Unit 7 Integrations

Structure:

7.1 Introduction

Most of the mathematical operations we come across occur in inverse pairs. For example, addition and subtraction, multiplication and division, squaring and taking square roots are such pairs. In this chapter we study integration as the inverse operation of differentiation. Integrals also have independent interpretation. It generalizes the process of summation. It can be used for evaluating the area under the graph of a function.

Objectives:

At the end of the unit you would be able to

- integrate standard functions
- apply the concept of definite integrals in the process of summation

7.2 Integration of standard function

Definition: If *f(x)* is a function of a real variable, then *g(x)* is the integral of f(x) if $\frac{d}{dx}(g(x)) = f(x)$. It is denoted by $\int f(x)dx$.

Remark: Integration is the operation of determining a function whose derivative is the given function.

Note: Clearly

$$f(x) = \frac{d}{dx}\left[\int f(x)dx\right] \quad \text{...... (7.1)}$$

$$f(x) = \int f'(x)dx \qquad \text{...... (7.2)}$$

(7.1) and (7.2) can be stated as follows:

Differential coefficient of integral of f(x) = integral of differential coefficient of *f(x) = f(x)*

Remark: If f(x) = x^2 and g(x) = x^2 + 2, then $f'(x) = g'(x) = 2x$. That is, the derivative of a function remains the same if a constant is added to it.

So while writing the integral of f(x), we need to add c, an arbitrary constant (by arbitrary constant we mean any real value).

In table 7.1 we list the integrals of some of standard function.

$$\int kdx = kx + c$$

$$\int xdx = \frac{x^2}{2} + c$$

$$\int x^2dx = \frac{x^3}{3} + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ when } n \neq -1$$

$$\int \frac{1}{x}dx = \log x + c$$

$$\int e^x\, dx = e^x + c$$

$$\int \sin xdx = -\cos x + c$$

$$\int \cos xdx = \sin x + c$$

$$\int \sec^2 xdx = \tan x + c$$

$$\int \text{cosec}^2 xdx = -\cot x + c$$

$$\int \sec x \tan xdx = \sec x + c$$

$$\int \text{cosec}x\cot xdx = -\text{cosec}\, x + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \log\left(x+\sqrt{x^2-1}\right)+c$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$$

Table 7.1: Table of integrals

We frequently apply constant function rule (Rule 1) and sum and difference rule (Rule 2) for integration which are similar to differentiation. However we do not have "product rule and quotient rule" in integration. So problems in integration are more difficult than the problems in differentiation.

Rule 1: (Constant function rule)

$\int cf(x)dx = c\int f(x)dx$ where c is a constant

Rule 2: (Sum and difference rule)

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example: Evaluate $\int f(x)dx$ when f(x) equals

a) x^{-4} b) $\frac{ax^2+bx+c}{x^3}$

c) $\left(x+\frac{1}{x}\right)^2$ d) $2\sin x + 3\cos x + e^x - \frac{1}{x}$

e) $\frac{3}{\sqrt{1-x^2}} + \frac{4}{\sqrt{1+x^2}} + \frac{5}{x\sqrt{x^2-1}}$

Solution:

a) $\int x^{-4}dx = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c$

$$= -\frac{1}{3x^3} + c$$

b) $\int \frac{ax^2 + bx + c}{x^3} dx = \int \left(\frac{ax^2}{x^3} + \frac{bx}{x^3} + \frac{c}{x^3} \right) dx$

$= \int \left(ax^{-1} + bx^{-2} + cx^{-3} \right) dx = a\int \frac{1}{x} dx + b \int \frac{1}{x^2} dx + c \int \frac{1}{x^3} dx$ (by rules 1 and 2)

$= \log x + \frac{bx^{-2+1}}{-2+1} + \frac{cx^{-3+1}}{-3+1} + k$

$= \log x - \frac{b}{x} - \frac{c}{2x^2} + k$

(**Note:** We use k for arbitrary constant since c appears in the function)

c) $\int \left(x + \frac{1}{x} \right)^2 dx = \int \left(x^2 + \frac{1}{x^2} + 2x.\frac{1}{x} \right) dx$

$= \int x^2 dx + \int \frac{1}{x^2} dx + \int 2dx$ (by rule 2)

$= \int x^2 \; dx + \int x^{-2} \; dx + 2\int dx$ (by rule 1)

$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + 2x + c$

$= \frac{x^3}{3} - \frac{1}{x} + 2x + c$

d) $\int \left(2\sin x + 3\cos x + e^x - \frac{1}{x} \right) dx$

$= \int 2\sin x dx + \int 3\cos x dx + \int e^x dx - \int \frac{1}{x} dx$ (by rule 2)

$= 2\int \sin x dx + 3\int \cos x dx + \int e^x dx - \int \frac{1}{x} dx$ (by rule 1)

$= 2(-\cos x) + 3(\sin x) + e^x - \log x + c$

$= -2\cos x + 3\sin x + e^x - \log x + c$

e) $\int \left(\frac{3}{\sqrt{1-x^2}} + \frac{4}{\sqrt{1+x^2}} + \frac{5}{x\sqrt{x^2-1}} \right) dx$

$$= \int \frac{3}{\sqrt{1-x^2}} dx + \int \frac{4}{\sqrt{1+x^2}} dx + \int \frac{5}{x\sqrt{x^2-1}} dx \qquad \text{(by rule 2)}$$

$$= 3\int \frac{dx}{\sqrt{1-x^2}} + 4\int \frac{dx}{\sqrt{1+x^2}} + 5\int \frac{dx}{x\sqrt{x^2-1}} \qquad \text{(by rule 1)}$$

$$= 3\sin^{-1} x + 4\log\left(x + \sqrt{1+x^2} \right) + 5\sec^{-1} x + c.$$

S.A.Q.1: Integrate $e^x + x^{-1} + x^{-2} + \sin x + \sec^2 x + \frac{1}{1+x^2}$

7.3 Rules of Integration

We have the following rules for integrating complicated function.

Rule 1: Constant function rule

Rule 2: Sum and difference rule

Rule 3: By substitution

Rule 4: Integration by parts

Sum and difference rules:

We have already given rules 1 and 2 and used it for integrating functions in section 7.2. In trigonometry, some functions can be expressed as sum or difference of two simpler trigonometric functions. The following formulas are useful in integration.

