Unit 10 Numerical Differentiation

Structure:

10.1 Introduction

Consider a function of single variable $y = f(x)$. If the function is known and simple, we can easily obtain its derivative (s) or can evaluate its definite integral. However, if we do not know the function as such or the function is complicated and is given in a tabular form at a set of points $x_0, x_1, \ldots, x_n$, we use only numerical method for differentiation or integration of the given function. Need for numerical differentiation and integration techniques arises quite extensively and regularly in engineering physics and other quantitative sciences. This unit tells you how to estimate the derivative or get the integral of a function by Numerical methods. The Interpolation methods of last two units provide the foundation for numerical differentiation and integration.

Objectives:

At the end of this unit you should be able to:

- Find the derivatives using interpolation formulas.
- Solve problems using Difference Equations.

10.2 Differentiation using Newton's Formulae

Numerical differentiation is the process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x from given set of data $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, when the actual relationship between x and y is not known. The procedure is to replace the exact relation $y = f(x)$ by the best approximating polynomial $y = \Phi(x)$ and then to differentiate the later. The interpolation formula to be used depends as usual on the particular value of x at which the value of $\frac{dy}{dx}$ is required.

If the value of x are equi-spaced and $\frac{dy}{dx}$ is required near the beginning of the table, we employ Newton's forward difference formula. If it is required near the end of the table, we use Newton's backward difference formula. If the values are not equispaced, we use Lagrange's interpolation or Newton's divided difference formula or Newton's general interpolation formula to represent the function.

10.2.1 Derivatives using Newton's forward difference formula

Consider Newton's forward difference formula,

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots, \text{ where } x = x_0 + ph.$$

or

$$y = y_0 + p\,\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \frac{(p^3-3p^2+2p)}{6}\Delta^3 y_0$$

$$+ \frac{(p^4-6p^3+11p^2-6p)}{24}\Delta^4 y_0 + \ldots, \text{ where } x = x_0 + ph. \qquad (1)$$

So, $p = \frac{x-x_0}{h}$

Differentiating (1) w.r.t. x we have, $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$

$$\frac{dy}{dx} = \Big[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0$$

$$+ \frac{(4p^3-18p^2+22p-6)}{24}\Delta^4 y_0 + \ldots\Big]\frac{1}{h} \qquad (2)$$

Take $x = x_0$ in $x = x_0 + ph$, we obtain $p = 0$ and hence (2) gives

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right] \quad -(3)$$

Differentiating (2) once again, we obtain

$$\frac{d^2 y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{1}{h}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\ \Delta^3 y_0 + \frac{(12p^2 - 36p + 22)}{24}\ \Delta^4 y_0 + \ldots\right] \quad -(4)$$

At x = x_0, p = 0, equation (4) becomes

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\ \Delta^4 y_0 - \ldots.\right] \quad -(5)$$

Similarly,

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\ \Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots.\right] \quad -(6)$$

Derivatives of higher order can similarly be obtained.

10.2.2 Derivatives using Newton's backward difference formula

We know that Newton's interpolation formula for backward differences is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \ldots. \quad -(7)$$

where $x = x_n + ph$

so, So, $p = \frac{x-x_n}{h}$

Differentiating (7) w.r.t. x, we get $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\ \nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n + \frac{(3p^2 + 6p + 2)}{6}\nabla^3 y_n\right.$$

$$+ \frac{(4p^3 + 18p^2 + 22p + 6)}{24} \nabla^4 y_n + \ldots \Bigg] \quad -(8)$$

At $x = x_n$, $p = 0$, we have

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right] \quad -(9)$$

Differentiating (8) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{1}{h}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6p+6}{6}\right)\nabla^3 y_n + \frac{(12p^2 + 36p + 22)}{24}\nabla^4 y_n + ..\right] \quad -(10)$$

At $x = x_n$, $p = 0$.

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right] \quad -(11)$$

Similarly we can find the higher order derivatives.

