Unit 9 Interpolation with Unequal Intervals

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Structure:

9.1 Introduction

In the previous chapter, we studied the finite difference with equal differences. The formulas like Newton's Forward, Newton's Backward interpolation, and Central difference formulas –Gauss forward and backward formula, Sterling's formula, and Bessel's formula, all are applied if the values of x are at equidistance. But there may be a case when the values are not at equidistance and we need to find the values of y for the in between values of x. In this unit we will discuss these cases and study different formula like Newton's divided difference formula, Lagrange Interpolation formula and also about inverse interpolation.

Objectives:

At the end of this unit the student should be able to:

- explain the concept of Finite Differences with unequal differences.
- Apply interpolation Formula

9.2 Lagrange's Interpolation Formula

Newton's interpolation formula described in the previous unit can be used only when the values of the independent variable x are equally spaced. Also the differences of y must ultimately become small. If the values of the independent variable are not given at equidistant intervals, then we consider the following.

If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$, then

$$y = f(x) = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + \ldots +$$

$$\frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n \quad \ldots\ldots\ldots \quad (1)$$

This is known as **Lagrange's Interpolation Formula** for unequal intervals.

Let y = f (x) be a function which takes the values (x_0, y_0), (x_1, y_1)… (x_n,y_n). Since there are (n + 1) pairs of values of x and y, we can represent f(x) by a polynomial in x of degree n.

Proof: Let this polynomial be of the form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots\ldots\ldots\ldots \quad (2)$$

Putting $x = x_0$, $y = y_0$, we get

$$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

That is, $a_0 = \dfrac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$

Similarly putting $x = x_1$, $y = y_1$ we get

$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}.$$

Proceeding the same way, we find $a_2, a_3, \ldots, a_n$.

Substituting the values of $a_0, a_1 \ldots, a_n$ in (2) we get (1).

Note that Lagrange's formula can be applied whether the values x_i are equally spaced or not.

Example: Find Lagrange's interpolation polynomial fitting the points

y(1) = -3, y(3) = 0, y(4) = 30, y(6) = 132. Hence find *y(5).*

Solution: The given data can be arranged as:

x	1	3	4	6
Y = f(x)	– 3	0	30	132

Using Lagrange's interpolation formula, we have

y = f (x)

$$= \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$= \frac{x^3-13x^2+54x-72}{-30}(-3) + \frac{x^3-11x^2+34x-24}{6}(0)$$

$$+ \frac{x^3-10x^2+27x-18}{-6}(30) + \frac{x^3-8x^2+19x-12}{30}(132)$$

On simplification, we get y(x) = $\frac{1}{2}(-x^3+27x^2-92x+60)$, which is the required Lagrange's interpolation polynomial.

Now y(5) = 75.

Example: Use Lagrange's interpolation formula to fit a polynomial for the data

x	0	1	3	4
y	– 12	0	6	12

Hence estimate *y* at *x = 2*

Solution: By data $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ $x_3 = 4$

$y_0 = -12$, $y_1 = 0$ $y_2 = 6$ $y_3 = 12$

We have *y = f (x)*

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\, y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\ y_3$$

i.e., $y = f(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + 0$

$$+ \frac{x(x-1)(x-4)}{(3)(2)(-1)}\ 6 + \frac{x(x-1)(x-3)}{(4)(3)(1)}\ 12$$

$y = f(x) = x^3 - 7x^2 + 18x - 12$

Now $f(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$

Therefore $f(2) = 4$.

Example: Apply Lagrange's interpolation formula to find a polynomial which passes through the points (0,-20), (1,-12), (3,-20), (4,-24).

