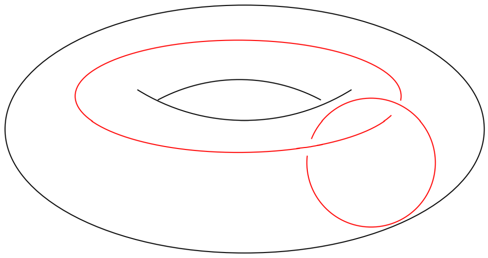

**Figure 4**: The two non-contractible loops on a torus.

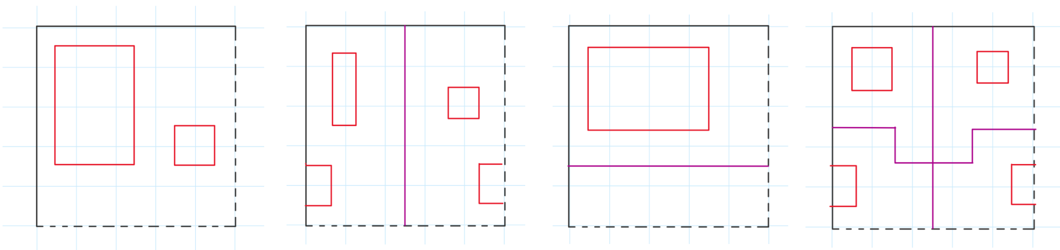

**Figure 5**: Representation of the four sectors of the ground-state manifold of the toric code, non-contractible loops in maroon.

This argument carries over to more complicated manifolds one could study the system on, where similarly the ground state degeneracy would be proportional to the number of non-contractible loops on that manifold.

The topological nature of the degeneracy is also apparent in its robustness. The degeneracy is not lifted upto high orders in perturbation theory. Considering a small but non-zero $-g\sum_i \sigma_i^x$ term as a perturbation to the $g = 0$ Hamiltonian, we readily see that the degeneracy is not lifted in first order degenerate perturbation theory, as $\langle 0|V_{(x,y)}\sum_i \sigma_i^x|0\rangle = 0$. In fact, it takes a macroscopic string of $\sigma^x$ operators to lift the degeneracy, and thus the ground states mix only at order $\sim \left(\frac{g}{K}\right)^L$.

### 2.2 Trivial limit, $g \to \infty$

The $g \to \infty$ is much tamer compared to the $g \to 0$ limit. The Hamiltonian is dominated by the transverse field term, and thus all spins must align with the $+x$ vector $|{\to}\rangle$ to minimize

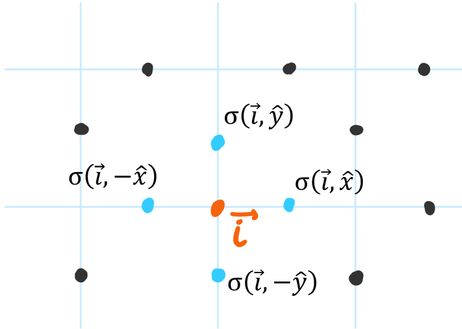


**Figure 6**: Naming the $\sigma$ operators on each vertex as per the direction of the edge they lie on.

the energy. We have the unique ground state

$$|\Rightarrow\rangle = \bigotimes_i |\rightarrow\rangle_i \, . \tag{2.7}$$

We note that unlike the toric code ground state, $|\Rightarrow\rangle$ is a tensor product state, and is manifestly not entangled. We can also see that placing the lattice on a torus introduces no new complications, as $V_{(x,y)}\,|\Rightarrow\rangle = +1\,|\Rightarrow\rangle$, and to go to the $-1$ eigenstate one has to spend a large amount of energy flipping a macroscopic number of spins against the transverse field. The ground state is still unique.

We see that the quantum model has a trivial (*confined*, $g \to \infty$) phase and a topological (*deconfined*, $g \to 0$) phase. We note in passing that the topological phase of the $\mathbb{Z}_2$ gauge theory describes a *gapped quantum spin liquid*. Its robust ground state degeneracy first showed us that topology can play a role in making quantum computations fault tolerant.

