Unit 1 Set Theory

Structure:

1.1 Introduction

The concept of set is basic in all branches of mathematics. It has proved to be of particular importance in the foundations of relations and functions, sequences, geometry, probability theory etc. The study of sets has many applications in logic philosophy, etc.

The theory of sets was developed by German mathematician Georg Cantor (1845 – 1918 A.D.). He first encountered sets while working on problems on trigonometric series. In this unit, we discuss some basic definitions and operations involving sets.

Objectives:

At the end of the unit you would be able to

- understand the concept of sets
- perform the different operations on sets
- write the Power set of a given set

1.2 Sets and their Representations

In every day life, we often speak of collection of objects of a particular kind such as pack of cards, a herd of cattle, a crowd of people, cricket team, etc. In mathematics also, we come across various collections, for example, collection of natural numbers, points in plane, prime numbers, etc. More specially, we examine the collections:

i) Odd natural numbers less than *10*, i.e., *1, 3, 5, 7, 9*
ii) The rivers of India
iii) The vowels in the English alphabet, namely *a, e, i , o, u*
iv) Prime factors of *210*, namely 2, 3, 5 and 7
v) The solutions of a equation $x^2 - 5x + 6 = 0$ viz, *2* and *3*

We note that each of the above collections is a well defined collection of objects in the sense that we can definitely decide whether a given object belongs to a given collection or not. For example, we can say that the river Nile does not belong to collection of rivers of India. On the other hand, the river Ganga does belong to this collection. However, the following collections are not well defined:

i) The collection of bright students in Class *XI* of a school
ii) The collection of renowned mathematicians of the world
iii) The collection of beautiful girls of the world
iv) The collection of fat people

For example, in (ii) above, the criterion for determining a mathematician as most renowned may vary from person to person. Thus, it is not a well defined collection.

We shall say that a set is a well defined collection of objects. The following points may be noted:

i) Objects, elements and members of a set are synonymous terms.
ii) Sets are usually denoted by capital letters *A, B, C, X, Y, Z* etc.
iii) The elements of a set are represented by small letters *a, b, c, x, y, z* etc.

If a is an element of a set *A*, we say that 'a belongs to *A'*. The Greek symbol $\in$ is used to denote the phrase 'belongs to'. Thus, we write $\alpha \in A$. If b is not an element of a set *A*, we write $b \notin A$ and read '*b* does not belong to *A'*. Thus, in the set V of vowels in the English alphabet, a $\in$ V but $l \notin$ V. In the set *P* of prime factors of *30*, $3 \in P$ but $15 \notin P$.

There are two methods of representing *a* set:

i) Roster or tabular form

ii) Set builder form.

i) In roster form, all the elements of a set are listed, the elements being separated by commas and are enclosed within braces *{ }.* For example, the set of all even positive integers less than *7* is described in roster form as *{2, 4, 6}.* Some more examples of representing a set in roster form are given below:

a) The set of all natural numbers which divide *42* is *{1, 2, 3, 6, 7, 14, 21, 42}.* Note that in roster form, the order in which the elements are listed is immaterial. Thus, the above set can also be represented as *{ 1, 3, 7, 21, 2, 6, 14, 42}.*

b) The set of all vowels in the English alphabets is *{a, e, i, o, u}.*

c) The set of odd natural numbers is represented by *{1, 3, 5,. . .}.* The three dots tell us that the list is endless.

It may be noted that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as distinct. For example, the set of letters forming the word "SCHOOL" is *{S, C, H, O, L}*.

ii) In set builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set *"{a, e, i, o, u}"* all the elements possess a common property, each of them is a vowel in the English alphabet and no other letter possesses this property. Denoting this set by *V*, we write *V* = {*x : x* is a vowel in the English alphabet}.

It may be observed that we describe the set by using a symbol *x* for elements of the set (any other symbol like the letters *y, z* etc. could also be used in place of *x*). After the sign of 'colon' write the characteristic property possessed by the elements of the set and then enclose the description within braces. The above description of the set *V* is read as 'The set of all *x* such that *x* is a vowel of the English alphabet'. In this description the braces stand for 'the set of all', the colon stands for 'such that'.

For example, the following description of a set

A = {x : x is a natural number and *3 < x < 10)* is read as “the set of all x such that x is a natural number and $3 < x < 10$”. Hence, the numbers $4, 5, 6, 7, 8$ and 9 are the elements of set A.

If we denote the sets described above in *(a), (b)* and *(c)* in roster form by A, B and C, respectively, then A, B and C can also be represented in set builder form as follows

$A = \{x : x$ is a natural number which divides $42\}$

$B = \{y : y$ is a vowel in the English alphabet$\}$

$C = \{z : z$ is an odd natural number$\}$.

Example: Write the set of all vowels in the English alphabet which precede q.

Solution: The vowels which precede q are a, e, i, o. Thus $A = \{a, e, i, o\}$ is the set of all vowels in the English alphabet which precede q.

Example: Write the set of all positive integers whose cube is odd.

Solution: The cube of an even integer is also an even integer. So, the members of the required set can not be even. Also, cube of an odd integer is odd. So, the members of the required set are all positive odd integers. Hence, in the set builder form we write this set as $\{x : x$ is an odd positive integer$\}$ or equivalently as

$\{2k + 1 : k \geq 0, k$ is an integer$\}$

Example: Write the set of all real numbers which can not be written as the quotient of two integers in the set builder form.

Solution: We observe that the required numbers can not be rational numbers because a rational number is a number in the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$. Thus, these must be real and irrational. Hence, in set builder form we write this set as

$\{x : x$ is real and irrational$\}$

Example: Write the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$ in the set builder form.