Solution: We have $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ $x_3 = 4$

$y_0 = -20$, $y_1 = -12$ $y_2 = -20$ $y_3 = -24$

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_3-x_2)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-20) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(-12) +$$

$$\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(-20) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-2)}(-24)$$

$$= x^3 - 8x^2 + 15x - 20.$$

Example: Using Lagrange's interpolation formula, find the value of y corresponding to x = 10 from the following data

x	5	6	9	11
f(x)	380	2	196	508

Solution: We have $x_0 = 5$, $x_1 = 6$, $x_2 = 9$ $x_3 = 11$

$y_0 = 380$, $y_1 = 2$ $y_2 = 196$, $y_3 = 508$

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_6$$

When x = 10,

$$Y = f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(380) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(2)$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(196) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(508)$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(380) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(2) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(196) + \frac{(5)(4)(1)}{(6)(5)(2)}(508)$$

= 63.33 – 0.667 + 163.33 + 169.33

= 395.33

Example: From the table evaluate f(x) at x = 9 using Lagrange's formula

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Solution: We have $x_0 = 5$, $x_1 = 7$, $x_2 = 11$ $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$ $y_2 = 1452$ $y_3 = 2366$, $y_4 = 5202$

So by Lagrange's interpolation formula, we have

We have $y = f(x)$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$+$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

Now, putting x = 9, we obtain

$$=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150)+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392)$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452)+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}(2366)$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202)$$

$$=\frac{(2)(-2)(-4)(-8)}{(-2)(-6)(-8)(-12)}(150)+\frac{(4)(-2)(-4)(-8)}{(2)(-4)(-6)(-10)}(392)$$

$$+\frac{(4)(2)(-4)(-8)}{(6)(4)(2)(-6)}(1452)+\frac{(4)(2)(-2)(-8)}{(8)(6)(2)(-4)}(2366)$$

$$+\frac{(4)(2)(-2)(-4)}{(12)(10)(6)4}(5202)$$

$$=-\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}-\frac{2366}{3}+\frac{578}{5}$$

= -16.67+ 209.067 + 1290.67 – 788.67 + 115.6

= 810

Example: Find $log_{10}656$ given that log_{10} 654 = 2.8156, $log_{10}658$ = 2.8182, $log_{10}659$ = 2.8189, $log_{10}661$ = 2.8202, using Lagrange's interpolation formula.

Solution: Let y = f(x) = $log_{10}x$ and $x_0 = 654$, $x_1 = 658$, $x_2 = 659$ $x_3 = 661$

$y_0 = 2.8156$, $y_1 = 2.8182$ $y_2 = 2.8189$ $y_3 = 2.8202$

By Lagrange's Interpolation formula, we have

y = f (x)

$$=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\ y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\ y_3$$

Now, put x = 656 and all above values, we get

$$Y = f(x) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}(2.8156)$$

$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}(2.8182)$$

$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}(2.8189)$$

$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}(2.8202)$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)}(2.8156)+\frac{(2)(-3)(-5)}{(4)(-1)(-3)}(2.8182)+\frac{(2)(-2)(-5)}{(5)(1)(-2)}(2.8189)$$

$$+\frac{(2)(-2)(-3)}{(7)(3)(2)}(2.8202)$$

$$= 0.6033 + 7.045 - 5.6378 + 0.8058 \quad = 2.8168$$

Example: Find the value of f(x) = xlogx at x = 5 from the following data, by using Lagrange's formula

x	3	7	9	12
Y=f(x)	1.4313	5.9156	8.5881	12.9501

Solution: We have $x_0 = 3$, $x_1 = 7$, $x_2 = 9$ $x_3 = 12$

$y_0 = 1.4313$, $y_1 = 5.9156$ $y_2 = 8.5881$, $y_3 = 12.9501$

By Lagrange's Interpolation formula, we have

$y = f(x)$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\ y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\, y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

y_3

Now, put x = 5, we get

$$y = f(x) = \frac{(5-7)(5-9)(5-12)}{(3-7)(3-9)(3-12)}(1.4313) + \frac{(5-3)(5-9)(5-12)}{(7-3)(7-9)(7-12)}(5.9156)$$

$$+ \frac{(5-3)(5-7)(5-12)}{(9-3)(9-7)(9-12)}(8.5881) + \frac{(5-3)(5-7)(5-9)}{(12-3)(12-7)(12-9)}(12.9501)$$

$$= \frac{(-2)(-4)(-7)}{(-4)(-6)(-9)}(1.4313) + \frac{(2)(-4)(-7)}{(4)(-2)(-5)}(5.9156) + \frac{(2)(-2)(-7)}{(6)(2)(-3)}(8.5881)$$

$$+ \frac{(2)(-2)(-4)}{(9)(5)(3)}(12.9501)$$

= 0.371077+ 8.28184 – 6.6793 + 1.5348 = 3.5084

SAQ 1. Estimate the value of f(4) by applying Lagrange’s formula to the following data:

x	0	2	3	6
f(x)	707	819	866	966

SAQ 2. Find Lagrange’s polynomial using the data

x	0	1	2	5
f(x)	2	3	12	147

Interpolate to find the value of f(2.25)

