15

1. (*15 pts*) Let $E = \sum_{n=1}^{\infty} 6e^{-2n}$.

Determine whether or not E converges, using the **Integral Test**, following the steps below. You may assume that $6e^{-2x}$ is continuous, positive, and decreasing for $1 \le x < \infty$.

a. (12 pts) Compute the needed integral *as a limit*. Find the antiderivative and show work step by step. If the integral converges, give its numerical value.

b. (3 pts) Give your conclusion (circle one), and explain briefly, using the integral test:
E Converges   E Diverges to $+\infty$ Test Inconclusive

$$E = \sum_{n=1}^{\infty} 6e^{-2n}$$

a)

$$\int_1^{\infty} 6e^{-2x}\,dx$$

$$= \lim_{K\to+\infty} \int_1^K 6e^{-2x}\,dx$$

$$\lim_{K\to+\infty} \int_1^K 6e^{-2x}\,dx$$

$$\lim_{K\to+\infty} 6\left(-\frac{e^{-2x}}{2}\right)\Big|_1^K$$

$$\lim_{K\to\infty} 6\left[\left(-\frac{e^{-2K}}{2}\right) - \left(-\frac{e^{-2}}{2}\right)\right]$$

$$\lim_{K\to\infty} \frac{6}{2}\left(-e^{-2K} + e^{-2}\right)$$

$$= 3\left(0 + e^{-2}\right)$$

$$= 3e^{-2}$$

$$= 0.406005847$$

$$\therefore \int_1^{\infty} 6e^{-2x}\,dx = 0.406$$

b) $\therefore \int_1^{\infty} 6e^{-2x}\,dx$ and $\sum_{n=1}^{\infty} 6e^{-2n}$ both converges where $f(n) = 0.406$.

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2. (*15 pts*) Let $C = \sum_{m=0}^{\infty} c_m = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, m}{4^m}$

a. (3 pts) Write down the terms c_1, c_2, c_3, and simplify.

b. (12 pts) Determine whether or not the series C converges, using the **Ratio Test**:

- Write down $\frac{c_{m+1}}{c_m}$ and simplify. Show work step by step.
- Compute the limit of $\left|\frac{c_{m+1}}{c_m}\right|$ as $m \to +\infty$. Explain briefly.
- Give your conclusion (circle one), and explain briefly, using the ratio test:

Converges Diverges to $+\infty$ Diverges to $-\infty$

Diverges and DNE Test Inconclusive

a) $c_1 = \frac{(-1)^{1-1}\,1}{4^1} = \frac{+1}{4}$ ✓

$c_2 = \frac{(-1)^{2-1}\cdot 2}{4^2} = -2/16$ ✓

$c_3 = \frac{(-1)^{3-1}\cdot 3}{4^3} = 3/64$ ✓

b) $\frac{c_{m+1}}{c_m}$, $c_m = \frac{(-1)^{m-1}\, m}{4^m}$

$c_{m+1} = \frac{(-1)^{m-1+1}\, m+1}{4^{m+1}} = \frac{(-1)^m\, m+1}{4^{m+1}}$

$\frac{c_{m+1}}{c_m} = \frac{(-1)^m\, m+1}{4^{m+1}} \cdot \frac{4^m}{(-1)^{m-1}\, m}$

$= \frac{(-1)^m\, m+1}{4^m \cdot 4} \cdot \frac{4^m}{\frac{(-1)^m}{-1}\cdot m}$

$= \frac{m+1 \cdot -1}{4m}$ ✓

$= \frac{-(m+1)}{4m}$

$\lim_{m\to\infty} \left|\frac{-(m+1)}{4m}\right|$ ✓

$\lim_{m\to\infty} \frac{m}{4m}$, using dominant term theorem.

$\lim_{m\to\infty} \frac{m}{4m} = \frac{1}{4}$ ✓

$\therefore \sum_{m=0}^{\infty} \frac{(-1)^{m-1}\, m}{4^m}$, converges.

~~where~~ because $\frac{1}{4} < 1$

15

3. (*15 pts*) Let $A = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2\sqrt{k}+8}{\sqrt{k}+4k}$.

Determine whether or not A converges, using the **Limit Comparison Test** (LCT), using the steps below. (The terms of A are all positive.)

a. (7 pts) Give a series $B = \sum_{k=1}^{\infty} b_k$ to compare to A.

Circle one: B Converges B Diverges to $+\infty$.

