BACHELOR OF COMPUTER APPLICATIONS

SEMESTER 3

DCA2101

COMPUTER ORIENTED NUMERICAL METHODS

Unit 2

Approximations and Round-off Errors

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1. INTRODUCTION

In the previous chapter, we studied different significant digits, types of error-inherent error, Numerical error, and Roundoff error. In this unit, we will proceed with the error study and will approximate a function with the help of the Taylors series and find the error approximation of it.

1.1 Objectives:

At the end of this unit the student should be able to:

- *Use the concept of Taylor's series for approximating a function.*
- *Know about error propagation of single and several variable functions.*
- *Understand different reasons for the error.*

2. TAYLOR SERIES

Taylor's Theorem was given by Brook Taylor (1685-1734) in the year 1715. Taylor's theorem allows us to represent a function in terms of polynomials with specified and boundable error.

Theorem: Suppose $f\ has$ (n+1) continuous derivative in $[a, b]$ and let $x, x_0 \epsilon [a, b]$. Then for every $x \epsilon [a, b]$, there exists a number $x\ and\ x_0$ with $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f'(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$R_n(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1}$$

Where $P_n(x)$ is called the nth Taylor polynomial for f about the point x_0 and $R_n(x)$ is called the remainder term or truncation error.

In terms of step function i.e $h = x - x_0$, Taylor series can be written as

$$f(x) = f(x_0) + f'(x_0)h + \frac{f'(x_0)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n(x)$$

Where the remainder $R_n(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}h^{n+1}$

Usually for finding Taylor's series of a function we use the following expression

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f'(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Example: Find Taylor's series of the function $f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$ at point c = 2.

Solution: We know Taylor's series expansion of function at the point x_0 is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

So at $x_0 = 2$, we have

$f(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x - 2)^n$.....(1)

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$
$$f(2) = 3(32) - 2(16) + 15(8) + 13(4) - 12(2) - 5 = 207$$
$$f'(x) = 15x^4 - 8x^3 + 45x^2 + 26x - 12$$
$$f'(2) = 15(16) - 8(8) + 45(4) + 26(2) - 12 = 396$$
$$f''(x) = 60x^3 - 24x^2 + 90x + 26$$
$$f''(2) = 60(8) - 24(4) + 90(2) + 26 = 590$$
$$f'''(x) = 180x^2 - 48x + 90$$
$$f'''(2) = 180(4) - 48(2) + 90 = 714$$
$$f^{iv}(x) = 360x - 48$$
$$f^{iv}(x) = 360(2) - 48 = 672$$
$$f^{v}(x) = 360$$
$$f^{v}(2) = 360$$
$$f^{vi}(x) = 0$$
$$f^{vi}(2) = 0$$

Putting these values in (1), we get

$$f(x) = 207 + 396(x-2) + \frac{590}{2!}(x-2)^2 + \frac{714}{3!}(x-2)^3 + \frac{672}{4!}(x-2)^4 + \frac{360}{5!}(x-2)^5$$

$$f(x) = 207 + 396(x-2) + 295(x-2)^2 + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5$$

Example: Find the Taylors Series for $f(x) = e^x$ about $x_0 = 0$

Solution: We know Taylor's series expansion of function at the point x_0 is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

So at $x_0 = 0$, we have

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n$.....(2)

$$f(x) = e^x$$
$$f(0) = e^0 = 1$$
$$f'(x) = e^x$$
$$f'(0) = 1$$
$$f''(x) = e^x$$
$$f''(0) = 1$$
$$\vdots$$
$$f^n(x) = e^x$$
$$f^n(0) = 1$$

Putting these values in (2), we get

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Example: Find the Taylors Series for $f(x) = \frac{1}{x^2}$ about $x_0 = -1$

Solution: We know Taylor's series expansion of function at the point x_0 is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

So, at $x_0 = -1$, we have

$f(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \cdots + \frac{f^{(n)}(-1)}{n!}(x+1)^n$ .... (3)

$$f(x) = \frac{1}{x^2}$$
$$f(-1) = 1$$
$$f'(x) = -2x^{-3}$$
$$f'(-1) = 2$$
$$f''(x) = 6x^{-4}$$
$$f''(-1) = 6$$
$$\vdots$$
$$f^n(x) = (-1)^{n+1}(n-1)!\, x^{-n-2}$$
$$f^n(-1) = (n+1)!$$