2Sin A Cos B = Sin (A + B) + Sin (A – B)

2Cos A Sin B = sin (A + B) – Sin (A – B)

2Cos A Cos B = Cos (A + B) – Cos (A – B)

2sin A Sin B = Cos (A – B) – Cos (A + B)

Worked Example: Evaluate $\int \sin 5x \cos 2x \, dx$

Solution:

$$\int \sin 5x \cos 2x \, dx = \frac{1}{2}\int 2\sin 5x \cos 2x \, dx$$

$$= \frac{1}{2}\int \left(\sin(5x+2x) + \sin(5x-2x) \right) dx$$

$$= \frac{1}{2}\int sin7x\,dx + \frac{1}{2}\int sin3x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) + \frac{1}{2}\left(\frac{-\cos 3x}{3}\right) + c$$

$$= \frac{-\cos 7x}{14} - \frac{\cos 3x}{6} + c$$

Worked Example: Integrate *Sin 10x Sin 2x w.r.t. x*

Solution:

$$\int \sin 10x \sin 2x\,dx = \frac{1}{2}\int 2\sin 10x \sin 2x\,dx$$

$$= \frac{1}{2}\int (\cos(10x - 2x) - \cos(10x + 2x))\,dx$$

$$= \frac{1}{2}\int (\cos 8x - \cos 12x)dx$$

$$= \frac{1}{2}\int \cos 8x\,dx - \frac{1}{2}\int \cos 12x\,dx$$

$$= \frac{1}{2}.\frac{\sin 8x}{8} - \frac{1}{2}.\frac{\sin 12x}{12} + c$$

$$\int \sin 10x \sin 2x\,dx = \frac{\sin 8x}{16} - \frac{\sin 12x}{24} + c$$

Worked Example: Evaluate $\int sin^2\ xdx$

Solution:

$$sin^2\ xdx = \int\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} - \frac{1}{2}.\frac{sin2x}{2} + c = \frac{x}{2} - \frac{sin2x}{4} + c$$

Integration by substitution:

Now we state rule 3.

Rule 3 (Substitution rule): If x can be written as a function g(t) of t, then

$$\int f(x)dx = \int f(g(t))\,g'(t)dt$$

Proof: As $x = g(t)$, $\frac{dx}{dt} = g'(t)$.

Hence $dx = g'(t)dt$ (Don't think that this step is obtained by cross multiplication. dx and dt are called differentials and $dx = g'(t)dt$ is the relation connecting the differentials).

So $\int f(g(t))g'(t)dt = \int f(x)dx$.

Example: Evaluate $\int (ax+b)^n dx, n \neq -1.$

Solution: Let $t = ax + b.$ Then $\frac{dt}{dx} = a.$ So $dt = adx$

$$dx = \frac{1}{a}dt$$

$$\int (ax+b)^n dx = \int t^n \frac{1}{a}dt = \frac{1}{a}\int t^n dt = \frac{1}{a}\frac{t^{n+1}}{n+1} + c$$

$$= \frac{(a+b)^{n+1}}{a(n+1)} + c$$

Note: If $\int f(x)\,dx = g(x) + c$ then

$$\int f(ax+b)dx = \frac{1}{a}g(ax+b) + c \qquad \text{………………….. (7.3)}$$

So using table (7.1) we get a list of formulas for $\int sin(ax+b)dx$ etc.

For example:

$$\int cos(a+b)dx = \frac{1}{a}sin(ax+b) + c$$

$$\int sec^2(ax+b)dx = \frac{1}{a}tan(ax+b) + c$$

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$$

Example: Integrate the following w.r.t. x

a) $(3-2x)^n$ b) $sin(3-2x)$

c) $sec^2(7-4x)$ d) $e^{\left(\frac{x-4}{3}\right)}$

Solution:

a) $\int (3-2x)^n \, dx = \dfrac{(3-2x)^{n+1}}{(-2).(n+1)} + c$

$$= -\frac{(3-2x)^{n+1}}{2(n+1)} + c$$

b) $\int sin(3-2x)dx = \left(\dfrac{1}{-2}\right)(-cos(3-2x)) + c$

$$= \frac{cos(3-2x)}{2} + c$$

c) $\int sec^2(7-4x)dx = \dfrac{1}{(-4)} tan(7-4x) + c$

$$= \frac{-\tan(7-4x)}{4} + c$$

d) $\int e^{\left(\frac{x-4}{3}\right)} dx = \dfrac{e^{\left(\frac{x-4}{3}\right)}}{\frac{1}{3}} + C$

$$= 3e^{\left(\frac{x-4}{3}\right)} + c$$

Working Example 4: Integrate the following w.r.t. x

a) $x\sqrt{x+a}$ b) $\dfrac{x}{\sqrt{a+bx}}$ c) $\text{Sin}^3 x$ d) $\dfrac{x^2}{(a+bx)^3}$ e) $\text{Sin}^2 3x$

Solution: Denote the required integrals by I

a) $\int x\sqrt{x+a}\, dx = I$ (say)

Put $x + a = t^2$ so $x = t^2 - a$, $dx = 2t\, dt$

$$\therefore I = \int (t^2 - a)t \;\; 2tdt = 2\int (t^4 - at^2)\; dt$$

$$= 2\left(\int t^4 dt - a\int t^2 \; dt\right) = 2\left(\frac{t^5}{5} - \frac{at^3}{3}\right) + c$$

$$= 2\left(\frac{(x+a)^{\frac{5}{2}}}{5} - \frac{a(x+a)^{\frac{3}{2}}}{3}\right) + c$$

b) $\int \frac{x}{\sqrt{a+bx}}dx = I$ (say)

Put $a + bx = t^2$ $x = \frac{t^2 - a}{b}, \ dx = \frac{2t\ dt}{b}$

$$I = \int \left(\frac{t^2 - a}{b}\right) \cdot \frac{1}{t} \cdot \frac{2t}{b} dt = \frac{2}{b^2} \int \left(t^2 - a\right) dt$$

$$= \frac{2}{b^2}\left(\int t^2\,dt - a\int 1 dt\right)$$

$$= \frac{2}{b^2}\left(\frac{t^3}{3} - at\right) + c$$

$$= \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{3}{2}}}{3} - a(a+bx)^{\frac{1}{2}}\right] + c$$

c) $\int sin^3\ xdx = \int sin^2 x \sin x\ dx$

$$= \int \left(1 - cos^2 x\right) \sin x\ dx$$

Let $t = cosx$; $dt = -sinx\ dx$

$$I = \int \left(1 - t^2\right)(-dt)$$

$$= -\int 1\,dt + \int t^2\ dt$$

$$= -t + \frac{t^3}{3} + c$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

d) $\int \frac{x^2}{(a+bx)^3}dx = I$ (say)

Put $a + bx = t$. So $x = \frac{(t-a)}{b}, \ dx = \frac{1}{b}dt$

$$\therefore I = \int \frac{\left(\frac{t-a}{b}\right)^2}{t^3} \frac{1}{b}dt = \frac{1}{b^3}\int \frac{(t-a)^2}{t^3}dt$$

$$= \frac{1}{b^3}\int \frac{\left(t^2 + a^2 - 2at\right)}{t^3} dt$$

$$= \frac{1}{b^3}\left(\int t^{-1} dt + a^2 \int t^{-3} dt - 2a \int t^{-2} dt\right)$$

$$= \frac{1}{b^3}\left(\log t + \frac{a^2 t^{-2}}{-2} - \frac{2at^{-1}}{-1}\right) + c$$

$$= \frac{1}{b^3}\left(\log(a+bx) - \frac{a^2}{2(a+bx)^2} + \frac{2a}{(a+bx)}\right) + c$$

e) $\int sin^2 3x\, dx = \int \frac{1}{2}(1 - cos(2.3x))\, dx$

$$= \frac{1}{2}\left(\int 1dx - \int cos\, 6x\, dx\right)$$

$$= \frac{1}{2}\left(x - \frac{1}{6} sin\, 6x\right) + c$$

Your skill in integration lies in spotting a suitable variable *t = g(x)* so that $dt = g'(x)dx$ can be "located" in f(x)dx. We illustrate the method of substitution by more examples, so that you can master this technique thoroughly.