Solution: Each member in the given set has the denominator one more than the numerator. Also, the numerators begin from 1 and do not exceed 6. Hence, in the set builder form the given set is

$$\left\{x : x = \frac{n}{n+1},\ \text{n is a natural number and } 1 \le n \ \le 6\right\}$$

Example: Match each of the sets on the left described in the roster form with the same set on the right described in the set builder form:

i) *{ L, I, T, E)*	a) {*x : x* is a positive integer and is a divisor of *18*}
ii) *{0)*	b) {*x : x* is an integer and $x^2 - 9 = 0$}
iii) *{1, 2, 3, 6, 9, 18}*	c) {*x : x* is an integer and $x + 1 = 1$}
iv) *{3, – 3}*	d) {*x : x* is a letter of the word LITTLE}

Solution: Since in (d), there are six letters in the word LITTLE and two letters *T* and *L* are repeated, so (i) matches (d). Similarly (ii) matches (c) as $x + 1 = 1$ implies $x = 0$. Also, *1, 2, 3, 6, 9, 18* are all divisors of *18*. So, (iii) matches *(a)*. Finally, $x^2 - 9 = 0$ implies. $x = 3, -3$. So, (iv) matches (b).

Example: Write the set $\{x : x$ is a positive integer and $x^2 < 40\}$ in the roster form.

Solution: The required numbers are *1, 2, 3, 4, 5, 6*. So, the given set in the roster form is *{1, 2, 3, 4, 5, 6}*.

1.3 The Empty Set

We will understand this concept with the help of example.

Consider the set $\{x : x$ is an integer, $x^2 + 1 = 0\}$. We know that there is no integer whose square is *–1*. So, the above set has no elements.

We now define set *B* as follows:

B = *{x : x* is a student presently studying in both Classes X and *XI}.*

We observe that a student cannot study simultaneously in both Classes *X* and *XI*. Hence, the set *B* contains no element at all.

Consider the set

A = *{x : x* is a student of Class *XI* presently studying in a school}

We can go to the school and count the number of students presently studying in Class *XI* in the school. Thus, the set *A* contains a finite number of elements.

Definition: A set which does not contain any element is called the empty set or the *null set* or the *void set*.

According to this definition *B* is an empty set while *A* is not. The empty set is denoted by the symbol 'ϕ'. We give below a few examples of empty sets.

i) Let $P = \{x: 1 < x < 2, x$ is a natural number }.

Then *P* is an empty set, because there is no natural number between *1* and *2*.

ii) Let $Q = \{x : x^2 - 2 = 0$ and x is rational}.

Then, *Q* is the empty set, because the equation $x^2 - 2 = 0$ is not satisfied by any rational number *x*.

iii) Let $R = \{x : x$ is an even prime number greater than $2\}$

Then *R* is the empty set, because *2* is the only even prime number.

iv) Let $S = \{x : x^2 = 4$, and x is an odd integer}. Then, *S* is the empty set, because equation $x^2 = 4$ is not satisfied by any value of *x* which is an odd integer.

1.4 Finite and Infinite Sets

Let $A = \{1, 2, 3, 4, 5)$, $B = \{a, b, c, d, e, f\}$ and $C = \{$men in the world}.

We observe that *A* contains *5* elements and *B* contains *6* elements. How many elements does *C* contain ? As it is, we do not know the exact number of elements in *C,* but it is some natural number which may be quite a big number. By number of elements of a set *A*, we mean the number of distinct elements of the set and we denote it by $n(A)$. If $n(A)$ is some finite number, then *A* is a finite set, otherwise the set *A* is said to be an infinite set. For example, consider the set, *N*, of natural numbers. We see that $n(N)$, i.e., the number of elements of *N* is not finite since there is no natural number which equals $n(N)$. We, thus, say that the set of natural number is an infinite set.

Definition: A set which is empty or consists of a definite number of elements is called finite. Otherwise, the set is called infinite.

We shall denote several set of numbers by the following symbols:

N : the set of natural numbers

Z : the set of integers

Q : the set of rational numbers

R : the set of real numbers

Z^+ : the set of positive Integers

Q^+ : the set of positive rational numbers

R^+ : the set of positive real numbers

We consider some examples:

i) Let ***M*** be the set of days of the week. Then *M* is finite.

ii) ***Q***, the set of all rational numbers is infinite.

iii) Let *S* be the set of solution (s) of the equation $x^2 - 16 = 0$. Then *S* is finite.

iv) Let *G* be the set of all points on a line. Then *G* is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not always possible to write all the elements of an infinite set within braces { } because the number of elements of such a set is not finite. However, we represent some of the infinite sets in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For instance, $\{1, 2, 3, 4, ... \}$ is the set of natural numbers, $\{1, 3, 5, 7, 9, .. . \}$ is the set of odd natural numbers and $\{..., -3, -2, -1, 0, 1, 2, 3, ... \}$ is the set of integers. But the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

1.5 Equal and Equivalent Sets

Given two sets *A* and *B*. If every element of *A* is also an element of *B* and if every element of *B* is also an element of *A*, the sets *A* and *B* are said to be equal. Clearly, the two sets have exactly the same elements.

Definition: Two sets *A* and *B* are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be *unequal* and we write $A \neq B$

We consider the following examples:

i) Let $A = \{1, 2, 3, 4, \}$ and $B = \{3, 1, 4, 2)$.
Then $A = B$.

ii) Let *A* be the set of prime numbers less than *6* and *P* the set of prime factors of *30*. Obviously, the set *A* and *P* are equal, since *2, 3* and *5* are the only prime factors of *30* and are less than *6*.

