Unit 8 Interpolation with Equal intervals

Structure:

8.1 Introduction

"Interpolation is the art of reading between the lines of a table".

Theile

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available. For instance, consider a function $y = f(x)$. As x takes values $x_0, x_1, x_2, \ldots, x_n$, let the corresponding values of y be $y_0, y_1, y_2, \ldots, y_n$. That is for a given table of values (x_k, y_k), $k = 0, 1, 2, 3, \ldots, n$; the process of estimating the value for y, for any intermediate value of x, is called *interpolation*. However, the method of computing the value of y, for given values of x, lying outside the table of values of x is known as *extrapolation*. It may be noted that if the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy. But, in practice, it may be difficult or sometimes impossible to know the function y= $f = f(x)$ in its exact form.

We will discuss a practical example; let us consider the computation of the trajectory of a rocket flight, where we solve the Euler's dynamical equations of motion *t* compute its position and velocity vectors at specified times during the flight. Under the same conditions, suppose, we require the position and velocity vector, at some other intermediate times; we need not compute the trajectory again by solving the dynamical equations. Instead, we can use the best known interpolation technique to get the desired values.

In numerical methods we come across functions defined by set of tabulated points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, though the explicit nature of the function y = f(x) is not known. In this case we consider a general form of the polynomial.

$y = \Phi(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, which agree at the above set of tabulated points. The function $\Phi(x)$ may not exactly define the function $y = f(x)$ as there may be a set of points other than (x_i, y_i), $i = 0, 1, 2, \ldots, n$, that satisfy the relation $y = f(x)$ but may not satisfy the general form. In this case we treat the general form of the polynomial as an approximation to $y = f(x)$. The polynomial $\Phi(x)$ is known *as interpolation polynomial.*

Example:

The value of the function $y = f(x) = 2^x$ is tabulated as shown in the following table:

x	0	1	2
$y = 2^x$	1	2	4

There are *3* tabulated value of *x* and *y*, that is, *(0, 1), (1, 2)* and *(2, 4).* We can construct an *interpolation polynomial of degree* 2 as

$$y = \phi(x) = a + bx + cx^2.$$

Solve for a, b and c using the tabulated points,

we get $y = 1 + \frac{x}{2} + \frac{x^2}{2} = \Phi(x)$

Consider the table:

X	0	1	2	4
$y = f(x) = 2^x$	1	2	4	16
$y = \phi(x) = 1 + \frac{x}{2} + \frac{x^2}{2}$	1	2	4	11

$f(x)$ and $\phi(x)$ agree at the set of tabulated points but not at $x = 4$. Thus $\phi(x) = 1 + \frac{x}{2} + \frac{x^2}{2}$ is the interpolation polynomial for the given data.

Objectives:

At the end of this unit the student should be able to:

- Apply interpolation Formulas with equal differences

8.2 Newton's Forward Difference Interpolation Formula

Given the set of $(n+1)$ values, $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of x and y, it is required to find a polynomial of the n^th^ degree $y_n(x)$ such that $y(x)$ and $y_n(x)$ agree at the tabulated points.

Let the values of x be equidistant, that is $x_i = x_0 + ih,\ i = 0,1,2,3 \ldots, n.$

Let $y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots$

$+ a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$ be a polynomial of degree n. **(1)**

Put $x = x_0, x_1, x_2 \ldots, x_n$ successively in *(1)*, we get

$y_n(x_0) = a_0$

$y_n(x_1) = a_0 + a_1(x_1 - x_0)$

$y_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

… and so on.

But $y_n(x_i) = y_i,\ i = 0, 1, 2, 3 \ldots, n.$

we get

$a_0 = y_0$ since $y_n(x_0) = y_0,\ y_n(x_1) = y_1$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= \frac{\Delta y_0}{h} \qquad \textit{(by taking } x_1 = x_0 + h)$$

Similarly $a_2 = \dfrac{\Delta^2 y_0}{h^2\ 2!}$

$$a_3 = \frac{\Delta^3 y_0}{h^3\ 3!}$$

…………

………..

$$a_n = \frac{\Delta^n\ y_0}{h^n\ n!}$$

Substituting these values in (1), we obtain

$$y_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{h^2\ 2!}(x - x_0)(x - x_1) + \ldots$$

$$\ldots\ldots\ldots + \frac{\Delta^n y_0}{h^n\ n!}(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{h-1}) \qquad -(2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph$$

$$x - x_1 = (x_0 + ph) - (x_0 + h) = (p - 1)\,h$$

$$x - x_2 = (x_0 + ph) - (x_0 + 2h) = (p - 2)\,h \text{ etc.}$$

Equation (2) becomes

$$y_n(x) = y_n(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$\ldots. + \frac{p\,(p-1)\,(p-2)\ldots..(p-n+1)}{n!}\Delta^n y_0 \qquad -(3)$$

Expression (3) is called *Newton's forward interpolation formula* as contains y_0

and the leading forward differences of y_0. It is useful for interpolation near the beginning of a set of tabular values.