Hereafter I stand for the integral to be evaluated.

Example: Integrate the following functions w.r.t. x

a) $x^{n-1}\, sin(x^n)$ b) $\frac{1}{x(log\, x)^n}$ c) $\frac{e^{\tan^{-1} x}}{1+x^2}$

Solution:

a) If $t = x^n$ then $dt = nx^{n-1}\, dx$ and this appears in $x^{n-1} \sin(x^n)$.

$$I = \int \sin(x^n)\, x^{n-1}\, dx = \int \sin t \frac{dt}{n} = -\frac{\cos t}{n} + c$$

$$= -\frac{\cos(x^n)}{n} + C$$

b) $\int \frac{1}{x(\log x)^n} dx = I$ (Say)

Put log $x = t$ so $\frac{1}{x} dx = dt$

$$\therefore I = \int \frac{1}{(\log x)^n} \frac{1}{x} dx = \int \frac{1}{t^n} dt = \int t^{-n} dt$$

$$= \frac{t^{-n+1}}{-n+1} + c = \frac{(\log x)^{-n+1}}{-n+1} + c$$

$$= -\frac{1}{(n-1)(\log x)^{n-1}} + c$$

c) $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = I$ (Say)

Put $\tan^{-1} x = t$ So $\frac{1}{1+x^2} dx = dt$

$$\therefore I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^{\tan^{-1} x} \frac{1}{1+x^2} dx$$

$$= \int e^t dt = e^t + c = e^{\tan^{-1} x} + c$$

Working Example: Integrate the following w.r.t. x

a) $\frac{x^2}{1+x^6}$ b) xe^{-x^2} c) $\frac{\sin\sqrt{x}}{\sqrt{x}}$

Solution:

a) $\int \frac{x^2}{1+x^6} dx = I$ (Say)

Put $x^3 = t$so $3x^2\, dx = dt;$ $x^2 dx = \frac{1}{3} dt$

$$I = \int \frac{x^2 dx}{1+(x^3)^2} = \int \frac{\frac{1}{3} dt}{1+t^2} = \frac{1}{3} \tan^{-1} t + c$$

$$= \frac{1}{3} \tan^{-1} x^3 + c$$

b) $\int xe^{-x^2} dx = I$ (Say)

$Put\ x^2 = t; 2xdx = dt;\ \ xdx = \frac{1}{2}dt$

$$I = \int e^{-x^2} xdx = \int e^{-t}\,\frac{1}{2}dt = \frac{1}{2}\int e^{-t}dt = \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + c$$

$$= \frac{-e^{-x^2}}{2} + c$$

c) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = I$ (Say)

Put $\sqrt{x} = t;\ \frac{1}{2\sqrt{x}}dx = dt;\ \frac{1}{\sqrt{x}}dx = 2dt$

$$I = \int \sin\sqrt{x}\,\frac{1}{\sqrt{x}}dx = \int \sin t\, 2dt = 2\int \sin t dt$$

$$= 2(-\cos t) + c = -2\cos\sqrt{x} + c$$

Working Example: Evaluate

a) $\int \tan\theta\, d\theta$ b) $\int \sec\theta\, d\theta$ c) $\int \frac{1+\cos x}{(x+\sin x)^2} dx$

d) $\int x^2\sqrt{a^3 + x^3}\ dx$ e) $\int \cot\theta\, d\theta$ f) $\int \operatorname{cosec}\theta\, d\theta$

Solution:

a) $\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta$

Let $y = \cos\theta;\ dy = -\sin\theta\ d\theta$

$$\therefore \int \tan\theta\, d\theta = \int \frac{-dy}{y} = -\int \frac{dy}{y} = -\log y + c$$

$$= -\log\cos\theta + c = \log(\cos\theta)^{-1} + c = \log\sec\theta + c$$

b) $\int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)} d\theta$

Let $y = \sec\theta + \tan\theta;\ dy = (\sec\theta\ \tan\theta\ + \sec^2\theta)\ d\theta$

$dy = \sec\theta(\sec\theta + \tan\theta)d\theta$

$$\int \sec\theta d\theta = \int \frac{dy}{y} = \log y + c$$
$$= \log(\sec\theta + \tan\theta) d\theta$$

c) $\int \frac{1+\cos x}{(x+\sin x)^2} dx = I$ (Say)

Put $y = x + \sin x$; $dy = (1 + \cos x)\, dx$

$$\therefore I = \int \frac{(1+\cos x)dx}{(x+\sin x)^2} = \int \frac{dy}{y^2} = \int y^{-2} dy = \frac{y^{-1}}{-1} + c$$
$$= -\frac{1}{(x+\sin x)} + c$$

d) $\int x^2 \sqrt{a^3 + x^3}\, dx = I$ (Say)

Put $a^3 + x^3 = t^2$; $3x^2\ dx = 2tdt$; $x^2 dx = \frac{2}{3} tdt$;

Also $\left(a^3 + x^3\right)^{\frac{1}{2}} = t$

$$I = \int \sqrt{a^3 + x^3}\ .\ x^2 dx = \int t \frac{2}{3} tdt = \frac{2}{3} \int t2\ dt$$
$$= \frac{2}{3} \frac{t^3}{3} + c = \frac{2}{9} t^3 + c$$
$$= \frac{2}{9}\left(a^3 + x^3\right)^{\frac{3}{2}} + c$$

e) $I = \int \cot\theta\, d\theta = \log\sin\theta + c$ (check it your self)

f) $I = \int \operatorname{cosec}\theta\, d\theta = -\log(\operatorname{cosec}\theta + \cot\theta) + c$ (Check)

Note: Remember the integrals of $\tan\theta$, $\sec\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$.

