Unit 5 Limits and Continuity

Structure:

5.1 Introduction

In this chapter you will be recalling the properties of number. You will be studying the limits of a function of a discrete variables, represented as a sequence and the limit of functions of a real variables. Both these limits describe the long term behaviour of functions. You will be studying continuity which is essential for describing a process that goes on without abrupt changes. You will see a good number of examples for understanding the concepts clearly. As mathematics is mastered only by doing, examples are given for practice.

You are familiar with numbers and using them in day – to – day life. Before introducing the concept of limits let us refresh our memory regarding various types of numbers.

Objectives:

At the end of the unit you would be able to

- understand the concept of limit.
- apply the concept of continuity in problems.
- find whether a given function is continuous or not.

5.2 The Real Number System

You are using numbers like $2, -3, \frac{3}{4}, \frac{-4}{7}, \pi, \; i, -i, \; 1+i, \; 2-3i$ etc.

The last two numbers $1 + i$ and $2 – 3i$ are complex numbers. The rest of them are real numbers.

The numbers *1, 2, 3, ……* are called natural numbers.

$N = \{1, 2, 3, \ldots\ldots\}$

The numbers …………………….. *– 3, –2, –1, 0, 1, 2, 3,* …… are integers.
$Z = \{0, \pm 1, \pm 2, \ldots\ldots\}$ or $\{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\ldots\ldots\}$

The set of quotients of two integers, the denominator not equal to 0 are called rational numbers and the set of rational numbers is denoted by Q.

Usually these numbers are represented as points on a horizontal line called the real axis. (Refer to Fig. 5.1)

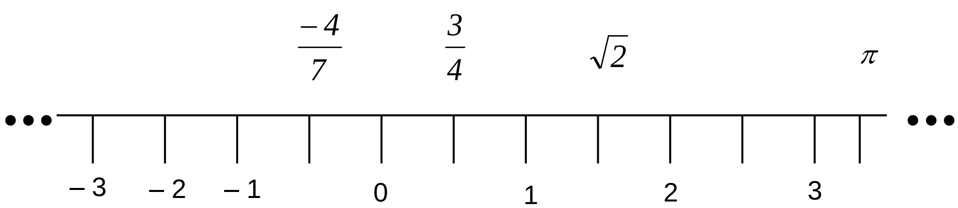


Fig. 5.1 Representation of numbers

After representing the integers and rational numbers. So there are no gaps in the real line and so it is called “continuous”.

We can also represent the relations “greater than” or “less than” geometrically. If $a < b$, then a lies to the left of b in the real line (and b lies to the right of *a).*

The modulus function

The modulus function simply represents the numerical value of a number. It is defined as follows:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

For example, $|2| = 2, |-2| = 2, |0| = 0$

Note: $|a+b| \leq |a| + |b|$

SAQ: (Self Assessment Questions).

Choose the right answer

1. If $a > b$, then $| a - b |$ *is*
   A) Positive
   B) Negative
   C) Zero
2. Choose the right answer
   $| a - b | + | b - a |$ is equal to
   A) $2 | a - b |$
   B) *0*
   C) *2 (a – b)*
   D) *2 (b – a)*

An Important Logical Symbol

In Mathematics, we use symbols instead of sentences. For example, *"3* is greater than *2"* is written as $3 > 2$. Throughout the test we used the symbol $\Rightarrow$ (read as "implies")

If $x > 2$, then $2x > 4$ is written as $(x > 2) \Rightarrow (2x > 4)$.

Generally 'If *P*, then *Q"* is written as

$P \Rightarrow Q$. (P is given and Q is the conclusion)

Note: $P \Rightarrow Q$ is different from $Q \Rightarrow P$. $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$.

The distance function

If *a* and *b* are two real numbers. Then the distance between a and b is defined as $| a - b|$. Refer Figure 5.2. Why we choose |a – b| as the distance between *a* and *b* should be clear from figure 5.2. When $b > a$, then the distance is $b - a$; when $a > b$, it is $a - b$. Both $a - b$ and $b - a$ are equal to $|a - b|$. So wherever *a* and *b* are on the real line, the distance is $|a - b|$.

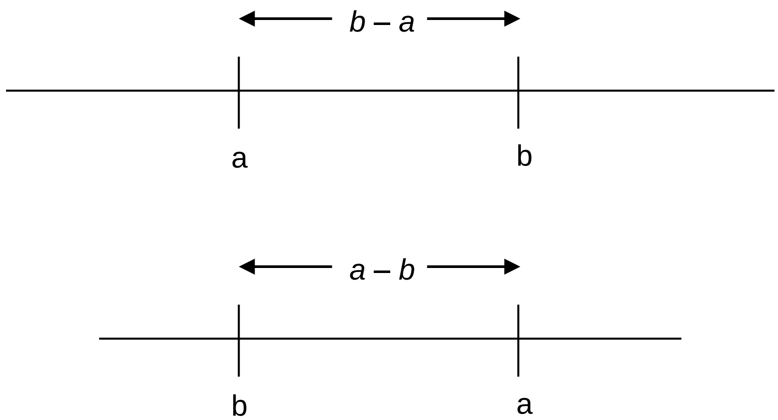


Fig. 5.2: Distance function

The distance function satisfies the following properties.

1. $|a-b| = 0 \Leftrightarrow a = b$
2. $|a-b| = |b-a|$
3. $|a-b| \leq |a-c| + |c-b|$

5.3 The Concept of Limit

In mathematics, the concept of a "**limit**" is used to describe the value that a function or sequence "approaches" as the input or index approaches some value. Limits are essential to calculus (and mathematical analysis in general) and are used to define continuity, derivatives and integrals.

Function of a discrete variable and a continuous variable

The Concept of limit is associated with functions. A function from a set A to a set *B* is a rule which assigns, to each element of *A* a unique (one and only one) element of *B*. An example of a function is $f(x) = 2x$, a function which associates with every number the number twice as large. Thus 5 is associated with 10, and this is written $f(5) = 10$.

There are two types of functions. **Discrete function and continuous function.**

Definition A function of a discrete variable or a discrete function is a function from *N* or a subset of *N* to the set *R* of all real numbers.

The second type of functions refer to functions from *R* to *R.* It is called a function of a continuous variable.

Definition: A function of a continuous real variable or simply a function of a real variable is a function from *R to R.*

Functions of a discrete variable

We have defined a function of a discrete variable as a function from *N {1, 2, 3, …}* or subset of *N* to the set *R* of real numbers. A convenient way to representing this function is by listing the images of 1*, 2, 3,* etc. If f denotes the function then the list.

f(1),f(2),f(3), ….. ………………… *(*)*

represents the function f usually *f(1), f(2)*,… are written as a_1, a_2,… etc. The list given in *(*)* is called a sequence.

In a sequence the order of the elements appearing in it is important. A common example of a sequence is a queue you see in a reservation counter. Then a_1 is the person standing in front of the counter getting his reservation done. a_2 is the person behind a_1 etc. the order of persons in the queue is important. You won't certainly be happy if the order of the persons in the queue is changed.

The limit of a sequence

From the above discussion, two points should be clear to you.

1. A sequence is an arrangement of real numbers as the first element, second element etc.
2. A sequence represents a function of a discrete variable.

We denote a sequence by (a_n) and a_n denotes the nth term.