9.3 Divided Differences

Lagrange’s interpolation formula has the disadvantage that if another interpolation point were inserted, then we have to recompute the interpolation coefficients. Hence it is desirable to have an interpolating formula that has the property that a polynomial of higher degree may be derived from it by simply adding new terms. *Newton's divided differences formula* is one such formula and this formula employ's the concept of divided differences.

Definition:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a given set of $(n + 1)$ points. The *first divided differences* are defined by the following relations

$$[x_0,\ x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_1,\ x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\ldots\ldots \qquad \ldots\ldots$$

$$\ldots\ldots \qquad \ldots\ldots$$

$$[x_{n-1},\ x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

The *second divided differences* are defined by

$$[x_0,\ x_1,\ x_2] = \frac{[x_1,\ x_2] - [x_0,\ x_1]}{x_2 - x_0} \text{ and so on.}$$

The *third divided differences* are defined by

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

In general the nth divided difference is defined by

$$[x_0, x_1, x_2, \ldots, x_n] = \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

Note that $[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = [x_1, x_0]$ The divided difference table is given below:

x	y	First div. diff.	Second div. diff.	Third div. diff	Fourth div. diff
x_0	y_0				
		$[x_0, x_1]$			
			$[x_0, x_1, x_2]$		
$x_1 y_1$				$[x_0, x_1, x_2, x_3]$	
		$[x_1, x_2]$			
			$[x_1, x_2, x_3]$		$[x_0, x_1, x_2, x_3, x_4]$
$x_2 y_2$				$[x_1, x_2, x_3, x_4]$	
		$[x_2, x_3]$			
			$[x_2, x_3, x_4]$		
$x_3 y_3$		$[x_3, x_4]$			

Example: The divided difference table for the following table

x	–1	0	2	4	5
y	0	1	9	65	126

Solution:

x	y	First div. diff.	Second div. diff.	Third div. diff	Fourth div. diff
–1	0				
		$\frac{1-0}{0-(-1)}=1$			
			$\frac{4-1}{2-(-1)}=1$		
0	1				
		$\frac{1-0}{1-(-1)}=1$ $\frac{9-1}{2-0}=4$		$\frac{6-1}{4-(-1)}=1$	
2	9		$\frac{28-4}{4-0}=6$		$\frac{1-1}{5-(-1)}=0$
		$\frac{65-9}{4-2}=28$			
				$\frac{11-6}{5-0}=1$	
4	65				
			$\frac{61-28}{5-2}=11$		
		$\frac{126-65}{5-4}=61$			
5	126				

Example: If $f(x) = \frac{1}{x}$, then find the divided difference $f(a,b)$& $f(a,b,c)$.

Solution: Here, we have $x = a, b, c$ and the corresponding y values are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Thus we have

$$f(a,b) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = \frac{a - b}{ab} \times \frac{1}{(b - a)} = -\frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c - a} = \frac{\frac{f(c)-f(b)}{c-b} - \frac{f(b)-f(a)}{b-a}}{c-a} = \frac{-\frac{1}{cb} + \frac{1}{ab}}{c-a} = \frac{\frac{-ab+cb}{abc}}{c-a} = \frac{b(c-a)}{(abc)^2(c-a)} = \frac{1}{abc}$$

Example: Find the third difference with the arguments 2, 4, 9, 10, of the function $f(x) = x^3 - 2x$.

Solution: Given $f(x) = x^3 - 2x$, so

x	y	First div. diff.	Second div. diff.	Third div. diff
2	4	26		
			15	
4	56	131		1
			23	
9	711	269		
10	980			

Therefore, the third difference is one

SAQ 3: If $f(x) = \frac{1}{x^2}$, find the divided difference $f(a,b)$, $f(a,b,c)$, $f(a,b,c,d)$.

