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(*
        Traits of a PolyEma Direct Differencer.

          The impulse response of a continuous--time direct differencer is
         the derivative of a reference low--pass impulse response.

          Key traits of a continuous--time direct differencer are:
            1) net pos/neg area after
       taking the derivative of a poly--ema impulse response
            2) gauge correction to convert from net gauge to unit gauge,
    and then to delta gauge
           3) effective wireframe difference interval
            4) gain correction of an integrated direct differencer

*)
(* mth order low-pass polyema with location correction (tau / m) *)
HPEma[s_, m_] := 1/(1 + s τ / m)^m;
HPEmaGainSq[ω_, τ_, m_] := 1/(1 + ((ω τ)/m)^2)^m;
HPEmaGainSqSimp[x_, m_] := 1/(1 + x/m^2)^m;

(* make lists  *)
HPEmas = List[HPEma[s, 2], HPEma[s, 3], HPEma[s, 4], HPEma[s, 5]];
```

$$\mathrm{HPEma}[s, m] = \frac{1}{(1 + s\tau/m)^m},\quad \mathrm{HPEmaGainSq}[\omega,\tau,m] = \frac{1}{\left(1+\left(\frac{\omega\tau}{m}\right)^2\right)^m},\quad \mathrm{HPEmaGainSqSimp}[x,m] = \frac{1}{(1+x/m^2)^m}$$

```
(*
    From the transfer function of continuous-time poly-emas:
      1) compute the impulse response
      2) find t_p, the time of the peak weight of the impulse response
*)

(* compute h(t) for each order *)
hPEmas = Map[Function[H, InverseLaplaceTransform[H, s, t]], HPEmas];
"Poly-ema impulse responses for orders 2:5 ->"
hPEmas

(* compute t_p such that h'(t_p) == 0 *)
tpeaksolns = Map[Function[h, Last[Solve[D[h, t] == 0, t]]], hPEmas];
tpeaks = t /. tpeaksolns;

"time t_p such that h[t_p] is maximum ->"
tpeaks

"weight at h[t_p] ->"
MapThread[Function[{x, y}, x /. y], {hPEmas, tpeaksolns}]
MapThread[Function[{x, y}, N[x /. y]], {hPEmas, tpeaksolns}]
```

Poly-ema impulse responses for orders 2:5 ->

$$\left\{\frac{4\, e^{-\frac{2t}{\tau}}\, t}{\tau^2}, \frac{27\, e^{-\frac{3t}{\tau}}\, t^2}{2\,\tau^3}, \frac{128\, e^{-\frac{4t}{\tau}}\, t^3}{3\,\tau^4}, \frac{3125\, e^{-\frac{5t}{\tau}}\, t^4}{24\,\tau^5}\right\}$$

time t_p such that $h[t_p]$ is maximum ->

$$\left\{\frac{\tau}{2}, \frac{2\,\tau}{3}, \frac{3\,\tau}{4}, \frac{4\,\tau}{5}\right\}$$

weight at $h[t_p]$ ->

$$\left\{\frac{2}{e\,\tau}, \frac{6}{e^2\,\tau}, \frac{18}{e^3\,\tau}, \frac{160}{3\,e^4\,\tau}\right\}$$

$$\left\{\frac{0.735759}{\tau}, \frac{0.812012}{\tau}, \frac{0.896167}{\tau}, \frac{0.976834}{\tau}\right\}$$

```
(*
    From the transfer function of continuous-time poly-emas:
      1) multiply by s
      2) find ωp, the non-zero frequency at which the gain spectrum is maximum
*)

(* multiply by s  *)
"Derivative transfer functions ->"
HDerivPEmas = Map[Function[H, s H], HPEmas]

(* replace s → ⅈ ω *)
HDerivPEmasOmega = Map[Function[H, H /. {s → ⅈ ω}], HDerivPEmas];

(* compute gain-squared spectra *)
HDerivPEmaGains =
 Map[Function[H, Simplify[H Conjugate[H], Assumptions → {ω ∈ Reals, τ ∈ Reals} ]],
  HDerivPEmasOmega]

(* find ωp such that (s H[s])./{s→ⅈω) is a rootSolve  *)
"frequencies ωsoln such that |s H[s]| = 0 ->"
wsolns = Map[Function[f, Solve[D[f, ω] == 0, ω]], HDerivPEmaGains]

"frequency ωp such that |s H[s]| is maximum ->"
wpeaksolns = Map[Function[l, Last[l]], wsolns];
wpeaksolns
N[wpeaksolns]