Working Example: Integrate the following w.r.t. x

a) $\frac{1}{\left(1+e^x\right)\left(1+e^{-x}\right)}$ b) $\frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}$

Solution:

a) $\displaystyle\int \frac{dx}{\left(1+e^{x}\right)\left(1+e^{-x}\right)} = \int \frac{dx}{\left(1+e^{x}\right)\left(1+\frac{1}{e^{x}}\right)}$

$$= \int \frac{dx}{\left(1+e^{x}\right)\frac{\left(e^{x}+1\right)}{e^{x}}} = \int \frac{e^{x}\ dx}{\left(1+e^{x}\right)^{2}} = I$$

Let $y = 1+e^{x}; \quad dy = e^{x}dx$

$$I = \int \frac{dy}{y^{2}} = \int y^{-2}\ dy = \frac{y^{-1}}{-1} + c$$

$$= -\frac{1}{\left(1+e^{x}\right)} + c$$

b) $\displaystyle\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\, dx = I$

$$\frac{d}{dx}\left(4\cos x + 5\sin x\right) = -4\sin x + 5\cos x$$

Let numerator = l (denominator) + $m\frac{d}{dx}$ (denominator)

Put $2\cos x + 3\sin x = l\left(4\cos x + 5\sin x\right) + m\left(-4\sin x + 5\cos x\right)$

We have $4l + 5m = 2$ and $5l - 4m = 3$

$20l + 25m = 10$

$20l - 16m = 12$

$$41m = -2, \quad m = -\frac{2}{41}$$

$$41 = 2 - 5m = 2 - 5\left(-\frac{2}{41}\right) = 2 + \frac{10}{41} + \frac{82+10}{41} = \frac{92}{41}$$

$$l = \frac{\overset{23}{\cancel{92}}}{41\times\cancel{4}}$$

$$l = \frac{23}{41} \text{ and } m = -\frac{2}{41}$$

$$I=\int\frac{\frac{23}{41}(4\cos x+5\sin x)-\frac{2}{41}\frac{d}{dx}(4\cos x+5\sin x)dx}{(4\cos x+5\sin x)}$$

$$=\frac{23}{41}\int dx-\frac{2}{41}\int\frac{d(4\cos x+5\sin x)}{(4\cos x+5\sin x)}$$

$$=\frac{23}{41}x-\frac{2}{41}\log(4\cos x+5\sin x)+c$$

S.A.Q.2: Integrate the following functions w.r.t. x

a) $cosec\,(4x+3)\cot(4x+3)$ b) $\frac{1}{(2x+3)^5}$ c) $\frac{1}{3x+2}$

d) $cosec^2\,(7-11x)$ e) e^{5-4x}

S.A.Q.3: Evaluate the following integrals

a) $\int\frac{\tan(3-4x)dx}{\cos(3-4x)}$ b) $\int\frac{1-\cos x}{1+\cos x}dx$

c) $\int\sqrt{1+\sin 2x\,dx}$ d) $\int 3^x\,dx$

(Hint: b) $1-\cos x=2\sin^2\frac{x}{2}$ etc c) $\sin 2x=2\sin x\cos x$

d) $3x=e^{\log 3^x}=e^{x\log^3}$

S.A.Q. 4: Integrate the following functions w.r.t. x

a) $\frac{\log x}{x}$ b) $e^x\sin(e^x)$ c) $x^3\sqrt{x^4+11}$ d) $\frac{e^{\tan x}}{\cos^2 x}$

S.A.Q.5: Integrate the following functions w.r.t. x

a) $\frac{e^x(1+x)}{\cos^2(xe^x)}$ b) $\frac{4x^3+2x}{1+x^4}$ c) $\frac{1}{x+x\log x}$ d) $\tan^3 x\sec x$

(Hint: b) Evaluate $\int\frac{4x^3}{1+x^4}dx+\int\frac{2x}{1+x^4}dx$ c) $t=1+\log x$ d) $t=\sec x$)

S.A.Q. 6: Integrate the following functions w.r.t. x

a) $\sin^{-1}x$ b) xe^x c) $x\log(x+1)$ d) $x\tan^{-1}x$

S.A.Q. 7: Integrate the following functions w.r.t. x

a) $\frac{log\,x}{x^2}$ b) $\frac{xe^x}{(1+x)^2}$ c) $\frac{log(x^2+a^2)}{x^2}$ d) $x^3\,log\,x$

Integration by parts

By now you might have noticed that there are not many general rules for integration as in the case of differentiation. Even simple functions like logx cannot be integrated with the rules we have come across so far.

In this section we give a method of integration called "Integration by parts". Using this method we can integrate many functions.

Rule 4: If *u* and *v* are functions of *x*,

$$\int udv = uv - \int vdu \qquad \text{….. (7.4)}$$

This follows by integrating the product rule

$$\frac{d}{dv}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides we get

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$= \int udv + \int vdu$$

(7.4) Follows from the above identity while applying integration by parts**Working Example:** Evaluate $\int log\,x\,dx$

Solution: Let $u = log\,x$ $dv = dx$

Then $\frac{du}{dx} = \frac{1}{x}$ or $du = \frac{dx}{x}$

$v = x.$

$$\int \underbrace{log\,x}_{u}\,\underbrace{dx}_{dv} = uv - \int v\,du$$

$$= x\,log\,x - \int x\frac{dx}{x}$$

$$= x\,log\,x - x + c$$

Working Example: Evaluate $\int \sqrt{a^2 + x^2}\ dx$

Solution: Let $u = \sqrt{a^2 + x^2}$ and $dv = dx$

$$\therefore v = x$$

$$\int \underbrace{\sqrt{a^2 + x^2}}_{u}\ \underbrace{dx}_{dv} = uv - \int v\ du$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x.2x}{2\sqrt{a^2 + x^2}}\ dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}}\ dx$$

$$= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\ dx + a^2 \int \frac{dx}{\sqrt{a^2 + x^2}} + c$$

$$\int \sqrt{a^2 + x^2}\ dx = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\ dx + a^2 \log\left(x + \sqrt{a^2 + x^2}\right) + c$$

$$2\int \sqrt{a^2 + x^2}\ dx = x\sqrt{a^2 + x^2} + a^2 \log\left(x + \sqrt{a^2 + x^2}\right) + c$$

$$\int \sqrt{a^2 + x^2}\ dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} log\left(x + \sqrt{a^2 + x^2}\right) + c$$

Working Example: Evaluate

a) $\int x^2 sin^{-1}\ x\ dx$ b) $\int x\, sin^2\ x\ dx$

Solution

a) $I = \int x^2 sin^{-1}\ x\ dx$

Let $u = sin^{-1} x$ $dv = x^2 dx$ $v = \dfrac{x^3}{3}$

$$I = \int udv = uv - \int vdu = \frac{x^3}{3} sin^{-1}\ x - \int \frac{x^3}{3} . \frac{1}{\sqrt{1 - x^2}}\ dx$$

$$I = \frac{x^3}{3}\,sin^{-1}\,x - \frac{1}{3}\int \frac{x^2}{\sqrt{1-x^2}}\,xdx$$

