BACHELOR OF COMPUTER APPLICATIONS

SEMESTER 3

DCA2101

COMPUTER ORIENTED NUMERICAL METHODS

Unit 3

Finite Difference Operator

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1. INTRODUCTION

We all know that Numerical Analysis has great importance in almost every field whether it is Engineering, Science, Technology, or any other field.

We get the results in numerical analysis by computing methods of the given data. The basics of numerical analysis are finite-difference which deals with the changes in dependent variables due to changes in independent variables. The change in the independent variable is not continuous but by finite jumps, it can be equal or unequal.

1.1 Objectives:

At the end of the unit the student should be able to :

- *explain the concept of finite difference*
- *use operator E(Shift operator),* $\Delta(forward\ operator), \nabla(backward)$ *,* δ *(central difference Operator)*
- *know factorial polynomial, difference of polynomials*

2. FINITE DIFFERENCE

Consider a function $y = f(x)$. Let $x_0, x_1 = x_0 + h, x_2 = x_1 + h, \dots, x_n = x_{n-1} + h$ (or $x_i = x_0 + ih, i = 0, 1, 2 \dots n$) be a set of points at a common interval h. Let the corresponding values of $y = f(x)$ be $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$.

The value of the independent variable x is called the *argument* and the corresponding functional value is known as the *entry.*

2.1 Forward Difference

For a given table of values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ equally space abscissas of function $y = f(x)$, we define the first *forward* difference operation Δ defined by $\Delta f(x)$ is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

Put $x = x_0$, we get

$$\Delta f(x_0) = f(x_0 + h) - f(x_0)$$

$$\Delta y_0 = f(x_1) - f(x_0) = y_1 - y_0,$$

that is, $\Delta y_0 = y_1 - y_0$. Where Δ is called the *forward difference operator.*

If $y_0, y_1, y_2, \dots, y_n$ denotes a set of values of y, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots, y_n - y_{n-1}$ are called the *first forward differences of* $y_0, y_1, y_2, \dots y_{n-1}$ and they are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \dots, \Delta y_{n-1}$ respectively. Now

$$\Delta y_0 = y_1 - y_0 = f(x_1) - f(x_0)$$

$$\Delta y_1 = y_2 - y_1 = f(x_2) - f(x_1)$$

$$\Delta y_2 = y_3 - y_2 = f(x_3) - f(x_2)$$

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$$\Delta y_{n-1} = y_n - y_{n-1} = f(x_n) - f(x_{n-1})$$

where $x_i = x_{i-1} + h = x_0 + ih, \quad i = 0, 1, 2, \dots, n.$

The differences of the first forward differences are called *Second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \Delta^2 y_3, \dots$

Thus

$$\begin{aligned} \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\ &= (y_2 - y_1) - (y_1 - y_0) \\ &= y_2 - 2y_1 + y_0 \\ \Delta^2 y_1 &= \Delta y_2 - \Delta y_1 \\ &= (y_3 - y_2) - (y_2 - y_1) \quad = y_3 - 2y_2 + y_1 \end{aligned}$$

Similarly, we can define *third forward differences, fourth forward differences,* etc. Thus

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^2 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$

$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = {}^4c_0 y_4 - {}^4c_1 y_3 + {}^4c_2 y_2 - {}^4c_3 y_1 + {}^4c_4 y_0$

In general, we arrive at the following:

$$\Delta^n y_0 = {}^n c_0 y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2} - {}^n c_3 y_{n-3} + \ldots + (-1)^n y_0$$

The coefficients occurring on the right-hand side are the binomial coefficient with an alternative sign.

The following table shows how the forward differences of all orders can be formed.

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	$\mathbf{y_0}$					
		$\mathbf{\Delta y_0}$				
x_1	y_1		$\mathbf{\Delta^2 y_0}$			
		Δy_1		$\mathbf{\Delta^3 y_0}$		
x_2	y_2		$\Delta^2 y_1$		$\mathbf{\Delta^4 y_0}$	
		Δy_2		$\Delta^3 y_1$		$\mathbf{\Delta^5 y_0}$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The first entries in the forward difference table namely, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$,….. are called the *leading forward differences of* y_0.

Properties:

1. Let f and g be two functions and let a & b are constants, then
$$\Delta(af + bg)(x) = \Delta\big(af(x) + bg(x)\big) = a\Delta f(x) + b\Delta g(x)$$
2. $\Delta\big(f(x).g(x)\big) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$ or
$$\Delta\big(f(x).g(x)\big) = f(x)\Delta g(x) + g(x+h)\Delta f(x)$$

Solution: L.H.S. :$\Delta[f(x)g(x)]$

By definition

$$\Delta\big(f(x).g(x)\big) = f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$$

$$= f(x+h)\{g(x+h) - g(x)\} + g(x)\{f(x+h) - f(x)\}$$

$$= f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Similarly, we can prove that

$$\Delta\big(f(x).g(x)\big) = f(x)\Delta g(x) + g(x+h)\Delta f(x)$$

3. $\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Solution:$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$

$$= \frac{g(x)f(x+h) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{g(x)\big(f(x+h) - f(x)\big) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

