14

1. (*15 pts*) Let $E = \sum_{n=1}^{\infty} 4e^{-3n}$.

Determine whether or not E converges, using the **Integral Test**, following the steps below. You may assume that $4e^{-3x}$ is continuous, positive, and decreasing for $1 \le x < \infty$.

11 a. (12 pts) Compute the needed integral *as a limit.* Find the antiderivative and show work step by step. If the integral converges, give its numerical value.

b. (3 pts) Give your conclusion (circle one), and explain briefly, using the integral test:
E Converges E Diverges to $+\infty$ Test Inconclusive

$$\sum_{n=1}^{\infty} 4e^{-3n} = 4\sum_{n=1}^{\infty} e^{-3n} \approx 4\int_1^{\infty} e^{-3x}\,dx$$

ok (approximately equal)

$$\lim_{m\to\infty} 4\int_1^{m} e^{-3x}\,dx$$

$u = -3x$

$\frac{du}{dx} = -3$

$\frac{du}{-3} = dx$

$$\lim 4\int_1^{m} e^{u}\,\frac{du}{-3}$$ $\boxed{1, m \to -3, -3m}$

$$\lim -\frac{4}{3}\int_1^{m} e^{u}\,du$$

oops

$$\lim_{m\to\infty} -\frac{3}{4}\left(e^{u}\right)\Big|_1^{m}$$

[problem here] (back on track)

$$\lim_{m\to\infty} -\frac{3}{4}\left(e^{-3x}\right)\Big|_1^{m}$$

ok

$$-\frac{3}{4}\left(e^{-3\infty} - e^{-3}\right)$$

$$+\frac{3}{4}e^{-3}$$

$\boxed{\frac{4}{3}e^{-3}}$

Since $\sum_{n=1}^{\infty} 4e^{-3n} \approx \int_1^{\infty} 4e^{-3x}\,dx$ and the integral converges, then $\sum_{n=1}^{\infty} 4e^{-3n}$ also converges

ok

15

$$\begin{array}{r} 36 \\ \times\ 6 \\ \hline 256 \end{array}$$

$C\cancel{26} \cdot \frac{36 \cdot \cancel{6}}{\cancel{3}} \cdot 2$

2. (*15 pts*) Let $C = \sum_{m=0}^{\infty} c_m = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, 6^m}{m}$

a. (3 pts) Write down the terms c_1, c_2, c_3, and simplify.

b. (12 pts) Determine whether or not the series C converges, using the **Ratio Test**:

- Write down $\frac{c_{m+1}}{c_m}$ and simplify. Show work step by step.
- Compute the limit of $\left|\frac{c_{m+1}}{c_m}\right|$ as $m \to +\infty$. Explain briefly.
- Give your conclusion (circle one), and explain briefly, using the ratio test:

Converges   Diverges to $+\infty$ Diverges to $-\infty$

Diverges and DNE   Test Inconclusive

ok

Note since were using the ratio test you can ignore $(-1)^{m-1}$ because of the absolute | | value

Plug in for the values ↳

a

$C_1 = \frac{(-1)^0\, 6^1}{1} = 6$

$C_2 = \frac{(-1)^1 \cdot 6^2}{2} = -18$

$C_3 = \frac{1 \cdot 6^3}{3} = 72$

b

$$\lim_{m\to\infty} \left| \frac{\left(\frac{6^{m+1}}{(m+1)}\right)}{\left(\frac{6^m}{m}\right)} \right| = \left| \frac{6^{\cancel{m+1}}}{m+1} \times \frac{m}{\cancel{6^m}} \right|$$

$$\left| \frac{6m}{m+1} \right|$$

$$\lim_{m\to\infty} \frac{6\frac{m}{m}}{\frac{m}{m} + \frac{1}{m}^{\,0}}$$

↓

$6 > 1$

↳ then $\sum_{m=1}^{\infty} c_m$ Diverges

Ay

AST DDT

not used

infinity over anything is zero ???

since the series is alternating between both postive and negative terms then the series ~~(based from the conclusion of the ratio test the series is divergent)~~ however alternating switches the infinity to $+\infty$, & $-\infty$.

Correct except for error in (a) leading to wrong conclusion.

14

3. (*15 pts*) Let $A = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2\sqrt{k}+12}{\sqrt{k}+6k^2}$

Determine whether or not A converges, using the **Limit Comparison Test** (LCT), using the steps below. (The terms of A are all positive.)

B 6 0*

a. (7 pts) Give a series $B = \sum_{k=1}^{\infty} b_k$ to compare to A.

Circle one: B Converges B Diverges to $+\infty$. X

Explain, using the convergence test of your choice.

✓ b. (5 pts) Compute the limit of $\frac{a_k}{b_k}$ as $k \to +\infty$. Justify and show work step by step.