SAQ 4. Construct a table of divided difference for the following data

x	0	2	3	5	6
f(x)	1	19	55	241	415

9.4 Newton's divided differences formula

Let $y(x)$ be a function taking values $y_0, y_1, y_2 \ldots, y_n$ corresponding values $x_0, x_1, \ldots, x_n$. Then the Newton's divided difference formula is given by

$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})[x_0, x_1, x_2 \ldots x_n]$.

Proof: we have $[x, x_0] = \dfrac{y - y_0}{x - x_0}$

Therefore $y = y_0 + (x - x_0)[x, x_0]$ – (1)

Also $[x, x_0, x_1] = \dfrac{[x, x_0] - [x_0, x_1]}{x - x_1}$

Therefore $[x, x_0] = [x_0\ x_1] + (x - x_1)[x, x_0, x_1]$ – (2)

Substituting (2) in (1) we have

$$y = y_0 + (x - x_0)([x_0, x_1) + (x - x_1)[x, x_0, x_1])$$
$$= y_0 + (x-x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$

Proceeding like this we get the result

$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})[x_0, x_1, x_2 \ldots x_n]$.

Example: Using the following table find y(x) as a polynomial in x.

x	0	1	2	3
y(x)	1	0	1	10

The divided difference table is given below.

x	**y**	**First div. diff.**	**Second div. diff.**	**Third div. diff**
$x_0 = 0$	$y_0 = 1$			
		$\frac{0-1}{1-0} = -\mathbf{1}$		
$x_1 = 1$	$y_1 = 0$		$\frac{1-(-1)}{2-0} = \mathbf{1}$	
		$\frac{1-0}{2-1} = 1$		$\frac{4-1}{3-0} = \mathbf{1}$
$x_2 = 2$	$y_2 = 1$		$\frac{9-1}{3-1} = 4$	
		$\frac{10-1}{3-2} = 9$		
$x_3 = 3$	$y_3 = 10$			

From the table we have

$$y_0 = 1, \ [x_0, x_1] = -1, \ [x_0, x_1, x_2] = 1, \text{ and } [x_0, x_1, x_2, x_3] = 1$$

Newton's divided difference formula is

$$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$

Therefore, $y(x) = 1 + (x - 0)(-1) + (x - 0)(x - 1)(1) + (x - 0)(x - 1)(x - 2)\,1$

$= 1 - x + (x^2 - x) + (x^3 - 3x^2 + 2x)$

Therefore, $y(x) = x^3 - 2x^2 + 1$

Example: Given the values

x	5	7	11	13	17
y	150	392	1452	2366	5202

Evaluate y(9) using Newton's divided differences formula.

Solution: Using the **Newton's divided difference formula**.

The divided difference table is given below

x	y	First div. diff.	Second div. diff.	Third div. diff	Fourth div. diff
$x_0 = 5$	$y_0 = 150$				
		121			
$x_1 = 7$	$y_1 = 392$		24		
		265		1	
$x_2 = 11$	$y_2 = 1452$		32		0
		457		1	
$x_3 = 13$	$y_3 = 2366$		42		
		709			
$x_4 = 17$	$y_4 = 5202$				

The Newton's divided difference formula is

$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$

$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$.

put $x = 9$. Then

$y(9) = 150 + (9 - 5)\,121 + (9 - 5)(9 - 7)\,24 + (9 - 5)(9 - 7)(9 - 11)1$

$= 150 + 484 + 192 - 16 = 810$

Therefore $y(9) = 810$.

Example: Find the function f(x) under suitable assumption from the following data

x	0	1	2	5
f(x)	2	3	12	147

Solution: Lets first make a divided difference table from the above data

x	f(x)	First Divided Difference	Second Divided Difference	Third Divided Difference
0	2			
		(3-2)/1-0=1		
1	3		(9-1)/2-0 =4	
		(12-3)/2-1 =9		(9-4)/5-0=1
2	12		(45-9)/5-1=9	
		(147-12)/5-2 =45		
5	147			

We have $x_0 = 0$, $f(x_0) = 2$, $[x_0,x_1] = 1$, $[x_0,x_1,x_2] = 4$, $[x_0,x_1,x_2] = 1$