Let us consider two sets $L = \{1, 2, 3, 4\}$ and $M = \{1, 2, 3, 8\}$. Each of them has four elements but they are not equal.

Definition: Two finite sets A and B are said to be *equivalent* if they have the same number of elements. We write $A \approx B$.

For example, let $A = \{a, b, c, d, e\}$ and $B = \{1, 3, 5, 7, 9\}$. Then A and B are equivalent sets.

Obviously, all equal sets are equivalent, but all equivalent sets are not equal.

Example: Find the pair of equal sets, if any, giving reasons:
$A = \{0\}$, $B = \{x : x > 15$ and $x < 5\}$, $C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$
$E = \{x : x$ is a positive integral root of the equation $x^2 – 2x – 15 = 0\}$

Solution: Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E. Therefore, $A \neq B, A \neq C, A \neq D, A \neq E$. $B = \phi$ but none of the other sets are empty. Hence $B \neq C, B \neq D$ and $B \neq E$. $C = \{5\}$, since $\{5, –5\} \in D$, hence $C \neq D$. Since $E = \{5\}$, $C = E$. $D = \{–5, 5\}$ and $E = \{5\}$. Therefore $D \neq E$. Thus, the only pair of equal sets are C and E.

1.6 Subsets

Consider the sets S and T, where S denotes the set of all students in your school and T denotes the set of all students in your class. We note that every element of T is also an element of S. We say that T is a subset of S.

Definition: If every element of a set A is also an element of a set B, then A is called a *subset* of B or A is contained in B. We write it as $A \subset B$.

If at least one element of A does not belong to B, then A is not a subset of B. We write it as $A \not\subset B$.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have $B \subset A$. In this case, A and B are the same sets so that we have $A \subset B$ and $B \subset A$ which implies $A = B$.

It follows from the definition that every set A is a subset of itself, i.e., $A \subset A$. Since the empty set ϕ has no elements, we agree to say that ϕ is a subset of every set. We now consider some examples

i) The set ***Q*** of rational numbers is a subset of the set ***R*** of real numbers and we write $\boldsymbol{Q} \subset \boldsymbol{R}$.

ii) If *A* is the set of all divisors of 56 and *B* the set of all prime divisors of *56*, then *B* is a subset of *A*, and we write $B \subset A$.

iii) Let $A = \{1, 3, 5\}$ and $B = \{x : x$ is an odd natural number less than $6\}$, then $A \subset B$ and $B \subset A$ and hence $A = B$.

iv) Let $A = \{a, e, i, o, u\}$, $B = \{a, b, c, d\}$. Then *A* is not a subset of *B*. Also *B* is not a subset of *A*. We write $A \not\subset B$ and $B \not\subset A$.

v) Let us write down all the subsets of the set $\{1, 2\}$. We know ϕ is a subset of every set. So ϕ is a subset of $\{1, 2\}$. We see that $\{1\}$, $\{2\}$ and $\{l, 2\}$ are also subsets of $\{1,2\}$. Thus the set $\{1,2\}$ has, in all, four subsets, viz. ϕ, $\{1\}$, $\{2\}$ and $\{1,2\}$.

Definition: Let *A* and *B* be two sets. If $A \subset B$ and $A \neq B$, then *A* is called a *proper subset* of *B* and *B* is called a *superset* of *A*. For example, $A= \{1, 2, 3\}$ is a proper subset of $B = \{1, 2, 3, 4\}$.

Definition: If a set *A* has only one element, then we call it a *singleton set*. Thus {a } is a singleton.

1.7 Power Set

In example *(v)* of Section *1.6*, we found all the subsets of the set $\{1, 2\}$, viz., ϕ, $\{1\}$, $\{2\}$ and $\{1, 2\}$. The set of all these four subsets is called the power set of $\{1, 2\}$.

Definition: The collection of all subsets of a set *A* is called the *power set* of *A*. It is denoted by *P(A).* In *P(A),* every element is a set.

Example *(v)* of section *1.6,* if $A = \{1, 2\}$, then $P(A)=\{\phi, \{1\}, \{2\}, \{1,2\}\}$. Also, note that, $n[P(A)] = 4 = 2^2$.

In general, if *A* is a set with $n(A) = m$, then it can be shown that
$n[P(A)] = 2^m > m = n(A)$.

1.8 Universal Set

If in any particular context of sets, we find a set U which contains all the sets under consideration as subsets of U, then set U is called the *universal set*. We note that the universal set is not unique.

For example, for the set $\boldsymbol{Z}$ of all integers, the universal set can be the set $\boldsymbol{Q}$ of rational numbers or, for that matter, the set $\boldsymbol{R}$ of real numbers.

For another example, in the context of human population studies, the universal set consists of all the people in the world.

Example: Consider the following sets : ϕ, $A = \{1, 3)$, $B = \{1, 5, 9\}$, $C = \{1, 3, 5, 7, 9\}$, Insert the correct symbol $\subset$ or $\not\subset$ between each pair of sets (i) ϕ — B, (ii) A — B (iii) A — C (iv) B — C.

Solution:

i) $\phi \subset B$ as ϕ is a subset of every set.
ii) $A \not\subset B$ as $3 \in A$ and $3 \notin B$.
iii) $A \subset C$ as $1, 3 \in A$ also belongs to C.
iv) $B \subset C$ as each element of B also belongs to C.

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$. Find all sets X such that

(i) $X \subset B$ and $X \subset C$ (ii) $X \subset A$ and $X \not\subset B$.