Putting these values in (3), we get

$$f(x) = 1 + 2(x+1) + \frac{6}{2!}(x+1)^2 + \cdots + \frac{(n+1)!}{n!}(x+1)^n$$

= $1 + 2(x+1) + 3(x+1)^2 + \cdots + (n+1)(x+1)^n$

Example: Find the Taylors Series for $f(x) = x^3 - 10x^2 + 6$ about $x_0 = 3$

Solution: We know Taylor's series expansion of function at the point x_0 is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

So at $x_0 = 3$, we have

$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \cdots + \frac{f^{(n)}(3)}{n!}(x-3)^n$ (4)

$$f(x) = x^3 - 10x^2 + 6$$
$$f(3) = -57$$

$$f'(x) = 3x^2 - 20x$$
$$f'(3) = -33$$

$$f''(x) = 6x - 20$$
$$f''(3) = -2$$
$$f'''(x) = 6$$
$$f^{n}(3) = 6$$
$$f^{iv}(x) = 0$$
$$f^{iv}(3) = 0, \qquad for\ all\ n \geq 4$$

Putting these values in (4), we get

$$f(x) = -57 - 33(x-3) - (x-3)^2 + (x-3)^3$$

Self-Assessment Questions – 1

1. Find the Taylors series expansion of the function $\frac{1}{1-x}$ at $x_0 = 2$
2. Find the Taylors series expansion of the function $\frac{x}{2+x}$ at $x_0 = -1$

Note: Taylor's series helps to predict values of the function at any point in the interval of its domain with the help of functional value and its derivative at another point.

3. ERROR PROPAGATION

After studying different types of errors in the previous section now in this section we will study how errors in numbers can propagate through mathematical functions. We know that numerical techniques consist of a series of calculations such as addition, multiplication, power, and division. So. If we perform these operations on any two numbers that have errors, then the result we will get also has the error.

It is necessary to know how an error in each case of computation is carried forward and finally results in the accumulated error

Let X and Y be two numbers and their approximate values are $X_A \; and \; Y_A$. Then, $e_X = X - X_A$, $e_Y = Y - Y_A$ respectively are errors in X and Y

Let's see how these errors propagate in the case of an addition, subtraction, multiplication, division, and power.

Addition

Let $Z = X + Y$ be the addition of two numbers X and Y and $Z_A = X_A + Y_A$ be the sum of approximated values of X and Y. So, the error in Z and Z_A is given as

$$e_Z = Z - Z_A$$

So, consider $Z = X + Y$

$$Z_A + e_Z = (X_A + e_X) + (Y_A + e_Y)$$

$$X_A + Y_A + e_Z = (X_A + e_X) + (Y_A + e_Y)$$

⇨ $e_Z = e_X + e_Y$

Thus, an error in the sum of two numbers is equal to the sum of the errors of their two numbers. Also, the same can be extended for n number of elements.

In a similar way, we can proceed with subtraction.

Multiplication

Let $Z = X.Y$ be the product of two numbers X and Y. Consider $Z_A = X_A.Y_A$ be the product of their approximates, so we have

$$Z - e_Z = (X - e_X).(Y - e_Y)$$

$= XY - (X\, e_Y + Y e_X) + e_X e_Y$

Now neglecting small quantity i.e., $e_X e_Y$, we have

$$e_Z \cong X\, e_Y + Y e_X$$

$$\frac{e_Z}{Z} \cong \frac{e_Y}{Y} + \frac{e_X}{X}$$

Division

Similarly, for the case of division, we have

$$e_Z \cong \frac{e_X}{Y} - \frac{X e_Y}{Y^2}$$

Functions of Single variable

Consider a function of single variable $f(x)$. Let x_a be an approximation of x. Now, lets find the change in values of $f(x)$ *and* $f(x_a)$.