3 c. (3 pts) Circle one: A Converges A Diverges to $+\infty$ Test Inconclusive

Explain briefly, using the LCT. If the test is inconclusive (see p. 1), try revising B.

$$\sum_{n=1}^{\infty} \frac{2\sqrt{k}+12}{\sqrt{k}+6k^2} \approx \sum_{n=1}^{\infty} \frac{2k^{\frac{1}{2}}}{6k^{\boxed{2}}} \qquad b_k = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$$

b_k is a p-series and diverges X because $\frac{1}{2} = p \le 1$

oops

$$\boxed{\text{oops}} \quad \frac{k^{\frac{1}{2}}}{k^2} = \frac{1}{k^{3/2}}$$

DDT

$$\lim_{n\to\infty} \frac{a_k}{b_k} = \lim_{n\to\infty} \frac{2\sqrt{k}+12}{\sqrt{k}+6k^2} \cdot \frac{6k^2}{2k^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{2\sqrt{k}}{6k^2} \cdot \frac{6k^2}{2\sqrt{k}} = 1$$

using LCT $L=1$ is between 0 & ∞. ~~and~~

Since ~~[illegible] A with LCT like b_k~~

$a_k \approx b_k$ and $\sum b_k$ diverges by p-series then $\sum a_k$ also diverges.

(however, since B converges, A converges)

*(3a) $b_K = \frac{2K^{\frac{1}{2}}}{6K^2} = \frac{1}{3K^{3/2}}$ so $B = \sum \frac{1}{3K^{3/2}}$ is a p-series with $p = 3/2$, $3/2 > 1$, so B converges.

OF (b)

4. (*15 pts*) Let R be the infinite region bounded by $y = \dfrac{3}{x+2}$ and the x axis, for $0 \le x \le \infty$. Let S be the solid (funnel) you get by rotating R about the x-axis.

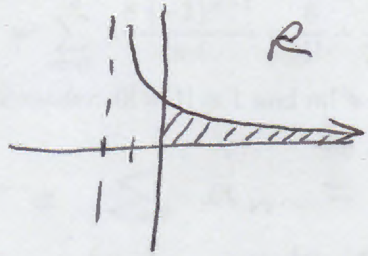
a. (2 pts) Sketch the region R.

b. (2 pts) Sketch the solid S in a separate picture.

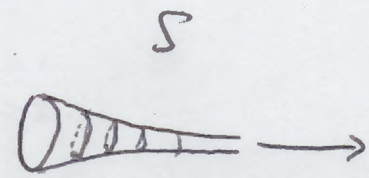
c. (4 pts) Write down an (infinite) integral for the volume of S. Explain briefly. *Reminder*: disk method.

d. (7 pts) Find the volume of S by evaluating the integral *as a limit*. Give the antiderivative and show work step by step.
Note: your answer must be either a positive number or $+\infty$.

(a)



(b)



(c) For each x, if r is disk radius, $\pi r^2 =$ area.

$$\text{vol} = \int_0^\infty \pi r^2\, dx = \int_0^\infty \pi \left(\frac{3}{x+2}\right)^2 dx$$

(d) $$\int \pi \left(\frac{3}{x+2}\right)^2 dx = \pi \int \frac{9}{(x+2)^2}\, dx = \pi \cdot 9 \int (x+2)^{-2}\, dx$$

$$= -9\pi\,(x+2)^{-1} \quad (+C) \quad \text{since } \int u^{-2}\,du = -u^{-1}\ (+C)$$

$$V = \int_0^\infty \pi\left(\frac{3}{x+2}\right)^2 dx = \lim_{M\to\infty} \int_0^M \pi \left(\frac{3}{x+2}\right)^2$$

$$= \lim_{M\to\infty} -9\pi \cdot \frac{1}{(x+2)}\Bigg|_0^M = \lim_{M\to\infty} -9\pi\left(\frac{1}{M+2} - \frac{1}{0+2}\right)$$

$$= -9\pi\left(0 - \frac{1}{2}\right) = \boxed{\frac{9\pi}{2}}$$

(as $M \to \infty$, $M+2 \to \infty$, so $\frac{1}{M+2} \to 0$).

10

5. (*10 pts*) Let G be the geometric series $\frac{2^4}{5^2} + \frac{2^5}{5^3} + \frac{2^6}{5^4} + \frac{2^7}{5^5} + \cdots$

$c \cdot r^n$

a. (5 pts) Rewrite G in standard form $\sum cr^n$, and give the values of c and r. Show work step by step. Then explain why G converges.

b. (5 pts) Calculate G step by step and write the answer as a simple fraction p/q.

$$\sum_{n=0}^{\infty} \frac{2^4}{5^2} \cdot \left(\frac{2}{5}\right)^n$$

$$c = \frac{2^4}{5^2} \quad r = \frac{2}{5}$$

ⓐ Converges to $\frac{c}{1-r}$ if $|r| <$ 1 and diverges if $|r| \geq 1$

My $r = \frac{2}{5}$ so it's less than 1 so it converges

~~geometric sum~~

$\frac{c}{1-r}$

KCF

$$\frac{\frac{2^4}{5^2}}{1-\frac{2}{5}} = \frac{2^4}{5^2} \cdot \frac{5}{3} = \frac{2^4 \cdot \cancel{5}}{\cancel{5^2} \cdot 3} = \frac{2^4}{15} = \frac{16}{15} = 1\frac{1}{15}$$

b?

