10

1. (*10 pts*) (Partial Fractions)

Let F be the integral (antiderivative) $\int \frac{3x}{(x-3)(x+1)}\,dx$.

a. (6 pts) Find numbers A and B where $\frac{3x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$.

Justify your answer or show work step by step.

b. (4 pts) Find the integral F.

$$\frac{A}{(x-3)} + \frac{B}{(x+1)} = \frac{3x}{(x-3)(x+1)}$$

$$\boxed{A(x+1) + B(x-3) = 3x}$$

To find A

$x=3$

$$A(3+1) + B\cancel{(3-3)} = 3(3)$$

$$\frac{A(4)}{4} + \cancel{B(0)} = \frac{9}{4}$$

$$\boxed{A = \frac{9}{4}}$$

To find B

$x=-1$

$$A\cancel{(-1+1)} + B(-1-3) = 3(-1)$$

$$\cancel{A(0)} + \frac{B(-4)}{-4} = \frac{-3}{-4}$$

$$\boxed{B = \frac{3}{4}}$$

$$\int \frac{\frac{9}{4}}{(x-3)} + \frac{\frac{3}{4}}{(x+1)}\,dx = \left[\frac{9}{4}\ln|x-3| + \frac{3}{4}\ln|x+1| + C\right]$$

The Integral F

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2. (*10 pts*) Let I be the integral $\int x^2 \ln(x)\, dx$.

Find I using Integration by Parts (IBP). Show work step by step.

Start by writing down "u" and "dv". *Reminder:* let $u = \ln(x)$.

$I = ux - \int du\,v$

1. use Integration by parts (pick the easiest)

2. Formula $uv - \int du\,v$

2. $u = \ln(x)$ $dv = x^2 dx$ ✓

$du = \frac{1}{x}$ $v = \frac{x^3}{3}$

$\ln(x)\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$

$\ln(x)\frac{x^3}{3} - \int \frac{x^3}{3x}\,dx$

$\ln(x)\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$ ✓

$\ln(x)\frac{x^3}{3} - \frac{x^3}{3(3)}$

$$\boxed{I = \frac{x^3\ln(x)}{3} - \frac{x^3}{9} \quad = \frac{3(x^3\ln(x)) - x^3}{9}}$$

$+ C$

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3. (*10 pts*) Let S be the integral $\int \frac{\cos w}{(\sin w - 2)^4}\, dw$.

Find S using integration by substitution.
Write down the substitution and show work step by step.
Note: $\cos w$ and $\sin w$ are the same as $\cos(w)$ and $\sin(w)$.

$u = \sin w - 2$

$du = \cos w \, dw \qquad dw = \frac{du}{\cos w}$

$$\int \frac{\cos w}{(\sin w - 2)^4}\, dw$$

$$= \int \frac{\cancel{\cos w}}{u^4} \cdot \frac{du}{\cancel{\cos w}}$$

$$= \int \frac{1}{u^4}\, du$$

$$= \int u^{-4}\, du$$

$$= -\frac{1}{3} u^{-3} + C.$$

$$= -\frac{1}{3} \cdot (\sin w - 2)^{-3} + C$$

$$\therefore = -\frac{1}{3(\sin w - 2)^3} + C.$$

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4. (*30 pts*) (Volume). A region (see picture) is bounded by $y = x^2$ and $y = 3x$.
Let S be the solid formed by rotating the region about the **x-axis.**
Read the whole problem before starting.
Change equations to the form $x = g(y)$ where necessary.

