Unit 11 Infinite Series

Structure:

11.1 Introduction

Infinite series: If u_n is a real sequence, then an expression of the form

$$u_1 + u_2 + u_3 + \ldots. + u_n + \ldots\ldots\ldots\ldots..$$

Which can be written also as $\sum_{n=1}^{\infty} u_n$ or $\sum_{1}^{\infty}$ or $\sum u_n$ is called an INFINITE SERIES.

Example, 1) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$

2) $1 + 2 + 3 + 4 + \ldots\ldots\ldots$

Objectives:

At the end of the unit you would be able to

- understand the properties of infinite series
- test the convergence or the divergence of an infinite series

Partial Sum

The expression $u_1 + u_2 + u_3 + \ldots + u_n + \ldots..$ involves addition of infinitely many term. To give meaning to this expression we define its sequence of partial sums 'S_n' by

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

We know that an infinite series is given by

$$\sum_{n=1}^{\infty} = u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots\ldots\ldots$$

$\therefore S_1 = u_{1,}\ \ S_2 = u_1 + u_2,\ \ S_3 = u_1 + u_2 + u_3, \ldots\ldots\ldots, S_n = u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n$ are called partial sums.

S_1 is called the 1^{st} partial sum, S_2 is called the 2^{nd} partial sum, ……. S_n is the nth partial sum.

$\therefore$ The sequence (S_n) is called the sequence of partial sums, then we say $\sum u_n$ converges, diverges, Oscillates according as its sequence of partial sums S_n, converges, diverges or oscillates.

Examples

1. The expression $1+(-1) + 1 + (-1) + \ldots\ldots + (-1)^{n+1} + \ldots\ldots$ *(i)*
   Or as it is usually written as $1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots$ is a series. The meaning of expression *(i)* is that from the terms $1, -1, +1, -1, \ldots\ldots$
   $(-1)^{n+1}, \ldots\ldots$ we form the partial sums,
   $S_1 = 1, S_2 = 1 - 1 = 0, S_3 = 1 - 1 + 1 = 1, \ldots\ldots.$
   $S_n = 1 - 1 + \ldots\ldots. + (-1)^{n+1} = \dfrac{1+(-1)^{n+1}}{2}$ ………… *(ii)*
2. The expression $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots+\left(\frac{1}{2}\right)^{n-1}+\ldots\ldots\ldots$ is a series.

   $\therefore$ the partial sums,
   $S_1 = 1, S_2 = 1\tfrac{1}{2},\ S_3 = 1\tfrac{3}{4}, \ldots\ldots$

3. General properties of Series

Following are the fundamental rules or properties of a series:

1. The coverage or divergence of an infinite series remains unaffected by the addition or removal of a finite number of the terms; for the sum of these terms being the finite quantity addition or removal doesnot change the nature of its sum.
2. If a series in which all the terms are positive is convergent, the series remain convergent even when some or all of its terms are negative; for the sum is clearly the greatest when all the terms are positive.

3. The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.

11.2 Convergence and divergence

Convergence of the infinite series

A series is called convergent if the sequence of its partial sums has a finite limit, this limit is termed as the sum of the convergent series.

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $lim_{n\to\infty}\ S_n = l$ where l is a unique real number.

Examples

1) Show that $\frac{1}{2}+\frac{1}{2^2}+......+\frac{1}{2^{n-2}}+......$ is convergent series.

$\lim_{n\to\infty} S_n = \frac{\frac{1}{2}}{1-\frac{1}{2}}$ a unique real number.

$\therefore$ The given series is a convergent series.

Divergence of the infinite series

If a sequence of its partial sums has no finite limit, then the series is called divergent. A divergent series has no sum.

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

If $\lim_{n\to\infty} S_n = \infty$.

Example

Show that $\sum_{n=1}^{\infty} u_n = 1+2+3+4+................+n+......................$ is a divergent

Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$

and hence the given series is divergent.

Necessary condition for convergence of a series

The series,

$u_1 + u_2 + u_3 + \ldots + \ldots\ldots\ldots + u_n + \ldots$ (1)

can converge only when the term u_n (the general term of the series) tends to zero.

i.e $\lim\limits_{n \to \infty} u_n = 0$

if the general term u_n does not tend to zero, then the series diverges.