Example:
Evaluate *f(15)*, given the following table of values:

x	10	20	30	40	50
y = f(x)	46	66	81	93	101

Solution: The value *x = 15* is near to the beginning of the table. We use Newton's forward difference interpolation formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
		20			
20	66		-5		
		15		2	
30	81		-3		-3
		12		-1	
40	93		-4		
		8			
50	101				

Using Newton forward difference formula

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$+\frac{p(p-1)(p-2)(p-4)}{4!}\Delta^4 y_0 \text{ where } x = x_0 + ph.$$

Here $h = 10$, $x_0 = 10$, $y_0 = 46$, $\Delta y_0 = 20$, $\Delta^2 y_0 = -5$, $\Delta^3 y_0 = 2$, $\Delta^4 y_0 = -3$. Therefore

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5.$$

Therefore

$$f(15) = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3) = 56.8672.$$

Example:

Find the missing term from the following table:

x	0	1	2	3	4
y	1	3	9	–	81

Solution: There are *4* tabulated values, that is, *(0, 1), (1, 3), (2, 9)* and *(4, 81)* are given, therefore its 4^{th} forward difference must be zero.

By constructing difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		a–19	
2	9		a–15		124 – 4a
		a–9		105–3a	
3	a (say)		90–2a		
		81 –a			
4	81				

Now we put $\Delta^4 y = 0$, we have $124 - 4a = 0$. This implies that $a = 31$

Thus $y = 31$ at $x = 3$

Observation of the given values of y indicates that $y = 3^x$.

Putting $x = 3$ in 3^x, we get *27*. This does not tally with the value obtained $y = 3$, reason is that 3^x is not a polynomial in x but we have assumed $y = f(x)$ to be a polynomial of degree *3*.

Example:

Find the cubic polynomial which takes the following values $y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$. Hence or otherwise, obtain $y(0.5)$.

Solution: Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3$
and $y_0 = 1,\ y_1 = 0,\ y_2 = 1\ \ y_3 = 10$

We form the difference table

x	y	Δ	Δ^2	Δ^3
$x_0 = 0$	$\mathbf{y_0 = 1}$			
		$\mathbf{\Delta y_0 = -1}$		
$x_1 = 1$	$y_1 = 0$		$\mathbf{\Delta^2 y_0 = 2}$	
		$\Delta y_1 = 1$		$\mathbf{\Delta^3 y_0 = 6}$
$x_2 = 2$	$y_2 = 1$		$\Delta^2 y_1 = 8$	
		$\Delta y_2 = 9$		
$x_3 = 3$	$y_3 = 10$			

From the above table we have,
$x_0 = 0,\ y_0 = 1,\ \Delta y_0 = -1,\ \Delta^2 y_0 = 2,\ \Delta^3 y_0 = 6$

Using Newton forward difference formula

$$y(x) = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2\,!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3\,!}\Delta^3 y_0 \text{ where } x = x_0 + ph$$

Here $h = 1,\ x_0 = 0,$ therefore $p = x$

$$y\,(x) = 1 + x\,(-1) + \frac{x\,(x-1)}{1.2}2 + \frac{x\,(x-1)\,(x-2)}{1.2.3}\ 6$$

$$= 1 - x + (x^2 - x) + (x^3 - 3x^2 + 2x)$$

$$y\,(x) = x^3 - 2x^2 + 1$$

which is the polynomial from which we obtained the above tabular values.

To compute y (0.5)

Here $x_0 + ph = x = 0.5$

$p = 0.5$ since $x_0 = 0$ and $h = 1$

$$y(0.5) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 1 + 0.5(-1) + \frac{(0.5)(0.5-1)}{2}\,2 + \frac{(0.5)(0.5-1)(0.5-2)}{6}\,6$$

$$= 0.625$$

which is the same value as that obtained by substituting $x = 0.5$ in the cubic polynomial.

Example:

From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

Solution: First we prepare the cumulative frequency table, as follows,

Marks less than (x)	40	50	60	70	80
No. of students y	31	73	124	159	190

Now the difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
40	**31**				
		42			
50	73		**9**		
		51		**–25**	
60	124		–16		**37**
		35		12	
70	159		– 4		
		31			
80	190				

We shall find the number of students with marks less than 45.

From the above table we have,

$y_0 = 31,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37$

Taking $x_0 = 40,\ x = 45$

We have

$x = x_0 + ph$

$45 = 40 + p10$

$p = 0.5$

Using Newton's forward interpolation formula, we get

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(45) = 31 + 0.5(42) + \frac{(0.5)(0.5-1)}{2!}(9)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{3!}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.87.$

Therefore $y(45) = 47.87$

The number of students with marks less than 45 is 47.87, that is 48. But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between
40 and 45 = 48 – 31 = 17 students.

SAQ 1: Find the cubic polynomial which takes the values y(1) = 24, y(3) = 120, y(5) = 336 & y (7) = 720. Hence obtain the value of y(8).

SAQ 2: Using Newton's forward difference formula, find the sum
$S_n = 1^3+2^3+3^3+\ldots+n^3$

8.3 Newton's Backward Difference Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the equally spaced values $x_0, x_1, x_2, \ldots, x_n$ of x. Suppose it is required to evaluate $y_n(x)$ for $x = x_n + ph$ where p is any real number. Then we have

$$y_n(x) = y_n(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n.$$

This is known as *Newton's Backward Difference Interpolation Formula* and it uses tabular values to the left of y_n. This formula is useful for interpolation near the end of the tabular values.

Example

The sales in a particular department store for five years are given in the following table:

Year	1974	1976	1978	1980	1982
Sales (in lkh.)	40	43	48	52	57

Estimate the sales for the year *1979*.

Solution: We construct the backward difference table is

x	y	∇	∇^2	∇^3	∇^4
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	–1	– 3	
1982	**57**	**5**	**1**	**2**	**5**

We shall find the number of students with marks less than *45*.