Put $1 - x^2 = t^2$ in the second term; $x^2 = 1 - t^2$; $2x\,dx = -2tdt$; $xdx = -tdt$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,xdx = \int \frac{(1-t^2)(-tdt)}{t} = \int(-1+t^2)dt$$

$$= \int -1\,dt + \int t^2 dt = -t + \frac{t^3}{3} + c$$

$$= \frac{t}{3}(t^2-3) + c = \frac{(1-x^2)^{\frac{1}{2}}}{3}(1-x^2-3) + c$$

Hence $$I = \frac{x^3}{3}\,sin^{-1}\,x - \frac{1}{3}\,\frac{(1-x^2)^{\frac{1}{2}}}{3}(-x^2-2) + c$$

$$= \frac{x^3}{3}\,sin^{-1}\,x + \frac{1}{9}\sqrt{1-x^2}\,(2+x^2) + c$$

b) $I = \int x\,sin^2\,xdx$

Let $u = x$ $dv = sin^2\,x\,dx$

Then $du = dx$

$$v = \int \sin^2\,xdx = \int \frac{1}{2}(1-\cos 2x)dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$$

$$v = \frac{x}{2} - \frac{sin2x}{4}$$

I = uv - $\int vdu$

$$I = \left(\frac{x}{2} - \frac{sin2x}{4}\right)x - \int\left(\frac{x}{2} - \frac{sin2x}{4}\right)dx$$

$$= \frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{4} - \frac{\cos 2x}{8} + c$$

$$= \frac{1}{4}\left(2x^2 - x\,sin2x - x^2 - \frac{cos\,2x}{2}\right) + C$$

$$= \frac{1}{4}\left(x^2 - x\,sin2x - \frac{cos\,2x}{2}\right) + c$$

7.4 More Formulas in Integration

Using the method of substitution or integration by parts, we can derive the following formulas (see table 7.2)

1)	$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
2)	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$
3)	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$
4)	$\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right)+c$
5)	$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right)+c$
6)	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$
7)	$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{2}+c$
8)	$\int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)+\frac{x\sqrt{a^2+x^2}}{2}+c$
9)	$\int \sqrt{x^2-a^2}\,dx = \frac{-a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+\frac{x\sqrt{a^2-x^2}}{2}+c$

Table 7.2: Additional formulas for integration

Now we are in a position to integrate functions having quadratic factors or square root of quadratic factors in numerator or denominator of functions.

Integral of functions of the form $\frac{a\cos x + b\sin x}{\cos x + d\sin x}$

Method

Step 1: Let numerator = A (Denominator) + $B\frac{d}{dx}$ (Denominator)

Step 2: Find the values of A and B

Step 3: Split the function and integrate

We have already seen this in W.E.

Completion of squares

All the subsequent methods use a technique called "Completing the square". It is simply writing a quadratic expression in the form, $a^2 + x^2$, $a^2 - x^2$, $x^2 - a^2$.

Example: Consider $x^2 + 3x + 2$. We can write

$$x^2 + 3x + 2 = x^2 + 2\left(\frac{3}{2}x\right) + 2$$

$$= x^2 + 2\left(\frac{3}{2}x\right) + \left(\frac{3}{2}\right)^2 + 2 - \left(\frac{3}{2}\right)^2$$

$$= \left(x + \frac{3}{2}\right)^2 + \frac{8-9}{4}$$

$$= \left(x + \frac{3}{2}\right)^2 - \frac{1}{4} = \left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

Example: Consider $2 - 7x - x^2$

$$2 - 7x - x^2 = 2 - \left(x^2 + 7x\right)$$

$$= 2 - \left(x^2 + 2\left(\frac{7}{2}\right)x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2\right)$$

$$= \left(2 + \frac{49}{4}\right) - \left(x + \frac{7}{2}\right)^2$$

$$= \frac{57}{4} - \left(x + \frac{7}{2}\right)^2$$

$$= \left(\frac{\sqrt{57}}{2}\right)^2 - \left(x + \frac{7}{2}\right)^2$$

Integration of functions of the form $\frac{lx+m}{ax^2+bx+c}$

Method:

Step 1: Write $lx+m = A\frac{d}{dx}\left(ax^2+bx+c\right)+B$

Step 2: Find the values of A and B

Step 3: Split the function and integrate

Working Example: Evaluate $\int \frac{x+4}{6x-7-x^2}\,dx$

Solution:

Here $\frac{d}{dx}\left(6x-7-x^2\right) = -2x+6$

Let $x + 4 = A\ (-2x + 6) + B = -2\ Ax + 6A + B$

$1 = -2A;\ 4 = 6A + B$

$$A = -\frac{1}{2};\ B = 4-6A = 4-6\left(-\frac{1}{2}\right) = 4+3 = 7$$

$$\int \frac{x+4}{6x-7-x^2}\,dx = -\frac{1}{2}\int \frac{-2x+6}{6x-7-x^2}\,dx + 7\int \frac{dx}{-7-\left(x^2-6x\right)}$$

$$= -\frac{1}{2}\int \frac{d\left(6x-7-x^2\right)}{\left(6x-7-x^2\right)} + 7\int \frac{dx}{\sqrt{-7-(x-3)^2+9}}$$

$$= -\frac{1}{2}\log\left(6x-7-x^2\right) + 7\int \frac{dx}{\left(\sqrt{2}\right)^2-(x-3)^2}$$

$$= -\frac{1}{2}\log\left(6x-7-x^2\right) + \frac{7}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+x-3}{\sqrt{2}-x+3}\right) + c$$

Integration of functions of the form $\frac{1}{ax^2+bx+c}$

Method: Write ax^2+bx+c in the form x^2+a^2 *or* x^2-a^2 *or* a^2-x^2 and integrate.

Working Example: Evaluate

a) $\int \frac{dx}{4x^2 - 4x + 2}$ b) $\int \frac{dx}{3x^2 + 13x - 10}$

Solution:

a) $\int \frac{dx}{4x^2 - 4x + 2}$

$$= \frac{1}{4} \int \frac{dx}{x^2 - x + \frac{1}{2}}$$

$$= \frac{1}{4} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4} \cdot \frac{1}{2} \tan^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}} \right) + c$$

$$= \frac{1}{8} \tan^{-1} (2x - 1) + c$$

b) $\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3} \int \frac{dx}{x^2 + \frac{13x}{3} - \frac{10}{3}}$

$$= \frac{1}{3} \int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \frac{10}{3} - \left(\frac{13}{6}\right)^2}$$

$$= \frac{1}{3} \int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \frac{10}{3} - \frac{169}{36}}$$

$$= \frac{1}{3} \int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$$

$$= \frac{1}{\cancel{3}} \cdot \frac{1}{\cancel{2}} \cdot \frac{1}{\frac{17}{\cancel{6}}} \log\left(\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right)+c$$

$$= \frac{1}{17} \log\left(\frac{x-\frac{2}{3}}{x+5}\right)+c$$

$$= \frac{1}{17} \log\frac{(3x-2)}{3(x+5)}+c$$

Integration of functions of the form $\frac{lx+m}{\sqrt{ax^2+bx+c}}$

Method:

Step 1: Write $lx+m = A\frac{d}{dx}\left(ax^2+bx+c\right)+B$

Step 2: Find the values of A and B

Step 3: Split the function and integrate

Working Example: Find $\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx$

Solution:

Let $6x+5 = A\frac{d}{dx}\left(6+x-2x^2\right)+B$

$6x+5 = A(1-4x)+B$

$6x+5 = -4Ax+A+B$

$6 = -4A;\ 5 = A+B$

$A=-\frac{3}{2};B=5-A=5+\frac{3}{2}=\frac{13}{2}$

$$\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx = \int \frac{-\frac{3}{2}(1-4x)+\frac{13}{2}}{\sqrt{6+x-2x^2}}dx$$

$$= -\frac{3}{2}\int \frac{(1-4x)}{\sqrt{6+x-2x^2}}dx + \frac{13}{2}\int \frac{dx}{\sqrt{6+x-2x^2}}$$

$$= -\frac{3}{2}\int \frac{d\left(6+x-2x^2\right)}{\left(6+x-2x^2\right)^{\frac{1}{2}}} + \frac{13}{2}\int \frac{dx}{\sqrt{2}\sqrt{3+\frac{x}{2}-x^2}}$$

$$= -\frac{3}{2}.2\left(6+x-2x^2\right)^{\frac{1}{2}} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{3+\frac{1}{16}-\left(x-\frac{1}{4}\right)^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{7}{4}}\right) + c$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c$$

Integrals of functions of the form $\frac{1}{\sqrt{ax^2+bx+c}}$

Method: Write $\sqrt{ax^2+bx+c}$ in form $\sqrt{a^2+x^2}$ *or* $\sqrt{a^2-x^2}$ *or* $\sqrt{x^2-a^2}$ and integrate

Working Example: Evaluate $\int \frac{dx}{\sqrt{3x^2+x-2}}$

Solution: $\int \frac{dx}{\sqrt{3x^2+x-2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2+\frac{x}{3}-\frac{2}{3}}}$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{1}{6}\right)^2-\frac{1}{36}-\frac{2}{3}}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x + \frac{1}{6}\right)^2 - \left(\frac{5}{6}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \log \left(x + \frac{1}{6} + \sqrt{\left(x + \frac{1}{6}\right)^2 - \left(\frac{5}{6}\right)^2} \right) + c$$

$$= \frac{1}{\sqrt{3}} \log \left(x + \frac{1}{6} + \sqrt{x^2 + \frac{x}{3} - \frac{2}{3}} \right) + c$$

$$= \frac{1}{\sqrt{3}} \log \left(x + \frac{1}{6} + \frac{1}{\sqrt{3}} \left(3x^2 + x - 2\right)^{\frac{1}{2}} \right) + c$$

Integration of functions of the form $(lx + m)\sqrt{ax^2 + bx + c}$

Method:

Step 1: Write $(lx + m) = A \frac{d}{dx}\left(ax^2 + bx + c\right) + B$

Step 2: Find the values of A and B

Step 3: Split the function and integrate

Working Example: Find $\int (3x - 2)\sqrt{x^2 + x + 1}\ dx$

Solution: Let $3x - 2 = A \frac{d}{dx}\left(x^2 + x + 1\right) + B$

$$3x - 2 = A(2x + 1) + B$$

$$3x - 2 = 2Ax + A + B$$

$$3 = 2A;\ -2 = A + B$$

$$A = \frac{3}{2};\ B = -2 - A = -2 - \frac{3}{2} = -\frac{7}{2}$$

$$\int (3x - 2)\sqrt{x^2 + x + 1}\, dx = \int \left\{ \frac{3}{2}(2x + 1) - \frac{7}{2} \right\} \sqrt{x^2 + x + 1}\, dx$$

$$= \frac{3}{2} \int \sqrt{x^2 + x + 1}\,(2x + 1)dx - \frac{7}{2} \int \sqrt{x^2 + x + 1}\, dx$$

$$= \frac{3}{2}\int (x^2+x+1)^{1/2}\, d(x^2+x+1) - \frac{7}{2}\int \left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^{1/2} dx$$

$$= \frac{\cancel{3}}{\cancel{2}}\frac{(x^2+x+1)^{3/2}}{\cancel{3}/\cancel{2}} - \frac{7}{2}\left(\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}+\frac{3}{4}\frac{1}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right)+c$$

$$= (x^2+x+1)^{3/2} - \frac{7}{8}(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c$$

Integration of functions of the form $\sqrt{ax^2+bx+c}$

Method: Write $\sqrt{ax^2+bx+c}$ in the form

$\sqrt{a^2-x^2}$ *or* $\sqrt{a^2+x^2}$ *or* $\sqrt{x^2-a^2}$ and integrate.

Working Example: Find $\int \sqrt{1+x-2x^2}\, dx$

Solution: $I = \sqrt{2}\int \left(\frac{1}{2}+\frac{x}{2}-x^2\right)^{1/2} dx$

$$= \sqrt{2}\int \left\{\frac{1}{2}+\frac{1}{16}-\left(x-\frac{1}{4}\right)^2\right\}^{\frac{1}{2}} dx$$

$$= \sqrt{2}\int \left\{\frac{9}{16}-\left(x-\frac{1}{4}\right)^2\right\}^{\frac{1}{2}} dx$$

$$= \sqrt{2}\int \left(\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2\right)^{\frac{1}{2}} dx$$

$$= \sqrt{2}\left(\frac{1}{2}\left(x-\frac{1}{4}\right)\left(\frac{9}{16}-\left(x-\frac{1}{4}\right)^2\right)^{\frac{1}{2}}+\frac{1}{2}.\frac{9}{16}\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{3}{4}}\right)\right)+c$$

$$= \sqrt{2}\left(\left(\frac{4x-1}{8}\right)\sqrt{\frac{1}{2}+\frac{x}{2}-x^2}+\frac{9}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)\right)+c$$

Integration using partial fractions

Consider the function $\frac{1}{(x+2)(x+1)}$. Expanding the denominator we can integrate the function using the earlier method. We can also integrate the same function using partial fractions.

$$\frac{1}{(x+2)(x+1)} = \frac{(x+2)-(x+1)}{(x+2)(x+1)} = \frac{1}{x+1} - \frac{1}{x+2}$$

Now it is easy to integrate $\frac{1}{x+1} - \frac{1}{x+2}$. We illustrate this method using an example.

Working Example: Evaluate $\int \frac{7x-6}{(x-1)(x-2)}\,dx$

Solution: Let us write the given function as $\frac{A}{x-1} + \frac{B}{x-2}$

Then $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

Multiplying both sides by $(x-1)(x-2)$, we get

$$7x - 6 = A(x-2) + B(x-1) \qquad \text{………………} (*)$$

Put $x = 1$ *in* (*). We get $7(1) - 6 = A(1-2)$

That is, $1 = -A$ of $\boxed{A = 1}$

Put $x = 2$ *in* (*). We get $7(2) - 6 = B(2-1)$

That is, $8 = B$ $\boxed{B = 8}$

$$\therefore I = \int\left(\frac{A}{x-1}+\frac{B}{x-2}\right)dx$$

$$= \int \frac{1}{x-1}\,dx + \int \frac{8}{x-2}\,dx$$

$$= \log(x-1) + 8\log(x-2) + c$$

7.5 Definite integrals

$\int f(x)\,dx$ is called the indefinite integral in integral calculus. You will be curious to know why it is called indefinite integral. Rieman defined a definite integral first. It is the form $\int_a^b f(x)dx$.

If $g(x) = \int f(x)dx$, then $\int_a^b f(x)dx = g(b) - g(a)$

Note that the definite integral $\int_a^b f(x)dx$ is a real number whereas $\int f(x)dx$ is a function.