"value of |s H[s]| at ωp ->"
MapThread[Function[{x, y}, Sqrt[x /. y]], {HDerivPEmaGains, wpeaksolns}]
MapThread[Function[{x, y}, N[Sqrt[x /. y]]], {HDerivPEmaGains, wpeaksolns}]
```

Derivative transfer functions ->

$$\left\{\frac{s}{\left(1+\frac{s\tau}{2}\right)^2},\ \frac{s}{\left(1+\frac{s\tau}{3}\right)^3},\ \frac{s}{\left(1+\frac{s\tau}{4}\right)^4},\ \frac{s}{\left(1+\frac{s\tau}{5}\right)^5}\right\}$$

$$\left\{\frac{16\,\omega^2}{(4+\tau^2\omega^2)^2},\ \frac{729\,\omega^2}{(9+\tau^2\omega^2)^3},\ \frac{65\,536\,\omega^2}{(16+\tau^2\omega^2)^4},\ \frac{9\,765\,625\,\omega^2}{(25+\tau^2\omega^2)^5}\right\}$$

frequencies ω_{soln} such that |s H[s]| = 0 ->

$$\left\{\left\{\{\omega\to 0\},\ \left\{\omega\to -\frac{2}{\tau}\right\},\ \left\{\omega\to \frac{2}{\tau}\right\}\right\},\ \left\{\{\omega\to 0\},\ \left\{\omega\to -\frac{3}{\sqrt{2}\,\tau}\right\},\ \left\{\omega\to \frac{3}{\sqrt{2}\,\tau}\right\}\right\},\right.$$
$$\left.\left\{\{\omega\to 0\},\ \left\{\omega\to -\frac{4}{\sqrt{3}\,\tau}\right\},\ \left\{\omega\to \frac{4}{\sqrt{3}\,\tau}\right\}\right\},\ \left\{\{\omega\to 0\},\ \left\{\omega\to -\frac{5}{2\,\tau}\right\},\ \left\{\omega\to \frac{5}{2\,\tau}\right\}\right\}\right\}$$

frequency ω_p such that |s H[s]| is maximum ->

$$\left\{\left\{\omega \to \frac{2}{\tau}\right\}, \left\{\omega \to \frac{3}{\sqrt{2}\ \tau}\right\}, \left\{\omega \to \frac{4}{\sqrt{3}\ \tau}\right\}, \left\{\omega \to \frac{5}{2\ \tau}\right\}\right\}$$

$$\left\{\left\{\omega \to \frac{2.}{\tau}\right\}, \left\{\omega \to \frac{2.12132}{\tau}\right\}, \left\{\omega \to \frac{2.3094}{\tau}\right\}, \left\{\omega \to \frac{2.5}{\tau}\right\}\right\}$$

value of |s H[s]| at ω_p ->

$$\left\{\sqrt{\frac{1}{\tau^2}}, \frac{2\sqrt{\frac{1}{\tau^2}}}{\sqrt{3}}, \frac{3}{4}\sqrt{3}\sqrt{\frac{1}{\tau^2}}, \frac{16\sqrt{\frac{1}{\tau^2}}}{5\sqrt{5}}\right\}$$

$$\left\{\sqrt{\frac{1}{\tau^2}}, 1.1547\sqrt{\frac{1}{\tau^2}}, 1.29904\sqrt{\frac{1}{\tau^2}}, 1.43108\sqrt{\frac{1}{\tau^2}}\right\}$$

```
(*
      From the transfer function of continuous-time poly-emas:
        1) multiply by s
        2) compute the impulse response, the derivative of the low-pass reference
        3) confirm that the gain of h'(t) is zero
*)

(* multiply by s *)
"s H[s] transfer functions ->"
HDerivPEmas

(* compute the inverse laplace transform to get d/dt h(t) = h'(t) *)
"h'(t) functions ->"
hderivPEmas = Map[Function[H, InverseLaplaceTransform[H, s, t]], HDerivPEmas]

(* confirm that at t_p, h'(t_p) is zero *)
"value of h'(t_p) ought to be zero ->"
MapThread[Function[{x, y}, x /. y], {hderivPEmas , tpeaksolns}]

(* confirm that the gain of h'(t) is zero *)
"gain of h'(t) ought to be zero ->"
netgains =
 Map[Function[h, Integrate[h, {t, 0, ∞}, Assumptions → {τ > 0}]], hderivPEmas]
```

s H[s] transfer functions ->

$$\left\{\frac{s}{\left(1+\frac{s\,\tau}{2}\right)^2}, \frac{s}{\left(1+\frac{s\,\tau}{3}\right)^3}, \frac{s}{\left(1+\frac{s\,\tau}{4}\right)^4}, \frac{s}{\left(1+\frac{s\,\tau}{5}\right)^5}\right\}$$

h′(t) functions ->

$$\left\{-\frac{4\,e^{-\frac{2t}{\tau}}\,(2\,t-\tau)}{\tau^3}, \frac{27\,e^{-\frac{3t}{\tau}}\,t\,(-3\,t+2\,\tau)}{2\,\tau^4}, \frac{128\,e^{-\frac{4t}{\tau}}\,t^2\,(-4\,t+3\,\tau)}{3\,\tau^5}, \frac{3125\,e^{-\frac{5t}{\tau}}\,t^3\,(-5\,t+4\,\tau)}{24\,\tau^6}\right\}$$

value of h′(t_p) ought to be zero ->

{0, 0, 0, 0}

gain of h′(t) ought to be zero ->

{0, 0, 0, 0}

```
(*
    Integrate h'(t) on [0, t_p]. Of course, this is just:

        ∫_0^{t_p} h'(t) ⅆt = h(t_p) - h(0)

    Poly-emas have it easy because there is no zero crossing of h(t),
as there is with Bessel filters.