Assume that you have a string of length *1 cm.* Denote it by a_1. Cut the string into two halves and throw away one half. Denote the remaining half by a_2.Then $a_2 = \frac{1}{2}$. Repeat the process indefinitely.

Then $a_3 = \left(\frac{1}{2}\right)^2$, $a_4 = \left(\frac{1}{2}\right)^3$ etc. After *(n + 1)* repetitions, you are left with a string of length $a_{n+1} = \left(\frac{1}{2}\right)^n$. Intuitively you feel that the string becomes smaller and smaller and you are left with a string whose length is nearly zero in the long run. At the same time you realize that you will have “some bit” of positive length at any time. Also you can make the string as small as you please provided you repeat the process sufficient number of times. In this case we say that *“a_n tends to 0 as n tends to infinity”* *“a_n tends to 0”* means $a_n \to 0$ is as small as we please *“n* lends to infinity” means we repeat the process sufficient number of times.

Now we are in a position to define the limit of a sequence (a_n)

Definition: Let (a_n) be a real sequence. Then (a_n) tends to a number a, if given a positive number $\in$, (pronounced as epsilon), there exists a natural number n_0 such that

$$|a_n - a| < \in \text{for all } n \geq n_0 \qquad \ldots\ldots\ldots\ldots\ldots\ (1.1)$$

In this case we write $a_n \to a$ *or* $\underset{n\to a}{lt}\ a_n = a$. We also say (a_n) converges to a.

Note: $|a_n - a|$ is the numerical value of $a_n - a$. For example $|2| = 2$ and $|-3| = 3$, *and* n_0 is a “stage”.

$n \geq n_0$ means after a certain stage, $|a_n - a| < \in$ simply means that a_n comes as close to a as we choose.

Example: Show that $(a_n) \to 0$ *where* $a_n = \frac{1}{n}$.

Solution: $a_n - 0 = \frac{1}{n} - 0 = \frac{1}{n}$. *So* $|a_n - 0| = \frac{1}{n}$.

Let $\in$ be a given positive number.

$|a_n - 0| = \frac{1}{n} < \in$ when $n > \frac{1}{\in}$.

Let n_0 be the smallest natural number $> \frac{1}{\in}$.

(For example, if $\frac{1}{\in} = 147.7$, take $n_0 = 148$). Then $\frac{1}{n_0} < \in$.

If $n > n_0$, then $\frac{1}{n} < \frac{1}{n_0} < \in$. Hence $|a_n - 0| < \in$ when $n \geq n_0$.

This proves (1.1) with 0 in place of a.

Hence $(a_n) \to 0.$

Example: Show that $(a_n) \to 0$ where $a_n = \frac{1}{2^n}$.

Solution: As in Example, $| a_n - 0 | = \frac{1}{2^n}.$

For $\frac{1}{2^n} < \in,$ we require $2^n > \frac{1}{\in}$ *or* $n > log_2\left(\frac{1}{\in}\right)$

By choosing n_0 to be the smallest natural number greater than $log_2\left(\frac{1}{\in}\right)$, we see that (1.1) is satisfied for n_0.

Hence $\left(\frac{1}{2^n}\right) \to 0.$

You can see that several similar sequences tend to 0. Some of them are

$\left(\frac{1}{n^2}\right), \left(\frac{1}{n^3}\right), \left(\frac{1}{n^4}\right), ... \left(\frac{1}{3^n}\right), \left(\frac{1}{4^n}\right), ... \frac{1}{log^n}, ..$ …… (1.2)

Subsequence: A **subsequence** is a sequence that can be derived from the given sequence by deleting some elements from the sequence without changing the order of the remaining elements. For example, ABD is a subsequence of ABCDEF.Formally, suppose that *X* is a set and that $(a_n)_{n \in N}$ is a sequence in *X*. Then, a subsequence of (a_n) is a sequence of the form (a_{n_r}) where (n_r) is a strictly increasing sequence in the index set N.

Example:

1. Consider the sequence,

$\langle A, C, B, D, E, G, C, E, D, B, G \rangle,$

Then $\langle B, C, D, G \rangle$ is a subsequence.

2. {1,1,1,1,....} is subsequence of {1,-1,1,-1,1,-1,....}. **Algebra of limits of sequences**

If (a_n) *and* (b_n) are two sequences, then we can get a new sequence by "adding them". Define $c_n = a_n + b_n$. Then (c_n) is a sequence and we can write $(c_n) = (a_n + b_n)$. We can also subtract one sequence (b_n) from another sequence (a_n), multiply two sequences, divide two sequences, etc. We can also multiply a sequence (a_n) by a constant k.

Let us answer the following questions.

1. What happens to the limit of sum of two sequences ?
2. What happens to the limit of difference, product, division of two sequences ?

We summarize the results as a theorem.

Theorem(∗): It (a_n) and (b_n) and two sequences converging to a and b respectively, then

a) $(a_n + b_n) \to a + b$

b) $(a_n - b_n) \to a - b$

c) $(ka_n) \to ka$

d) $(a_nb_n) \to ab$

e) $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ provided $b_n \neq 0$ for all n and $b \neq 0$.

Proof: We prove only a) Let $\in > 0$ be a positive number.

As $(a_n) \to a$, we can apply (1.1) by taking $\frac{\in}{2}$ in place of $\in$. Thus we get a natural number n_0 such that

$$|a_n - a| < \frac{\in}{2} \text{ for all } n \geq n_0 \qquad \text{………….. (1.3)}$$

Similarly, using the convergence of (b_n), we can get n_1 such that

$$|b_n - b| < \frac{\in}{2} \text{ for all } n \geq n_1 \qquad \text{…………….. (1.4.}$$

Let m = maximum of n_0 and n_1. Then (1.3) and (1.4) are simultaneously true for $n \geq m$.

Thus,

$$|a_n - a| < \frac{\in}{2}, |b_n - b| < \frac{\in}{2} \text{ for all } n \geq m \qquad \text{………………… (1.5)}$$

From (1.5) we get

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\in}{2} + \frac{\in}{2} = \in$$

for $n \geq m$.

Thus (1.1) holds good for the sequence $(a_n + b_n)$ in place of (a_n) and $a + b$ (in place of a)

Hence $(a_n + b_n) \to a + b$

Note: The choice of m may puzzle you. When $|\, a_n - a \,| < \frac{\in}{2}$ for all $n \geq 1000$, then certainly $|\, a_n - a \,| < \frac{\in}{2}$ for all $n \geq 1001, 1002$ etc. So $|\, a_n - a \,| < \frac{\in}{2}$ for all $n > m$, m being greater than 1000.

Remark: The other subdivisions can be proved similarly. As you are more interested in applications you need not get tied down by the technical details of the proof.Convergence of Sequence: **A sequence is said to be convergent if it has unique limit point.**

Example:

1. (1/n) has 0 as its only limit point so it is convergent.
2. (1,-1,1,-1,1,….) has 1 and -1 as its limit pointAs the limit of its subsequence (1,1,1,…) is 1 and limit of subsequence(-1,-1,-1,….) is -1. So this sequence is not convergent. Such sequences are called oscillatory sequences.

Ailter Proof:

We can write the sequence as (a_n) where

$$a_n = \begin{cases} 1 \text{ when } n \text{ is odd} \\ -1 \text{ when } n \text{ is even} \end{cases}$$

Suppose $(a_n) \to a$ for some real number a. The number has to satisfy one and only one of the following conditions: $a < -1$, $-1 \leq a \leq 1$, $a > 1$.