## 3 Universality class of the confinement-deconfinement transition

Finally, let us consider the deconfinement transition from the point of view of the quantum model. In this section, we will derive a Kramers-Wannier duality that will show that this transition lies in the 3d Ising universality class. In the quantum Hamiltonian eq. 2.1, we label the Pauli matrices acting on any edge connected to a vertex $\vec{i}$ as $\sigma(\vec{i}, \pm\hat{x})$ or $\sigma(\vec{i}, \pm\hat{y})$ where the arguments denote the direction the link connects to the vertex (figure 6). Then, we can utilise the gauge condition $G_i = 1$ to write

$$\sigma^x(\vec{i}, \hat{y})\sigma^x(\vec{i}, -\hat{y})\sigma^x(\vec{i}, \hat{x})\sigma^x(\vec{i}, -\hat{x}) = 1 \tag{3.1}$$

Or equivalently,

$$\sigma^x(\vec{i}, \hat{y}) = \sigma^x(\vec{i}, -\hat{y})\sigma^x(\vec{i}, \hat{x})\sigma^x(\vec{i}, -\hat{x}). \tag{3.2}$$

Similarly, we can use the gauge condition on the vertex $\vec{i} - \hat{y}$ to eliminate $\sigma^x(\vec{i}, -\hat{y})$ from the right hand side, and keep going till all $\sigma^x(\hat{y})$ are eliminated:

$$\sigma^x(\vec{i}, \hat{y}) = \prod_{n\,\geq\,0} \sigma^x(\vec{i} - n\hat{y}, \hat{x})\sigma^x(\vec{i} - n\hat{y}, -\hat{x}) \tag{3.3}$$

Once all $\sigma^x$ operators on the $\hat{y}$ edges are removed by fixing the gauge, we can proceed to fix the $\sigma^z(\hat{y})$ operators as well, as there are no longer any operators that don't commute with them in $H$. Thus without loss of generality, we can set

$$\sigma^z(\vec{i}, \hat{y}) = +1 \tag{3.4}$$

for all vertices $\vec{i}$. This reduces the four-$\sigma^z$ operator terms in the Hamiltonian to two operator terms, hinting towards a formulation in terms of the 2d Ising model. Now the only operators left are those that live on $\hat{x}$ edges. In this gauge condition, consider a duality mapping to spins on the dual lattice $\vec{j}$ with

$$\tau^x(\vec{j}) = \prod_{\text{plaquette around } \vec{j}} \sigma^z, \tag{3.5}$$

and

$$\tau^z(\vec{j}) = \prod_{n\, \geq\, 0} \sigma^x(\vec{i} - n\hat{y}, \hat{x}) \tag{3.6}$$

where $\vec{i}$ is (one of) the closest primal lattice points to $\vec{j}$. Based on these definitions, we can see that the dual spin operators $\tau^x, \tau^z$ satisfy the same commutation relations as any Pauli spin operators: $(\tau^z)^2 = (\tau^x)^2 = 1$, and

$$\{\tau^x(\vec{j}), \tau^z(\vec{j}')\} = \delta(\vec{j} - \vec{j}'). \tag{3.7}$$

From the definition of $\tau^z$,

$$\tau^z(\vec{j})\tau^z(\vec{j} - \hat{y}) = \Big(\sigma^x(\vec{i}, \hat{x})\sigma^x(\vec{i} - \hat{y}, \hat{x}) \ \ldots\Big) \Big(\sigma^x(\vec{i} - \hat{y}, \hat{x}) \ \ldots\Big) = \sigma^x(\vec{i}, \hat{x}) \tag{3.8}$$

and also

$$\tau^z(\vec{j})\tau^z(\vec{j} - \hat{x}) = \Big(\sigma^x(\vec{i}, \hat{x})\sigma^x(\vec{i} - \hat{y}, \hat{x}) \ \ldots\Big) \Big(\sigma^x(\vec{i}, -\hat{x})\sigma^x(\vec{i} - \hat{y}, -\hat{x}) \ \ldots\Big) = \sigma^x(\vec{i}, \hat{y}) \tag{3.9}$$

due to the gauge condition eq. 3.3. Finally, in terms of the $\tau$ operators, the quantum Hamiltonian $H$ looks like

$$H = -K \sum_j \tau^x(\vec{j}) - g \sum_{j, \langle j, j' \rangle} \tau^z(\vec{j})\tau^z(\vec{j}') \tag{3.10}$$

which is exactly the two-dimensional transverse field Ising model with the couplings reversed. The topological phase is dual to the field-aligned phase in terms of the $\tau$ spins, and the trivial phase of the gauge theory is dual to the symmetry-breaking phase. With this Kramers-Wannier duality, we conclude that the phase transition in the quantum Ising gauge theory eq. 2.1 is in the 2d quantum Ising universality class which itself is the quantum version of the *classical Ising model* in three dimensions. It is interesting to note that in this exercise, moving through the quantum Hamiltonians allowed us to establish a duality between two non-trivial classical theories in three dimensions. Our equivalent conclusion is that

> *The deconfinement phase transition in 3d Ising lattice gauge theory lies in the universality class of the 3d Ising model.*

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