Working Example 18: Evaluate $\int_0^1 \frac{dx}{1+x-x^2}$

Hint: write $1+x-x^2 = (\frac{\sqrt{5}}{2})^2 - (x-\frac{1}{2})^2$

Solution:

$$\int \frac{dx}{1+x-x^2} = \frac{1}{\sqrt{5}} \log\left(\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right) + c \text{ (Check)}$$

Hence

$$\int_0^1 \frac{dx}{1+x-x^2} = \left[\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}-1+2(1)}{\sqrt{5}+1+2(1)}\right)+c\right] - \left[\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}-1+2(0)}{\sqrt{5}+1-2(0)}\right)+c\right]$$

$$= \frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}+3}\right) - \frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)$$

Note: The arbitrary constant c gets cancelled while finding *g(1)-g(0)*.

S.A.Q 8: Integrate the following functions w.r.t. x

a) $\frac{4x-3}{x^2+3x+8}$ b) $\frac{3x+2}{x^2+x+1}$

S.A.Q 9: Integrate the following functions w.r.t. x

a) $\dfrac{1}{\sqrt{8+x-x^2}}$ b) $\dfrac{4x-3}{\sqrt{x^2+2x-1}}$

S.A.Q. 10: Integrate the following functions w.r.t. x

a) $\dfrac{1}{\sqrt{8-x-x^2}}$ b) $\dfrac{1}{\sqrt{x^2+8x-20}}$

S.A.Q. 11: Integrate the following functions w.r.t. x

a) $\sqrt{3-2x-x^2}$ b) $\sqrt{9x^2+16}$

S.A.Q. 12: Integrate the following functions w.r.t. x

a) $\dfrac{1}{2-3x+x^2}$ b) $\dfrac{x}{(x-1)^2\,(x+2)}$

$\left(\text{(Hint b) Write the given function as } \dfrac{A}{x+2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}\right)$

S.A.Q. 13: Evaluate the following definite integrals

a) $\displaystyle\int_0^1 \frac{x+1}{(x+2)(x+3)}\,dx$ b) $\displaystyle\int_0^1 \left[5x^4+3(2x+3)^4\right]dx$

7.6 Summary

In this unit, we studied the standard forms of integration. The different rules of integration is given and explained with the help of standard examples. All the formula concerned with integration is followed by good examples. The concept of definite integral and its applications are explained clearly with good examples.

7.7 Terminal Questions

1. Integration of the following w.r.t. x

   a) $\dfrac{1}{1+x^2}+\sin x$ b) $\dfrac{x+7}{2x}+\cos x+\sec^2 x$

   c) $\dfrac{3x^2+4x+5}{4x}$ d) $\left(\sqrt{x+\dfrac{1}{\sqrt{x}}}\right)^2$

   e) $\sec x\tan x+\cos x$

2. Evaluate the following integrals

   a) $\int 3\cos^2 5x\, dx$ b) $\int \sqrt{1+\sin x\, dx}$

   c) $\int \cos 5x \cos 3x\, dx$ d) $\int \sin 7x \cos 5x\, dx$

3. Integrate the following functions w.r.t. x

   a) $\dfrac{1}{\sin^2 x \cos^2 x}$ b) $\dfrac{\sin^2 x}{1+\cos x}$ c) $\sqrt{1-\sin^2 x}$ d) $\left(e^x - 1\right)^2 e^{-4x}$

   (Hint: a) $\sin^2 x + \cos^2 x = 1$ b) $\sin^2 x = 1 - \cos^2 x$ c) $\sin 2x = 2 \sin x \cos x$ d) Expand and integrate.)

4. Integrate the following functions w.r.t. x

   a) $3x^2 \cos\left(x^3\right)$ b) $\dfrac{\sin(\log x)}{x}$

   c) $\dfrac{1}{x \log x}$ d) $\dfrac{e^x(1+x)}{\cos^2\left(xe^x\right)}$

5. Integrate the following functions w.r.t. x

   a) $\dfrac{6e^{\frac{1}{x}}}{x^2}$ b) $\dfrac{e^x}{4+9e^{2x}}$

   c) $\dfrac{\sec x}{(\sec x + \tan x)^n}\, n > 0$ d) $\dfrac{\cos^3 x}{\sqrt{\sin x}}$

6. Integrate the following functions w.r.t. x

   a) $\cos^{-1}\left(\dfrac{x}{a}\right)$ b) x sin2x

   c) $\tan^{-1} x$ d) $x^n \log x$

   (Hint: a) $u = \cos^{-1}\left(\dfrac{x}{a}\right)$ b) $u = x$ c) $u = \tan^{-1} x$ d) $u = \log x$)

7. Integrate the following functions w.r.t. x

   a) $x \tan^2 x$ b) $e^x(\sin x + \cos x)$ c) $x^2 \sin x$

   (Hint: a) $u = x$ b) use integration by parts to $\int e^x \sin x dx = \int \sin x d\left(e^x\right)$ and proceed c) take $u = x^2$, evaluate $\int x \sin x\, dx$

8. Integrate the following functions w.r.t.x

   a) $\dfrac{3x+1}{2x^2+x+3}$ b) $\dfrac{5x+1}{x^2-2x-35}$

9. Integrate the following functions w.r.t. x

   a) $\dfrac{1}{3x^2+13x-10}$ b) $\dfrac{1}{4x^2+4x+10}$

10. Integrate the following functions w.r.t. x

    a) $\dfrac{2x}{\sqrt{3+4x-x^2}}$ b) $\dfrac{6x+7}{\sqrt{(x-4)(x-5)}}$

11. Integrate the following functions w.r.t. x

    a) $\dfrac{1}{\sqrt{x^2+3x+10}}$ b) $\dfrac{1}{\sqrt{1+x-x^2}}$

12. Integrate the following function w.r.t. *x*

    a) $\sqrt{x^2-4x+6}$ b) $\sqrt{(2-x)(1-x)}$

13. Evaluate the following integrals

    a) $\displaystyle\int\frac{2x+3}{(2x+1)(1-3x)}$ b) $\dfrac{x^2+1}{(x^2-1)(2x+1)}$

    (Hint: $x^2 - 1 = (x+1)(x-1)$. Write the given function as $\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{2x+1}$)

14. Evaluate the following integrals

    a) $\displaystyle\int_0^\infty \frac{3x}{1+2x^4}dx$ b) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x^8}}$