Solution:

i) $X \subset B$ means that X is a subset of B, and the subsets of B are ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$. $X \subset C$ means that X is a subset of C, and the subsets of C are ϕ, $\{2\}$, $\{4\}$ and $\{2, 4\}$. Thus, we note that $X \subset B$ and $X \subset C$ means that X is a subset of both B and C. Hence, $X = \phi, \{2\}$.
ii) $X \subset A$, $X \not\subset B$ means that X is a subset of A but X is not a subset of B. So, X is one of these $\{4\}$, $\{1,2,4\}$, $\{2,3,4\}$, $\{l,3,4\}$, $\{1,4\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3,4\}$.

Note: *A* set can easily have some elements which are themselves sets. For example, $\{1, \{2,3\}, 4\}$ is a set having $\{2,3\}$ as one element which is a set and also elements $1,4$ which are not sets.

Example: Let A, B and C be three sets. If $A \in B$ and $B \subset C$, is it true that $A \subset C$? If not, give an example.

Solution: No. Let $A = \{1\}$, $B = C = \{ \{ 1 \}, 2\}$. Here $A \in B$ as $A = \{1\}$ and $B = C$ implies $B \subset C$. But $A \not\subset C$ as $1 \in A$ and $1 \notin C$.

Note that an element of a set can never be a subset of it.

1.9 Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams. Figures representing sets in the form of enclosed region in the plane are called *Venn diagrams* named after British logician John Venn *(1834 –1883 A.D.)*. The universal set U is represented by the interior of a rectangle. Other sets are represented by the interior of circles.

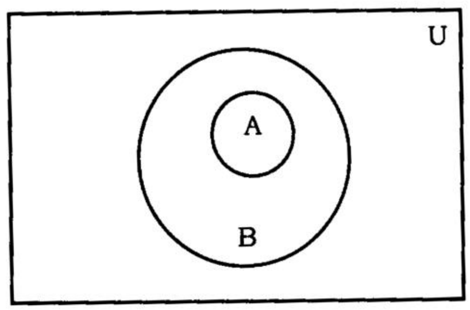

Fig. 1.1

Fig. 1.1 is a Venn diagram representing sets A and B such that $A \subset B$.

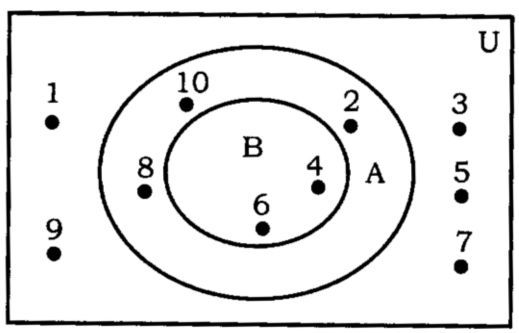

Fig. 1.2

In Fig.1.2, $U = \{1, 2, 3, ..., 10\}$ is the universal set of which $A = \{2,4,6,8,10\}$ and $B = \{4,6\}$ are subsets. It is seen that $B \subset A$. The reader will see an extensive use of the Venn diagrams when we discuss the operations on sets.

1.10 Complement of a Set

Let the universal set U be the set of all prime numbers. Let A be the subset of U which consists of all those prime numbers that are not divisors of 42. Thus $A = \{ x : x \in U$ and x is not a divisor of $42\}$. We see that $2 \in U$ but $2 \notin A$, because 2 is a divisor of 42. Similarly $3 \in U$ but $3 \notin A$, and $7 \in U$ but $7 \notin A$. Now $2, 3$ and 7 are the only elements of U which do not belong to A. The set of these three prime numbers, i.e., the set $\{2, 3, 7\}$ is called the complement of A with respect to U, and is denoted by A'. So we have $A' = \{2, 3, 7\}$. Thus, we see that $A' = \{x : x \in U \text{ and } x \notin A)$. This leads to the following definition.

Definition: Let U be the universal set and A is a subset of U. Then the complement of A with respect to (w.r,t.) U is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus $A' = \{x : x \in U$ and $x \notin A\}$. It can be represented by Venn diagram as

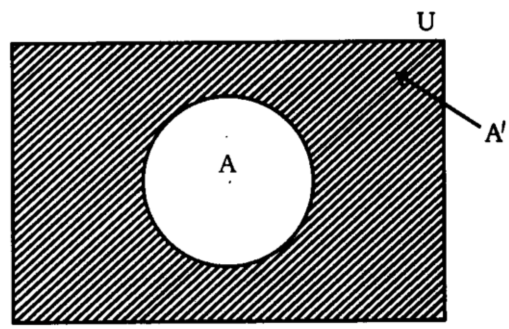

Fig. 1.3

The shaded portion in Fig. 1.3 represents A'.

Example: Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ *and* $A = \{1,3,5,7,9\}$. *Find* A'.

Solution: We note that $2, 4, 6, 8, 10$ are the only elements of U which do not belong to A. Hence $A' = \{2, 4, 6, 8, 10\}$.

Example: Let U be the universal set of all the students of Class *XI* of a co-educational school. Let A be the set of all girls in the Class *XI*. Find A'.

Solution: As A is the set of all girls, hence A' is the set of all boys in the class.

1.11 Operations on Sets

In earlier classes, you learnt how to perform the operations of addition, subtraction, multiplication and division on numbers. You also studied certain properties of these operations, namely, commutativity, associativity, distributivity etc. We shall now define operations on sets and examine their properties. Henceforth, we shall refer all our sets as subsets of some universal set.

a) ***Union of Sets:*** Let *A* and *B* be any two sets. The union of *A* and *B* is the set which consists of all the elements of *A* as well as the elements of *B*, the common elements being taken only once. The symbol '$\cup$' is used to denote the union. Thus, we can define the union of two sets as follows.