From the table, we find, with $x_n = 1982$, $y_n = 57$, $\nabla y_n = 5$, $\nabla^2 y_n = 1$, $\nabla^3 y_n = 2$, $\nabla^4 y_n = 5$.

$$p = \frac{1979 - 1982}{2} = -1.5.$$

By the Newton's interpolation formula,

$$f(1979) = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1) + \frac{(-1.5)(-0.5)(0.5)}{6}(2) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5) = 50.1172.$$

Example:

Using Newton's backward interpolation formula, find the interpolating polynomial that approximates the function given by the following table:

x	*0*	*1*	*2*	*3*
f (x)	*1*	*3*	*7*	*13*

Hence find *f (2.5)*

Solution: For the given data, the backward difference table is as shown below:

x	y	∇	∇^2	∇^3
$x_0 = 0$	1			
		2		
$x_1 = 1$	3		2	
		4		**0**
$x_2 = 2$	7		**2**	
		6		
$x_3 = 3$	**13**			

From the table, we find, with $x_n = x_3 = 3$, $y_n = 13$, $\nabla y_n = 6$, $\nabla^2 y_n = 2$, $\nabla^3 y_n = 0$

For $x = x_n + ph$, we have

$$p = \frac{x - x_n}{n} = \frac{x-3}{1} = x - 3$$

and the Newton's backward interpolation formula gives

$$y(x) = 13 + 6(x-3) + \frac{2}{2!}(x-3)(x-3+1) + 0$$

$$= x^2 + x + 1 .$$

Thus $y(x) = x^2 + x + 1$ is the interpolating polynomial that approximates the given function.

Hence, $f(2.5) \simeq y(2.5) = (2.5)^2 + 2.5 + 1 = 9.75$

Example: Estimate the value of sin x at $x = 25^0$ using Newton's Backward difference formula with the help of following table:

x (in degree)	10	20	30	40	50
Sinx	0.1736	0.3420	0.5000	0.6428	0.7660

Solution: the backward difference formula for the above table is

x	Sinx	∇	∇^2	∇^3	∇^4
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.1580		-0.0048	
30	0.5000		-0.0152		0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			

50	0.7660				

By Newton's backward interpolation formula, we have

$$y_n(x) = y_n(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n. \text{ Where } p = \frac{x-x_n}{h}$$

Here $x_n = 50$, $x = 25$, $h = 10$, $p = \frac{x-x_n}{h} = \frac{25-50}{10} = -2.5$

$$y_n(x) = 0.7660 + (-2.5)(0.1232) + \frac{(-2.5)(-2.5+1)}{2!}(-0.0196) + \frac{(-2.5)(-2.5+1)(-2.5+2)}{3!}(-0.0044)$$

$$+ \frac{(-2.5)(-2.5+1)(-2.5+2)(-2.5+3)}{4!}(0.0004)$$

$$= 0.7660 - 0.308 - 0.03675 + 0.000125 - 0.0000156$$

$$= 0.42136$$

Example: The following data gives the melting point of an alloy of lead and zinc, where y is the temperature in degree "c" and P is the percentage of lead in the alloy

P	40	50	60	70	80	90
y	184	204	226	250	276	304

Find the melting point of alloy containing 84% lead.

Solution: The value of 84 is near the end of the table, so we will use Newton's Backward interpolation formula.

We have $x_n = 90$, $x = 84$, $h = 10$

$$p = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6$$

The difference table is

P	y	∇	∇^2	∇^3	∇^4	∇^5
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

$y_n(x) = y_n(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$

$\ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n$.

$y_n(x) = 304 + (-0.6)(28) + \frac{(-0.6)(-0.6+1)}{2!}(2) + 0$

$= 304 - 16.8 - 0.24$

$= 287.44$

SAQ 3: Find the value of f(7.5) for the table

x	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

SAQ 4: Compute f(1.38) for the table

x	1.1	1.2	1.3	1.4
f(x)	7.831	8.728	9.627	10.744

SAQ 5: Find the value of y when x = 0.37, using the given values

x	0.000	0.10	0.20	0.30	0.40
f(x)	1.000	1.2214	1.4918	1.8221	2.2255

SAQ 6: Find f(2.8) from the following table

x	0	1	2	3
f(x)	1	2	11	34

8.4 Central Difference Formula

For interpolation near the middle of the table, central difference formula is preferred

8.4.1 Gauss Forward Formula

Let y = f(x) be a given function. Let $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$are equidistant set of observations with common difference h and let $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ are their corresponding values then

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots,$ where $p = \frac{x-x_0}{h}$

Proof: Consider the polynomial equation by using the arrow marks as shown in the table below

Let $y_p = y_0 + G_1\Delta y_0 + G_2\Delta^2 y_{-1} + G_3\Delta^3 y_{-1} + G_4\Delta^4 y_{-2} + \cdots,$ (1)

Where $G_1, G_2, G_3, G_4, \dots$ are unknowns

Now, $y_p = y_{p+0} = E^0 y_0 = (1+\Delta)^p y_0$

$= \left(1 + C_1^p \Delta + C_2^p \Delta^2 + C_3^p \Delta^3 + \cdots + C_p^p \Delta^p\right) y_0$

$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \Delta^p y_0$ (2)

So, $y_{-1} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0$

Implies $y_{-1} = y_0 - \Delta y_0 + \Delta^3 y_0 - \cdots$

$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \cdots$ (3)

$\Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^3 y_0 + \cdots$ (4)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
.	.				
.	.				
.	.				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				
.	.				
.	.				
.	.				