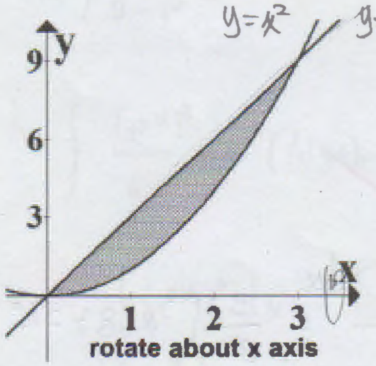
a. (2 pts) Sketch the solid S roughly, in a separate picture. Briefly describe its shape.

b. (10 pts) Using the **washer method**, set up ~~an integral~~ for the volume of S. Give the geometric meaning of each term in the integral (relate to the washer).

c. (10 pts) Using the **cylindrical shell method**, set up an integral for the volume of S. Give the geometric meaning of each term in the integral (relate to the shell).

d. (8 pts) Find the volume of S by evaluating ONE of the integrals above. Give the antiderivative, showing work step by step. (More space on last pages.)

Give your answer in the form $\boxed{N\pi}$ where N is a number.



a.

A cone which has empty space inside.

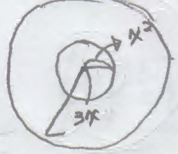
(cone - funnel)

b. washer method.

$$V = \int_0^3 \pi \left((3x)^2 - (x^2)^2 \right) dx$$

→ Area of the $\boxed{\text{circle}}$ washer

Outer radius = $3x$
inner radius = x^2

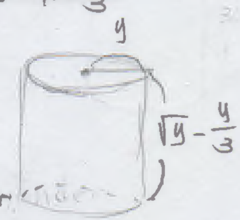


c. cylindrical shell method $x = \sqrt{y}$ $x = \frac{y}{3}$

$$V = \int_0^9 2\pi y \left(\sqrt{y} - \frac{y}{3} \right) dy$$

radius of cylinder

height of the cylinder



d. $\int_0^3 \pi \left((3x)^2 - (x^2)^2 \right) dx$

$= \pi \int_0^3 (9x^2 - x^4)\, dx$

$= \pi \left[3x^3 - \frac{1}{5}x^5 \right]_0^3$

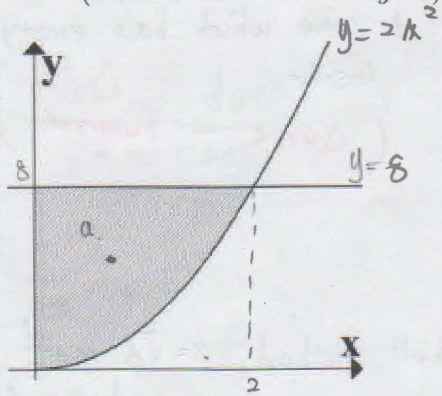
$\therefore = \pi \left(81 - \frac{3^5}{5} \right) = \boxed{\frac{162}{5}\pi}$

5. (*20 pts*) (Center of mass.) Let R be the region (see picture) bounded by the y-axis and the graphs of $y = 2x^2$ and $y = 8$.
In the questions below, *Area* is the area of R, which is $32/3$ units^2 $\frac{32}{3}$

a. (2 pts) Mark the approximate location of the center of mass of the region (balance point) with a dot (•).

b. (9 pts) The x-value of the center of mass (COM) is $x_{\text{cm}} = \dfrac{\int_a^b xh(x)\,dx}{Area}$,
where $h(x)$ is the height of the slice (rectangle) at x.
Find a and b and write down the integral for x_{cm}. (Be careful!)
(*Don't evaluate the integral.*)

c. (9 pts) The y-value of the COM is similar: $y_{\text{cm}} = \dfrac{\int_c^d yh(y)\,dy}{Area}$.
Find c and d and write down an integral for y_{cm}.
(*Don't evaluate the integral.*)



b. $2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$$\therefore x_{cm} = \frac{\int_0^2 x(8-2x^2)\,dx}{\text{Area}} = \frac{3}{32}\int_0^2 x(8-2x^2)\,dx$$

$\boxed{a = 0,\ b = 2}$

c. $2x^2 = y \qquad x = \sqrt{\frac{y}{2}}$

$x^2 = \frac{y}{2}$

$x = \pm\sqrt{\frac{y}{2}}$

$$\therefore y_{cm} = \frac{\int_0^8 y\left(\sqrt{\frac{y}{2}}\right)dy}{\text{Area}} = \frac{3}{32}\int_0^8 y\left(\sqrt{\frac{y}{2}}\right)dy$$

$\boxed{c = 0,\ d = 8}$

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6. (*20 pts*) (10 pts each.) Find **any two** of the following integrals.
State the main method used—Guess-and-Check (G&C), Algebra, Substitution (Sub), Partial Fractions (PF), Integration by Parts (IBP), or Other (state).
Show work step by step (or show the "check").
Circle the labels of the problems you want graded (or I'll grade the first two tried).
(Extra space on the last pages.)