Definition: The union of two sets *A* and *B* is the set *C* which consists of all those elements which are either in *A* or in *B* (including those which are in both).

Symbolically, we write $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and usually read as '*A* union *B*'.

The union of two sets can be represented by a Venn diagram as shown in Fig. 1.4.

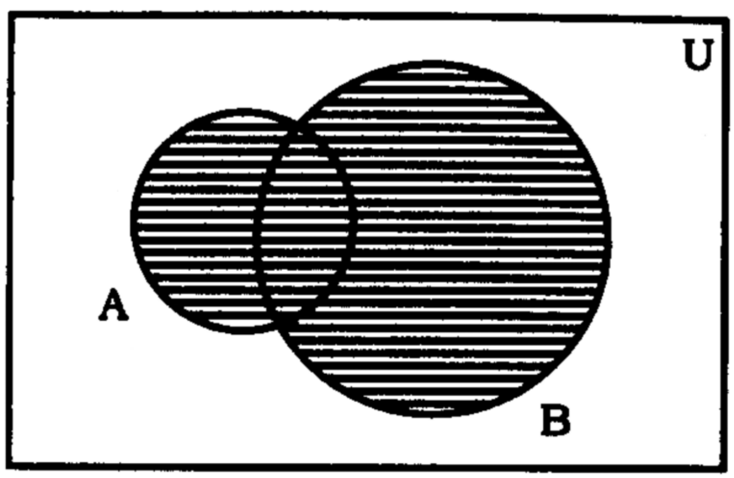

Fig. 1.4

The shaded portion in Fig. 1.4 represents $A \cup B$.

Example: Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$. Find $A \cup B$.

Solution: We have $A \cup B = \{2, 4, 6, 8, 10, 12\}$.

Note that the common elements *6* and *8* have been taken only once while writing $A \cup B$.

Example: Let $A = \{a, e, i, o, u\}$ and $B = \{a, i, u\}$. Show that $A \cup B = A$.

Solution: We have $A \cup B = \{a, e, i, o, u\} = A$.
This example illustrates that the union of a set *A* and its subset *B* is the set *A* itself, i.e., if $B \subset A$, then $A \cup B = A$.

Example: Let *X* = *{Ram, Shyam, Akbar}* be the set of students of Class *XI* who are in the school Hockey team. Let *Y* = {Shyam, David, Ashok} be the set of students from Class *XI* who are in the school Football team. Find $X \cup Y$ and interpret the set.

Solution: We have $X \cup Y$ = {Ram, Shyam, Akbar, David, Ashok}. This is the set of students from Class *XI* who are either in the Hockey team or in the Football team.

Example: Find the union of each of the following pairs of sets:

i) $A = \{1, 2, 3, 4\}; B = \{2, 3, 5\}$
ii) $A = \{x : x \in Z^+ \text{ and } x^2 > 7\}; B = \{1, 2, 3\}$
iii) $A = \{x : x \in Z^+ \}; B = \{x : x \in Z \text{ and } x < 0\}$
iv) $A = \{x : x \in N \text{ and } 1 < x \leq 4\}; B = \{x : x \in N \text{ and } 4 < x < 9\}$

Solution:

i) $A \cup B = \{1, 2, 3, 4, 5\}$
ii) $A = \{3, 4, 5,... \}$, $B = \{1, 2, 3\}$. So, $A \cup B = \{1, 2, 3, 4, 5,... \} = Z^+$
iii) $A=\{1, 2, 3, . . .\}$, $B =\{x:x$ is a negative integer$\}$ = $\{-1,-2,\ldots\}$. So $A \cup B=\{x:x \in Z, x \neq 0\} = \{...,-2,-1,1,2,...\}$.
iv) $A = \{2, 3, 4\}$, $B = \{5, 6, 7, 8\}$. So, $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$.

b) *Intersection* of Sets: The intersection of sets *A* and *B* is the set of all elements which are common to both *A* and *B*. The symbol $\cap$ is used to denote the intersection.

Thus, we have the following definition.

Definition: The intersection of two sets *A* and *B* is the set of all those elements which belong to both A and B. Symbolically, we write

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ and read as 'A intersection B'.

The intersection of two sets can be represented by a Venn diagram as shown in Fig. 1.5.

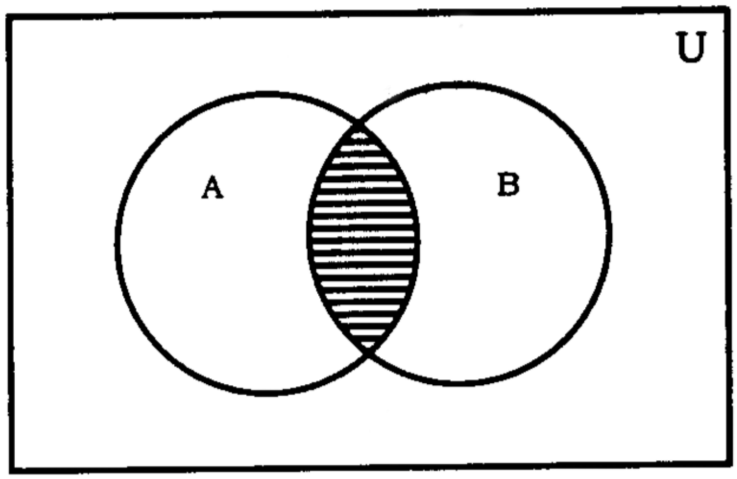

Fig. 1.5

The shaded portion represents $A \cap B$.