4. If c is constant, then $\Delta c = 0$

Solution: We have f(x) = c, f(x+h) = c

So, $\Delta f(x) = f(x+h) - f(x) = c - c = 0$

5. If m, n are positive integers then $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

Solution:$\Delta^m \Delta^n f(x) = (\Delta \times \Delta \times \ldots .m - times)(\Delta \times \Delta \times \ldots .n - times)f(x)$

$$= (\Delta \times \Delta \times \Delta \times \ldots .m + n - times)f(x)$$

$$= \Delta^{m+n} f(x)$$

Example: Construct a forward difference table from the following values:

x	0	1	2	3	4
y = f (x)	1	2	4	5	9

Solution: ***Forward Difference Table***

x	y	Δ	Δ^2	Δ^3	Δ^4
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 1$			
$x_1 = 1$	$y_1 = 2$		$\Delta^2 y_0 = 1$		
		$\Delta y_1 = 2$		$\Delta^3 y_0 = -2$	
$x_2 = 2$	$y_2 = 4$		$\Delta^2 y_1 = -1$		$\Delta^4 y_0 = 6$
		$\Delta y_2 = 1$		$\Delta^3 y_1 = 4$	
$x_3 = 3$	$y_3 = 5$		$\Delta^2 y_2 = 3$		
		$\Delta y_3 = 4$			
$x_4 = 4$	$y_4 = 9$				

From the table, we have $\Delta^3 y_0 = -2$

This is also defined as,

$$\Delta^3 y_0 = {}^3c_0 y_3 - {}^3c_1 y_2 + {}^3c_2 y_1 - {}^3c_3 y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$= 5 - 3 \times 4 + 3 \times 2 - 1$$
$$= 5 - 12 + 6 - 1$$
$$\Delta^3 y_0 = -2$$

Example: Evaluate

(i) Δa^x (ii) $\Delta^2 x^3,\ h = 1$ (iii) $\Delta\,(tan^{-1}x)$ (iv) $\Delta\,(Sinx)$ (v) $\Delta(lnx)$ (vi) $\Delta^2 e^x$

Solution: (i) By definition w.k.t $\Delta f(x) = f(x+h) - f(x)$

$$\Delta a^x = a^{x+h} - a^x$$
$$= a^x(a^h - 1)$$

(ii) $\Delta^2 x^3 = \Delta(\Delta x^3) = \Delta(x^{3+h} - x^3)$

at h=1 we have

$$= \Delta(x^{3+1} - x^3)$$
$$= \Delta(x^4 - x^3)$$
$$= \Delta x^4 - \Delta x^3$$
$$= (x^5 - x^4) - (x^4 - x^3)$$
$$= x^5 - 2x^4 + x^3$$

(iii) $\Delta\,(tan^{-1}x) = tan^{-1}(x+h) - tan^{-1}x$

$$= tan^{-1}\frac{x+h-x}{1+(x+h)x} = tan^{-1}\frac{h}{1+(x+h)x}$$

(iv) $\Delta\,(Sinx) = Sin(x+h) - Sinx$

$= 2\cos(\frac{x+h+x}{2})\sin(\frac{x+h-x}{2})$

$= 2\cos(x+\frac{h}{2}) sin\frac{h}{2}$

(v) $\Delta(lnx) = \ln(x+h) - lnx$

$= \ln\left(\frac{x+4}{x}\right) = \ln(1+\frac{h}{x})$

(vi) $\Delta^2 e^x = \Delta(\Delta e^x) = \Delta(e^{x+h} - e^x)$

$= \Delta(e^h - 1)e^x$

$= (e^h - 1)\Delta e^x$

$= (e^h - 1)(e^{x+h} - e^x)$

$= (e^h - 1)(e^h - 1)e^x$

$= (e^h - 1)^2 e^x$

Example: If $f(x) = 3x^2 + 1$,evaluate $\Delta^2 f(x)$& $h = 1$.

Solution: We have $\Delta f(x) = f(x+1) - f(x)$

$= \{3(x+1)^2 + 1\} - \{3x^2 + 1\}$

$= 3(x^2 + 1 + 2x) + 1 - 3x^2 - 1$

$= 6x + 3$

Now, $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(6x+3)$

$= \{6(x+1) + 3\} - 6x - 3$

$= 1$

Example: Evaluate $\Delta[(x+1)(x+2)]$

Solution: By using formula $\Delta(f(x).g(x)) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

$$\Delta[(x+1)(x+2)] = (x+h+1)\Delta(x+2) + (x+2)\Delta(x+1)$$

$= (x+h+1)\{(x+h+2) - (x+2)\} + (x+2)\{(x+h+1) - (x+1)\}$

$= (x+h+1)h + (x+2)h$

$= h(x+h+1+x+2)$

$= h(2x+h+3)$

Example: Let a function $f(x)$ is given at a point (0,7), (4,43), (8,367) then find the forward difference of the function at x = 4.

Solution: We know$\Delta f(x) = f(x+h) - f(x)$

So, $\Delta f(4) = f(4+h) - f(4)$

By the data given in the question h = 4

⇨ $\Delta f(4) = f(8) - f(4) = 367 – 43 = 324$

Example: If $f(x) = x^2 + ax + b$, where a & b are real constants, calculate $\Delta^r f(x)$.