So by Newton's divided difference formula, we have

$y = f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$

$\quad + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$

$= 2 + (x-0)1 + (x-0)(x-1)4 + (x-0)(x-1)(x-2)1$

$= x^3 + x^2 - x + 2.$

Example: Derive the equation of the interpolating polynomial for the data given in the table below

x	0	1	2	3	4	5
f(x)	3	2	7	24	59	118

Solution: Lets first make a divided difference table from the above data

x	y	First div. diff.	Second div. diff.	Third div. diff	Fourth div. diff
$x_0 = 0$	$y_0 = 3$				
		-1			
$x_1 = 1$	$y_1 = 2$		3		
		5		1	
$x_2 = 2$	$y_2 = 7$		6		0
		17		1	
$x_3 = 3$	$y_3 = 24$		9		0
		35		1	
$x_4 = 4$	$y_4 = 59$		12		
		59			
$x_5 = 5$	$y_5 = 118$				

Using Newton's divided difference formula, the interpolating formula is:

$$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3].$$

$$y(x) = 3 + (x - 0)(-1) + (x-0)(x-1)(3) + (x-0)(x-1)(x-2)(1) + (x-0)(x-1)(x-2)(x-3)(0)$$

$$= 3 - x + (3x^2-3x) + (x^3-3x^2+2x)$$

$$= x^3-2x+3$$

Example: Find the interpolating polynomial for the following data:

x	0	1	2	4
f(x)	1	1	2	5

Solution:

The divided difference table is given below.

x	**y**	**First div. diff.**	**Second div. diff.**	**Third div. diff**
$x_0 = 0$	$y_0 = 1$			
		0		
			½	
$x_1 = 1$	$y_1 = 1$			-1/12
		1		
			1/6	
$x_2 = 2$	$y_2 = 2$			
		3/2		
$x_3 = 4$	$y_3 = 5$			

By Newton's divided difference formula, we have

$$y(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] .$$

$$= 1 + (x-0)(0) + (x-0)(x-1)1/2 + (x-0)(x-1)(x-2)(-1/12)$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2x}{30} + 1$$

SAQ 5. By means of Newton's divided difference formula, find the value of f(8), f(9), f(15) form the table given below

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

SAQ 6 : By means of newton's divided difference formula, find the value of f(6) from the following table:

x	3	5	8	9	12
f(x)	24	120	504	720	1716

9.5 Inverse Interpolation- Lagrange's Inverse Interpolation

In case of inverse interpolation, we find the argument x corresponding to the intermediate value of y.

Lagrange's Formula for inverse interpolation

We have by Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots.(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n \quad \ldots..(1)$$

Now, interchanging x & y in eq (1), we can express x a function of y as follows

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\ldots(y_1-y_n)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)} x_2 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})} x_n$$

This is called inverse Lagrange interpolation.

Example: From the table given below find the value of x corresponding to the values of y = 8 by applying Lagrange's inverse interpolation formula

x	-2	-1	1	2
f(x)	-7	2	0	11

Solution: Here

$x_0 = -2, \; x_1 = -1, \; x_2 = 1 \; \; x_3 = 2$

$y_0 = -7, \; y_1 = 2, \; y_2 = 0, \; y_3 = 11$

and $y = 8$

By inverse Lagrange interpolation, we have

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\ldots(y_1-y_n)}x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)}x_2 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})}x_n$$

$$= \frac{(8-2)(8-0)(8-11)}{(-7-2)(-7-0)(-7-11)}(-2) + \frac{(8+7)(8-0)(8-11)}{(2+7)(2-0)(2-11)}(-1)$$

$$+ \frac{(8+7)(8-2)(8-11)}{(0+7)(0-2)(0-11)}(1) + \frac{(8+7)(8-2)(8-0)}{(11+7)(11-2)(11-0)}(2)$$

$$= \frac{(6)(8)(-3)}{(-9)(-7)(-18)}(-2) + \frac{(15)(8)(-3)}{(9)(2)(-9)}(-1) + \frac{(15)(6)(-3)}{(7)(-2)(-11)}(1) + \frac{(15)(6)(8)}{(18)(9)(11)}(2)$$

$$= -\frac{288}{1134} - \frac{360}{162} - \frac{270}{154} + \frac{1440}{1782}$$

= -0.254 - 2.222 – 1.753 + 0.808

= -3.421

Example: Determine x for y = 7 from the following data

x	1	3	4
f(x)	4	12	19

Solution: Here

$x_0 = 1$, $x_1 = 3$, $x_2 = 4$

$y_0 = 4$, $y_1 = 12$, $y_2 = 19$

and $y = 7$

By inverse Lagrange interpolation, we have

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\ldots(y_1-y_n)}x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)}x_2 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})}x_n$$

$$= \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

$$= \frac{(-5)(-12)}{(-8)(-15)} + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$