If $A \cap B = \phi$, then A and B are said to be disjoint sets. For example, let $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$. Then, A and B are disjoint sets, because there is no element which is common to A and B. The disjoint sets can be represented by Venn diagram as shown in Fig. 1.6.

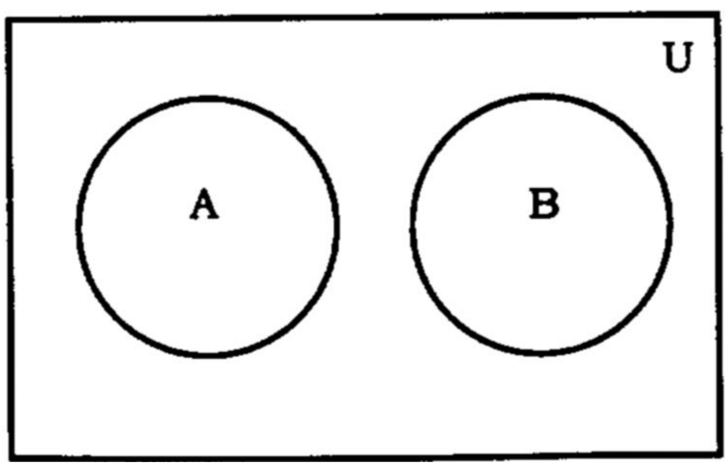

Fig. 1.6

Example: Let $A = \{2, 4, 6, 8\}$ *and* $B = \{6, 8, 10, 12\}$. *Find* $A \cap B$.

Solution: We see that *6, 8* are the only elements which are common to both the sets *A* and *B*. Hence $A \cap B = \{6, 8\}$.

Example: Let *X = {Ram, Shyam, Akbar}* be the set of students of Class *XI* who are in the school Hockey team. Let *Y* = {Shyam, David, Ashok} be the set of students from Class *XI* who are in the school Football team. Find $X \cap Y$.

Solution: We see that the element "Shyam" is the only element common to both the sets *X* and *Y*. Hence, $X \cap Y = \{ Shyam \}$.

SAQ 1: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$ and prove that $A \cap B = B$.

SAQ 2: Let $A = \{x : x \in Z^+\}$; $B = \{x : x$ *is a multiple of 3,* $x \in Z\}$:

C = {x:x is a negative integer}; D = {x:x is an odd integer}. Find (i) $A \cap B$, (ii) $A \cap C$, (iii) $A \cap D$, (iv) $B \cap C$, (v) $B \cap D$, (vi) $C \cap D$.

c) *Difference* of Sets: The difference of sets *A* and *B*, in this order, is the set of elements which belong to *A* but not to *B*. Symbolically, we write *A — B* and read as '*A* difference *B*'. Thus $A — B = \{x : x \in A$ and $x \notin B\}$ and is represented by Venn diagram in Fig.1.7. The shaded portion represents *A — B.*

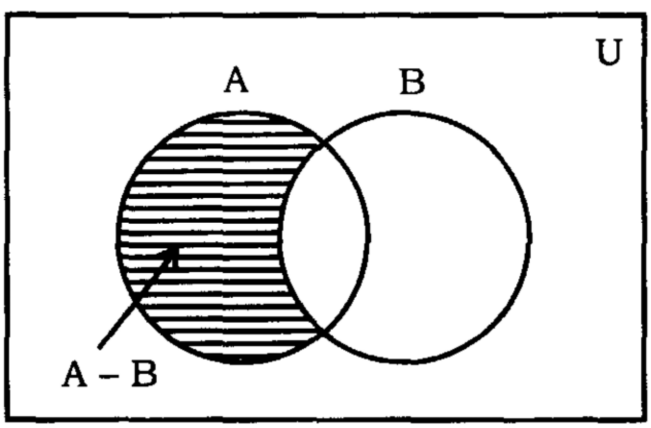

Fig. 1.7

SAQ 3: Let $V = \{a, e, i, o, u\}$ and $B = \{a, i, k, u\}$. Find $V – B$ and $B – V$.

SAQ 4: Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$. Find $A - B$ and $B - A$.

1.12 Applications of Sets

Let A and B be finite sets. If $A \cap B = \phi$, then

$$n(A \cup B) = n(A) + n(B) \quad (1)$$

The elements in $A \cup B$ are either in A or in B but not in both as $A \cap B = \phi$. So (1) follows immediately.

In general, if A and B are finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \quad (2)$$

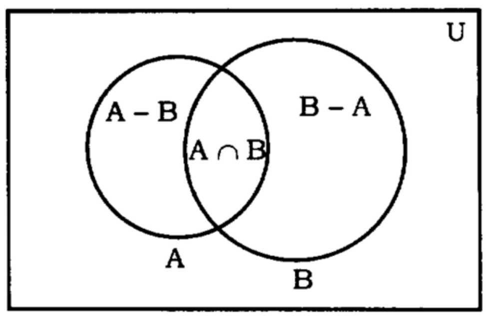

Fig. 1.8

Note that the sets $A - B$, $A \cap B$ *and* $B - A$ are disjoint and their union is $A \cup B$ (Fig 1.8). Therefore

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(A \cap B) + n(B - A) \\ &= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B). \end{aligned}$$

which verifies (2).