Solution: We have $\Delta f(x) = f(x+h) - f(x)$

$$= ((x+h)^2 + a(x+h) + b) - (x^2 + ax + b)$$

$$= x^2 + h^2 + 2xh + ax + ah + b - x^2 - ax - b$$

$$= 2xh + h^2 + ah$$

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = (2(x+h)h + h^2 + ah) - (2xh + h^2 + ah)$$

$$= 2h^2$$

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x)) = \Delta(2h^2) = 0$$

$$\Delta^r f(x) = 0, for\ all\ r \geq 3$$

Example: By constructing a difference table and taking the second-order difference as constant find the sixth term of the series 8,12,19,29,42.

Solution: let "a" be the sixth term of the series. The difference table is

x	y	Δ	Δ^2	Δ^3
1	8			
		4		
2	12		3	
		7		0
3	19		3	
		10		0
4	29		3	
		13		a-55-3=0
5	42		a-55	
		a-42		
6	a			

Since the second differences are constant third difference must be zero

Hence, a-55-3 = 0

⇨ a = 58

Self-Assessment Questions – 1

1. If $f(x) = e^{ax}$, show that $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$
2. Find $\Delta f(x), \Delta^2 f(x), \Delta^3 f(x)$ for the function $x^2 + 2x + 3$ with h = 2.
3. Construct the forward difference table for $y = f(x) = x^3 + 2x + 1$ for x = 1,2,3,4,5.

2.2 Backward Difference

The first backward difference of $f(x)$ denoted by $\nabla f(x)$ is defined as

$\nabla f(x) = f(x) - f(x-h)$

Put $x = x_1$

$\nabla f(x_1) = f(x_1) - f(x_1 - h) = f(x_1) - f(x_0)$

$\nabla y_1 = y_1 - y_0.$

Thus, the differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ are called the first backward differences and they are denoted by $\nabla y_1, \nabla y_2, \nabla y_3 \ldots \nabla y_n$ respectively, where ∇ is the backward difference operator. Similarly, we can define backward differences of higher orders.

Thus we obtain

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - 2y_1 + y_0$

$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0.$

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$			
		∇y_3			$\nabla^4 y_4$	
x_3	y_3			$\nabla^3 y_4$		$\nabla^5 y_5$
			$\nabla^3 y_4$			
		∇y_4			$\nabla^5 y_5$	
x_4	y_4			$\nabla^3 y_5$		
			$\nabla^3 y_5$			
		∇y_5				
x_5	y_5					

Example: Let a function f(x) is given at point (0,2), (2,10), (4,15), (6,18), (8,22), (10,10). Find $\nabla y_4, \nabla^3 y_3$.

Solution: By definition $\nabla y_1 = y_1 - y_0 = 10 - 2 = 8$

$$\nabla y_2 = y_2 - y_1 = 15 - 10 = 5$$

$$\nabla y_3 = y_3 - y_2 = 18 - 15 = 3$$

$$\nabla y_4 = y_4 - y_3 = 22 - 18 = 4$$

$$\nabla^3 y_3 = \nabla^2(y_3 - y_2) = \nabla(\nabla y_3 - \nabla y_2)$$

$$= \nabla(y_3 - y_2 - y_2 + y_1)$$

$$= \nabla(y_3 - 2y_2 + y_1)$$

$$= \nabla y_3 - 2\nabla y_2 + \nabla y_1$$

$$= y_3 - y_2 - 2(y_2 - y_1) + y_1 - y_0$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$= 18 - 3(15) + 3(10) - 2 = 1$$

Example: For the following set of values find the backward difference table

(9,5.0), (10, 5.4), (11, 6.0), (12, 6.8), (13, 7.5), (14, 8.1)

Solution: *Backward Difference Table*

x	y	∇	∇^2	∇^3	∇^4	∇^5
9	5.0					
10	5.4	0.4				
11	6.0	0.6	0.2			
12	6.8	0.8	0.2	0.0		
13	7.5	0.7	-0.1	-0.3	-0.3	
14	8.1	0.6	0.1	0.2	0.5	0.8

Example: If $f(x) = x^2 + ax + b$, where a & b are real constants. Calculate $\nabla^r f(x)$.

Solution: We first calculate $\nabla f(x)$ as follows

$$\nabla f(x) = f(x) - f(x-h) = (x^2 + ax + b) - ((x-h)^2 + a(x-h) + b)$$

$= -h^2 + ah + 2xh$

$$\nabla^2 f(x) = \nabla(\nabla f(x)) = (-h^2 + ah + 2xh) - (-h^2 + ah + 2(x-h)h) = 2h^2$$

$$\nabla^3 f(x) = \nabla(\nabla^2 f(x)) = 2h^2 - 2h^2 = 0$$

Thus, $\nabla^r f(x) = 0, \ \forall\, r \geq 3$

Self-Assessment Questions – 2

4. Obtain the backward difference for the function $f(x) = x^3$ for x = 1, 1.01, 1.02, 1.03, 1.04, 1.05.

2.3 Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent the central difference operator, and the subscript of δy for any difference is the average of the subscripts of the two members of the difference. We write,

$$y_1 - y_0 = \delta y_{\frac{1}{2}}$$

$$y_2 - y_1 = \delta y_{\frac{3}{2}}$$

……..