Examples:

a) The series, $0.0 + 0.44 + 0.444 + 0.4444 + \ldots$ diverges because the general term u_n does not tend to zero.

b) The series $1 - 1 + 1 - 1 + \ldots\ldots$ diverges because the general term u_n does not tend to zero (and has no limit at all).

The remainder of a series

Let us consider an infinite series

$u_1 + u_2 + u_3 + \ldots + \ldots\ldots + u_m + u_{m+1} + u_{m+2} + \ldots$ ----------- (I)

If we discard first m terms of a series, we get the series,

$u_{m+1} + u_{m+2} + \ldots\ldots$ ----------- (II)

which converges (or diverges) if the series *(I)* converges (or diverges) . Therefore, while finding the convergence of a series we can distinguish between a few terms.

When the series *(I)* converges, the sum $R_m = u_{m+1} + u_{m+2} + \ldots$ of series *(II)* is called the remainder or (remainder term) of the first series. ($R_1 = u_2 + u_3 + \ldots$ is the first remainder. $R_2 = u_3 + u_4 + \ldots$ is the second, etc.) the remainder R_m is the error committed by substituting the partial sum S_m (or the sum S of the series *(I)*). The sum S of the series and the remainder R_m are connected by

$S = S_m + R_m$.

As $m \to \infty$ the reminder term of the series approaches to zero. It is of practical importance that this approach be “sufficiently rapid”, that is, that the remainder R_m should become less than the permissible error, for m not too great. Then we say that the series *(I)* converges rapidly, otherwise that series is said to converge slowly.

For example consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+..........$ converges very slowly. Summing the first *20* terms, we get the value of the sum of the series only to within 0.5×10^{-1}; to attain the accuracy up to 0.5×10^{-4}; we have to take at leat *19,999* terms.

Examples

1. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+.............$

Solution:

$$S_n=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...............\frac{1}{n(n+1)}$$

Here $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$$\therefore u_1=\frac{1}{1}-\frac{1}{2},\ u_2=\frac{1}{2}-\frac{1}{3},\ u_3=\frac{1}{3}-\frac{1}{4},...........u_n=\frac{1}{n}-\frac{1}{n+1}$$

$$\therefore S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+......\frac{1}{n}-\frac{1}{n+1}$$

$=1-\frac{1}{n+1}$ [$\therefore$ all the other terms cancel]

$\therefore lim_{n\to\infty} S_n=lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$ which is a unique finite quantity.

$\therefore$ The given series is convergent.

2. Show that the given series divergent. $\sum_{n=1}^{\infty}(-1)^n=-1+1-1+1-1.....$

Solution: $S_n = -1+1 - 1+1 - 1+1$ …….. to n terms
$= 0$ *or* -1 according as n is even or odd.
$lim_{n\to\infty}\ S_n=0\ \text{ or }\ -1$
$\therefore$ The given series oscillates between *2* finite values *0* and *–1.*

3. Test for divergence of the following series: $1^2 + 2^2 + 3^2 + \ldots\ldots +$

Solution: $S_n=1^2+2^2+3^2+.......+n^2=\frac{n(n+1)(2n+1)}{6}$

$$lim_{n\to\infty}\ S_n=lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6}\right)=\infty$$

$\therefore$ given series diverges to $+\infty$.

4. Test for divergence of the following series: $1 + 2 + 3 + \ldots.. + n + \ldots\ldots.. \infty$

   Solution: $S_n = 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \left(\dfrac{n(n+1)}{2}\right)$

   $\therefore lim_{n\to\infty}\ S_n = lim_{n\to\infty}\left(\dfrac{n(n+1)}{2}\right) = \dfrac{1}{2}\ lim_{n\to\infty}(n(n+1)) \to \infty$

   $\therefore$ given series in divergent.

5. Show that the series $1 + r + r^2 + r^3 + \ldots\ldots\ldots. + \infty$
   (i) converges if $| r | < 1$
   (ii) diverges if $r \geq 1$ and
   (iii) oscillates if $r \leq 1$

   Solution: Let $S_n = 1 + r + r^2 + r^3 + \ldots\ldots\ldots. + r^{n-1}$

   Case (i), when $| r | < 1$, $\lim_{n\to\infty} r^n = 0$, since it is a G.P. series,