Substituting (2), (3) & (4) in (1), we get

$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \Delta^p y_0 =$

$$y_0 + G_1 \Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \cdots) + G_3(\Delta^3 y_0 - \Delta^3 y_0 + \cdots) + \cdots$$

Now, comparing the corresponding coefficients, we get

$$G_1 = p, \qquad G_2 = \frac{p(p-1)}{2},$$

Also, $-G_2 + G_3 = \frac{p(p-1)(p-2)}{3!}$

Implies $G_3 = \frac{p(p+1)(p-1)}{3!}$,

Similarly, $G_4 = \frac{p(p+1)(p-1)(p-2)}{4!}, \ldots$

Substituting these values of , $G_1, G_2, G_3, G_4, \ldots$ in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Note: This formula is used to interpolate the values of the function for the values of p (0 < p < 1) measured forwardly from the origin.

Example: Apply Gauss forward formula to obtain f(x) = x Sinx+2 at x = 3.5 from the table

x	2	3	4	5
f(x)	3.818	2.423	-1.027	-2.794

Solution: Here $x_0 = 3$, x = 3.5, h = 1,

$$p = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$$

The difference table for this is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	-1	3.818			
			-1.395		
3	0	2.423		-2.055	
			-3.45		3.738
4	1	-1.027		1.683	
			-1.767		
5	2	-2.794			

So by Gauss Forward formula, we have

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 2.423 + (0.5)(-3.45) + \frac{0.5(0.5-1)}{2!}(-2.0555) +$$

$$\frac{0.5(0.5+1)(0.5-1)}{3!}(3.738)$$

= 2.423 – 1.725 + 0.2569 –0.2336

= 0.72131

Example: Find the value of y at x = 0 using Gauss Forward formula from the table below

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Solution: The difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	−2	18.4708				
			−0.6564			
25	−1	17.8144		−0.0510		
			−0.7074		−0.0054	
29	0	17.1070		−0.0564		−0.0022
			−0.7638		−0.0076	
33	1	16.3432		−0.0640		
			−0.8278			
37	2	15.5154				

Here $x_0 = 29$, $x = 30$, $h = 4$,

$$p = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25$$

So by Gauss Forward formula, we have

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$= 17.1070 + (0.25)(-0.7638) + \frac{0.25(0.25-1)}{2!}(-0.0564) + \frac{0.25(0.25+1)(0.25-1)}{3!}(-0.0076) + \frac{0.25(0.25+1)(0.25-1)(0.25-2)}{4!}(-0.0022)$

$= 17.1070 - 0.19095 + 0.0052875 + 0.0003 - 0.00004$

$= 16.9215$

Example: Apply Gauss forward formula to find the value of f(x) at x = 3.75 form the following table

x	2.5	3.0	3.5	4.0	4.5	5.0
f(x)	24.145	20.043	20.225	18.644	17.262	16.047

Solution: The difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	−2	24.145					
			−2.102				
3.0	−1	22.043		0.284			
			−1.818		−0.047		
3.5	0	20.225		0.237		0.009	
			−1.581		−0.038		−0.003
4.0	1	18.644		0.199		0.006	
			−1.382		−0.032		
4.5	2	17.262		0.167			
			−1.215				
5.0	3	16.047					

Here $x_0 = 3.5$, $x = 3.75$, $h = 0.5$,

$$p = \frac{x - x_0}{h} = \frac{3.75 - 3.5}{0.5} = 0.5$$

So by Gauss Forward formula, we have

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$= 20.225 + (0.5)(-1.581) + \frac{0.5(0.5-1)}{2!}(0.237) + \frac{0.5(0.5+1)(0.5-1)}{3!}(-0.038) + \frac{0.5(0.5+1)(0.5-1)(0.5-2)}{4!}(0.009)$

$+ \frac{0.5(0.5+1)(0.5-1)(0.5-2)(0.5+2)}{5!}(-0.003)$

$= 20.225 - 0.7905 - 0.02963 + 0.002375 + 0.0002109$

$- 0.00003516$

$= 19.4074$

SAQ7: Find the value of f(35)**,** by applying Gauss forward formula from the following table

x	20	30	40	50
f(x)	51203	43931	34563	24348

SAQ 8: Apply Gauss forward formula to find the value of f(9) if f(0) = 14, f(4) = 24, f(8) = 32, f(16) = 40

8.4.2 Gauss Backward Formula

Let y = f(x) be a given function. Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ are given a set of observations with common difference h and $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ are their corresponding values. Then

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$

$+\frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots,$ where $p = \frac{x-x_0}{h}$

Proof: From the difference table, we have

x	y	Δ	Δ^2	Δ^3	Δ^4
.	.				
.	.				
.	.				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				
.	.				
.	.				
.	.				