That is, $\Delta f(x_a) = |f(x) - f(x_a)|$

Now, we will find the value of $f(x)$where x is unknown, this we can easily do with the help of the Taylors Theorem**.** If x_a is the value close to x then by Taylor's theorem we can estimate $f(x)$as

$$f(x) = f(x_a) + f'(x_a)(x - x_a) + \frac{f''(x_a)}{2!}(x - x_a)^2 + \cdots$$

Neglecting the second and higher-order terms, we have

$$f(x) - f(x_a) \cong f'(x_a)(x - x_a)$$

Or $\Delta f(x_a) = |f'(x_a)|(x - x_a) = |f'(x_a)|\Delta x_a$ ...(5)

Where $\Delta f(x_a)$ represents an estimate of the error function and Δx_a represents an error of x.

Eq. (5) gives the capability to approximate the error in (x).

Functions of several variables

If $f = f(x_1, x_2, x_3, \dots, x_n)$ be a function of n- variables $x_1, x_2, x_3, \dots, x_n$.

Let Δx_i represents the error in each x_i. So the error in f is

$$f + \Delta f = f(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, \dots, x_n + \Delta x_n)$$

$$= f(x_1, x_2, x_3, \dots, x_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i + O(\Delta x_i{}^2)$$

If the error in x_i is small so the second and higher powers of Δx_i can be ignored.

Hence

$$\Delta f_{max} = |\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i| = |\frac{\partial f}{\partial x_1}\Delta x_1| + |\frac{\partial f}{\partial x_2}\Delta x_2| + \cdots + |\frac{\partial f}{\partial x_n}\Delta x_n|$$

The relative error E_r is given by

$$E_r = \frac{\Delta f_{max}}{f} = \frac{\partial f}{\partial x_1}\frac{\Delta x_1}{f} + \frac{\partial f}{\partial x_2}\frac{\Delta x_2}{f} + \cdots + \frac{\partial f}{\partial x_n}\frac{\Delta x_n}{f}$$

Example: Consider a function $f(x) = x^3$. Estimate the error in $f(x)$, given that $x_a = 2.5\ and\ \Delta x_a = 0.01$.

Solution: we know that

$$\Delta f(x_a) = |f'(x_a)|\Delta x_a$$

$f'(x_a) = 3{x_a}^2 = 3(2.5)^2 = 18.75$

$=>\Delta f(x_a) = (18.75)(0.01) = 0.1875$

Example: Determine the relative error for the function

$$f(x, y, z) = 3x^2y^2 + 5y^2z^2 - 7x^2z^2 + 38$$

Where x = y = z = 1and $\Delta x = -0.05, \Delta y = 0.001, \Delta z = 0.02$

Solution: Given that $f(x, y, z) = 3x^2y^2 + 5y^2z^2 - 7x^2z^2 + 38$

At $(x, y, z) = (1,1,1)$

$$f(x, y, z) = 3 + 5 - 7 + 38 = 38$$

$$\frac{\partial f}{\partial x} = 6xy^2 - 14xz^2$$

$$\frac{\partial f}{\partial y} = 6x^2y + 10yz^2$$

$$\frac{\partial f}{\partial z} = 10y^2z - 14x^2z$$

$$|\Delta f| = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i = |\frac{\partial f}{\partial x_1}\Delta x_1| + |\frac{\partial f}{\partial x_2}\Delta x_2| + \cdots + |\frac{\partial f}{\partial x_n}\Delta x_n|$$

$= |(6xy^2 - 14xz^2)\Delta x| + |(6x^2y + 10yz^2)\Delta y| + |(10y^2z - 14x^2z)\Delta z)|$
$= |(-14)(0.05)| + |(16)(0.001)| + |(-4)(0.02)| = 0.796$

$$E_r = \frac{\Delta f_{max}}{f} = \frac{0.796}{38} = 0.0209$$

Self-Assessment Questions – 2

3. Find the relative maximum error in F where $F = \frac{5x^2y}{z^3}$. Given that $\Delta x = \Delta y = 0.001$ and x = y = z = 1

4. TOTAL NUMERICAL ERRORS

Total numerical error is defined as the sum of truncating error and round-off errors.