   $S_n = \dfrac{1-r^n}{1-r} = \dfrac{1}{1-r} - \dfrac{r^n}{1-r}$

   $\therefore lim_{n\to\infty}\ S_n = \dfrac{1}{1-r}$

   $\therefore$ given series is convergent.
   Case (ii), when $r = 1,$
   $S_n = 1 + 1 + 1 + 1 + \ldots\ldots. + 1 = n$
   And, $lim_{n\to\infty}\ S_{n\to\infty}$
   $\therefore$ given series is divergent.
   Case (iii), when $r = -1,$ the series becomes
   $S_n = 1 - 1 + 1 - 1 + \ldots\ldots.$
   Which is an oscillatory series.
   (ii) when $r < -1,$ let $r = -p,$ so that $p > 1,$
   then $r^n = (-1)^n p^n$

   and $S_n = \dfrac{1-r^n}{1-r} = \dfrac{1-(-1)^n\ p^n}{1+p}$

   as $lim_{n\to\infty}\ P^n \to \infty$

   $\therefore lim_{n\to\infty}\ S_n \to -\infty\ or\ +\infty$, from this n is even or odd.
   Hence the series oscillates.

11.3 Series of Positive terms

If the terms of a series of $\sum u_n$ are positive, then its sequence of partial sums

$S_n = u_1 + u_2 + u_3 + ... + u_n$ is monotonically increasing for $S_{n+1} - S_n = u_{n+1} > 0$ for all the values of n.

$\therefore$ Σ u_n of positive terms converges or diverges to according as S_n is bounded or unbounded.

Theorem1: A positive term series either converges to a positive number or diverges to ∞, according as its sequence of partial sums is bounded or not.

Proof:

Let $\sum a_n = S_n = a_1 + a_2 + a_3 + ... + a_n$

$S_{n+1} = a_1 + a_2 + a_3 + ... + a_n + a_{n+1}$

$S_{n+1} - S_n = a_{n+1} > 0$

$\therefore \{S_n\}$ is monotonically increasing.

According to $\{S_n\}$ we have the following 2 possibilities.

(i) $\{S_n\}$ is bounded, or

(ii) $\{S_n\}$ is unbounded above

(i) If $\{S_n\}$ is bounded, then $\{S_n\}$ is bounded above. Hence $\{S_n\}$ is a monotonically increasing sequence which is bounded above.

$\therefore \{S_n\}$ is convergent.

$\therefore lim_{n\to\infty}$ $S_n = a$ unique real number.

$\therefore \sum a_n$ is convergent.

(ii) If $\{S_n\}$ is unbounded above

The $\{S_n\}$ is a monotonically increasing sequence which is unbounded above

$\therefore \{S_n\}$ diverges to $+\infty$

$\therefore lim_{n\to\infty}$ $S_n = +\infty$

$\therefore \sum a_n$ diverges to $+\infty$

Therefore $\sum a_n$ converges to diverges to ∞

Theorem 2: Necessary condition for the convergence of a series of a positive terms. If *a* series $\Sigma\, a_n$ is convergent then $lim_{n\to\infty}\ a_n = 0.$ The converse is not true.

Theorem 3: The nature of the series is not altered by the multiplication of all the terms of the series by the same non-zero constant C

Theorem 4: The nature of the series is not altered by addition of a finite number of terms to the series or by removing a finite number of terms from the beginning.

Theorem 5: If $\sum a_n$ and $\sum b_n$ are *2* series which converge to l and m respectively then the series $\sum (a_n \pm b_n)$ converges to $l \pm m.$

11.4 Binomial Series

According to the Binomial theorem, we have,

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...x^n$$

If the right hand side is extended to ∞,

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...to\ \infty$$

Become infinite series and this series is called as Binomial series.

The Binomial series is absolutely convergent if $|\ x\ | < 1,$ and when the series is convergent, the sum of the finite series is $(1 + x)^n.$

Replacing *x* by – *x,*

$$(1-x)^n = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...to\ \infty$$

While finding the sum of the Binomial series we can use some of the following cases.