(See Fig. 5.3 representing these cases).

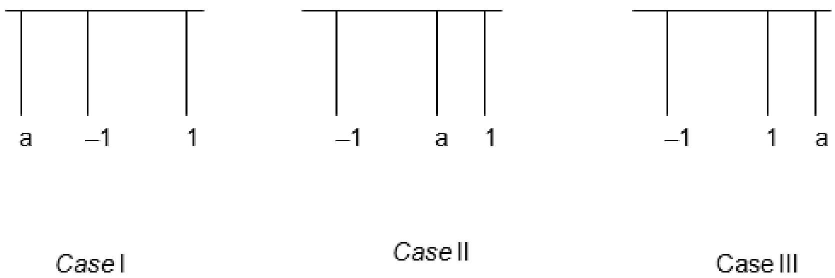


Fig. 5.3: Illustration of W.E.

In case $1, |\, a_n - a \,| > 2$ if n is odd. So we cannot prove *(1.1)* in this case *(for* $\in > 2$*).*

In case $3, |\, a_n - a \,| > 2$ if n is even and we cannot prove *(1.1)* for $\in > 2$.

In case *2,* if a is closer to *1,* then $|\, a_n - a \,| > 2$ for even n. If a is close to *–1,* then $|\, a_n - a \,| > 2$ for odd *n.* If $a = 1$, then $|\, a_n - a \,| = 2$ for even *n.* So *(1.1)* cannot hold good for $\in > 2$ or $\in = 2$. So the given sequence is not convergent.

Worked Examples

W.E.: Show that a constant sequence is convergent (A sequence (a_n) is a constant sequence if $a_n = k$ for all n).

Solution Since $a_n = k$ for all n. Consider $|\, a_n - k \,|$.

$$|\, a_n - k \,| = |\, k - k \,| = 0$$

As $0 < \in$, for all $n \geq 1$, that is, $n_0 = 1$ and I for every positive number $\in > 0$ *,the sequence converges.* So a constant sequence converges to its constants value.

W.E.: Find the nth term of the sequence $3, 2, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \ldots\ldots\ldots\ldots$ **and find its limit, if it exists.**

Solution: To discover a pattern in the terms of the sequence start from the third term.

$$a_3 = \frac{5}{3} = \left(\frac{(2+3)}{3}\right) \qquad a_4 = \frac{6}{4} = \frac{(2+4)}{4}$$

$a_5 = \frac{7}{5} = \frac{(2+5)}{5}$ $a_6 = \frac{8}{6} = \frac{(2+6)}{6}$

So a positive choice is $a_n = \frac{(2+n)}{n}$, when a_n is written in this way

$a_1 = \frac{(2+1)}{1} = 3,\ a_2 = \frac{(2+2)}{2},\ a_3 = \frac{(2+3)}{3} = \frac{5}{3}$, etc.

To find the limit of the sequence (if it exists), with $a_n = \frac{2}{n} + \frac{n}{n} = \frac{2}{n} + 1$

Taking $b_n = \frac{1}{n}$ and $c_n = 1$, we get $a_n = 2b_n + c_n$

As $(b_n) = \left(\frac{1}{n}\right) \to 0$ and $(c_n) = (1) \to 1$,

$(a_n) \to 2(0) + 1 = 1$, as n tends to infinity Hence the given sequence converges to 1.

W.E.: Evaluate $\underset{n\to\infty}{lt} \frac{2+n+n^2}{2+3n+4n^2}$

Solution As we know that $\left(\frac{1}{n}\right) \to 0$ and $\left(\frac{1}{n^2}\right) \to 0$, we try to write the n^{th} term of the given sequence in terms of $\frac{1}{n}$ and $\frac{1}{n^2}$.

$$a_n = \frac{2+n+n^2}{2+3n+4n^2}$$

$$= \frac{\frac{(2+n+n^2)}{n^2}}{\frac{(2+3n+4n^2)}{n^2}}$$

$$= \frac{\frac{2}{n^2} + \frac{1}{n} + 1}{\frac{2}{n^2} + \frac{3}{n} + 4}$$

As $\left(\frac{1}{n^2}\right), \left(\frac{1}{n}\right) \to 0$ and $(1) \to 1$,

$$\underset{n\to\infty}{lt}\left(\frac{2}{n^2}+\frac{1}{n}+1\right)=2lt\left(\frac{1}{n^2}\right)+lt\left(\frac{1}{n}\right)+lt1=2(0)+0+1=1$$

Similarly,

$$\underset{n\to\infty}{lt}\left(\frac{1}{n^2}+\frac{3}{n}+4\right)=2(0)+3(0)+4=4$$

Hence

$$\underset{n\to\infty}{lt}\ \frac{2+n+n^2}{2+3n+4n^2}=\frac{1}{4}$$

S.A.Q. 3: Which of the following sets can be arranged as a sequence?

a) The passengers in a *3* ties coach
b) The people attending a meeting in a beach
c) The people living in Karnataka
d) The students of M.Sc. Biotechnology in a college

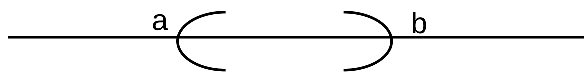
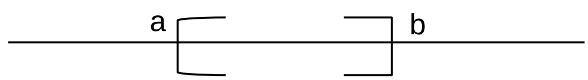
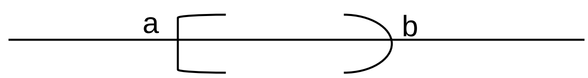
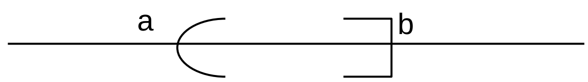
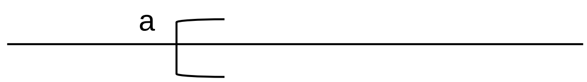
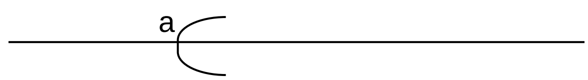
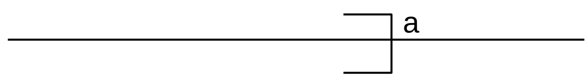
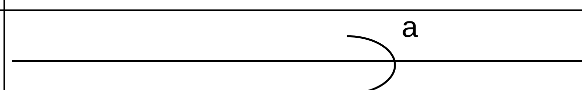
The Limit of a Function of a Real Variable

You are now familiar with natural numbers and real numbers. The natural numbers appear as "discrete" points along the real line and we are able to fix some element say *1* as the first natural number, *2* as the second natural number etc. So the natural numbers appear as the terms of a sequence. But it is not possible to arrange the real numbers as a sequence. If a real number a is the n^{th} element and b is the $(n+1)^{th}$ element, where will you place $\frac{a+b}{2}$. It appears between the n^{th} element and the $(n+1)^{th}$element. If you take $\frac{a+b}{2}$ as the $(n+1)^{th}$ element, where will you place $\frac{1}{2}\left(a+\frac{(a+b)}{2}\right)$?

So you feel initiatively that real numbers can not be arranged as a sequence.