Let us assume a polynomial equation by using the arrow marks, we get

$y_p = y_0 + G_1 \Delta y_{-1} + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-2} + G_4 \Delta^4 y_{-2} + \cdots,$ (1)

Where $G_1, G_2, G_3, G_4, \ldots$ are unknowns

Also, we know that

$$y_p = y_{p+0} = E^0 y_0 = (1+\Delta)^p y_0$$

$= \left(1 + C_1^p \Delta + C_2^p \Delta^2 + C_3^p \Delta^3 + \cdots + C_p^p \Delta^p\right) y_0$

$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \Delta^p y_0$ (2)

Now, $y_{-1} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0$

Implies $y_{-1} = y_0 - \Delta y_0 + \Delta^3 y_0 - \cdots$

Therefore, $\Delta y_{-1} = \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \cdots$ (3)

$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \cdots$ (4)

Also, $y_{-2} = E^{-2} y_0 = (1+\Delta)^{-2} y_0$

$= (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots) y_0$

$= y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - 4\Delta^3 y_0 + \cdots$

$\Delta^3 y_{-2} = \Delta^3 y_0 - 2\Delta^4 y_0 + \cdots$ (5)

Substituting (2), (3), (4)&(5) in (1), we get

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \Delta^p y_0 =$$

$$y_0 + G_1(\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \cdots) + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \cdots) + G_3(\Delta^3 y_0 - 2\Delta^4 y_0 + ..)$$

Comparing corresponding coefficients, we get

$$G_1 = p, -G_1 + G_2 = \frac{p(p-1)}{2!}, implies\ G_2 = \frac{p(p+1)}{2!}$$

Also, $G_1 - G_2 + G_3 = \frac{p(p-1)(p-2)}{3!}$

Implies $G_3 = \frac{p(p+1)(p-1)}{3!}$,

Similarly, $G_4 = \frac{p(p+1)(p-1)(p+2)}{4!}, \ldots$

Substituting these values of , $G_1, G_2, G_3, G_4, \ldots$ in (1), we get

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots, \quad \text{where } p = \frac{x - x_0}{h}$$

Note: This formula is used to interpolate the value of the function for negative values of p (-1 < p < 0)

Example: Interpolate by employing the Gauss backward formula, the sales of the firm for the year 1966, given that

Year	1931	1941	1951	1961	1971	1981
Sales (in lakhs)	12	15	20	27	39	52

Solution: So far Gauss backward formula, we will choose x_0 = 1971,
x = 1966, h = 10,

$$p = \frac{x - x_0}{h} = \frac{1966 - 1971}{10} = -0.5$$

Difference table is:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1931	−4	12					
			3				
1941	−3	15		2			
			5		0		
1951	−2	20		2		3	
			7		3		−10
1961	−1	27		5		−7	
			12		−4		
1971	0	39		1			
			13				
1981	1	52					

By Gauss backward formula, we have

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots,$$

$$= 39 + (-0.5)(12) + [\tfrac{(-0.5+1)(-0.5)}{2}](1)$$

$$+ [\tfrac{(-0.5+1)(-0.5)(-0.5-1)}{6}](-4) + \ldots$$

= 39 – 6 - 0.125 - 0.2

= 32.625

Therefore, Sales in the year 1966 is Rs 32.625

Example: The area A of a circle of diameter d is given for the following values

x	80	85	90	95	100
f(x)	5026	5674	6362	7088	7854

Calculate the area of the circle of diameter 105

Solution: Here x_0 = 100, x = 105, h = 5, $p = \frac{x-x_0}{h} = \frac{105-100}{5} = 1$

The difference table is:

d	p	A	Δ A	Δ^2 A	Δ^3 A	Δ^4 A
80	−4	5026				
			648			
85	−3	5674		40		
					−2	
			688			
						4
90	−2	6362				
				38		
					2	
			726			
95	−1	7078		40		
			766			
100	0	7854				

By Gauss backward formula, we have

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots,$$

= 7854 + 766 + 40

= 8660.

Example: Given the table

x	1.5	2.5	3.5	4.5
Y= xe^x	8.963	24.364	66.340	180.034

Find the value of y at x = 2.25 by Gauss Backward formula.

Solution: Here h = 1, x_0 = 2.5, x = 2.25

$$p = \frac{x - x_0}{h} = \frac{2.25 - 2.5}{1} = -0.25$$

The difference table is:

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.5	-1	8.963			
			15.401		
				26.465	
2.5	0	24.364			
					45.473
			41.866		
3.5	1	66.230		71.938	
			113.804		
4.5	2	180.034			

So by Gauss Backward formula, we get

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots$$

= 24.364 + (-0.25)15.401 + $\frac{[(-0.25)(-0.25+1)]}{2!}$(26.465)

= 24.364 - 3.85025 - 2.4811

= 18.03265

Example: Apply the best Gauss interpolation formula to compute the population of a town for the year 1974, given that

Year	1939	1949	1959	1969	1979	1989
Sales(in lakhs)	**12**	**15**	**20**	**27**	**39**	**52**

Solution: Since the year 1974 of x lies in the middle of the year 1969 to 1979. So we can use any one of the Gauss interpolation formula.

Here x_0 = 1969, x = 1974, h = 10, $p = \frac{x-x_0}{h} = \frac{1974-1969}{10} = 0.5$

Difference table is:

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1939	−3	12					
			3				
1949	−2	15		2			
			5		0		
1959	−1	20		2		3	
			7		3		−10
1969	0	27		5		−7	
			12		−4		
1979	1	39		1			
			13				
1989	2	52					

By Gauss backward formula, we have

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots,$$

$= 27 + (0.5)(7) + [\frac{(0.5+1)(0.5)}{2}](5)$

$+ [\frac{(0.5+1)(0.5)(0.5-1)}{6}](3) + [\frac{(0.5+1)(0.5)(0.5-1)(0.5+2)}{24}](-7)$

$+ [\frac{(0.5+1)(0.5)(0.5-1)(0.5+2)(0.5-2)}{120}](-10)$

= 27+ 3.5 + 1.875 – 0.1875 + 0.2743 – 0.1172

= 32.345

SAQ 9: Given that $\sqrt{12500} = 111.803399, \sqrt{12510} = 111.848111$, $\sqrt{12520} = 111.892806, \sqrt{12530} = 111.937483$, show by Gauss Backward formula that $\sqrt{12516} = 111.874930$

SAQ 10: Use Gauss formula to get y_{30} given that y_{21} = 18.4708, y_{25} = 17.8144, y_{29} = 17.1070, y_{33} = 16.3432, y_{37} = 15.5154.

