Unit - 13

Numerical Solution of Ordinary differential equation - II

* Runge Kutta method forth order :-

Consider initial value problem $\frac{dy}{dx} = f(x, y)$ where $y(x_0) = y_0$.

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf\left[x_n + (h/2),\ y_n + (K_1/2)\right]$$

$$K_3 = hf\left[x_n + \frac{h}{2},\ y_n + \frac{K_2}{2}\right]$$

$$K_4 = hf\left[x_n + h,\ y_n + K_3\right]$$

$$y_{n+1} = y_n + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

* Given $\frac{dy}{dx} = x + y^2$, initial value $y(0) = 1$, then find $y(0.2)$ when $h = 0.1$ using Runge Kutta of forth order method.

Sol:- Here $f(x, y) = x + y^2$ and $y(0) = 1$, so $x_0 = 0$ and $y_0 = 1$

Here $h = 0.1$ is given, so $x_1 = x_0 + h$, $x_2 = x_1 + h$

$$x_0 = 0$$
$$x_1 = 0.1$$
$$x_2 = 0.2$$

now put $n=0$ in Runge-Kutta forth order method.

$K_1 = hf(x_0, y_0) = h(x_0 + y_0^2) = 0.1(0+1) = 0.1$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right) = h\left[\left(x_0+\frac{h}{2}\right)+\left(y_0+\frac{K_1}{2}\right)^2\right]$

$= 0.1\left[\left(0+\frac{0.1}{2}\right)+\left(1+\frac{0.1}{2}\right)^2\right] = 0.11525$

$K_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right] = h\left[\left(x_0+\frac{h}{2}\right)+\left(y_0+\frac{K_2}{2}\right)^2\right]$

$= 0.1\left[\left(0+\frac{0.1}{2}\right)+\left(1+\frac{0.11525}{2}\right)^2\right] = 0.1169$

$K_4 = hf[x_0+h, y_0+K_3] = h[(x_0+h)+(y_0+K_3)^2]$

$= 0.1[(0+0.1)+(1+0.1169)^2] = 0.1347$

$y_1 = y_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$y_1 = 1+\frac{1}{6}[0.1+2(0.11525)+2(0.1169)+0.1347]$

$\boxed{y_1 = 1.1165}$

now put $n=1$ in Runge Kutta forth order,

$K_1 = hf(x_1, y_1) = h(x_1+y_1^2) = 0.1[0.1+(1.1165)^2] = 0.1347$

7 am

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{K_1}{2}\right)^2\right]$$

8

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1165 + \frac{0.1347}{2}\right)^2\right] = 0.1552$$

9

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{K_2}{2}\right)^2\right]$$

10

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1165 + \frac{0.1552}{2}\right)^2\right] = 0.1576$$

11

$$K_4 = hf(x_1 + h, y_1 + K_3) = h\left[(x_1 + h) + (y_1 + K_3)^2\right]$$

12

$$= 0.1\left[(0.1 + 0.1) + (1.1165 + 0.1576)^2\right] = 0.1823$$

1 pm

2

$$y_2 = y_1 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

3

$$= 1.1165 + \frac{1}{6}\left[0.1347 + 2(0.1552) + 2(0.1576) + 0.1823\right]$$

4

$$\boxed{y_2 = 1.2737}$$

5

So, $x_0 = 0$ $\quad y_0 = 1$

6 $x_1 = 0.1$ $\quad y_1 = 1.1165$

$x_2 = 0.2$ $\quad y_2 = 1.2737$

7 So, $y(0.2) = 1.2737$.

* Milne's method:-

Consider initial value problem $\frac{dy}{dx} = f(x,y)$,

$y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$.

where x_0, x_1, x_2, x_3 are equidistemce value's of x with interval h.

Milne's predictor formula

$$y_4^p = y_0 + \frac{4}{3}h(2f_1 - f_2 + 2f_3)$$

Milne's corrector formula.

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p).$$

* Find y when $x = 0.8$ given $\frac{dy}{dx} = x - y^2$,

$y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$

Sol:- Here $f(x,y) = x - y^2$, $h = 0.2$ interval of x.

x_n	y_n	$f_n = x - y^2$
0	0	$f_0 = x_0 - y_0^2 = 0 - 0 = 0$
0.2	0.02	$f_1 = x_1 - y_1^2 = (0.2) - (0.02)^2$
0.4	0.0795	$f_2 = x_2 - y_2^2 = (0.4) - (0.0795)^2$
0.6	0.1762	$f_3 = x_3 - y_3^2 = (0.6) - (0.1762)^2$
0.8	$y_4^p = ?$	$f_4^p = x_4 - (y_4^p)^2 = ?$

now milne predictor formula,

$$y_4^p = y_0 + \frac{4}{3}h\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4}{3}(0.2)\left[2(0.1996) - 0.3937 + 2(0.5689)\right]$$

$$\boxed{y_4^p = 0.3049}$$

Now $f_4^p = x_4 - (y_4^p)^2 = (0.8) - (0.3049)^2 = 0.707.$

now milne corrector formula,

$$y_4^c = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^p\right]$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.707\right]$$

$$\boxed{y_4^c = 0.3046}$$

* Adam's predictor and corrector method:-

Adam's Predictor formula:-

$$y_4^p = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

Adam's Corrector formula:-

$$y_4^c = y_3 + \frac{h}{24}\left[9f_4^p + 19f_3 - 5f_2 + f_1\right]$$

All process of Adam are same as Minle, only formula chang.

* Runge-Kutta Second order method:-

Consider $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + K_1)$$

$$y_{n+1} = y_n + \frac{1}{2}[K_1 + K_2]$$