When we consider numbers between *a* and *b.* We consider points lying between the points representing the numbers a and *b.* The numbers lying between two numbers *a* and *b* form an "interval". So "interval" on the real line is the basic concept. Usually we define a function of a real variable on an interval. We define various types of "intervals" as follows (refer to Table 5.1)

Table 5.1: Intervals

Set notation	Interval	Graphical representation
$\{x \in R \mid a < x < b\}$	(a, b)	
$\{x \in R \mid a \le x \le b\}$	[a, b]	
$\{x \in R \mid a \le x < b\}$	[a, b)	
$\{x \in R \mid a < x \le b\}$	(a, b]	
$\{x \in R \mid a \le x\}$	[a, ∞)	
$\{x \in R \mid a < x\}$	(a, ∞)	
$\{x \in R \mid x \ge a\}$	(–∞, a]	
$\{x \in R \mid x > a\}$	(–∞, a)	
R	(–∞,∞)	

Note: [Here (represents inclusion of all numbers > *a.*[represents the inclusion of all numbers ≥ *a.* $-\infty$ is not a number. It simply represents the inclusion of all "large" –*ve* numbers. $+\infty$ represents the inclusion of all "large" positive real numbers.

(a, b) and *[a, b]* are called open interval and closed interval.]

So it is natural to represent R as the interval $(-\infty,\infty)$S.A.Q.4

a) Find all natural numbers in the intervals *[3, ∞), (3, ∞), (– ∞, 3)* and *(– ∞, 3]*
b) Find all integers lying in the intervals given above
c) Find all numbers in *[2, 2], (2, 2)*

Example; Represent $\{x \in R / | x - 3 | < 2\}$ as an interval

Solution: $| x - 3 | < 2$ *represents two inequalities x – 3 < 2, – (x – 3) < 2.*

Whenx – 3 < 2, x < 5

When *– (x – 3) < 2, x – 3 > – 2, or x > 3 – 2 = 1*

Hence the given interval is *(1, 5)*

We can also arrive at this interval geometrically (see Fig 5.4)

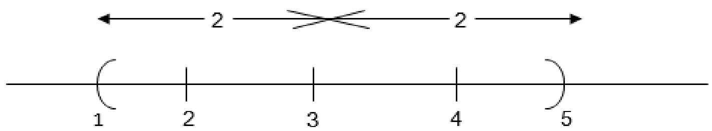


Fig. 5.4

S.A.Q. 5: Represent the sets $\{x \in R / | x - 3 | \le 2\}$,

$\{x \in R / | x - 3 | > 2\}$, $\{x \in R / | x - 3 | \ge 2\}$as intervals.

Now we have enough background to define the limit of a function f of a real variable x. In the case of functions of a discrete variable or sequence, we defined $\underset{n \to \infty}{lt} a_n$ *or* $\underset{n \to \infty}{lt} f(n)$. This limit represented the long term behaviour of f. In the case of a functions of a real variables, we can discuss the behaviour of *f(x),* when the variable x comes close to a real number a. In other ways we will be defining $\underset{x \to a}{lt} f(x).$ We want to write the statement *"x* comes close to a" rigorously. The geometric representation of real numbers can be used for this purpose. When do you say that your house is near your college? When the distance between your house and your college is small. In the same way, we can say that *"x* is close to a "when | *x – a*| is small. If "smallness" is defined by a distance of say 0.1, then x is close to a if |*x – a*| < *0.1*. Of course the measure of "smallness" is relative. For a person living in Mangalore, Manipal is not near Mangalore. For a person living in US, Mangalore and Manipal are near to each other. So "smallness" is decided by the choice of a positive number $\in$ (This was done in defining the limit of a sequence also)

Before giving a rigorous definition of limit, let us consider two examples.

Consider *f(x) = 1 + x.* Let us try to see what happens when *x* is close to 1. We evaluate *f(x),* when *x = 0.9, 0.99, 0.999, 1.1, 1.01, 1.001*

f(0.9) = 1.9	*f(0.99) = 1.99*	*f(0.999) = 1.999*
f(1.1) = 2.1	*f(1.01) = 2.01*	*f(1.001) = 2.001*

Note all these values are near the value *2*.

Consider another function $g(x) = \frac{x^2 - 1}{x - 1}, x \neq 1.$

(Why don't we define g*(1)* ? If we put x = 1 in $\frac{x^2 - 1}{x - 1}$, then we get $\frac{0}{0}$ which is not defined).

As in the case of *f(x),* we compute some function values.

$g(0.9) = \frac{(0.9)^2 - 1}{0.9 - 1} = 1.9 \qquad g(0.99) = 1.99$

g(0.999) = 1.999	*g(1.1) = 2.1*
g(1.01) = 2.01	*g(1.001) = 2.001*

Note: These values are close to *2.* Hence we can say that *x* close to $1 \Rightarrow$ both *f(x)* and *g(x)* are closed to *2* and we can take *2* as $\underset{x \to 1}{lt} f(x)$ or $\underset{x \to 1}{lt} g(x)$.

Now let us formulate a rigorous definition of $\underset{x \to a}{lt} f(x)$

Definition: Let f be a function of a real variable. Then $\underset{x \to a}{lt} f(x) = I$ if given a positive number $\in$, there exists a positive number δ such that

$$0 < | x - a | < \delta, | f(x) - I | < \in \qquad \text{............ (1.6)}$$

Let us analyze the definition.

We have two choice of positive numbers ($\in$and δ) and two conditions $(0 < | x - a | < \delta \text{ and } | f(x) - I | < \in)$ Given any positive number $\in$, there exists a positive number δ such that the condition P:"$0 < | x - a | < \delta$" Implies the condition Q :"$| f(x) - I | < \in$"

The choice of δ depends on the given number $\in$. The function *f* need not be defined at $x = a$.

The condition *P* says that *x* is close to a.

The condition *Q* says that *f(x)* is close to *l*

We can also express the definition geometrically.

Given $\in > 0$, there exists $\delta > 0$ such that

$$x \in (a-\delta, a) \cup (a, a+\delta) \Rightarrow f(x) \in (I-\in, I+\in)$$

In figure 5.5, the point *(a, l)* is shown as *0*, meaning that the functional value of *f* at *x = a* is not known or defined.

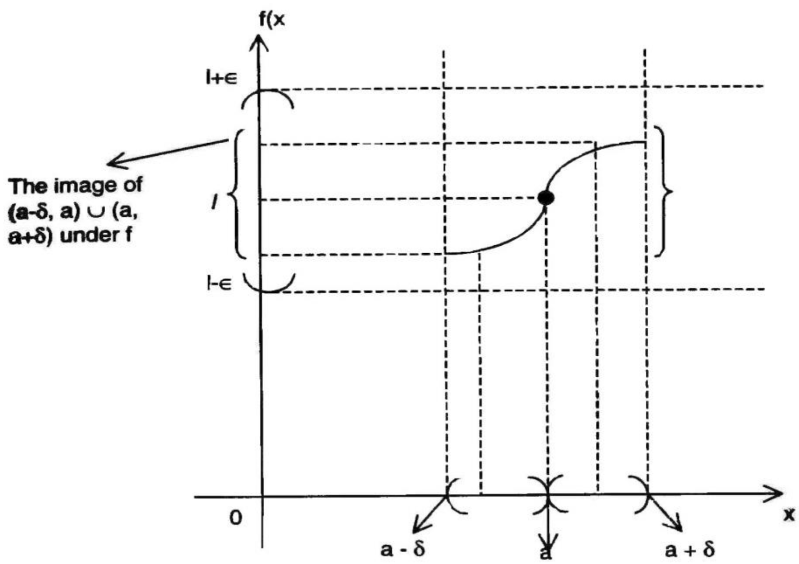


Fig. 5.5: Definition of limit of function

The images of points in $(a-\delta, a) \cup (a, a+\delta)$ under the function f is a subset of the interval $(l - \in, l + \in)$ along the vertical axis.