If A, B and C are finite sets, then

$$\begin{aligned} n(A \cup B \cup C) = {} & n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) \\ & - n(A \cap C) + n(A \cap B \cap C) \quad (3) \end{aligned}$$

In fact, we have

$$\begin{aligned} n[A \cap (B \cup C)] &= n(A) + n(B \cup C) - n(A \cap (B \cup C)) \\ & \text{[by (2)]} \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap (B \cup C)) \quad \text{[by (2)]} \end{aligned}$$

Since $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we get

$$
\begin{aligned}
n[A \cap (B \cup C)] &= n[(A \cap B) \cup (A \cap C)] \\
&= n(A \cap B) + n(A \cap C) - n[A \cap B \cap A \cap C)] \{\text{by } (1)\} \\
&= n(A \cap B) + n(A \cap C) - n[A \cap B \cap C)]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) \\
&\quad - n(A \cap C) + n(A \cap B \cap C).
\end{aligned}
$$

This proves (3).

Example: If X and Y are two sets such that $n(X \cup Y) = 50$, $n(X) = 28$ and $n(Y) = 32$, find $n(X \cap Y)$.

Solution: By using the formula

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y),$$

we find that

$$
\begin{aligned}
n(X \cap Y) &= n(X) + n(Y) - n(X \cup Y) \\
&= 28 + 32 - 50 = 10..
\end{aligned}
$$

Alternatively, suppose $n(X \cap Y) = k$, then

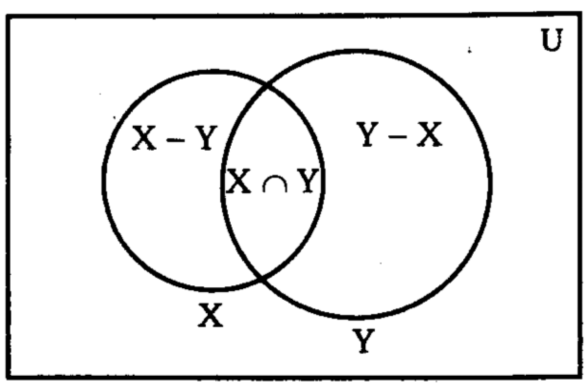

Fig. 1.9

$n(X - Y) = 28 - k$, $n(Y - X) = 32 - k$. (by Venn diagram in Fig 1.9)

This gives $50 = n(X \cup Y) = (28 - k) + k + (32 - k)$.

Hence, $k = 10$

Example: In a school there are 20 teachers who teach mathematics or physics. Of these, *12* teach mathematics and *4* teach physics and mathematics. How many teach physics?

Solution: Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach physics. We are given that $n(M \cup P) = 20, n(M) = 12, n(M \cap P) = 4$. Therefore

$n(P) = n(M \cup P) – n(M) + n(M \cap P) = 20 – 12 + 4 = 12.$

Hence, *12* teachers teach physics.

SAQ 5: In a group of *50* people, *35* speak Hindi, *25* speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi ? How many people speak English?

1.13 Cartesian Product of Sets

Let A, B be two sets. If $a \in A$, $b \in B$, then (a, b) denotes an *ordered pair* whose first component is a and the second component is b. Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$.

In the ordered pair (a, b), the order in which the elements a and b appear in the bracket is important. Thus (a, b) and (b, a) are two distinct ordered pairs if $a \neq b$. Also, an ordered pair (a, b) is not the same as the set $\{a, b\}$.

Definition: The set of all ordered pairs (a, b) of elements $a \in A$, $b \in B$ is called the *Cartesian Product* of sets A and B and is denoted by A x B. Thus

$$A \times B = \{(a, b): a \in A, b \in B\}.$$

Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$. To write the elements of A x B, take $a_1 \in A$ and write all elements of B with a_1, i.e., $(a_1, b_1), (a_1, b_2), (a_1, b_3)$. Now take $a_2 \varepsilon A$ and write all the elements of B with a_2, i.e., $(a_2, b_1), (a_2, b_2), (a_2, b_3)$. Therefore, A x B will have six elements, namely, $(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)$.

Remarks:

i) If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

ii) If $A \neq \phi$ and $B \neq \phi$, then $A \times B \neq \phi$. Thus, $A \times B \neq \phi$ if and only if $A \neq \phi$ and $B \neq \phi$. Also, $A \times B \neq B \times A$.

iii) If the set A has m elements and the set B has n elements, then $A \times B$ has mn elements.

iv) If A and B are non-empty sets and either A or B is an infinite set, so is A x B.

v) If $A = B$, then $A \times B$ is expressed as A^2.

vi) We can also define, in a similar way, ordered triplets. If A, B and C are three sets, then (a, b, c), where $a \in A$, $b \in B$ and $c \in C$, is called an ordered triplet. The Cartesian Product of sets A, B and C is defined as

$A \times B \times C = \{(a, b, c): a \in A, b \in B, c \in C\}$. An ordered pair and ordered triplet are also called *2-tuple* and *3-tuple*, respectively. In general, if $A_1, A_2, \ldots, A_n$ are n sets, then $(a_1, a_2, \ldots, a_n)$ is called an *n-tuple* where $a_i \in A_i$, $i = 1, 2, \ldots, n$ and the set of all such *n-tuples*, is called the Cartesian product of $A_1, A_2, \ldots, A_n$. It is denoted by $A_1 \times A_2 \times \ldots \times A_n$. Thus

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n): a_i \in A_i, 1 \le i \le n\}.$$

Example: Find x and y if $(x + 2, 4) = (5, 2x + y)$.

Solution: By definition of equal ordered pairs, we have

$$x + 2 = 5 \qquad (1)$$

$$2x + y = 4 \qquad (2)$$

Solving *(1)* and *(2)*, we get $x = 3$, $y = -2$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Find $A \times B$ and $B \times A$ and show that $A \times B \neq B \times A$.

