10

1. (*10 pts*) (Partial Fractions)

Let F be the integral (antiderivative) $\int \frac{2x}{(x+2)(x-4)}\,dx$.

a. (6 pts) Find numbers A and B where $\frac{2x}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$.

Justify your answer or show work step by step.

b. (4 pts) Find the integral F.

a. $\frac{A(x-4)+B(x+2)}{(x+2)(x-4)}$

$2x = A(x-4)+B(x+2)$

$x=4$ $\quad 2(4) = A(4-4)+B(4+2) \Rightarrow 8 = 6B \Rightarrow B = \frac{8}{6} = \frac{4}{3}$

$x=-2$ $\quad 2(-2) = A(-2-4)+B(-2+2) \Rightarrow -4 = -6A \Rightarrow A = \frac{4}{6} = \frac{2}{3}$

b. $\int \frac{\frac{2}{3}}{x+2}\,dx + \int \frac{\frac{4}{3}}{x-4}\,dx = \frac{2}{3}\int \frac{1}{x+2}\,dx + \frac{4}{3}\int \frac{1}{x-4}\,dx = \frac{2}{3}\ln|x+2| + \frac{4}{3}\ln|x-4| + C$

$u = x+2$ $\quad u = x-4$

$\frac{du}{dx} = 1$ $\quad \frac{du}{dx} = 1$

10

2. (*10 pts*) Let I be the integral $\int x^4 \ln(x)\, dx$.

Find I using Integration by Parts (IBP). Show work step by step.
Start by writing down "u" and "dv". *Reminder*: let $u = \ln(x)$.

$$\int u v' dx = uv - \int u' v\, dx \qquad u = \ln(x) \quad v' = x^4 \qquad u' = \frac{1}{x} \quad v = \frac{1}{5}x^5$$

$$= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\int \frac{1}{x} x^5\, dx$$

$$\frac{1}{5}\int \cancel{\frac{1}{x}} x^{\cancel{5}4}\, dx = \frac{1}{5}\int x^4\, dx = \frac{1}{5}\left(\frac{1}{5}x^5\right) = \frac{1}{25}x^5 \quad ✓$$

$$\int x^4 \ln(x)\, dx = \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + C \quad ✓$$

10

3. (*10 pts*) Let S be the integral $\int \frac{\sin w}{(\cos w - 1)^3}\, dw$.

Find S using integration by substitution.
Write down the substitution and show work step by step.
Note: $\cos w$ and $\sin w$ are the same as $\cos(w)$ and $\sin(w)$.

$$\int \frac{\sin(x)}{(\cos(x)-1)^3}\, dx \qquad u = \cos x - 1 \qquad dx = \frac{du}{-\sin x} \qquad = \int \frac{\cancel{\sin(x)}}{u^3} \frac{du}{\cancel{-\sin x}}$$

$$= -\int \frac{1}{u^3}\, du = -\int u^{-3}\, du = -\left(-\frac{1}{2}u^{-2}\right) = -\left(-\frac{1}{2}(\cos x - 1)^{-2}\right)$$

$$= -\left(-\frac{1}{2} \cdot \frac{1}{(\cos x - 1)^2}\right) = \frac{1}{2(\cos x - 1)^2} + C$$

30

4. (*30 pts*) (Volume). A region (see picture) is bounded by $y=\sqrt{x}$ and $y=\frac{x}{2}$.
Let S be the solid formed by rotating the region about the **x-axis.**
Read the whole problem before starting.
Change equations to the form $x=g(y)$ where necessary.

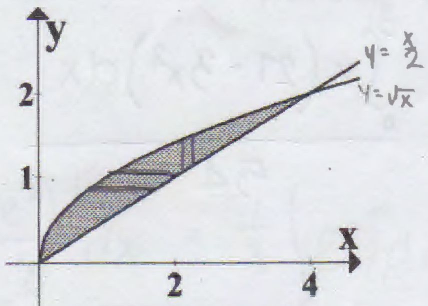
a. (2 pts) Sketch the solid S roughly, in a separate picture. Briefly describe its shape.

b. (10 pts) Using the **washer method**, set up an integral for the volume of S. Give the geometric meaning of each term in the integral (relate to the washer).

c. (10 pts) Using the **cylindrical shell method**, set up an integral for the volume of S. Give the geometric meaning of each term in the integral (relate to the shell).