W.E.: Evaluate $\underset{x \to 3}{lt} (2x-6)$

Solution: Here $f(x) = 2x - 6$

Choose any $\in > 0$. We have to choose a δ such that *(1.6)* is satisfied. We have to guess the value of *l*. When *x* comes close to *3,2x – 6* should come close to *2(3) – 6 = 0. Take l = 0.* Then

$$|\,f(x)-0\,|<\in \Leftrightarrow |\,2x-6-0\,|<\in$$
$$\Leftrightarrow |\,2(x-3)\,|<\in$$
$$\Leftrightarrow |\,x-3\,|<\frac{\in}{2}$$

So, choose $\delta=\frac{\in}{2}$, then

$$0<|\,x-3\,|<\frac{\in}{2} \Rightarrow |\,f(x)-0|<\in$$

Hence, $\underset{x\to 3}{lt}\,(2x-6)=0$

Note: *f(3) = 2(3) – 6 = 0*. In this case the function f is defined at *x = 3* and *f(3)* coincides with $\underset{x\to 3}{lt}\,f(x)$.

W.E.: Evaluate $\underset{x\to 0}{lt}\,(2x^2+1)$

Solution: Let $\in>0$. As in the previous problem, we can guess the value of l it is $2(0)^2+1=1$

$$|\,f(x)-1\,|<\in \Leftrightarrow \left|\,2x^2+1-1\,\right|<\in$$
$$\Leftrightarrow \left|\,2x^2\,\right|<\in$$
$$\Leftrightarrow |\,x\,|<\frac{\sqrt{\in}}{\sqrt{2}}$$

Hence for a given $\in>0$, the corresponding δ is chosen as $\frac{\sqrt{\in}}{\sqrt{2}}$ and condition (1.6) holds good. Hence $\underset{x\to 0}{lt}\,(2x^2+1)=1$

W.E.: Evaluate $\underset{x\to 0}{lt}\,\sqrt{x}$

Solution: Proceed as in the previous example. In this problem $\delta=\in^2$ and $\underset{x\to 0}{lt}\,\sqrt{x}=0$

General Remark While evaluating $\underset{x \to a}{lt} f(x)$, if *f(a)* is defined or it is not of the form $\frac{0}{0}$, and *f(x)* is defined by a single expression, it will turn out that $\underset{x \to a}{lt} f(x) = f(a)$. If *f(a)* is not defined or f is given by two different expressions. Then we have to guess the value of the limit and prove condition (1.6).

In some problems, *f(x)* may be given as quotient of two expressions but it may reduce to an easier function on simplification. In such cases the problem will reduce to an easier one.

W.E.: Evaluate $\underset{x \to 2}{lt} \frac{2x^2 - 3x - 2}{x - 2}$

Solution: When $x = 2$, $f(x)$ is of the form $\frac{0}{0}$. So we try to see when $x - 2$ is a factor of $2x^2 - 3x - 2$.

$2x^2 - 3x - 2 = 2x^2 - 4x + x - 2 = 2x(x - 2) + (x - 2) =$

$(2x + 1)(x - 2)$.

When $x \to 2$, x does not assume the value 2. So,

$$\frac{2x^2 - 3x - 2}{x - 2} = \frac{(2x + 1)(x - 2)}{(x - 2)}$$

$= 2x + 1$ on canceling $x - 2$, *since* $x - 2 \neq 0$.

So the given limit reduces to

$$\underset{x \to 2}{lt} (2x + 1) = 5$$

Algebra of Limits of Functions

It is not necessary that we use the $\in\!-\delta$ definition for every problem. We study important properties of limits of functions as a theorem. We can evaluate limits using this theorem.

Theorem I:

a) $\underset{x \to a}{lt}\ x = a$

b) $\underset{x \to a}{lt}\ k = k$, where k is a constant

c) $\underset{x \to a}{lt}\ x^2 = a^2$

d) $\lim\limits_{x \to a} x^3 = a^3$

e) $\lim\limits_{x \to a} x^n = a^n$

f) $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$ when $a > 0$

Theorem 2: Let k be a constant, f and g functions having limit at a and n a positive integer. Then the following hold good.

a) $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$

b) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

c) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

d) $\lim\limits_{x \to a} [f(x) . g(x)] = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} g(x)$

e) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, provided $\lim\limits_{x \to a} g(x) \neq 0$

f) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

g $\lim\limits_{x \to a} \sqrt{f(x)} = \sqrt{\lim\limits_{x \to a} f(x)}$ provided $\lim\limits_{x \to a} f(x)$ is positive

You need not prove these results. It is enough if you clearly understand the theorems and apply them for evaluating limits.

Self Assessment Questions

SAQ 6: Evaluate the following limits

a) $\lim\limits_{n \to \infty} \dfrac{2n+1}{2n}$ b) $\lim\limits_{n \to \infty} \dfrac{n}{n+1}$ c) $\lim\limits_{n \to \infty} (100)$

d) $\lim\limits_{n \to \infty} \dfrac{3n^2+4n+7}{5+3n+2n^2}$ e) $\lim\limits_{n \to \infty} \dfrac{n+n^2}{1+n+n^2}$ f) $\lim\limits_{n \to \infty} \dfrac{(1+n)^3}{(1-n)^3}$

g) $\lim\limits_{n \to \infty} \dfrac{2n^2}{1+n+n^2}$ h) $\lim\limits_{n \to \infty} \dfrac{2+3n+9n^2}{n^2}$

SAQ 7: Evaluation the following limits

a) $\underset{x\to 1}{lt}\left(1+x+x^2\right)$ b) $\underset{x\to 1}{lt}\dfrac{1}{1+x+x^2}$ c) $\underset{x\to 0}{lt}\,2+3x+4x^2$

d) $\underset{x\to 2}{lt}\dfrac{x^2-4}{x-2}$ e) $\underset{x\to 1}{lt}\dfrac{x^3-1}{x-1}$ f) $\underset{x\to 1}{lt}\dfrac{2x+3x^2}{1+x+x^2}$

g) $\underset{x\to 2}{lt}\dfrac{x^2-3x+2}{x-2}$ h) $\underset{x\to 1}{lt}\dfrac{x-1}{x^2-1}$

S.A.Q.8: If $\underset{x\to a}{lt}\,f(x)=l$ *and* $\underset{x\to a}{lt}\,g(x)=m,$ evaluate the following

a) $\underset{x\to a}{lt}\,2f(x)+3g(x)$ b) $\underset{x\to a}{lt}\,[f(x)+g(x)]\,[f(x)-g(x)]$

c) $\underset{x\to a}{lt}\sqrt{(f(x))^2+(g(x))^2}$ d) $\underset{x\to a}{lt}\dfrac{f(x)}{\sqrt{1+(g(x))^2}}$

e) $\underset{x\to a}{lt}\dfrac{f(x)+2g(x)}{4g(x)}$ if m is positive f) $\underset{x\to a}{lt}\sqrt{g(x)+2f(x)}$ *if l, m > 0.*

5.4 Concept of Continuity

In mathematics and sciences, we use the word "continuous" to describe a process that goes on without abrupt changes. For example, the growth of a plant, the water level in a tank and the speed of a moving car in a four-base highway are exhibiting continuous behaviour.