$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7} = 1.8571$

Example: Find the value of x corresponding to y = 100, by using inverse interpolation from the following data

x	3	5	7	9
f(x)	6	24	58	108

Solution: Here

$x_0 = 3$, $x_1 = 5$, $x_2 = 7$, $x_3 = 9$

$y_0 = 6$, $y_1 = 24$, $y_2 = 58$, $y_3 = 108$

and $y = 100$

By inverse Lagrange interpolation, we have

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\ldots(y_1-y_n)}x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)}x_2 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\ldots(y_n-y_{n-1})}x_n$$

$$= \frac{(100-24)(100-58)(100-108)}{(6-24)(6-58)(6-108)}(3) + \frac{(100-6)(100-58)(100-108)}{(24-6)(24-58)(24-108)}(5)$$

$$+ \frac{(100-6)(100-24)(100-108)}{(58-6)(58-24)(58-108)}(7) + \frac{(100-6)(100-24)(100-58)}{(108-6)(108-24)(108-58)}(9)$$

$$= \frac{(76)(42)(-8)}{(-18)(-52)(-102)}(3) + \frac{(94)(42)(-8)}{(18)(-34)(-84)}(5) + \frac{(94)(76)(-8)}{(52)(34)(-50)}(7) + \frac{(94)(76)(42)}{(102)(84)(50)}(9)$$

$$= \frac{76608}{95472} - \frac{157920}{51408} + \frac{400064}{88400} + \frac{2700432}{428400}$$

= 0.802413 – 3.071895 + 4.52561 + 6.30352

= 8.559657

SAQ 7: From the following table, find x for y = -0.510, using Lagrange's inverse formula.

9.6 Summary

In this unit, various methods for constructing interpolation polynomials for unequally spaced values of arguments are discussed. Methods of building different types of difference tables and how to use them for estimating function values at a point are discussed.

We also discussed Lagrange's interpolation formula which is a better approximation since the values need not be equidistant. Further, we dealt with Newton's divided difference formula. Also, we studied the inverse method- Lagrange's inverse interpolation for finding the values of x from the given values of y.

9.7 Terminal Questions

1. Use Lagrange's formula to find the value of f (8) given

x	4	5	7	10	11	13
f (x)	48	100	294	900	1210	2028

2. By Newton's divided difference formula find the equation of the cubic curve which passes through the points *(4, – 43), (7, 83), (9, 327)* and *(12, 1053).* Hence find *f(10).*

3. Given Sin 45° = 0.7071, Sin 50° = 0.7660, Sin 55° = 0.8192, Sin 60° = 0.8660, find sin 57° using appropriate interpolation formula.

4. Use Lagrange's inverse interpolation formula to obtain the value of t, when A = 85 from the following table

t	2	5	8	14
A	94.8	87.9	81.3	68.7

5. Use Lagrange's inverse interpolation formula to obtain the value of x, when F(x) = 0.3887 form the following table

x	21^0	23^0	25^0
F(x)	0.3706	0.4068	0.4433

9.8 Answers

Self Assessment Questions

1. 906.44
2. $x^3 + x^2 - x + 2$, 16.203125
3. $-\frac{a+b}{a^2b^2}, \frac{ab+bc+ca}{a^2b^2c^2}, -\frac{abc+bcd+acd+abd}{a^2b^2c^2d^2}$
4. 75
5. f(8) = 448, f(9) = 648, f(15) = 3150
6. 207
7. x = 0.6

Terminal Questions

1. 448
2. f(x) = $x^3 - 4x^2 - 7x - 15$, 515
3. 0.8387
4. 6.5928
5. 22^0