30

6. (*30 pts*) (15 pts each) ***Do any two of the following***: circle the letters of the problems you want graded.

In each case, determine whether the series Converges or Diverges, using one or more convergence tests. (More space on back.)

State the tests used and show where and how they are used.

Check that the requirements of the chosen test hold. Show calculations step by step.

a. $S = \sum_{k=2}^{\infty} \frac{\ln(k)}{7k}$

b. $T = \sum_{n=1}^{\infty} \frac{6}{5n + 4 + 3^n}$

c. $U = \sum_{m=1}^{\infty} \frac{8\,(-1)^{m-1}}{m!} = \frac{8}{1!} - \frac{8}{2!} + \frac{8}{3!} - \frac{8}{4!} + \cdots$

Reminder: $0! = 1! = 1$ and $m! = m(m-1)(m-2)\cdots 2\cdot 1$ for $m \geq 2$.

LCT

a) $\underbrace{\sum_{k=2}^{\infty} \frac{\ln(k)}{7k}}_{S_k} \approx \underbrace{\sum \frac{1}{7k}}_{b_k}$ $\quad b_k = \frac{1}{7}\sum\frac{1}{k}$ b_k diverges because p-series diverges when $p \leq 1$

$\lim_{k\to\infty} \frac{\ln k}{7k} \cdot \frac{7k}{1} = \lim_{k\to\infty} \ln(k) = \infty$ by LCT

Since $\lim_{k\to\infty} \frac{S_k}{b_k} = \infty$ and b_k diverges then $\cancel{S_k}$ also diverges

ok

c) ratio test

$U_m = \sum_{m=1}^{\infty} \frac{8(-1)^{m-1}}{m!}$

$\lim_{m\to\infty}\left|\frac{U_{m+1}}{U_m}\right| = \lim_{m\to\infty}\left|\frac{8(-1)^m}{(m+1)!}\cdot\frac{m!}{8(-1)^{m-1}}\right|$

$= \lim_{m\to\infty}\left|\frac{\cancel{8}\cdot\cancel{(-1)^m}}{(m+1)\cancel{(m!)}}\cdot\frac{\cancel{m!}}{\cancel{8}\cdot\cancel{(-1)^m}\cdot(-1)^{-1}}\right| = \lim_{m\to\infty}\left|\frac{-1}{(m+1)}\right| =$

shorthand – be careful

$= \left|\frac{-1}{\infty+1}\right| = 0$

"Big U"

Since $\lim_{m\to\infty}\left|\frac{U_{m+1}}{U_m}\right| < 1$, then $\cancel{U_m}$ converges by ratio test

6. (*30 pts*) (15 pts each) ***Do any two of the following***: circle the letters of the problems you want graded.

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State the tests used and show where and how they are used.
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a. $S = \sum_{k=2}^{\infty} \frac{\ln(k)}{7k}$

b. $T = \sum_{n=1}^{\infty} \frac{6}{5n + 4 + 3^n}$

c. $U = \sum_{m=1}^{\infty} \frac{8\,(-1)^{m-1}}{m!} = \frac{8}{1!} - \frac{8}{2!} + \frac{8}{3!} - \frac{8}{4!} + \cdots$

Reminder: $0! = 1! = 1$ and $m! = m(m-1)(m-2)\cdots 2 \cdot 1$ for $m \geq 2$.

b) $T = \sum a_n = \sum \frac{6}{5n+4+3^n}$

By LCT (DCT works also)

• Let $\sum b_n = \sum \frac{1}{3^n}$. This is a G-series with $r = \frac{1}{3}$. $|r| < 1$ so $\sum b_n$ converges.

• $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{5n+4+\boxed{3^n}} \cdot 3^n \underset{\text{DTT}}{=} \lim_{n\to\infty} \frac{6 \cdot 3^n}{3^n} = 6$

$0 < 6 < \infty$, so, since $\sum b_n$ converges, T also converges.

OR by ratio test

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{6}{5(n+1)+4+\boxed{3^{n+1}}} \cdot \frac{5n+4+\boxed{3^n}}{6}$

$\underset{\text{DTT}}{=} \lim_{n\to\infty} \boxed{\frac{6}{6}} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{3^n}{3^n \cdot 3} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$

$\rho = \frac{1}{3}$ so $|\rho| < 1$. $\therefore$ $T = \sum a_n$ converges.