8.4.3 Stirling's Formula

Let y = f(x) be a given function. Let ….., $x_{-2,}x_{-1,}x_{0,}x_{1,}x_{2,}$ …. are given set of observations with common difference h and ….., $y_{-2,}y_{-1,}y_{0,}y_{1,}y_{2,}$ …. are their corresponding values. Then

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots,$ where $p = \frac{x-x_0}{h}$

Proof: Stirling's formula is obtained from the average of Gauss forward and Gauss backward formula

As we know from Gauss forward formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots,$ …(1)

Also, form Gauss Backward formula,

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2}$$

$+\frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots,$ …(2)

So, Stirling's formula = $\frac{1}{2}$ (Gauss forward formula + Gauss Backward formula)

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots,$$

Note: Stirling's formula is useful when p lies in between -1/2 to ½. But it gives best estimate when p lies in -1/4 to 1/4.

Example: Use Stirling's formula to show that $\tan 16^0 = 0.2867$ from the following table

x (in degrees)	0^0	5^0	10^0	15^0	20^0	25^0	30^0
Y= tanx	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Solution: Taking$x_0 = 15^0$, $x = 16^0$, $h = 5$,

$$p = \frac{x - x_0}{h} = \frac{16^0 - 15^0}{5} = 0.2$$

The difference table is

x	p	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
0^0	−3	0.000						
			0.0875					
5^0	−2	0.0875		0.0013				
			0.0888		0.0015			
10^0	−1	0.1763		0.0028		0.0002		
			0.0916		0.0017		−0.0002	
15^0	0	0.2679		0.0045		0.000		0.0011
			0.0961		0.0017		0.0009	
20^0	1	0.3640		0.0062		0.0009		
			0.1023		0.0026			
25^0	2	0.4663		0.0088				
			0.1111					
30^0	3	0.5774						

By Stirling's formula, we have

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Putting p = 0.2 and other values and other values from the table, we get

$$y_{0.2} = 0.2679 + (0.2)(\frac{0.0961 + 0.0916}{2}) + \frac{(0.2)^2}{2!}(0.0045)$$

$$+\frac{(0.2)(0.004-1)}{3!}(\frac{0.0017+0.0017}{2}) + 0 + \cdots$$

= 0.2679 + 0.01877 + 0.00009 – 0.00005

= 0.28671

Example: Apply Stirling's formula to find the value of f(1.22) from the following table which gives the values of $f(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{x^2}{2}}dx$, at intervals of x = 0.5 from x = 0 to 2.

Solution: Takingx$_0$ = 1, x = 1.22, h = 0.5,

$$p = \frac{x - x_0}{h} = \frac{1.22 - 1.00}{0.5} = 0.44$$

The difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	-2	0				
			191			
0.5	-1	191		-41		
			150		-17	
1	0	341		-58		27
			92		10	
1.5	1	433		-48		
			44			
2.0	2	477				

By Stirling's formula, we have

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Putting p = 0.44 and other values and other values from the table, we get

$$y_{0.44} = 341 + (0.44)(\frac{92+150}{2}) + \frac{(0.44)^2}{2!}(-58)$$

$$+\frac{(0.44)(0.1936-1)}{3!}(\frac{-17+10}{2}) + \frac{(0.1936)(0.1936-1)}{4!}(-27) + \cdots$$

= 341 + 53.24 – 5.6144 + 0.206976 + 0.1755

= 389.00807

Example: Find f(0.41) using Stirling's formula, if f(0.30) = 0.1179, f(0.35) = 0.1368, f(0.40) = 0.1558, f(0.45) = 0.1736, f(0.50) = 0.1915.

Solution: The difference table is

x	p	$10^{-4}y$	$10^{-4}\Delta y$	$10^{-4}\Delta^2 y$	$10^{-4}\Delta^3 y$	$10^{-4}\Delta^4 y$
0.30	-2	1179				
			189			
0.35	-1	1368		-3		
			186		-1	
0.40	0	1554		-4		2
			182		1	
0.45	1	1736		-3		
			179			
0.50	2	1915				

Takingx_0 = 0.40, x = 0.41, h = 0.05,

$$p = \frac{x-x_0}{h} = \frac{0.41-0.40}{0.05} = 0.2$$

By Stirling formula, we have

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Putting p = 0.2 and other values and other values from the table, we get

$$y_{0.2} = 10^{-4}\{1544 + (0.2)(\frac{186+182}{2}) + \frac{(0.2)^2}{2!}(-4)$$

$$+\frac{(0.2)(0.04-1)}{3!}(\frac{-1+1}{2}) + \frac{(0.04)(0.04-1)}{4!}(2) + \cdots\}$$