……..

$$y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

Higher-order central differences are defined as

$$\delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = \delta^2 y_1$$

$\delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} = \delta^2 y_2$ and

$\delta^2 y_2 - \delta^2 y_1 = \delta^2 y_{\frac{3}{2}}$ and so on.

These differences are shown in the following table:

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\underline{5}}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{\frac{7}{2}}$			
x_4	y_4				

It is clear from the three tables that for a particular numerical problem if the same numbers occur in the same positions then we get $y_1 - y_0$ whether we use forward, backward or central differences. that is

$$\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}} = y_1 - y_0 .$$

2.4 Shift Operator

Let $y = f(x)$ be a function of x, and let x takes the consecutive values x, $x + h$, $x + 2h$, …,etc. Define the shift operator E as the operation of increasing the argument x by h so that

$$Ef(x) = f(x + h)$$

$$E^2 f(x) = E[E\, f(x)] = E[f(x + h)] = f(x + 2h)$$

That is, $Ey_0 = y_1$

$$Ey_r = y_{r+1}$$

$$E^2 y_r = Ey_{r+1} = y_{r+2}$$

In general $E^n y_r = y_{r+n}$

The inverse operator E^{-1} is defined by

$E^{-1} f(x) = f(x - h)$ and similarly, $E^{-n} f(x) = f(x - nh)$

2.5 Average Operator μ:

The average operator is defined as

$$\mu f(x) = \frac{1}{2}[f\left(x+\frac{h}{2}\right)+f(x-\frac{h}{2})]$$

That is,

$$\mu\, y = \frac{1}{2}[y_{\frac{1}{2}} + y_{-\frac{1}{2}}]$$

2.6. Differential Operator (D)

It is denoted by D and is defined as

$$Df(x) = \frac{d}{dx}f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2}f(x) = f''(x)$$

$$D^3 f(x) = \frac{d^3}{dx^3}f(x) = f'''(x)$$

and so on.

Relationship between E and ∆:

we have

$\Delta y_0 = y_1 - y_0 = Ey_0 - y_0$

$= (E - 1)\, y_0$

$\Delta y_0 = (E - 1)\, y_0$. This means $\Delta = E - 1$

That is $\boldsymbol{E = 1 + \Delta}$

Similarly, the higher order forward differences are

i) $\Delta^3 y_0 = (E-1)^3 y_0 = (E^3 - 3E^2 + 3E - 1)y_0 = E^3 y_0 - 3E^2 y_0 + 3Ey_0 - y_0$.

Therefore $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$.

In general, $\Delta^k y_0 = (E-1)^k y_0$.

ii) We have $y_2 = E^2 y_0 = (1 + \Delta)^2 y_0 = (1 + 2\Delta + \Delta^2)\, y_0$

That is, $y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$ and

$y_3 = E^3 y_0 = (1 + \Delta)^3 y_0 = (1 + 3\Delta + 3\Delta^2 + \Delta^3)\, y_0$

That is, $y_3 = y_0 + 3\,\Delta y_0 + 3\,\Delta^2 y_0 + \Delta^3 y_0$.

In general,

$y_k = E^k y_0 = (1 + \Delta)^k y_0$

$$y_k = y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

This formula enables us to write an expression for every value of y_k in terms of y_0 and the leading forward differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ etc.

iii) **Relation between the shift operator (E) and the differential operator (D):**

E = e^hD^. $E = e^{hD}$.

we have

$E\,y(x) = y(x+h)$

$$= y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots\ \infty \text{[by Taylor's series]}$$

$$= y(x) + h\,D\,y(x) + \frac{h^2}{2!}D^2 y(x) + \ldots\ \infty$$

$$E\,y(x) = \left[1 + hD + \frac{h^2}{2!}D^2 + \ldots\right] y(x) = e^{hD} y(x)$$

Therefore $E = e^{hD}$.

iv) **Relationship between E and ∇:**

We have by definition of $\nabla f(x)$

$$\nabla f(x) = f(x) - f(x-h)$$

$= f(x) - E^{-1}f(x)$

$= (1 - E^{-1})f(x)$

Therefore,

$$\nabla = (1 - E^{-1})$$

$$E^{-1} = 1 - \nabla$$

$$(E^{-1})^{-1} = E = (1-\nabla)^{-1}$$

(v) **Relation between** Δ, $\boldsymbol{\nabla}$ & E:

We have

$$E\nabla = E(1 - E^{-1}) = E - 1 = \Delta$$

Similarly, $\boldsymbol{E^{-1}}\Delta = \boldsymbol{\nabla}$

(vi) **Relation between** $\boldsymbol{E}$, $\boldsymbol{\delta}$ & μ:

We have

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f(x - \frac{h}{2})$$

$$= E^{\frac{1}{2}}f(x) - E^{-\frac{1}{2}}f(x)$$

$$\therefore \boldsymbol{\delta} = \boldsymbol{E}^{\frac{1}{2}} - \boldsymbol{E}^{-\frac{1}{2}} = \boldsymbol{E}^{-\frac{1}{2}}(\boldsymbol{E}-\mathbf{1}) = \boldsymbol{E}^{-\frac{1}{2}}\Delta$$

$$= E^{\frac{1}{2}}(1-E^{-1}) = \boldsymbol{E}^{\frac{1}{2}}\boldsymbol{\nabla}$$

Also,

$$\mu f(x) = \frac{1}{2}[f\left(x+\frac{h}{2}\right) + f(x-\frac{h}{2})]$$

$$= \frac{1}{2}(E^{\frac{1}{2}}f(x) + E^{-\frac{1}{2}}f(x))$$

$$\therefore \boldsymbol{\mu} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{E}^{\frac{1}{2}} + \boldsymbol{E}^{-\frac{1}{2}})$$

Example: Show that $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

Solution: By definition

$$\mu\delta = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) = \frac{1}{2}(E-E^{-1})$$

Also, $(E-E^{-1}) = (E-1) + (1-E^{-1}) = \Delta + \nabla$

Hence, $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

Example: Show that

(i) $\Delta - \nabla = \Delta\nabla = \delta^2$

(ii) $\Delta^2 = E^2 - 2E + 1 = E\Delta\nabla$

(iii) $\delta = \nabla(1-\nabla)^{-\frac{1}{2}}$

(iv) $\mu = [1+\frac{\delta^2}{4}]^{\frac{1}{2}}$

Solution: (i) $\Delta - \nabla = (\mathrm{E}-1) - (1-E^{-1})$

$= E - 2 + E^{-1}$ (A)

$$\Delta\nabla = (\mathrm{E}-1)(1-E^{-1})$$

$= E - 1 - 1 + E^{-1}$

$= E - 2 + E^{-1}$ (B)

Also, $\delta^2 = [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2$

$= E^1 + E^{-1} - 2$ (C)

From (A), (B), (C), we get

$$\Delta - \nabla = \Delta\nabla = \delta^2$$

(ii) $\Delta^2 = (E-1)(E-1)$

$= E^2 + 1 - 2E$

$= E(E - 2 + E^{-1})$

$= E(\Delta - \nabla)$ (from A)

$= E\Delta\nabla$

Hence proved

(iii) Consider the RHS

$\nabla(1-\nabla)^{-\frac{1}{2}} = (1\text{-}E^{-1})(1-(1-E^{-1}))^{-\frac{1}{2}}$

$= (1\text{-}E^{-1})(1-1+E^{-1})^{-\frac{1}{2}}$

$= E^{\frac{1}{2}}(1-E^{-1})$

$= E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \delta$

Hence,$\delta = \nabla(1-\nabla)^{-\frac{1}{2}}$

(iv)We have $\delta^2 = [E^{\frac{1}{2}} - E^{-\frac{1}{2}}]^2$

$= E^1 + E^{-1} - 2$

$$\therefore [1+\frac{\delta^2}{4}]^{\frac{1}{2}} = [1+\frac{1}{4}(E^1+E^{-1}-2)]^{\frac{1}{2}}$$

$= \frac{1}{2}[E+E^{-1}+2]^{\frac{1}{2}}$

$= \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$

$= \mu$

= LHS.

Example: Evaluate $\left(\frac{\Delta^2}{E}\right)x^3$

Solution: Let h be the interval of differencing

$$\left(\frac{\Delta^2}{E}\right)x^3 = (\Delta^2 E^{-1})x^3$$

$= (E-1)^2 E^{-1} x^3$

$= (E^2 - 2E + 1)E^{-1}x^3$

$= (E - 2 + E^{-1})x^3$

$= (x+h)^3 - 2x^3 + (x-h)^3$

$= 6xh^2$

Example: Prove that $e^x = \frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x}$, the interval of differencing being h.

Solution: We know that

$$Ef(x) = f(x+h)$$

So, $Ee^x = e^{x+h}$

Also, $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

⇨ $\Delta^2 e^x = e^x.(e^h-1)^2$

Hence, $(\frac{\Delta^2}{E})e^x = (\Delta^2 E^{-1})e^x = \Delta^2 e^{x-h} = e^{-h}(\Delta^2 e^x) = e^{-h}e^x(e^h-1)^2$

(because$\Delta^2 e^x = e^x(e^h-1)^2$)

And $\frac{Ee^x}{\Delta^2 e^x} = \frac{e^{x+h}}{e^x(e^h-1)^2}$

So, $\frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x} = e^{-h} e^x (e^h - 1)^2 \frac{e^{x+h}}{e^x (e^h-1)^2} = e^x$ = LHS

Example: Show that $\Delta^r y_x = \nabla^r y_{x+r}$

Solution: $\nabla^r y_{x+r} = (1 - E^{-1})^r y_{x+r}$

$$= (\frac{E-1}{E})^r y_{x+r}$$

$$= (E-1)^r E^{-r} y_{x+r}$$

$$= (E-1)^r y_x$$

$$= \Delta^r y_x$$

Hence proved.