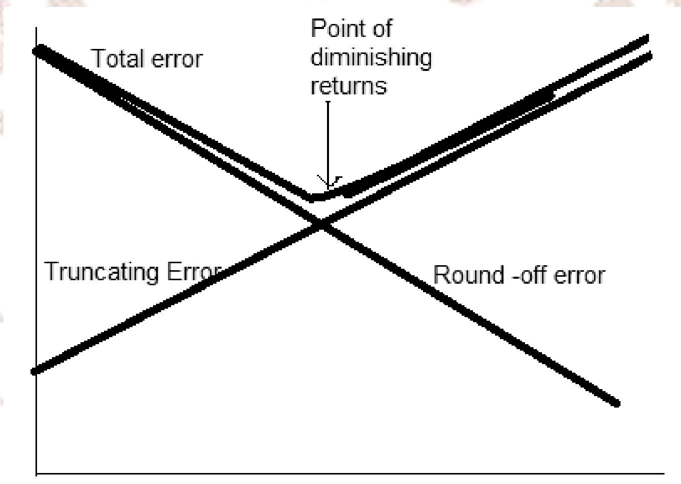
That is, Total Numerical Error = Truncating error + Round off error

Note that round-off errors decrease by increasing the number of significant digits and it increases due to subtractive cancellation i.e., increasing the step size or increasing the number of computations. Truncating error can be reduced by decreasing the step size. Thus, if the truncating error decreases, then the round-off error will increase.

So, if we are trying to decrease one error the other error will increase. Hence our main focus is to determine an appropriate step size for a particular computation. We would choose a large step size to decrease the number of calculations and round off errors without increasing large truncating errors.

We would like to obtain a point called the point of diminishing returns where round-off errors decreases as the step size decreases as shown in the figure

But in practical cases, such situations are very rare.



Note: To reduce numerical error avoid the following:

1. Subtracting two nearly equal numbers
2. When adding or subtracting two numbers, sort them and start with the smallest one.

5. BLUNDERS

These errors can be either due to human imperfection or computer malfunctioning. It can occur at any stage of the mathematical modeling process and can contribute to all the other components of error.

Blunders are usually not considered when we discuss the different types of errors this is just because mistakes are to some extent unavoidable no matter how efficiently we are going to work.

Blunders can be avoided only by a sound knowledge of the subject.

5.1 Formulation Errors

Formulation error also known as the model error is due to incomplete mathematical models. So, if we are working with some poorly conceived model, then no numerical method will provide adequate results.

5.2 Data Uncertainty

This type of error is also known as noise**.** Uncertainty error is due to uncertainty in physical data upon which a model is based. This error shows both inaccuracy and imprecision. If the given data has n significant digits of accuracy, then the result obtained from it will contain n significant digits of accuracy. For example if a = 2.467 and b = 0.03241 both have 4 significant digits of accuracy then $a - b = 2.43459$. Although $a - b$ has 6 significant digits the correct answer will have four significant digits only which is 2.434 so the answer will be 2.434.

6. SUMMARY

In this unit, we studied the total numerical error, Taylors series, error propagation, and reasons and types of different errors with the help of suitable examples.

7. TERMINAL QUESTIONS & ANSWERS

1. Find the Taylors series of $7x^3 - 3x + 13$ at $x_0 = -1$
2. Find the Taylors series of $5x^5 - x^4 + 2x^3 + x^2 - 2$ at $x_0 = -1$
3. Find the maximum error in y where $y = \frac{FL^4}{8EI}$ where $F = 50, L = 30, E = 1.5 \times 10^8, I = 0.06, \Delta F = 2, \Delta L = 0.1, \Delta E = 0.01 \times 10^8, \Delta I = 0.0006$

Answers

Self Assessment Questions

1. $\frac{1}{5}\left[1 - \frac{(x-2)}{5} + \left(\frac{x-2}{5}\right)^2 - \left(\frac{x-2}{5}\right)^3 + \cdots\right]$
2. $-1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots$
3. 0.004

Terminal Questions

1. $23 - 17(x+1) + 7(x+1)^2$
2. $-7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$
3. 0.039375.