Explain, using the convergence test of your choice.

b. (5 pts) Compute the limit of $\frac{a_k}{b_k}$ as $k \to +\infty$. Justify and show work step by step.

c. (3 pts) Circle one: A Converges A Diverges to $+\infty$ Test Inconclusive

Explain briefly, using the LCT. If the test is inconclusive (see p. 1), try revising B.

a. $b_k = \frac{1}{2k^{\frac{1}{2}}}$

$B = \sum_{k=1}^{\infty} \frac{1}{2} \cdot \frac{1}{k^{\frac{1}{2}}}$

$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} = +\infty$

$p = \frac{1}{2} < 1$ diverges (p-series) ok

b. $\frac{a_k}{b_k} = \frac{\frac{2\sqrt{k}+8}{\sqrt{k}+4k}}{\frac{1}{2\sqrt{k}}} = \frac{4k+16\sqrt{k}}{\sqrt{k}+4k}$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{4k+16\sqrt{k}}{4k+\sqrt{k}} = 1$.

DTT

ok — clarify

$= \lim_{k\to\infty} \frac{4k}{4k} = 1$

c. since $B = \sum_{k=1}^{\infty} \frac{1}{2k^{\frac{1}{2}}}$ diverges and $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$, A also diverges by limit comparison test.

15

4. (*15 pts*) Let R be the infinite region bounded by $y = \dfrac{2}{x+3}$ and the x axis, for $0 \le x \le \infty$. Let S be the solid (funnel) you get by rotating R about the x-axis.

a. (2 pts) Sketch the region R.

b. (2 pts) Sketch the solid S in a separate picture.

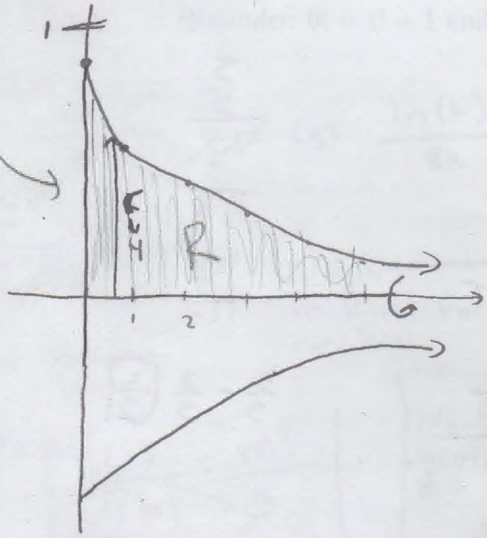
c. (4 pts) Write down an (infinite) integral for the volume of S. Explain briefly.
Reminder: disk method.

d. (7 pts) Find the volume of S by evaluating the integral *as a limit*. Give the antiderivative and show work step by step.

Note: your answer must be either a positive number or $+\infty$.

Sketch

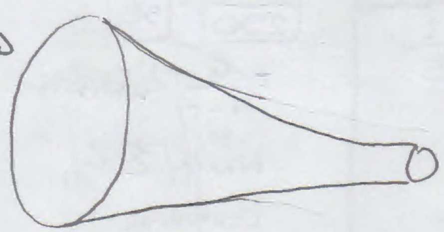
(a)



x	y
0	2/3
1	1/2
2	2/5
3	1/3
4	2/7
5	2/8
6	2/9
7	2/10

Vols

(b)



πr^2 = disk method

r = radius which = y and $y = \frac{2}{x+3}$

(c)

$$\lim_{n\to\infty}\int_0^\infty \pi\left(\frac{2}{x+3}\right)^2 dx$$

$$\lim_{n\to\infty} \pi\int_0^n \frac{4}{(x+3)^2}$$

$$\lim_{n\to\infty} 4\pi\int_0^n \frac{1}{(x+3)^2}$$

$$\lim_{n\to\infty} 4\pi\left(-\frac{1}{x+3}\right)\Bigg|_0^n$$

$u = x+3$
$du = 1dx$

$$\int \frac{1}{(x+3)^2}dx$$

$$\int \frac{1}{u^2}\,du$$

$$\int u^{-2}\,du$$

$$\frac{u^{-1}}{-1}$$

$$\boxed{-\frac{1}{u}}$$

$$\infty \leftarrow \frac{-4\pi}{(n+3)} - \left(\frac{-4\pi}{0+3}\right)$$

clarify

$$0 + \frac{4\pi}{3}$$

(d) $\boxed{S \text{ Volume} = \frac{4\pi}{3}}$

$\cancel{\left(\frac{2}{2+3}\right)^2}$ $\cancel{\frac{4}{25}}$ $\frac{4}{25}$

10

5. (*10 pts*) Let G be the geometric series $\frac{3}{5^3} + \frac{3^2}{5^4} + \frac{3^3}{5^5} + \frac{3^4}{5^6} + \cdots$

a. (5 pts) Rewrite G in standard form $\sum cr^n$, and give the values of c and r. Show work step by step. Then explain why G converges.

b. (5 pts) Calculate G step by step and write the answer as a simple fraction p/q.

a. $\boxed{G = \sum_{n=0}^{\infty} \frac{3}{125}\left(\frac{3}{5}\right)^n \quad c = \frac{3}{125},\ r = \frac{3}{5}}$