10.2.3 Derivatives using Sterling formula

$$y_p = y_0 + p(\frac{\Delta y_0 + \Delta y_{-1}}{2}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2})$$

$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots,$ …(12)

where$p = \frac{x-x_0}{h}$ …(13)

Differentiating (12) w.r.t p, we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2P}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \cdots \quad (14)$$

Now, differentiating (13) w.r.t. x, we get

$\frac{dp}{dx} = \frac{1}{h}$….(15)

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{h}\left\{\frac{dy}{dp}\right\}$$

$$= \frac{1}{h}\left\{\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2P}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \cdots\right\} (16)$$

At $x = x_0$, $p = 0$

So, $\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \cdots\right\}$ ….(17)

Again differentiating (16), we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left(\Delta^2 y_{-1} + p\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{12p^2-2}{24}\Delta^4 y_{-2} + \cdots\right)$$

At $x = x_0$, we get

$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right)$….(18)

Similarly, we can obtain higher derivatives.

Example

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1$ and $x = 0$ from the following data:

x	0	2	4	6	8
y	7	13	43	145	367

Solution: The forward difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 7$				
		6			
$x_1 = 2$	$y_1 = 13$		24		
		30		48	
$x_2 = 4$	$y_2 = 43$		72		0
		102		48	
$x_3 = 6$	$y_3 = 145$		120		
		222			
$x_4 = 8$	$y_4 = 367$				

We have

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_n + \frac{(3p^2-6p+2)}{6}\Delta^3 y_n + \ldots\right]$$

where $x = x_0 + ph$

Let us take $x = 1$ in $x_0 + ph = x$

Here $x_0 = 0$, $h = 2$, $x = 1$,

Therefore $p = \frac{1}{2}$, Hence by

$$\frac{dy}{dx} = \left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0\right.$$

$$\left. + \frac{(4p^3-18p^2+22p-6)}{24}\Delta^4 y_0 + \ldots\right]\frac{1}{h}, \text{ we have}$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2}\left[6 + \frac{\left(2.\frac{1}{2}-1\right)}{2}.24 + \frac{3\left(\frac{1}{2}\right)^2 - 6(\frac{1}{2}) + 2}{6}48 + 0\right]$$

$$= \frac{1}{2}\left[6 + 0 + \left(\frac{3}{4} - 3 + 2\right)\frac{48}{6}\right]$$

$$= \frac{1}{2}\left[6 + \left(\frac{3}{4} - 1\right)\frac{48}{6}\right] = \frac{1}{2}[6-2] = 2$$

$$\left.\frac{dy}{dx}\right|_{at\ x=1} = 2$$

To find $\frac{d^2y}{dx^2}$ at x = 1

We use the formula (4).

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{(12p^2-36p+22)}{24}\Delta^4 y_0 + \ldots\right],$$

We have

$$\left.\frac{d^2y}{dx^2}\right|_{at\ x=1} = \frac{1}{2^2}[24-24] = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{at\ x=1} = 0$$

To find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 0, we use the formula (3) and (5).

$$\left.\frac{dy}{dx}\right|_{at\ x=0} = \frac{1}{2}\left[6 - \frac{24}{2} + \frac{1}{3}\times 48\right]$$

$$= \frac{1}{2}[10] = 5$$

$$\left.\frac{d^2y}{dx^2}\right|_{at\ x=0} = \frac{1}{2^2}[24-48] = \frac{1}{4}[-24] = -6$$

$$\left.\frac{dy}{dx}\right|_{at\ x=0} = 5,\ \left.\frac{d^2y}{dx^2}\right|_{at\ x=0} = -6$$

Example: Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) x = 1.1 (b) x = 1.6

Solution: The difference table is,

x	y	Δ	Δ²	Δ³	Δ⁴	Δ⁵	Δ⁶
1.0 1.1 1.2 1.3 1.4 1.5 1.6	7.989 8.403 8.781 9.129 9.451 9.750 10.031	0.414 0.378 0.348 0.322 0.299 0.281	-0.036 -0.030 -0.026 -0.023 -0.018	0.006 0.004 0.003 0.005	-0.002 -.0001 0.002	0.001 0.003	0.002

As we have to find the derivative at x = 1.1 and 1.6 and these two values are from the table, so we can assume x_0 = 1.1 in first case

To find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.1, we use Newton's Forward differentiation formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right]$$