1. When $n = -1,$

   $1 - x + x^2 - x^3 + ...\infty = (1+x)^{-1}$

2. When $n = -1$, and x is changed to $-x$,
   $1 + x + x^2 + x^3 + \ldots\ldots\ \infty = (1 + x)^{-1}$
3. When $n = -2$,
   $1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots\infty = (1 + x)^{-2}$
4. When $n = -2$, and x is changed to $-x$,
   $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots\infty = (1 + x)^{-2}$
5. When $n = \frac{p}{q}$ where p and q are integers and $q \neq 0$, we get

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ldots to\ \infty$$

6. When $n = \frac{p}{q}$, x is replaced by $-x$, we get,

$$(1-x)^{\frac{p}{q}} = \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ldots.to\ \infty$$

7. When $n = -\frac{p}{q}$, we get,

$$(1+x)^{\frac{-p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ldots to\ \infty$$

8. When $n = -\frac{p}{q}$, x is replaced *by* $-x$, we get,

$$(1-x)^{\frac{-p}{q}} = 1 - \frac{p}{1!}\frac{x}{q} + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots\ldots\ldots to\ \infty$$

Examples

1. Solve the following: $\frac{3}{50} + \frac{3.18}{50.100} + \frac{3.18.33}{50.100.150} + \ldots\ldots\ldots$

   Solution: Let $S = \frac{3}{50} + \frac{3.18}{50.100} + \frac{3.18.33}{50.100.150} + \ldots\ldots\ldots.$

   S can be written as,

$$S = \frac{3.1}{1!.50} + \frac{3.18}{2!.}\left(\frac{1}{50}\right)^2 + \frac{3.18.33}{3!}\left(\frac{1}{50}\right)^3 + \ldots\ldots..$$

$$\therefore S - 1 = (1-x)^{\frac{-p}{q}}$$

$$\therefore P = 3, q = 15, \frac{x}{q} = \frac{1}{50} \Rightarrow x = \frac{15}{50} = \frac{3}{10}$$

$$S - 1 = \left(1 - \frac{3}{10}\right)^{\frac{-3}{15}} = \left(\frac{7}{10}\right)^{\frac{-1}{5}} = \left(\frac{10}{7}\right)^{\frac{1}{5}}$$

$$\therefore S = \left(\frac{10}{7}\right)^{\frac{1}{5}} - 1$$

2. **Solve:** $\frac{3}{1} + \frac{3.5}{1.2}.\frac{1}{3} + \frac{3.5.7}{1.2.3}\frac{1}{3^2} + \ldots\ldots\ldots\ldots\infty$

Solution: Comparing the given series with one of the general Binomial series, we get

$$p = 3, q = 2, \frac{x}{q} = \frac{1}{3}$$

$$\therefore x = \frac{2}{3}$$

But the power of $\frac{1}{3}$ is not equal to the number factors. Hence we have to multiply and divide by 3,

Let $S = \frac{3}{1} + \frac{3.5}{1.2}.\frac{1}{3} + \frac{3.5.7}{1.2.3}\frac{1}{3^2} + \ldots\ldots\ldots\infty$

$$S = 3\left[\frac{3}{1!}\frac{1}{3} + \frac{3.5}{2!}.\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + \ldots\ldots\ldots\ldots\right]$$

$$= 3\left[-1 + 1 + \frac{3}{1!}\frac{1}{3} + \frac{3.5}{2!}.\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + \ldots\ldots\ldots\ldots\right]$$

$$= -3 + 3\left[1 + \frac{3}{1!}\frac{1}{3} + \frac{3.5}{2!}.\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + \ldots\ldots\ldots \text{ to } \infty.\right]$$

$$= -3 + 3\left[(1 - x)^{\frac{-p}{q}}\right]$$

$$= -3 + 3\left[\left(1 - \frac{2}{3}\right)^{\frac{-3}{2}}\right]$$

$$= -3 + 3\left[\frac{1}{3}\right]^{\frac{-3}{2}}$$

$$= -3 + 3(3)^{\frac{3}{2}}$$

$$\therefore S = -3 + (3)^{\frac{5}{2}}$$

11.5 Exponential series

The exponential function e^x expressed as an infinite series in the form

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

This series is convergent for all values of x.

In finding the sum of the exponential series, the following are to be used.

I. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

II. Putting $x = 1$, in form *(I)* we get,

$$e = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...$$

III. By changing x to – x, in form (I) we get,

$$e^{-x} = \sum_{n=0}^{\infty} \frac{\left(-x^n\right)}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$$

IV. Putting $x = -1$ in form *(I),* it gives,

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...$$

V. By adding *(I)* and *(III),* results in,

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

VI. Subtracting *(III)* from *(I)*,

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots = \sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

VII. Putting $x = 1$ *in (V)*,

$$\frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\ldots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

VIII. Putting $x = 1$ *in (VI)*,

$$\frac{e^1 - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\ldots\ldots\ldots\ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

Note: In exponential series it should be carefully observed whether the summation is from 0 to ∞ or 1 to ∞ or 2 to ∞ etc.