Before defining continuous functions, let us look at the graphs of three functions.

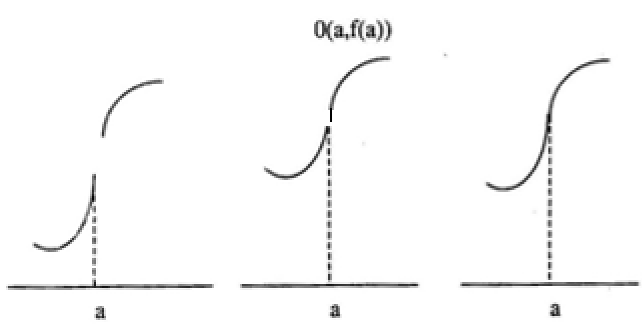


Fig. 5.6: Two discontinuous functions and a continuous function

The first graph has a break at $x = a$ in the second graph also there is a break at $x = a.$ If you ignore the point corresponding to $x = a,$ there is no break for the break occurs at $x = a$ the third function has no break. So it should be intuitively clear to you that the first two functions are not continuous while the third function is continuous.

Let us formulate a rigorous definition of continuity.

Definition: Let f be a function of a real variable defined in an open interval containing a. Then f is continuous at a if $\underset{x \to a}{lt} f(x) = f(a)$

Note: In order to define continuity at a, we need three conditions.

1) $\underset{x \to a}{lt} f(x)$ exists

2) $f(a)$ is defined

3) $\underset{x \to a}{lt} f(x) = f(a)$

Even if any one of them fails, then the function f is not continuous at a.

Now look at Fig. 5.6. The first function, say f, has no limit at a, i.e., $\underset{x \to a}{lt} f(x)$ does not exist. For the second function, $\underset{x \to a}{lt} f(x)$ exists but $\underset{x \to a}{lt} f(x) \neq f(a)$. The third function is continuous.

Example: Define f as follows:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & if \quad x \neq 2 \\ 3 & if \quad x = 2 \end{cases}$$

Is f continuous at 2?

Solution:

$$\underset{x \to 2}{lt} \frac{x^2 - 4}{x - 2} = \underset{x \to 2}{lt} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \underset{x \to 2}{lt} (x+2)$$

$$= 4$$

So $\underset{x \to 2}{lt} f(x)$ exists.

But $f(2) = 3 \neq 4 = \underset{x \to 2}{\mathit{lt}} f(x)$

Hence the function f is not continuous *2.*

Example: Define *f* as follows.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & if \quad x \neq 2 \\ 4 & if \quad x = 2 \end{cases}$$

Is *f* continuous at *2?*

Solution: From above example, we have $\underset{x \to 2}{\mathit{lt}} f(x) = 4$*. As f(2) = 4, f is* continuous at *2.*

Sometimes function may be defined by two different expressions. In such cases the following method of proving continuity will be useful. For that we need the concept of left limit and right limit.

Definition: Let a function f be defined in an interval *(b, a)* where *b < a).* Then $\underset{x \to a-}{\mathit{lt}} f(x) = I$ (called the left limit) if for given $\in > 0$, there exists $\delta > 0$ such that $x \in (a - \delta, a) \Rightarrow |f(x) - I| < \in$ ………… *(1.7).*

Definition; Let a function f be defined in an interval (a, b), where b > a. Then $\underset{x \to a+}{\mathit{lt}} f(x) = l$ called the right limit if given $\in > 0$, there exists $\delta > 0$ such that $x \in (a, a + \delta) \Rightarrow |f(x) - I| < \in$ …………... *(1.8)*

Note: We can prove that *(1.7)* and *(1.8)* implies *(1.6).* Hence if the left and right limits exist and are equal then

$$\underset{x \to a-}{\mathit{lt}} f(x) = \underset{x \to a+}{\mathit{lt}} f(x) = \underset{x \to a}{\mathit{lt}} f(x)$$

When a function is defined by two different expressions, we have to evaluate the left and right limits.

$\underset{x \to a}{\mathit{lt}} f(x) = I$ if both the left and right limits exist and are equal.

Worked Examples:

W.E.: Test whether f is continuous at $x = 3$ where f is defined by

$$f(x) = \begin{cases} -3x+4 & if \quad x \le 3 \\ -2 & if \quad x > 3 \end{cases}$$

Solution: As f is defined by two expressions one for $(-\infty, 3]$ and another for $(3, \infty)$, we evaluate the left and right limits.

$$\underset{x\to 3-}{lt} f(x) = \underset{x\to 3-}{lt} (-3x+4) = -3(3)+4 = -5$$

$$\underset{x\to 3+}{lt} f(x) = \underset{x\to 3+}{lt} (-2) = -2$$

As the left and right limits are not equal, $\underset{x\to 3}{lt} f(x)$ does not exist. Hence f is not continuous at *3.*

When a function f is continuous at every point of an interval *(a, b)* we say that the function is continuous on *(a, b).* In particular, if a function is continuous at every real number. Then we say that a function is continuous on *R.*

Using Theorem 1, we can prove that the functions, *k (a constant),* x, x^2, ……..x^n, where $n > 2$ are continuous on *R.*

The function $\sqrt{x}$ is continuous on $(0, \infty)$

The function $\frac{1}{x}$ is continuous on $(-\infty, 0) \cup (0, \infty)$

Using Theorem 2, we can prove the following theorem.

Theorem:

a) If *f* and *g* are continuous at a, then

   i) $f(x) + g(x)$ ii) $f(x) - g(x)$ iii) $f(x)\, g(x)$ are continuous at a

b) If *f* and *g* are continuous at a and $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a

c) If *f* is continuous at *a*, $f(x) > 0$ *for x* in an open interval containing a, $\sqrt{f(x)}$ is continuous at a.

Worked Examples

W.E.: Test the continuity of the function f at all real points where f is defined by

$$f(x)=\begin{cases} x^2 & \text{for} \quad x \le 3 \\ 2x+1 & \text{for} \quad x > 3 \end{cases}$$

Solution: If $a < 3$, then $f(x)$ is defined by the expression x^2 in an open interval containing a. So f is continuous for all $a < 3$.
So it remains to test continuity only at 3.

$$\underset{x\to 3-}{\text{lt}} f(x) = \underset{x\to 3-}{\text{lt}} x^2 = 3^2 = 9$$

$$\underset{x\to 3+}{\text{lt}} f(x) = \underset{x\to 3+}{\text{lt}} (2x+1) = 2(3)+1 = 7$$

As $\underset{x\to 3-}{\text{lt}} f(x) \ne \underset{x\to 3+}{\text{lt}} f(x)$, $\underset{x\to 3}{\text{lt}} f(x)$ does not exist. So f is not continuous at 3.

Thus f is continuous at all real points except 3.