    Normalize h'(t) for unit gauge.
*)

(* Definite integral of net areas  *)
"net area of h'(t) on [0, t_p] ->"
netareas = MapThread[Function[{h, tlim},
   Integrate[h, {t, 0, tlim}, Assumptions → {τ > 0}]], {hderivPEmas, tpeaks}]

(* Normalize h'(t) for unit gauge *)
"normalized h'(t) for unit gauge -> "
hderivPEmasUnitGauge =
 MapThread[Function[{h, netarea}, h / netarea], {hderivPEmas, netareas}]
```

net area of h'(t) on [0, t_p] ->

$$\left\{\frac{2}{e\,\tau}, \frac{6}{e^2\,\tau}, \frac{18}{e^3\,\tau}, \frac{160}{3\,e^4\,\tau}\right\}$$

normalized h'(t) for unit gauge ->

$$\left\{-\frac{2\,e^{1-\frac{2t}{\tau}}\,(2\,t-\tau)}{\tau^2}, \frac{9\,e^{2-\frac{3t}{\tau}}\,t\,(-3\,t+2\,\tau)}{4\,\tau^3}, \frac{64\,e^{3-\frac{4t}{\tau}}\,t^2\,(-4\,t+3\,\tau)}{27\,\tau^4}, \frac{625\,e^{4-\frac{5t}{\tau}}\,t^3\,(-5\,t+4\,\tau)}{256\,\tau^5}\right\}$$

```
(*
    Compute first moment of h[t] over [0, tp].
    Compute difference of net positive and net negative arms to get tinterval.
*)

"check that net arm areas are equal:"
"∫_0^tp h'(t)dt ->"
MapThread[Function[{h, tlim}, Integrate[h, {t, 0, tlim}, Assumptions → {τ > 0}]],
 {hderivPEmasUnitGauge, tpeaks}]
"∫_tp^∞ h'(t)dt ->"
MapThread[
 Function[{h, tlim}, -Integrate[h, {t, tlim, ∞}, Assumptions → {τ > 0}]],
 {hderivPEmasUnitGauge, tpeaks}]

"first moment of pos arm ->"
M1PosArm = MapThread[
  Function[{h, tlim}, Integrate[t h, {t, 0, tlim}, Assumptions → {τ > 0}]],
  {hderivPEmasUnitGauge, tpeaks}]

"first moment of neg arm ->"
M1NegArm = MapThread[
  Function[{h, tlim}, -Integrate[t h, {t, tlim, ∞}, Assumptions → {τ > 0}]],
  {hderivPEmasUnitGauge, tpeaks}]

"temporal interval of direct differencer ->"
tinterval = MapThread[
   Function[{posarm, negarm}, - Simplify[posarm - negarm]], {M1PosArm, M1NegArm}];

"temporal interval of the wireframe components of a direct differencer ->"
tinterval
Map[Function[v, N[v]], tinterval]

(* Map[Function[h, Simplify[LaplaceTransform[h, t, s]]], hderivPEmasUnitGauge] *)
```

check that net arm areas are equal:

$\int_0^{t_p} h'(t)\,dt$ ->

{1, 1, 1, 1}

$\int_{t_p}^{\infty} h'(t)\,dt$ ->

{1, 1, 1, 1}

first moment of pos arm ->

$$\left\{-\frac{1}{2}\,(-3+e)\,\tau,\ -\frac{1}{6}\left(-9+e^{2}\right)\tau,\ \frac{1}{36}\left(53-2\,e^{3}\right)\tau,\ -\frac{3}{160}\left(-77+e^{4}\right)\tau\right\}$$

first moment of neg arm ->

$$\left\{\frac{3\,\tau}{2},\ \frac{3\,\tau}{2},\ \frac{53\,\tau}{36},\ \frac{231\,\tau}{160}\right\}$$

temporal interval of direct differencer ->

temporal interval of the wireframe components of a direct differencer ->

$$\left\{\frac{e\,\tau}{2},\ \frac{e^{2}\,\tau}{6},\ \frac{e^{3}\,\tau}{18},\ \frac{3\,e^{4}\,\tau}{160}\right\}$$

$\{1.35914\,\tau,\ 1.23151\,\tau,\ 1.11586\,\tau,\ 1.02372\,\tau\}$