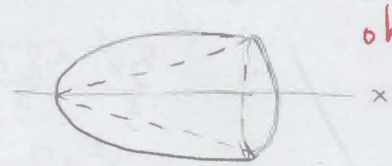
d. (8 pts) Find the volume of S by evaluating ONE of the integrals above. Give the antiderivative, showing work step by step. (More space on last pages.)

Give your answer in the form $N\pi$ where N is a number.



rotate about x axis

a)



ok

$y=\frac{x}{2} \rightarrow x=2y$

$y=\sqrt{x} \quad x=y^2$

b) $R=\sqrt{x}$ (outer)

$r=\frac{x}{2}$ (inner)

$$\int_a^b \pi\left(R(x)^2-r(x)^2\right)dx$$

$$\int_0^4 \pi\left[\underbrace{\left(\sqrt{x}\right)^2}_{\text{outer radius}}-\underbrace{\left(\frac{x}{2}\right)^2}_{\text{inner radius}}\right]\underbrace{dx}_{\text{width}}$$

ok

c) $$\int_a^b 2\pi\, r(x)h(x)\,dx$$ dy

ok

$$\int_0^2 2\pi \underbrace{y}_{\text{radius}}\underbrace{(2y-y^2)}_{\text{height}}\underbrace{dy}_{\text{width}}$$

$y=\frac{x}{2} \rightarrow x=2y$

$y=\sqrt{x} \rightarrow x=y^2$

(show)

d) $$\pi\int_0^4 x-\frac{x^2}{4}\,dx = \int x\,dx-\int\frac{x^2}{4}dx$$

$$=\pi\left(\frac{1}{2}x^2-\frac{1}{4}\cdot\frac{1}{3}x^3\right)=\pi\left(\frac{1}{2}x^2-\frac{1}{12}x^3\right)\Big|_0^4$$

$$\pi\left(\frac{1}{2}(4)^2-\frac{1}{12}(4)^3\right)-\underbrace{\cancel{\pi\left(\frac{1}{2}(0)^2-\frac{1}{12}(0)^3\right)}}_{0}$$

$$\pi\left(\frac{1}{2}(16)-\frac{1}{12}(64)\right)=\pi\left(8-\frac{64}{12}\right)=\frac{8}{3}\pi$$

20

5. (*20 pts*) (Center of mass.) Let R be the region (see picture) bounded by the y-axis and the graphs of $y = 3x^2$ and $y = 27$.
In the questions below, *Area* is the area of R, which is 54 units^2.

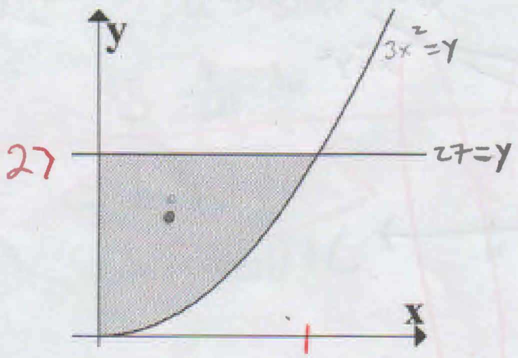
a. (2 pts) Mark the approximate location of the center of mass of the region (balance point) with a dot (•).

b. (9 pts) The x-value of the center of mass (COM) is $x_{\text{cm}} = \dfrac{\int_a^b x h(x)\,dx}{Area}$, where $h(x)$ is the height of the slice (rectangle) at x.

Find a and b and write down the integral for x_{cm}. (Be careful!)
(*Don't evaluate the integral.*)

c. (9 pts) The y-value of the COM is similar: $y_{\text{cm}} = \dfrac{\int_c^d y h(y)\,dy}{Area}$.

Find c and d and write down an integral for y_{cm}.
(*Don't evaluate the integral.*)



$3x^2 = 27$
$x^2 = 9$
$x = 3$

$54 = \int_0^3 (27 - 3x^2)\,dx$ ok

$x_{cm} = \dfrac{\int_a^b x(27-3x^2)\,dx}{54}$

$b = 3 \quad a = 0$

$x_{cm} = \dfrac{\int_0^3 x(27-3x^2)\,dx}{54}$

$y_{cm} = \dfrac{\int_c^d y\left(\sqrt{\frac{y}{3}}\right)dy}{54}$

$d = 27 \quad c = 0$

$y_{cm} = \int_0^{27} y\left(\sqrt{\frac{y}{3}}\right)dy$

Find **any two** of the following integrals.
Guess-and-Check (G&C), Algebra, Substitution (Sub),
tion by Parts (IBP), or Other (state).
w the "check").
s you want graded (or I'll grade the first two tried).