Examples

Problem 1. $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots\ldots\ldots.\text{to } \infty$

The solution is:

$$S = \sum_{n=1}^{\infty} \frac{2n}{(2n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{2n+1-1}{(2n+1)!}$$

$$= \sum_{n=1}^{\infty} \left[\frac{2n+1}{(2n+1)!} - \frac{1}{(2n+1)!} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n)!} - \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$$

$$= \left(\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots \right) - \left(\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots\ldots \right)$$

$$= \left(\frac{e + e^{-1}}{2} - 1 \right) - \left(\frac{e - e^{-1}}{2} - 1 \right)$$

$$= e^{-1}$$

$$\text{or } s = \frac{1}{e}$$

Problem 2. $\frac{1.2.3}{2!} + \frac{2.4.5}{3!} + \frac{3.6.7}{4!} + ...$

Solution: The given series can be written as,

$$S = \sum_{n=1}^{\infty} \frac{n(2n)(2n+1)}{(n+1)!} = \sum_{n=1}^{\infty} \frac{4n^3 + 2n^2}{(n+1)!}$$

Consider $4n^3 + 2n^2 = a + b(n+1) + c(n+1)n + d(n+1)\,n(n-1)$

$$= a + bn + b + cn^2 + cn + dn^3 - dn$$

$$= dn^3 + cn^2 + (b + c - d)n + (a + b)$$

$$\therefore d = 4, c = 2, b + c - d = 0 \Rightarrow b + 2 - 4 = 0 \Rightarrow b = 2.$$

$$\therefore a + b = 0 \Rightarrow a + 2 = 0 \text{ or } a = -2$$

$$\therefore 4n^3 + 2n^2 = -2 + 2(n+1) + 2(n+1)n + 4(n+1)\,n(n-1)$$

$$\therefore \sum_{n=1}^{\infty} \frac{4n^3 + 2n^2}{(n+1)!} = \sum_{n=1}^{\infty} \frac{-2 + 2(n+1) + 2(n+1)n + 4(n+1)n(n-1)}{(n+1)!}$$

$$= -2\sum_{n=1}^{\infty} \frac{1}{(n+1)!} + 2\sum_{n=1}^{\infty} \frac{1}{n\,!} + 2\sum_{n=1}^{\infty} \frac{1}{(n+1)!} + 4\sum_{n=1}^{\infty} \frac{1}{(n-2)!}$$

$$= -2\left[e - \left(1 + \frac{1}{1!}\right)\right] + 2[e - 1] + 2e + 4e$$

$S = 6e + 2$

11.6 Logarithmic Series

The series:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + ...$$

is called the logarithmic series and is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$

Theorem: If Σu_n is a positive term series and $lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = x$ (finite or infinite).

Then the series $\sum_{n=1}^{\infty} u_n$

(i) Converges if $x > 1$

(ii) Diverges if $x < 1$

(iii) May converge or diverge if x = 1

The logarithmic series convergent if $-1 < x < 1$. When it is convergent, the sum of the logarithmic series is given by,

1. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots = \log_e (1+x)$
2. Replacing x by $-x$, in *(1)* we get,

   $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \log_e (1-x)$

   $\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\log_e (1-x)$
3. Adding *(1)* and *(2)* we get,

   $2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\ldots\ldots\ldots\ldots\ldots\right) = \log_e (1+x) - \log_e (1-x)$

   $\therefore 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\ldots\ldots\ldots\ldots\right) = \log_e \left(\frac{1+x}{1-x}\right)$
4. Subtracting *(2)* from *(1)* we get,

   $-2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\ldots\ldots\right) = \log_e (1+x) + \log_e (1-x)$

   Therefore, $2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{5}\ldots\ldots\ldots\ldots\ldots\right) = -\log_e\left(1-x^2\right)$ where $x^2 < 1$
5. Putting $x = 1$ *in (1)* it gives,

   $\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\right) = \log_e 2$

Note: Usually the sum of the logarithmic series is found by resolving the nth term into partial fractions.