Self-Assessment Questions – 3

5. Show that $\Delta \log f(x) = \log[1 + \frac{\Delta f(x)}{f(x)}]$

   (Hint: $f(x+h) = Ef(x) = (\Delta + 1)f(x) = \Delta f(x) + f(x)$

   =>$\frac{f(x+h)}{f(x)} = \frac{\Delta f(x)}{f(x)} + 1 .Then\ taking \log of\ both\ side\ \ and\ proceed$)

6. Show that$E = e^{hD}, where\ D = \frac{d}{dx}$.

7. Prove that$\Delta^3 = E^3 - 3E^2 + 3E - 1$

8. Evaluate$\Delta(3x + e^{2x} + Sinx)$

9. Evaluate $\Delta^3(1-x)(1-2x)(1-3x), h = 1$

3. DIFFERENCES OF POLYNOMIALS

Let $f(x)$ be a polynomial of degree n, then

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

So,

$$f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_n$$

Therefore,

$$\begin{aligned} f(x+h) - f(x) \\ &= a_0\{(x+h)^n - x^n\} + a_1\{(x+h)^{n-1} - x^{n-1}\} + a_2\{(x+h)^{n-2} - x^{n-2}\} + \cdots \\ &= a_0\left\{x^n + nx^{n-1}h + \tfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots - x^n\right\} \\ &\quad + a_1\{x^{n-1} + (n-1)x^{n-2}h + \tfrac{(n-1)(n-2)}{2}x^{n-3}h^2 + \cdots - x^{n-1}\} \\ &\quad + a_2\{x^{n-2} + (n-2)x^{n-3}h + \tfrac{(n-2)(n-3)}{2}x^{n-4}h^2 + \cdots - x^{n-2}\} \\ &\quad + \ldots \\ &= a_0(nh)x^{n-1} + a_1'x^{n-2} + \cdots + a_n' \end{aligned}$$

Where $a_1', a_2', \ldots, a_n'$ are the new coefficients.

That is,

$$\Delta f(x) = a_0(nh)x^{n-1} + a_1'x^{n-2} + \cdots + a_n'$$

Thus, the first difference of a polynomial of the nth degree is a polynomial of degree n-1. Similarly, the second difference of polynomial will be a polynomial of degree (n-2), and the coefficient of the term x^{n-2} will be $a_0n(n-1)\,h^2$. The nth difference will be a constant $a_0n!\ h^n$ and the $(n+1)^{th}$ and the higher-order difference will be zero.

Conversely, if the n^{th} difference of a polynomial is constant and all the higher $(n+1)^{th}$, $(n+2)^{th}$, …differences vanishes, then the polynomial is of degree n.

Note: It helps to approximate a function by a polynomial if its differences of the same order become nearly constant.

Theorem

(Differences of a polynomial): If $f(x)$ be a polynomial of nth degree in x, then the nth difference of $f(x)$ is constant and $\Delta^{n+1}f(x) = 0$

The converse of this theorem is also true, that is, if the *n^{th} differences of a function tabulated at equally spaced intervals are constant, then the function is a polynomial of degree n.* It should be noted that these results hold good only if the values of x are equally spaced.

Example: Estimate the missing term from the following table:

x	0	1	2	3	4
f(x)	4	3	4	a	12

Solution: Since we are given four values, so the third differences are constant and the fourth differences are zero.

Hence, $\Delta^4 f(x) = 0$, for all values of x

⇨ $(E-1)^4 f(x) = 0$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)f(x) = 0$$

$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$$

$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

where the interval of difference is 1.

Now, substituting x = 0, we obtain

$$f(4) + 4f(3) + 6f(2) - 4f(1) + f(0) = 0$$

Or, $12 + 4a + 6(4) - 4(3) + 4 = 0$

Or, a = 7

Example: Let $y = f(x) = x^2 + x - 1$ be a polynomial of degree 2.

We form a table of difference of function $f(x) = x^2 + x - 1$ for $x = 0, 1, 2, 3, 4, 5$ as

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$x_0 = 0$	$y_0 = -1$	2	2	0	0	0
$x_1 = 1$	$y_1 = 1$	4	2	0	0	
$x_2 = 2$	$y_2 = 5$	6	2	0		
$x_3 = 3$	$y_3 = 11$	8	2			
$x_4 = 4$	$y_4 = 19$	10				
$x_5 = 5$	$y_5 = 29$					

Here $\Delta^2 y = 2$ a constant and $\Delta^3 y = \Delta^4 y = \Delta^5 y_0 = 0$, since $y = f(x)$ is a polynomial of degree 2.

Observation: If there are $(n+1)$ equally spaced tabulated values of (x, y) then we can form the maximum of n^{th} forward / backward differences.