6a $\int \frac{y}{2y-3}\,dy$

it works! ☺

6 $\boxed{a}$ $\boxed{PF}$

a) $\int \frac{y}{2y-3} = \int \frac{y}{2(y-1.5)}$

$\int \frac{y}{2(y-1.5)} = \frac{A}{2} + \frac{b}{y-1.5}$

$y = A(y-1.5) + b(2)$

let y=1.5 $1.5 = 0 + 2b$

$\boxed{\frac{3}{4} = b}$

sub b back into get A alone

let y=0 $0 = -1.5A + \frac{3}{4}(2)$

$0 = -1.5A + 1.5$

$-1.5 = -1.5A$

$\boxed{-1 = A}$ ← oops

$A = +1$

$\int \frac{y}{2y-3} = \int \frac{-1}{2} + \frac{\frac{3}{4}}{y-1.5}$

$= \boxed{\ominus\frac{1}{2}y + \frac{3}{4}\ln|y-1.5|}$ + C

+

6. (*20 pts*) (10 pts each.) F
State the main method used—Gu
Partial Fractions (PF), Integratio
Show work step by step (or show
Circle the labels of the problems
(Extra space on the last pages.)

a. $\int \frac{y}{2y-3}\,dy$

10 b. $\int \frac{(e^x)^4}{8}\,dx$ algebra

c. $\int w\sqrt{3+w}\,dw$

10 d. $\int \frac{t}{t^2-36}\,dt$

$\boxed{\text{method?}}$

b) $\int \frac{e^{\cancel{x^4}\ 4x}}{8}\,dx$ X

$= \frac{1}{8}\int e^{4x}\,dx$ ✓

$u = 4x$

$\frac{du}{dx} = 4$

$\frac{du}{4} = dx$

$\frac{1}{8}\int e^u \frac{du}{4}$

$\frac{1}{8}\cdot\frac{1}{4}\int e^u\,du$

$\frac{1}{32}\, e^u$

$\frac{1}{32}e^{4x} + C$

d) $\int$

$\int$

t

$t =$

$t = 6$

6

$t = -6$

20

6. *(20 pts)* (10 pts each.) Find **any two** of the following integrals.
State the main method used—Guess-and-Check (G&C), Algebra, Substitution (Sub), Partial Fractions (PF), Integration by Parts (IBP), or Other (state).
Show work step by step (or show the "check").
Circle the labels of the problems you want graded (or I'll grade the first two tried).
(Extra space on the last pages.)

a. $\int \frac{y}{2y-3}\,dy$

b. $\int \frac{(e^x)^4}{8}\,dx$

10 c. $\int w\sqrt{3+w}\,dw$ Substitution

10 d. $\int \frac{t}{t^2-36}\,dt$ partial fraction

b) $\int \frac{(e^x)^4}{8}dx = \frac{1}{8}\int (e^x)^4 dx = \frac{1}{8}\int e^{4x} dx$

c) $\int w\sqrt{3+w}\,dw$ $\quad u=3+w \quad \frac{du}{dx}=1 \quad u-3=w \quad \int (u-3)\sqrt{u}\,du = \int (u-3)(u)^{1/2}du$

$= \int u^{3/2} - 3u^{1/2}du = \frac{2}{5}u^{5/2} - 2u^{3/2} = \frac{2}{5}(3+w)^{5/2} - 2(3+w)^{3/2}$ $+C$

d) $\int \frac{t}{(t+6)(t-6)}dt = \frac{A}{(t+6)} + \frac{B}{(t-6)} \rightarrow A(t-6)+B(t+6)=t$

$t=6 \quad \cancel{A(0)} + B(12) = 6$

$B = \frac{6}{12} = \frac{1}{2}$

$t=-6 \quad A(-12) + \cancel{B(0)} = -6$

$A = \frac{-6}{-12} = \frac{1}{2}$

$\int \frac{\frac{1}{2}}{(t+6)}dt + \int \frac{\frac{1}{2}}{(t-6)}dt = \frac{1}{2}\ln|t+6| + \frac{1}{2}\ln|t-6| + C$