$= 10^{-4}(1544 + 36.8 – 0.08 +0 – 0.0032)$

$= 10^{-4}(1580.7168) = 0.1580$

Example: Use Stirling's formula to compute f(12.2) from the following data

x	10	11	12	13	14
Y = 10^5logx	23967	28060	31788	35209	38368

Solution: The difference table is

x	p	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^5 y$
10	-2	23967				
			4093			
11	-1	28060		-365		
			3728		58	
12	0	31788		-307		-13
			3421		45	
13	1	35209		- 262		
			3159			
14	2	38368				

Takingx_0 =12, x = 12.2, h = 1,

$$p = \frac{x-x_0}{h} = \frac{12.2-12}{1} = 0.2$$

By Stirling's formula, we have

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Putting p = 0.2 and other values from the table, we get

$$y_{0.2} = 10^5 \mathrm{f}(12.2) = 31788 + (0.2)(\frac{3421 + 3728}{2}) + \frac{(0.2)^2}{2!}(-307)$$

$$+\frac{(0.2)(0.04-1)}{3!}(\frac{45+58}{2}) + \frac{(0.04)(0.04-1)}{4!}(-13) + \cdots$$

= 31788 + 714.9 – 6.14 – 1.64 + 0.208

= 32495.33

f(12.2) = 0.32495

SAQ 11: Using Stirling's formula find f (1.63) from the following table

x	1.5	1.6	1.7	1.8	1.9
f(x)	17.609	20.412	23.045	25.27	27.875

SAQ 12 : Find y, when x = 35, using Stirling's formula. Given

x	20	30	40	50
y	512	439	346	243

SAQ 13: Use Stirling's formula to find f(28), given that f(20)= 49225, f(25) = 48316, f(30) = 47236, f(35) = 45926, f(40) = 44306.

8.4.4 Bessel's Formula

Let y = f(x) be a given function. Let …., $x_{-2,}x_{-1,}x_{0,}x_{1,}x_{2,}$ …. are given set of observations with common difference h and …., $y_{-2,}y_{-1,}y_{0,}y_{1,}y_{2,}$ …. are their corresponding values. Then

$$y_p = \frac{y_0 + y_1}{2} + (p - \frac{1}{2})\Delta y_0 + \frac{p(p-1)}{2!}(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2})$$

$$+[\frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}]\Delta^3 y_{-1} + \left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots, \text{where } p = \frac{x - x_0}{h}$$

Solution: We know by Gauss forward formula that

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$+\frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots,$ .....(1)

Also, from forward difference table, we have

$$y_1 - y_0 = \Delta y_0$$

Implies, $y_0 = y_1 - \Delta y_0$

Also, $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

$$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$$

Similarly, $\Delta^2 y_{-2} = \Delta^4 y_{-1} - \Delta^3 y_{-2}$ etc.,

So eq(1) can be written as

$$y_p = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + p\Delta y_0 + \frac{1}{2}\left[\frac{p(p-1)}{2!}\right]\left(\frac{\Delta^2 y_{-1}}{2} + \frac{\Delta^2 y_{-1}}{2}\right) + \cdots$$

Now, using the above results, we obtain

$$y_p = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + p\Delta y_0 + \frac{1}{2}\left[\frac{p(p-1)}{2!}\right]\Delta^2 y_{-1}$$

$$+\frac{1}{2}\left[\frac{p(p-1)}{2!}\right](\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{y_0+y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \frac{1}{2}\left(\frac{p(p-1)}{2!}\right)(\Delta^2 y_{-1} + \Delta^2 y_0)$$

$$+\left[\frac{p(p-1)}{2!}\right]\left(-\frac{1}{2} + \frac{p+1}{3}\right)\Delta^3 y_{-1} + \cdots$$

$$= \frac{y_0+y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left(\frac{p(p-1)}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+\left[\frac{(p-\frac{1}{2})p(p-1)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

Note: Bessel's formula is more useful when p lies between 1 / 4 to 3/4

Example: Apply Bessel's formula to compute f(12.3) from the following table, when f(x) = 2x + Cosx

x	10	11	12	13	14
f(x)	20.9848	22.9816	24.9781	26.9743	28.9703

Solution: Takingx_0 =12, x = 12.3, h = 1,

$$p = \frac{x - x_0}{h} = \frac{12.3 - 12}{1} = 0.3$$

The difference table for this is

x	p	f(x)	Δ	Δ^2	Δ^3	Δ^4
10	-2	20.9848				
			1.9968			
11	-1	22.9816		-0.0003		
			1.9965		0	
12	0	24.9781		-0.0003		0.0001
			1.9962		0.0001	
13	1	26.9743		-0.0002		
			1.996			
14	2	28.9703				

By Bessel's formula, we have

$$y_p = \frac{y_0+y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left(\frac{p(p-1)}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \left[\frac{(p - \frac{1}{2})p(p-1)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

$$= \left(\frac{24.9781+26.9743}{2}\right) + \left(0.3 - \frac{1}{2}\right)(1.9962) + \left(\frac{(0.3)(0.3-1)}{2!}\right)\left(\frac{-0.003-0.002}{2}\right) + \left(\frac{\left(0.3-\frac{1}{2}\right)(0.3)(0.3-1)}{3!}\right)(0.0001)$$

= 25.9762 – 0.39924 +0.0002625 +0.0000007

= 25.577

Example: Probability distribution function values of a normal distribution are given below

x	0.2	0.6	1.0	1.4	1.8
f(x)	0.39104	0.33322	0.24197	0.14973	0.07895

Using suitable interpolation formula, find the value of f(1.2).