7.8 Answers

Self Assessment Question

1. $e^x+\log x-\dfrac{2}{x^3}-\cos x+\tan x+\tan^{-1}x$

2. a) $-\dfrac{\operatorname{cosec}(4x+3)}{4}$ b) $\dfrac{-1}{8(2x+3)^4}$

c) $\dfrac{log(3x+2)}{3}$ d) $\dfrac{cot(7-11x)}{11}$

e) $\dfrac{-e^{5-4x}}{4}$

3. a) $\dfrac{-sec(3-4x)}{4}$ b) $2\tan\dfrac{x}{2}-x$

c) $\sin x-\cos x$ d) $\dfrac{3x}{\log 3}$

4. a) $\dfrac{(\log x)^2}{2}$ b) $-\cos\left(e^x\right)$

c) $\dfrac{1}{6}\left(x^4+11\right)^{\frac{3}{2}}$ d) $e^{\tan x}$

5. a) $\tan\left(xe^x\right)$ b) $\log\left(1+x^4\right)+\tan^{-1}\left(x^2\right)$

c) $\log(1+\log x)$ d) $\dfrac{sec^{3x}}{3}-\sec x$

6. a) $u=\sin^{-1}x;\ \ x\sin^{-1}x+\sqrt{1-x^2}$ b) $u=x;\ e^x(x-1)$

c) $u=\log(x+1);\ \dfrac{1}{2}\left(x^2-1\right)\log(x+1)-\dfrac{x^2}{4}+\dfrac{x}{2}$

d) $\dfrac{1}{2}\left[\left(x^2+1\right)\tan^{-1}x-x\right]$

7. a) $u=\log x;\ \dfrac{(1+\log x)}{x}$

b) $u=xe^x,\ dv=\dfrac{dx}{1+x^2};$ Answer $\dfrac{-e^x}{1+x}$

c) $u=\log\left(x^2+a^2\right);\ \dfrac{-\log\left(x^2+a^2\right)}{x}+\dfrac{2}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ d) $\dfrac{x^4\log x}{4}-\dfrac{x^4}{16}$

8. a) $2\log\left(x^2+3x+8\right)-\dfrac{18}{\sqrt{23}}\tan^{-1}\left(\dfrac{2x+3}{\sqrt{23}}\right)$

b) $\dfrac{3}{2}\log\left(x^2+x+1\right)+\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$

9. a) $\frac{2}{\sqrt{3}} tan^{-1}\left(\frac{2x+5}{\sqrt{3}}\right)$ b) $\frac{1}{\sqrt{29}} log\left(\frac{2x-7-\sqrt{29}}{2x-7+\sqrt{29}}\right)$

10. a) $sin^{-1}\left(\frac{2x+1}{\sqrt{33}}\right)$ b) $log\left[x+4+\sqrt{x^2+8x-20}\right]$

11. a) $\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+2sin^{-1}\left(\frac{x+1}{2}\right)$ b) $\frac{1}{6}\left[3x\sqrt{9x^2+16}+16\,log\left\{3x+\sqrt{9x^2+16}\right\}\right]$

12. a) $log\left(\frac{x-2}{x-1}\right)$ b) $\frac{2}{9} log\left(\frac{x-1}{x+2}\right)-\frac{1}{3}.\frac{1}{x-1}$

13. a) The indefinite integral is 2 log (x + 3) – log (x + 2) + c
Answer: 2 log 4 – 3 log 3 + log 2

b) Answer: $\left\{1^5-\frac{3}{10}[2(1)+3]^5\right\}-\left\{0^5-\frac{3}{10}[2(0)+5]\right\}=\frac{5}{2}-\frac{3(625)}{2}$

Terminal Questions:

1. a) $tan^{-1}x-cos\,x$ b) $\frac{x}{2}+\frac{7}{2}log\,x+sin\,x+tan\,x$

c) $\frac{3}{8}x^2+x+\frac{5}{4}log\,x$ d) $\frac{x^2}{2}+2x+log\,x$

e) $sec\,x+sin\,x$

2. a) $\frac{3}{2}x+\frac{3}{20}sin10x$ b) $-2\sqrt{2}\,cos\left(\frac{\pi}{4}+\frac{x}{2}\right)$

c) $\frac{1}{10}\sin 8x+\frac{1}{4}\sin 2x$ d) $\frac{-cos12x}{12}-\frac{cos2x}{2}$

3. a) $tan\,x-cot\,x$ b) $x-sin\,x$

c) $-\cos x+\sin x$ d) $\frac{e^{-2x}}{2}+\frac{2e^{-3x}}{3}-\frac{e^{-4x}}{4}$

4. a) sin (x^3) b) – cos (log x)

c) log (log x) d) tan (xe^x)

5. a) $-6e^{\frac{1}{x}}$ b) $\frac{1}{6}tan^{-1}\left(\frac{3e^x}{2}\right)$

c) $\frac{1}{n(sec\,x+tan\,x)^n}$ d) $2\sqrt{sin\,x}\left(1-\frac{sin^2 x}{5}\right)$

6. a) $x\cos^{-1}\left(\frac{x}{a}\right)-\sqrt{a^2-x^2}$ b) $-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}$

   c) $x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)$ d) $\frac{x^{n+1}\log x}{n+1}-\frac{x^{n+1}}{(n+1)^2}$

7. a) $x\tan x+\log\cos x-\frac{x^2}{2}$ b) $e^x \sin x$

   c) $-x^2\cos x+2x\sin x+2\cos x$

8. a) $\frac{3}{4}\log\left(2x^2+x+3\right)+\frac{1}{2\sqrt{23}}\tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right)$

   b) $\frac{5}{2}\log\left(x^2-2x-35\right)+\frac{1}{2}\log\left(\frac{x-7}{x+5}\right)$

9. a) $\frac{1}{17}\log\left(\frac{3x-2}{3x+15}\right)$ b) $\frac{1}{6}\tan^{-1}\left(\frac{2x+1}{2}\right)$

10. a) $-2\sqrt{3+4x-x^2}+4\sin^{-1}\left(\frac{x-2}{\sqrt{7}}\right)$

    b) $6\sqrt{x^2-9x+20}+34\log\left[x-\frac{9}{2}+\sqrt{x^2-9x+20}\right]$

11. a) $\log\left[x+\frac{3}{2}+\sqrt{x^2+3x+10}\right]$ b) $\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

12. a) $\frac{(x-2)}{2}\sqrt{x^2-4x+6}+\log\left[x-2+\sqrt{x^2-4x+6}\right]$

    b) $\frac{1}{8}\left[2(2x-1)\sqrt{2+x-x^2}+9\sin^{-1}\left(\frac{2x-1}{3}\right)\right]$

13. a) $\frac{2}{5}\log(2x+1)-\frac{11}{15}\log(1-3x)$

    b) $\frac{1}{3}\log(x-1)+\log(x+1)-\frac{5}{6}\log(12x+1)$

14. a) The indefinite integral is $\frac{3\sqrt{2}}{4} tan^{-1}\left(\sqrt{2}x^2\right)+c$ (Put t = x^2 and integrate).

Answer:

$$\frac{3\sqrt{2}}{4} tan^{-1}(\infty)-\frac{3\sqrt{2}}{4} tan^{-1}(0)=\frac{3\sqrt{2}}{4}.\frac{\pi}{2}-\frac{3\sqrt{2}}{4}.0=\frac{3\pi\sqrt{2}}{8}.$$

b) $\frac{1}{4} sin^{-1}\left(x^4\right)+c$ is the indefinite integral (Put $t = x^4$ and integrate).

Answer:

$$\frac{1}{4} sin^{-1}(1)-\frac{1}{4} sin^{-1}(0)=\frac{1}{4}.\frac{\pi}{2}-0=\frac{\pi}{8}.$$