W.E.: Test the continuity of the function f where f is defined by

$$f(x)=\begin{cases} \dfrac{x-2}{|x-2|} & if \quad x \ne 2 \\ 7 & if \quad x = 2 \end{cases}$$

Solution: When $x < 2$, $|x-2|$ is negative. So $|x-2| = -(x-2)$

$$f(x) = \frac{x-2}{-(x-2)} = -1$$

When $x > 2$, $|x-2|$ is positive. So $|x-2| = x-2$

$$f(x) = \frac{x-2}{x-2} = 1$$

$f(2) = 7$. Thus

$$f(x)=\begin{cases} -1 & if \quad x < 2 \\ 7 & if \quad x = 2 \\ 1 & if \quad x > 2 \end{cases}$$

As in the previous worked example, f is continuous for all $a < 2$ and all $a > 2$.

$$\underset{x\to 2-}{\text{lt}} f(x) = \underset{x\to 2-}{\text{lt}} (-1) = -1$$

$$\underset{x\to 2+}{\text{lt}} f(x) = \underset{x\to 2+}{\text{lt}} (1) = 1$$

As $\underset{x\to 2-}{\text{lt}} f(x) \ne \underset{x\to 2+}{\text{lt}} f(x)$, $\underset{x\to 2}{\text{lt}} f(x)$ does not exist. So f is continuous at all points except 2.

Note. In some cases $\underset{x \to a}{lt} f(x)$ may not exist. You may ask a question: how to establish that $\underset{x \to a}{lt} f(x)$ does not exist. One simple case is where the left and right limits exist but are not equal. We have examples for this case. The worst case is when neither of the two limits exist. For proving the non-existence of limits, we use the following theorem.

Theorem: A function f is continuous at a if and only if the following conditions holds good.

$$x_n \to a \Rightarrow f(x_n) \to f(a)$$

To prove the non existence of the limit it is enough to construct a sequence (x_n) converging to a such that f(x_n) does not converge to *f(a).*

W.E. : Show that the function f defined by

$f(x) = \begin{cases} \frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$ is not continuous at 0.

Solution: Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ (obvious) $f(x_n) = n$ and so $f(x_n)$ does not converge to 0 since $f(x_n)$ indefinitely increases and so cannot approach 0.

Self Assessment Questions

S.A. Q. 9: Verify whether the following functions f is continuous at a

a) $f(x) = \begin{cases} 2x + 3 & if \quad x \leq 1 \\ 5 & if \quad x > 1 \end{cases}$ $\qquad a = 1$

b) $f(x) = \begin{cases} 4x + 5 & if \quad x \leq 1 \\ 9 & if \quad x > 1 \end{cases}$ $\qquad a = 1$

c) $f(x) = \begin{cases} 1 + x & if\ x \leq 2 \quad a = 2 \\ -1 + x^2 & if \quad x > 2 \end{cases}$ $\qquad a = 2$

d) $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & if \ \ x \neq 3 \\ 1 \ \ if \quad x = 3 \end{cases}$ $\qquad a = 3$

e) $f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if \quad x\neq 3\\ 6 \quad if & x=3\end{cases} \qquad a=3$

f) $f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if \quad x\neq 3\\ 7 \quad if & x=3\end{cases} \qquad a=2$

g) $f(x)=\begin{cases}\dfrac{x-3}{x^2-9} & if \quad x\neq 3\\ 6 \quad if & x=3\end{cases} \quad a=3$

S.A.Q. 10: Show that the function f defined by

$f(x)=\begin{cases}\dfrac{1}{x-1} & if \quad x\neq 1\\ 0 \quad if & x=1\end{cases}$ is not continuous at 1.

S.A.Q. 11: Show that the following functions are continuous on R.

a) $f(x)=\begin{cases}x^2-6x+5 & if \quad x\leq 1\\ 2x^2-2 & if \quad x<1\end{cases}$

b) $f(x)=\begin{cases}1+x+x^2+x^3 & if \quad x\leq 1\\ 4 & if \quad x>1\end{cases}$

c) $f(x)=\begin{cases}1+4x^3 & if \quad x\leq 1\\ 4+x^2 & if \quad x>1\end{cases}$

d) $f(x)=\begin{cases}x^2-3x+2 & if \quad x\leq 1\\ x^3-1 & if \quad x>1\end{cases}$

5.5 Summary

In this unit, we studied the basics of real number system then the concept of limit was discussed which was further extended to the concept of continuity. All definitions and properties of the above mentioned concepts is given very clearly with sufficient number of examples wherever necessary.

5.6 Terminal Questions

1. Find all natural numbers in the following intervals

   a) *[4, 9)* b) *(4, 9]* c) *(4, 9)* d) *[f, g]*

   e) $(4, \infty)$ f) $(9, \infty)$ g) $[9, \infty)$ h) $(4, \infty) \cap (9, \infty)$

2. Find $\underset{n\to\infty}{lt}\ a_n$ when

   a) $a_n = \frac{3}{2^n}$ b) $a_n = \frac{2}{3^n} + 1$ c) $a_n = \frac{3}{2^n} - \frac{2}{3^n}$

   d) $a_n = \frac{1}{n\,!}$ e) $a_n = \frac{1}{n+2}$

3. Show $\underset{n\to\infty}{lt}\ a_n$ does not exist when

   a) $a_n = n$ b) $a_n = 2^n$ c) $a_n = n!$

4. Evaluate $\underset{n\to\infty}{lt}\ a_n$ when

   a) $a_n = \frac{3n-2}{4n+7}$ b) $a_n = \frac{n+1}{n}$ c) $a_n = \frac{2+3n}{n}$

   d) $a_n = \frac{2+n+n^2}{3+4n+7n^2}$ e) $a_n = \frac{1-n+n^2}{1+n-n^2}$ f) $a_n = \frac{1+n+n^2}{n^2}$

   g) $a_n = \sqrt{\frac{2+n}{3+2n}}$ h) $a_n = \sqrt{\frac{1+2n+3n^2}{3+2n+n^2}}$

   i) $a_n = \frac{(n^2+3)(n+1)}{(n+4)(2n^2+3)}$ j) $a_n = \frac{n^2+n}{n^3+2}$ k) $a_n = \sqrt{n+1} - \sqrt{n}$

5. Evaluate $\underset{n\to\infty}{lt}\ a_n$ when *f(x)* is equal to

   a) $2x^3$ b) $x^3 + x^4$ c) $2x^3 - x^4 + x^5$

   d) $(2+3x)(4+5x^2)$ e) $\frac{2+3x}{4+5x^2}$ f) $\frac{\sqrt{x^2+9}}{4}$

6. Evaluate the following limits

   a) $\underset{x\to-1}{lt}(2x^2 - 1)$ b) $\underset{x\to1}{lt}(2x+1)(3-2x)$

   c) $\underset{x\to1}{lt}\frac{4+7x}{5+x^2}$ d) $\underset{x\to3}{lt}\sqrt{3x-3}$

   e) $\underset{x\to1}{lt}\frac{1}{\sqrt{2x+3}}$ f) $\underset{x\to3}{lt}\frac{x^2+2x+1}{x^2-3x+2}$

g) $\underset{x \to 1}{lt} \dfrac{x^3+1}{x^2+1}$ h) $\underset{x \to 1}{lt} \dfrac{x^2+x-2}{x^2-1}$