Solution: Taking x_0 =1, x = 1.2, h = 0.4,

$$p = \frac{x - x_0}{h} = \frac{1.2 - 1}{0.4} = 0.5$$

This shows that Bessel's formula is the most suitable one

The difference table is

x	p	Y	Δ	Δ^2	Δ^3	Δ^4
0.2	-2	0.39104				
			-0.05782			
0.6	-1	0.33322		-0.03348		
			-0.09125		0.03244	
1.0	0	0.24197		-0.00099		-0.05291
			-0.09224		-0.02047	
1.4	1	0.14973		-0.02146		
			-0.07078			
1.8	2	0.07895				

By Bessel's formula, we have

$$y_p = \frac{y_0+y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left(\frac{p(p-1)}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \left[\frac{(p - \frac{1}{2})p(p-1)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

$$= \left(\frac{0.24197+0.14973}{2}\right) + \left(0.5 - \frac{1}{2}\right)(-0.09224) + \left(\frac{(0.5)(0.5-1)}{2!}\right)\left(\frac{-0.00099-0.02146}{2}\right) +$$

$$\left(\frac{\left(0.5-\frac{1}{2}\right)(0.5)(0.5-1)}{3!}\right)(-0.02047)$$

= 0.19585 + 0 + 0.001403+0

= 0.197

Example: Apply Bessel's formula to obtain f(25) given that f (20) = 2854, f(24) = 3162, f(28) = 3544, f (32) = 3992

Solution: Taking x_0 =24, x = 25, h = 4,

$$p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

The difference table is

x	p	f(x)	Δ	Δ^2	Δ^3
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

By Bessel's formula, we have

$$y_p = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left(\frac{p(p-1)}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \left[\frac{(p - \frac{1}{2})p(p-1)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{(p+1)p(p-1)(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

$$= \left(\frac{3162+3544}{2}\right) + \left(0.25 - \frac{1}{2}\right)(382) + \left(\frac{(0.25)(0.25-1)}{2!}\right)\left(\frac{74+66}{2}\right) +$$

$$\frac{\left(0.25 - \frac{1}{2}\right)(0.25)(0.25-1)}{3!})(-8)$$

= 3353 – 95.5 – 6.5625 – 0.0625

= 3250.875

SAQ 14: Using Bessel's formula estimate $3\sqrt{46.24}$, given the following table of $y = 3\sqrt{x}$

x	41	45	49	53
y	3.4482	3.5569	3.6593	3.7563

SAQ 15: Using Bessel's formula, find f(5)

x	0	4	8	12
y	143	158	177	199

SAQ 16: Apply Bessel's formula to get the value of y (45), given

x	40	44	48	52
y	51.08	63.24	70.88	79.84

8.5 Summary

In this unit, various methods for constructing interpolation polynomials for equally spaced values of arguments are discussed. Methods of building different types of difference tables and how to use them for estimating function values at any point are discussed.

8.6 Terminal Questions

1. The population of a town in the decennial census was as given below. Estimate the population for the years 1895 and 1925.

Year: x	1891	1901	1911	1921	1931
Population: y (in thousands)	46	66	81	93	101

2. Find the value of a) tan0.12 b) tan0.26 from the values of tan x given in the table

x	0.10	0.15	0.20	0.25	0.30
y = tanx	0.1003	0.1511	0.2027	0.2553	0.3093

3. The area A of a circle of diameter d is given for the following values

x	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find approximate values for the areas of the circle of diameter 82 and 92 resp.

4. A population of a town is given below . Apply Gauss backward formula to get the population in 1926

Year(x)	1911	1921	1931	1941	1951
Population in thousands (y)	15	20	27	39	52

5. Using Gauss interpolation formula, find a) Sin 61.24, b) Sin63.48 from the following table

x	60	61	62	63	64	65
Sinx	0.86603	0.87462	0.88295	0.89101	0.89879	0.90631

6. Using Bessel's formula, estimate f(25) given that

x	20	24	28	32
f(x)	24	32	35	40

7. Apply Bessel's formula to find the value of f(27.4) from the table

x	25	26	27	28	29	30
f(x)	4.000	3.846	3.704	3.571	3.448	3.333

8. Apply Bessel's formula to find the value of y = f(x) at x = 3.75, given that

x	2.5	3.0	3.5	4.0	4.5	5.0
f(x)	24.145	22.043	20.2250	18.644	17.262	16.047

9. Find the value of f(15) using Bessel's formula if f(10) = 2845, f(14) = 3162, f(18) = 3544, f(22) = 3992

10. Use Bessel's formula to find the values of f at x = 1.95 given that

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
f(x)	2.979	3.144	3.283	3.391	3.463	3.997	4.491

11. Apply Stirling formula to find the value of f(1.22) from the table

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
fx	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.999570	0.99385	0.9385

8.7 Answers

Self Assessment Questions

1. $x^3+6x^2+11x+6$, 990
2. $[\frac{n(n+1)}{2}]^2$
3. 421.87
4. 10.963
5. 2.0959
6. 27.992
7. 39431
8. 33(approx.)
9. 10.15.5154
10. 21.21933
11. 395
12. 47692
13. 3.58931
14. 162.4
15. 65.0175

Terminal Questions

1. 54.85, 96.84 thousands
2. 0.1205, 0.2662
3. 5281, 6504
4. 22.898
5. 0.87798 b) 0.89259
6. 32.945
7. 3.6497
8. 19.407
9. 3251
10. 3.346
11. 0.93910