4. FACTORIAL POLYNOMIALS

A polynomial of the form $f(x) = y_x = x^{(n)} = x(x-1) \dots (x-n+1)$

where n is a positive integer, is known as factorial polynomials.
If n = 1,2,3,4,…

$$f(1) = y_1 = x^{(1)} = x$$
$$f(2) = y_2 = x^{(2)} = x(x-1)$$
$$f(3) = y_3 = x^{(3)} = x(x-1)(x-2)$$

and so on
Similarly, we have

$$f(x+1) = y_{x+1} = (x+1)^{(n)} = (x+1)x(x-1) \dots$$

First finite difference of the factorial polynomial:
By definition

$$\Delta x^{(n)} = (x+1)^{(n)} - x^{(n)}$$

$$= [(x+1)x(x-1) \dots (x-n+2)] - [x(x-1)(x-2) \dots (x-n+1)]$$

$$= [(x+1) - (x-n+1)]x(x-1)(x-2) \dots (x-n+2) = nx^{(n-1)}$$

Similarly, higher finite differences will be

$$\Delta^2 x^{(n)} = nx^{(n-1)}$$
$$\Delta^n x^{(n)} = n!$$

Note: $1.x^{(0)}$ is defined as 1.

1. $\Delta^{r+1} x^{(r)} = 0$
2. If h = 1 then the successive differences of $x^{(r)}$ can be obtained by ordinary successive differentiation of $x^{(r)}$
3. If r is a positive integer, then

$$x^{(-r)} = \frac{1}{(x+h)(x+2h) \dots .(x+rh)}$$
$$if\ h = 1\ ,$$
$$x^{(-r)} = \frac{1}{(x+1)(x+2) \dots .(x+r)}$$

Example: Consider a function $y_k = k(k-1)(k-2)$.Then find Δy_k

Solution: By definition

$$\Delta y_k = y_{k+1} - y_k$$

= $(k+1)k(k-1) - k(k-1)(k-2)$

= $3k(k-1)$

Representation of a polynomial using factorial notation

Let $f(x) = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + \cdots + a_n x^{(n)}$ ...(1)

be a polynomial of degree n where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_0 \neq 0$, then

$$\Delta f(x) = \Delta(a_0 + a_1 x^{(1)} + a_2 x^{(2)} + \cdots + a_n x^{(n)})$$

= $a_1 + 2a_2 x^{(1)} + \cdots + a_n n x^{(n-1)}$

$$\Delta^2 f(x) = \Delta(a_1 + 2a_2 x^{(1)} + \cdots + a_n n x^{(n-1)})$$

= $2a_2 + 6a_3 x^{(1)} + \cdots + a_n n(n-1) x^{(n-2)}$

Similarly, $\Delta^n f(x) = a_n n(n-1) \ldots .2.1 x^{(0)}$

= $a_n n!$

Now, putting x = 0 in the above equations, we get

$$f(0) = a_0, \quad \Delta f(0) = a_1, \quad \frac{\Delta^2 f(0)}{2!} = a_2, \ldots, \frac{\Delta^n f(0)}{n!} = a_n$$

Now substituting these values of $a_0, a_1, a_2, \ldots, a_n$ in (1), we get

$$f(x) = f(0) + \Delta f(0) x^{(1)} + \frac{\Delta^2 f(0)}{2!} x^{(2)} + \cdots + \frac{\Delta^n f(0)}{n!} x^{(n)}$$

Example: Show that

(i) $k^2 = k^{(1)} + k^{(2)}$

(ii) $k^3 = k^{(1)} + 3k^{(2)} + k^{(3)}$

(iii) $k^4 = k^{(1)} + 7k^{(2)} + 6k^{(3)} + k^{(4)}$

Solution: (i) Consider $k^{(1)} + k^{(2)} = k + k(k-1)$

$$= k + k^2 - k = k^2$$

(ii) $k^{(1)} + 3k^{(2)} + k^{(3)} = k + 3\big(k(k-1)\big) + k(k-1)(k-2)$

= $k + 3k^2 - 3k + k^3 - 3k^2 + 2k$

= k^3

(iii) $k^{(1)} + 7k^{(2)} + 6k^{(3)} + k^{(4)}$

$$= k + 7\big(k(k-1)\big) + 6k(k-1)(k-2) + k(k-1)(k-2)(k-3)$$

= $k + 7k^2 - 7k + 6k^3 - 18k^2 + 12k + k^4 - 6k^3 + 11k^2 - 6k$

= k^4

Example: We know that $\binom{k}{n} = \frac{k!}{n!(k-n)!}$ and in terms of factorial notation, it is

$$\binom{k}{n} = \frac{k^{(n)}}{n!}$$

Then prove that

$$\binom{k+1}{n+1} = \binom{k}{n+1} + \binom{k}{n}$$

Solution: Consider

$$\binom{k+1}{n+1} - \binom{k}{n+1} = \frac{(k+1)^{(n+1)}}{(n+1)!} - \frac{k^{(n+1)}}{(n+1)!}$$

$$= \frac{\Delta(k)^{(n+1)}}{(n+1)!} = \frac{(n+1)k^{(n)}}{(n+1)!} \quad (\text{because } \Delta(k)^{(n+1)} = (n+1)k^{(n)})$$

$$= \frac{k^{(n)}}{(n)!} = \binom{k}{n}$$

Hence

$$\binom{k+1}{n+1} = \binom{k}{n+1} + \binom{k}{n}$$

Self-Assessment Questions – 4

10. Express $f(x) = 3x^3 + x^2 + x + 1$, in the factorial notation keeping h = 1.

4.1. Difference of Zero

Theorem: If n and s are two positive integers and the interval of differencing (h)is equal to 1, then

$$\Delta^n 0^s = n^s - \binom{n}{1}(n-1)^s + \binom{n}{2}(n-2)^s - \cdots + \binom{n}{n-1}(-1)^{n-1}$$