10 a. $\int \frac{y}{4y-1}\,dy$

b. $\int \frac{(e^x)^3}{6}\,dx$

c. $\int w\sqrt{2-w}\,dw$

10 d. $\int \frac{t}{9-t^2}\,dt$

a. $\int \frac{y}{4y-1}\,dy$ ~~IBP~~ X using algebra

$\frac{1}{4} + \frac{\frac{1}{4}}{4y-1}$

$4y-1 \,/\, y$

$-(y-\frac{1}{4})$

$\frac{1}{4}$

$\int \frac{1}{4} + \frac{\frac{1}{4}}{4y-1}\,dy$

$\int \frac{1}{4}\,dy + \int \frac{\frac{1}{4}}{4y-1}\,dy$

$\frac{1}{4}y + \frac{1}{4}\int \frac{1}{4y-1}\,dy$

let $u = 4y-1$

$du = 4\,dy$

$dy = \frac{1}{4}\,du$

$\frac{1}{4}y + \frac{1}{4}\int \frac{1}{u}\cdot\frac{1}{4}\,du$ ✓

$= \frac{1}{4}y + \frac{1}{4}\cdot\frac{1}{4}\int \frac{1}{u}\,du$

$= \frac{1}{4}y + \frac{1}{16}\int \frac{1}{u}\,du$

$= \frac{1}{4}y + \frac{1}{16}\cdot \ln|u| + C$

$= \boxed{\frac{1}{4}y + \frac{1}{16}\ln|4y-1| + C}$ ✓

$= \boxed{\frac{1}{4}\left(y + \frac{1}{4}\ln|4y-1|\right) + C}$

d. $\int \frac{t}{9-t^2}\,dt$ $\boxed{\text{substitution}}$

let $u = 9-t^2$

$du = -2t\,dt$

$dt = -\frac{1}{2t}\,du$ ✓

$\int \frac{t}{u}\cdot -\frac{1}{2t}\,du$

$= \int -\frac{1}{2u}\,du$

$= -\frac{1}{2}\int \frac{1}{u}\,du$

$= -\frac{1}{2}\ln|u| + C$ ✓

$= \boxed{-\frac{1}{2}\ln|9-t^2| + C}$

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6. (*20 pts*) (10 pts each.) Find **any two** of the following integrals.
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(Extra space on the last pages.)

a. $\int \frac{y}{4y-1}\,dy$

10 b. $\int \frac{(e^x)^3}{6}\,dx$

10 c. $\int w\sqrt{2-w}\,dw$

d. $\int \frac{t}{9-t^2}\,dt$

b. $\int \frac{(e^x)^3}{6}\,dx$ (Algebra)

$= \frac{1}{6}\int (e^x)^3\,dx$

$= \frac{1}{6}\int e^{3x}\,dx$

$= \frac{1}{6}\left[\frac{1}{3}e^{3x}\right] + C$

$\therefore = \frac{1}{18}e^{3x} + C$

c. $\int w\sqrt{2-w}\,dw$ (substitution)

$u = 2-w \quad w = 2-u$
$du = -dw$

$\int (2-u)\sqrt{u}\cdot(-du)$

$= -\int 2u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= -\left[\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right] + c$

$= -\frac{4}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}} + c$

$\therefore = -\frac{4}{3}(2-w)^{\frac{3}{2}} + \frac{2}{5}(2-w)^{\frac{5}{2}} + c$