Solution: We have

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

and $$B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$$

Note that $(1, 4) \in A \times B$ and $(1, 4) \notin B \times A$. Therefore, $A \times B \neq B \times A$.

Example: Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$. Find

i) $A \times (B \cap C)$ ii) $(A \times B) \cap (A \times C)$

iii) $A \times (B \cup C)$ iv) $(A \times B) \cup (A \times C)$

Solution:

i) We have $B \cap C = \{4\}$. Therefore, $A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$.

ii) We note that

$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

and $A \times C = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

Therefore $(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

iii) Clearly $B \cup C = \{3, 4, 5, 6\}$. Thus

$A \times (B \cup C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

iv) In view of (ii), we see that

$(A \times B) \cup (A \times C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

In view of the assertion in above Example , we note that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

and $$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

SAQ 6: Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If $(a_1, 2), (a_2, 3), (a_3, 2), (a_4, 3), (a_5, 2)$ are in $A \times B$ and a_1, a_2, a_3, a_4 and a_5 are distinct. Find A and B.

1.14 Summary

This unit tells us about sets and their representations. We study the concepts of Empty sets, Finite and Infinite sets, Equal sets, universal sets. All the concepts discussed is well illustrated by standard examples. The different operations on sets like complement of Set, Operation on Sets and Applications of sets are also discussed here.

1.15 Terminal Questions

1. Which of the following pairs of sets are equal ? Justify your answer.
   i) A, the set of letters in “ALLOY” and B, the set of letters in “LOYAL”
   ii) $A = \{n : n \in Z \text{ and } n^2 \leq 4\}$ and $B = \{x : x \in \mathbf{R} \text{ and } x^2 - 3x + 2 = 0\}$.
2. State which of the following sets are finite and which are infinite:
   i) $\{x : x \in \mathbf{N} \text{ and } (x - 1)(x - 2) = 0\}$
   ii) $\{x : x \in \mathbf{N} \text{ and } x^2 = 4\}$
   iii) $\{x : x \in \mathbf{N} \text{ and } 2x - 1 = 0\}$
   iv) $\{x : x \in \mathbf{N} \text{ and } x \text{ prime}\}$
   v) $\{x : x \in \mathbf{N} \text{ and } x \text{ odd}\}$
3. If A and B are two non-empty sets such that $A \times B = B \times A$, show that $A = B$

1.16 Answer

Self Assessment Questions

1. We have $A \cap B = \{2, 3, 5, 7\} = B$.

   We note that if $B \subset A$, then $A \cap B = B$.

2. $A = \{x:x$ is a positive integer}, $B = \{3n : n \in Z\}$; *C = {x:x is a negative integer}; D = {x:x* is an odd integer}.
   i) $A \cap B = \{3, 6, 9, 12,...\} = \{3n:n \in Z^+\}$.
   ii) $A \cap C = \phi$
   iii) $A \cap D = \{1, 3, 5, 7,...\}$
   iv) $B \cap C = \{-3, -6, -9,. . . \} = \{3n : n$ is a negative integer}
   v) $B \cap D = \{. . ., -15, -9, -3, 3, 9, 15,...\}$
   vi) $C \cap D = \{-1, -3, -5, -7,...\}$

3. We have $V - B = \{e, o\}$, since the only elements of V which do not belong to B are e and o. Similarly $B - V = \{k\}$

4. We have $A - B = \{1, 3, 5\}$, as the only elements of A which do not belong to B are 1, 3 and 5. Similarly, $B - A = \{8\}$.
   We note that $A - B \neq B - A$

5. Let H denote the set of people speaking Hindi and E the set of people speaking English. We are given that $n(H \cup E) = 50$, $n(H) = 35$, $n(H \cap E) = 25$. Now

   $n(H \cup E) = n(H) + n(E - H)$.

   So $50 = 35 + n(E - H)$, i.e. , $n(E - H) = 15$.

   Thus, the number of people who speak only English but not Hindi is 15.
   Also, $n(H \cup E) = n(H) + n(E) - n(H \cap E)$ implies

   $50 = 35 + n(E) - 25$,

   which gives $n(E) = 40$.

   Hence, the number of people who speak English is 40.

6. Since $a_1, a_2, a_3, a_4, a_5 \in A$ and $n(A) = 5$, $A = \{a_1, a_2, a_3, a_4, a_5\}$. Also $2, 3 \in B$ and $n(B) = 2$. Therefore, $B = \{2, 3\}$.

Terminal Quesitons

1. i) A = {A, L, L, O, Y}, B = {L, O, Y, A, L}. Then A, B are equal sets as repetition of elements in a set do not change a set. Thus A = {A, L, O, Y} = B.

   ii) A = {–2, –1, 0, 1, 2,}, B = (1, 2). Since $0 \in A$ and $0 \notin B$, A and B are not equal sets.

2. i) Given set = {1, 2}. Hence, it is finite.

   ii) Given set = {2}. Hence, it is finite.

   iii) Given set = ϕ. Hence, it is finite.

   iv) The given set is the set of all prime numbers and since the set of prime numbers is infinite, hence the given set is infinite.

   v) Since there are infinite number of odd numbers, hence the given set is infinite

3. Let $a \in A$. Since $B \neq \phi$, there exists $b \in B$. Now, $(a, b) \in A \times B = B \times A$ implies $a \in B$. Therefore, every element in A is in B giving $A \subset B$. Similarly, $B \subset A$. Hence A = B.