Examples

1. Solve $\left(1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+..........\ to\ \infty\right)$

Solution:

Let $S=1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+.......\ to\ \infty$

$$=1+\frac{(\sqrt{x})^2}{3}+\frac{(\sqrt{x})^4}{5}+\frac{(\sqrt{x})^6}{7}+......\ to\ \infty$$

$$=\frac{1}{\sqrt{x}}\left[\sqrt{x}+\frac{(\sqrt{x})^3}{3}+\frac{(\sqrt{x})^5}{5}+\frac{(\sqrt{x})^7}{7}+.......\ to\ \infty\right]$$

$$=\frac{1}{\sqrt{x}}\left[\frac{1}{2}\log_e\left[\frac{1+\sqrt{x}}{1-\sqrt{x}}\right]\right]$$

$$S=\frac{1}{2\sqrt{x}}\left[\log_e\left[\frac{1+\sqrt{x}}{1-\sqrt{x}}\right]\right]$$

2. $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+........$

Solution:

Let $S=\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+.........$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n(2n+1)}$$

Consider, $\frac{1}{(2n-1)\,2n(2n+1)}=\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}$

therefore, $1 = A\,(2n)\,(2n+1) + B(2n-1)\,(2n+1) + C(2n)\,(2n-1)$

Put $n = 0,\ 1 = B(-1) \Rightarrow B = -1$

Put $n = ½,\ 1 = 2A \text{ or } A = \frac{1}{2}$.

Put $n = -1/2$, then $C = (-1)\,(-2) = 2 \text{ or } 2C = 1.$

Substituting these vales in $\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n(2n+1)}$, we get,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n(2n+1)}=\sum_{n=1}^{\infty}\frac{1}{2(2n-1)}-\frac{1}{2n}+\frac{1}{2(2n+1)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$$

Splitting $\frac{1}{2n}$ as $\frac{1}{2}\frac{1}{2n}+\frac{1}{2}\frac{1}{2n}$

$$S=\frac{1}{2}\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+........\right]-\frac{1}{2}\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+......\right]$$

$$=\frac{1}{2}\log 2-\frac{1}{2}(1-\log 2)$$

$$=\log 2-\frac{1}{2}$$

Self Assessment Questions

1. Solve $1-\frac{2}{6}+\frac{2(-1)}{72}-\frac{2(-1)(-4)}{1296}+......\infty$
2. Show that harmonic series of order p, $\sum\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+......+\infty$ converges for $p>1$ and diverges for $p\le 1$

11.7 Summary

In this unit, initially we discussed about the partial sum and general properties of series. Then we studied different rules for convergence or divergence of series. Lastly in this unit we studied binomial series, exponential series, logarithmic series with properly illustrated examples.

11.8 Terminal Questions

1. Write the general properties of a series
2. Explain the binomial series

11.9 Answers

Self Assessment Questions

1. Let $S=1-\frac{2}{6}+\frac{2(-1)}{72}-\frac{2(-1)(-4)}{1296}+...........\infty$

S can be written as

$$S = 1 - \frac{2}{1!}\left(\frac{1}{6}\right) + \frac{2(-1)}{2!}\left(\frac{1}{6}\right)^2 - \frac{2(-1)(-4)}{3!}\left(\frac{1}{6}\right)^3 + \ldots\ldots\ldots\infty$$

Comparing with the expansion of $(1-x)^{p/q}$,

$$p = 2, q = 3, \frac{x}{q} = \frac{1}{6}$$

$$\therefore x = \frac{1}{2}$$

$$S = (1-x)^{-p/q}$$

$$= \left(1 - \frac{1}{2}\right)^{2/3} = \left(\frac{1}{2}\right)^{2/3} = \frac{1}{\sqrt[3]{4}}$$

$$\therefore S = \frac{1}{\sqrt[3]{4}}$$

2. But the above test, this series will converge or diverge according as $\int_1^{\infty} \frac{dx}{x^p}$ is finite. If $p \neq 1, \int_1^{\infty} \frac{dx}{x^p} = \lim_{n\to\infty} \int_1^{n} \frac{dx}{x^p} = \lim_{n\to\infty}\left(\frac{n^{1-p}-1}{1-p}\right)$

   $= \frac{1}{1-p}$, i.e. finite for $p > 1$

   $= \infty$ for $p < 1$

   If $p = 1, \int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \log x \to \infty$

   Therefore the series converges for $p > 1$ and $p \leq 1.$

Terminal Questions

1. Refer to Section 11.1.3
2. Refer to Section 11.4