Proof: We know, $\Delta^n x^s = (E-1)^n x^s$

$$= [E^n - \binom{n}{1}E^{n-1} + \binom{n}{2}E^{n-2} + \cdots + (-1)^n]x^s$$

$$= E^n x^s - \binom{n}{1}E^{n-1}x^s + \binom{n}{2}E^{n-2}x^s + \cdots + (-1)^n x^s$$

$$=(x+n)^s - \binom{n}{1}(x+n-1)^s + \binom{n}{2}(x+n-2)^s$$

$$+ \cdots + \binom{n}{n-1}(-1)^{n-1}(x+1)^s + (-1)^n x^s$$

Put x = 0, we get

$$\Delta^n 0^s = (n)^s - \binom{n}{1}(n-1)^s + \binom{n}{2}(n-2)^s + \cdots + \binom{n}{n-1}(-1)^{n-1}$$

Note: 1. $\Delta 0^s = 1^s = 1$

2. $\Delta^n 0^n = n!$

Example: Evaluate $\Delta^2 0^3$

Solution: By the above theorem, we have

$$\Delta^n 0^s = n^s - \binom{n}{1}(n-1)^s + \binom{n}{2}(n-2)^s - \cdots + \binom{n}{n-1}(-1)^{n-1}$$

So, $\Delta^2 0^3 = 2^3 - \binom{2}{1}(2-1)^3 + \binom{2}{2}(2-2)^3$

$= 8 - 2 = 6$

Self-Assessment Questions – 5

11. Evaluate $\Delta^3 0^6$, $\Delta^5 0^6$, $\Delta^6 0^6$

5. SUMMARY

In this unit, we studied about forward, backward, and central differences. Also, the relationship between different operators like forward, backward, central, shift, and average was studied with the help of suitable examples. The concept of factorial function was also introduced.

6. TERMINAL QUESTIONS

1. By constructing a difference table and taking the second-order difference as constant find the sixth term of the series 8, 12, 19, 29, 42, ….
2. Evaluate $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$ &$\Delta^n(\frac{1}{x})$ taking 1 as the interval of differencing.
3. Show that $(1+\Delta)(1-\nabla)=1$
4. Prove that $\frac{\Delta}{\nabla}-\frac{\nabla}{\Delta}=\Delta+\nabla$
5. Given $u_0=1, u_1=11, u_2=21, u_3=28, u_4=29$ find $\Delta^4 u_0$ without forming difference table.
6. Find $\Delta^3(1-3x)(1-2x)(1-x)$
7. Find the first term of the series whose second and sequent terms are 8,3,0,-1,0. (Hint: f(1) = $f(1) = E^{-1}f(2) = (1+\Delta)^{-1}f(2) = (1-\Delta+\Delta^2-\Delta^3+\cdots)f(2)$)
8. (i) Let $y_k = k^{(n)}$, then show that $\Delta^2 y_k = n(n-1)k^{(n-2)}$. (Hint:$\Delta^2 y_k = \Delta\left(\Delta k^{(n)}\right)$)

   (ii) Show that $\Delta^n k^{(n)} = n!$ and $\Delta^{n+1}k^{(n)} = n!$ (Hint: we know that $k^{(0)} = 1$ and also since n! is constant so its differentiation is zero.
9. Calculate $k^{(n)}$ for k = 1/3 and n = 2,3
10. Express $f(x) = 3x^3 - 4x^2 + 3x - 11$, in the factorial notation keeping h = 1.

7. ANSWERS

Self-Assessment Questions

2. 4x+8, 8, 0
3.

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		9		
2	13		12	
		21		6
3	34		18	
		39		6
4	73		24	
		63		
5	136			

4. *Backward Difference Table*

x	*y*	∇	∇^2	∇^3	∇^4	∇^5
1.00	*1*					
		.03				
1.01	*1.03*		*0.001*			
		.031		*-0.001*		
1.02	*1.061*		*0.000*		*0.002*	
		.031		*0.001*		*-0.003*
1.03	*1.092*		*0.001*		*-0.001*	
		.032		*0.000*		
			0.001			
1.04	*1.124*	*.033*				
1.05	*1.157*					

8. $3h + e^{2x}(e^{2h} - 1) + 2\cos(x + \frac{h}{2})Sin(\frac{h}{2})$

9. -36

10. $\frac{18}{3!}x^{(3)} + \frac{20}{2!}x^{(2)} + 5x^{(1)} + 1$

11. 540, 1800, 6!

Terminal Questions

1. K = 58

2. $\frac{-2}{(x+2)(x+3)} - \frac{3}{(x+3)(x+4)}, \quad \frac{(-1)^n}{x(x+1)(x+2)\ldots(x+n)}$

5. 0

6. -36

7. 15

9. $-\frac{2}{9}, \frac{10}{27}$

10. $3x^{(3)} + 5x^{(2)} + 2x^{(1)} - 11$