$\overset{g_0}{\cancel{n_0}} = \frac{3}{5^3} \quad \overset{g_1}{\cancel{n_1}} = \frac{3^2}{5^4} \quad n_2 = \frac{3^3}{5^5}$

$\frac{3^2}{5^4} \div \frac{3}{5^3} = \frac{3^2}{5^4} \cdot \frac{5^3}{3} = \frac{3}{5} = r$

$\cancel{\frac{a}{1-r} = \frac{\frac{3}{5^3}}{1-\frac{3}{5}} = \frac{\frac{3}{5^3}}{\frac{2}{5}} = \frac{3}{5^3} \cdot \frac{5}{2} = \frac{3}{50}}$ $\quad c = \frac{3}{5^3} = \frac{3}{125}$

G converges $\cancel{\frac{\frac{3}{5^3}}{1-\frac{3}{5}} = \frac{3}{50}}$ because $|r| = \frac{3}{5}$ which is less than 1.

b. $G = \frac{a}{1-r} = \frac{\frac{3}{5^3}}{1-\frac{3}{5}} = \frac{\frac{3}{5^3}}{\frac{2}{5}} = \frac{3}{5^3} \cdot \frac{5}{2} = \frac{3}{5^2 \cdot 2}$

$= \frac{3}{25 \cdot 2} = \boxed{\frac{3}{50}}$

30

6. (*30 pts*) (15 pts each) ***Do any two of the following***: circle the letters of the problems you want graded.

In each case, determine whether the series Converges or Diverges, using one or more convergence tests. (More space on back.)
State the tests used and show where and how they are used.
Check that the requirements of the chosen test hold. Show calculations step by step.

✓ a. $S = \sum_{k=2}^{\infty} \frac{\ln(k)}{8k}$

✓ b. $T = \sum_{n=1}^{\infty} \frac{5}{4 + 3n + 2^n}$

c. $U = \sum_{m=1}^{\infty} \frac{3\,(-1)^m}{m!} = -\frac{3}{1!} + \frac{3}{2!} - \frac{3}{3!} + \frac{3}{4!} - \cdots$

Reminder: $0! = 1! = 1$ and $m! = m(m-1)(m-2)\cdots 2\cdot 1$ for $m \geq 2$.

a) $a_k = \frac{\ln(k)}{8k}$ $b_k = \frac{1}{8k}$ $\sum b_k$ diverges by p-series $(p=1\leq 1)$

$\frac{a_k}{b_k} = \frac{\ln(k)}{8k} \cdot \frac{8k}{1} = \ln(k)$ ✓

$\lim_{k\to\infty} \ln(k) = +\infty$ S diverges since b_k diverges and $\lim_{k\to\infty} \frac{a_k}{b_k} = +\infty$
LCT

b)
$a_n = \frac{5}{4+3n+2^n}$, $b_n = \frac{5}{2^n}$, $0 \leq a_n \leq b_n$ ✓

$\sum 5\left(\frac{1}{2}\right)^n$ converges by G-series $(r = \frac{1}{2} < 1)$ $(|r| < 1)$

So T converges since b_n converges and $0 \leq a_n \leq b_n$
DCT

6. (*30 pts*) (15 pts each) ***Do any two of the following***: circle the letters of the problems you want graded.

In each case, determine whether the series Converges or Diverges, using one or more convergence tests. (More space on back.)
State the tests used and show where and how they are used.
Check that the requirements of the chosen test hold. Show calculations step by step.

$n=1 \qquad 4+3n+2$

15 ✓

c. $U = \displaystyle\sum_{m=1}^{\infty} \frac{3\,(-1)^m}{m!} = -\frac{3}{1!} + \frac{3}{2!} - \frac{3}{3!} + \frac{3}{4!} - \cdots$

Reminder: $0! = 1! = 1$ and $m! = m(m-1)(m-2)\cdots 2\cdot 1$ for $m \geq 2$.

test used? (ratio test)

(c) $u = \displaystyle\sum_{m=1}^{\infty} \frac{3\,(-1)^m}{m!} = a_n$

$$\left| a_{n+1} \cdot \frac{1}{a_n} \right| = \left| \frac{\cancel{3}\,\cancel{(-1)^m}\cdot(-1)^1}{(m+1)\,\cancel{m!}} \cdot \frac{\cancel{m!}}{\cancel{3}\,\cancel{(-1)^m}} \right| = \left| \frac{-1}{m+1} \right|$$

$$\lim_{m\to\infty} \left| \frac{-1}{m+1} \right| = 0$$

Since $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ & $L < 1$, $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converg ✓

Since $\displaystyle\lim_{m\to m} \left| \frac{a_{m+1}}{a_m} \right| = 0 < 1$; The ~~original~~ series also converges