7. If $\underset{n \to a}{lt} f(x) = l$ and $\underset{n \to a}{lt} g(x) = m,$ evaluate

a) $\underset{n \to a}{lt} f(x)g(x)$ b) $\underset{n \to a}{lt} [f(x)+2g(x)][2f(x)-g(x)]$

c) $\underset{n \to a}{lt} \sqrt{2f(x)+3g(x)}$ *(when l, m, > 0)*

d) $\underset{n \to a}{lt} \dfrac{f(x)+g(x)}{(g(x))^2}$ when$m > 0$

8. Show that the following functions are continuous at a

a) $f(x) = 1 + x^2 + x^2$ for all x *in R,* $a = 0$

b) $f(x) = \dfrac{1}{1+x^2}$ for all x *in R* $a = 0$

c) $f(x) = \dfrac{2+3x}{1+2x^2}$ for all x *in R* $a = 0$

9. Show that the following functions are continuous at a

a) $f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & if \quad x \neq 4 \\ 8 & if \quad x > 4 \end{cases}$ $a = 4$

b) $f(x) = \begin{cases} 2+3x+4x^2 & if \quad x \le 1 \\ 4+3x+2x^2 & if \quad x > 1 \end{cases}$ $a = 1$

c) $f(x) = \begin{cases} 1-x^2 & if \quad x \le 1 \\ 1-x^3 & if \quad x > 1 \end{cases}$ $a = 1$

10. Show that the following functions are not continuous at a

a) $f(x) = \begin{cases} \dfrac{2x^2-8}{x-2} & if \quad x \neq 2 \\ 7 & if \quad x = 2 \end{cases}$ $a = 2$

b) $f(x) = \begin{cases} \dfrac{1-x^2}{1-x} & if \quad x \neq 1 \\ 1 & if \quad x = 1 \end{cases}$ $a = 1$

c) $f(x) = \begin{cases} 1+3x+4x^2 & if \quad x \le 1 \\ -3+x+4x^2 & if \quad x > 1 \end{cases}$ $a = 1$

d) $f(x) = \begin{cases} 1 & if \quad x \le 1 \\ -1 & if \quad x > 1 \end{cases} \quad a = 1$

e) $f(x) = \begin{cases} -1 & if \quad x < 0 \\ 0 & if \quad x = 0 \\ 1 & if \quad x > 0 \end{cases} \quad a = 0$

5.7 Answers

Self Assessment Questions

1. *A*
2. *A*
3. a) and d) can be arranged as a sequence according to chart and attendance register
4. (a) i) {3,4,5,…}, ii) {4,5,6,…}, iii) {1,2}, iv) {1,2,3}
   b) (i) *{3, 4, 5, …..} (ii){4, 5, 6, ……},iii){……. –3, –2, –1, 0, 1, 2,},*
   iv) {…. –3, –2, –1, 0, 1, 2, 3}
   (C) i) φ, ii) φ
5. [1, 5], $(-\infty, 1) \cup (5, \infty)$, $(-\infty, 1] \cup [5, \infty)$
6. a) *1* b) *1* c) *100* d) $\frac{3}{2}$
   e) *1* f) *–1* g) *2* h) *9*
7. a) *3* b) $\frac{1}{3}$ c) *2* d) *4*
   e) *3 (Hint:* $x^3 - 1 = (x - 1)(x^2 + x + 1)$*)* f) $\frac{5}{3}$ g) *1*
   h) $\frac{1}{2}$
8. a) $2l + 3m$ b) $l^2 - m^2$ c) $\sqrt{l^2 + m^2}$ d) $\frac{1}{\sqrt{1 + m^2}}$
   e) $\frac{l + 2m}{4m}$ f) $\sqrt{m + 2l}$
9. a), b), c) continuous d) discontinuous, e),f) continuous,
   g) not continuous

10. Take $x_n = 1 + \frac{1}{n}\left(1 + \frac{1}{n}\right) \to 0,$ but $f(x_n) = n$, $f(x_n)$ does not converge. So f is not continuous at *1.*

11. a) For $a < 1$, $f(x) = x^2 - 6x + 5$. So f is continuous for $a < 1$. As $f(x) = 2x^2 - 2$, f is continuous for $a > 1$.

$\underset{x \to 1-}{lt} f(x) = 1 - 6 + 5 = 0,\ \underset{x \to 1+}{lt} f(x) = 2 - 2 = 0.$ Hence $\underset{x \to 1}{lt} f(x) = 0.$

Also $f(1) = 1 - 6 + 5 = 0$

b), c), d) Similar.

Terminal Questions

1. a) $\{4, 5, 6, 7, 8\}$ b) $\{5, 6, 7, 8, 9\}$ c) $\{5, 6, 7, 8\}$
d) $\{f,g\}$
e) $\{5, 6, \ldots..\}$ f) $\{11, 12, \ldots..\}$ g) $\{9, 10, 11, \ldots..\}$
h) $\{10, 11, \ldots..\}$

2. a) 0 b) 1 c) 0 d) 0 e) 0

3. In all the case, As n increases a_n increases. So $|a_n - k|$ cannot be made less than a fixed number $\in$. So $\underset{n \to \infty}{lt} a_n$ does not exists.

4. a) $\frac{3}{4}$ b) 1 c) 3 d) $\frac{1}{7}$
e) -1 f) 1 g) $\sqrt{\frac{1}{2}}$ h) $\sqrt{3}$

i) Write a_n as $\left(\frac{1+\frac{3}{n^2}}{2+\frac{3}{n^2}}\right) \cdot \left(\frac{1+\frac{1}{n}}{1+\frac{4}{n}}\right) . = \underset{n \to \infty}{lt} a_n = \frac{1(1)}{2(1)} = \frac{1}{2}$

j) $a_n = \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{2}{n^3}} . = \underset{n \to \infty}{lt} a_n = 0 + \frac{0}{1+0} = \frac{0}{1+0} = 0$

k) $a_n = \sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}}$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n}\left(1+\sqrt{\frac{n+1}{n}}\right)} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{n+1}{n}}}$$

$$= \underset{n\to\infty}{lt}\ a_n = \frac{0}{1+1} = 0$$

5. a) $2a^3$ b) $a^3 + a^4$ c) $2a^3 - a^4 + a^5$ d) $(2 + 3a)(4 + 5a^2)$

   e) $\frac{2+3a}{4+5a^2}$ f) $\sqrt{\frac{a^2+9}{4}}$

6. a) $2(-1)^2 - 1 = 1$ b) $(2 + 1)(3 - 2) = 3$ c) $\frac{11}{6}$ d) $\sqrt{6}$

   e) $\frac{1}{\sqrt{5}}$ f) $\frac{(9+6+1)}{9-9+2} = 8$ g) 1

   h) $\frac{x^2+x-2}{x^2-1} = \frac{(x-1)(x+2)}{(x-1)(x+1)} = \frac{x+2}{x+1}$. Hence answer is $\frac{3}{2}$.

7. a) lm b) $(l + 2m)(2l - m)$ c) $\sqrt{2l+3m}$ d) $\frac{(l+m)}{m^2}$

8. a) $\frac{(2x^2-8)}{x-2} = \frac{2(x+2)(x-2)}{x-2}\ \underset{x\to 2}{lt}\ f(x) = 8\, f(2) = 7.$ So f is not continuous at x=2.

   b) $\underset{x\to 1}{lt}\ f(x) = \underset{x\to 1}{lt}\ \frac{(1-x^2)}{1-x} = \underset{x\to 1}{lt}\ (1+x) = 2.$ But $f(1) = 1$

   c) Left limit at 1 = 8; right limit at 1 = 2

   d) The left and right limits are 1 and –1 respectively.

   e) The left and right limits are –1 and 1.