Here h = 0.1, x_0 = 1.1, p = 0. The above formula can be rewritten as

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_1 - \frac{1}{2}\Delta^2 y_1 + \frac{1}{3}\Delta^3 y_1 - \frac{1}{4}\Delta^4 y_1 + \frac{1}{5}\Delta^5 y_1 - \frac{1}{6}\Delta^6 y_1 + \ldots\right] \text{(i)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_1} = \frac{1}{h^2}\left[\Delta^2 y_1 - \Delta^3 y_1 + \frac{11}{12}\Delta^4 y_1 - \frac{5}{6}\Delta^5 y_1 + \frac{137}{180}\Delta^6 y_1 + \ldots\right] \quad \text{(ii)}$$

Substituting $\Delta y_1 = 0.378$, $\Delta^2 y_1 = -0.030$, $\Delta^3 y_1 = 0.004$, $\Delta^4 y_1 = -0.001$, $\Delta^5 y_1 = 0.003$ in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.1}\left[0.378 = \frac{1}{2}(-0.030) + \frac{1}{3}(-0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right]$$

= 3.947

Therefore $\left(\frac{dy}{dx}\right)_{at\ x=1.1} = 3.947$

$$\left(\frac{d^2y}{dx^2}\right)_{at\ x=1.1} = \frac{1}{(0.1)^2}\left[-0.03 - 0.004 + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.545$$

Therefore $\left(\frac{d^2y}{dx^2}\right)_{at\ x=1.1} = -3.546$

To find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ at x = 1.6, we have to use Newton's backward differentiation formula, that is,

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad \ldots \text{(iii)}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots \text{(iv)}$$

We use the above difference table and backward difference operator ∇ instead of Δ.

Here $h = 0.1$, $x_n = x_6 = 1.6$ and $\nabla y_6 = 0.281$, $\nabla^2 y_6 = -0.018$, $\nabla^3 y_6 = 0.005$, $\nabla^4 y_6 = -0.002$, $\nabla^5 y_6 = 0.003$, $\nabla^6 y_6 = 0.002$.

Putting these values in (iii) and (iv), by taking $n = 6$, we get,

$$\left(\frac{dy}{dx}\right)_{at\,x=1.6} = \frac{1}{0.1}\left[\ 0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\ \right]$$

$$= 2.728$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\,x=1.6} = \frac{1}{(0.1)^2}\left[\ -0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\ \right]$$

$$= -1.703$$

Therefore $\left(\frac{dy}{dx}\right)_{at\,x=1.6} = 2.728$

and

$\left(\frac{d^2y}{dx^2}\right)_{at\,x=1.6} = -1.703$.

Example: From the following table of values of x and y, obtain $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ for x = 1.2.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Here $x_0 = 1.2$, $y_0 = 3.3201$ and $h = 0.2$.

$$\left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$= 3.3205$.

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{22}(0.0080) + \frac{5}{6}(0.0014)\right] = 3.318$$

Example:

The population of a certain town is shown in the following table

Year x	1931	1941	1951	1961	1971
Population y	40.62	60.80	79.95	103.56	132.65

Find the rate of growth of the population in 1961.

Solution: Here h = 10. Since the rate of growth of population is $\frac{dy}{dx}$, we have to find $\frac{dy}{dx}$ at $x = 1961$, which lies nearer to the end value of the table. Hence we choose the origin at $x = 1971$ and we use Newton's backward interpolation formula for derivative, where $p = \frac{x - x_0}{10} = \frac{1961 - 1971}{10} = -1.$

The backward difference table.

Year x	Population y	∇y	∇²y	∇³y	∇⁴y
1931 1941 1951 1961 1971	40.62 60.80 79.95 103.56 132.65	20.18 19.15 23.61 **29.09**	-1.03 4.46 **5.48**	5.49 **1.02**	**-4.47**

So by Newton's backward interpolation formula for derivative, we have

$$\frac{dy}{dx}=\left[\nabla y_4+\frac{(2p+1)}{2}\nabla^2 y_4+\frac{(3p^2+6p+2)}{6}\nabla^3 y_4+\frac{(2p^3+9p^2+11p+3)}{12}\nabla^4 y_4+\ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{p=-1}=\frac{1}{10}\left[\ 29.09-\left(\frac{1}{2}\right)(5.48)+\frac{(3(-1)^2+6(-1)+2)}{6}\times 1.02\right.$$
$$\left.+\frac{(2(-1)^3+9(-1)^2+11(-1)+3)}{12}(-4.47)\ \right]$$

$$=\frac{1}{10}\,[29.00-2.74-0.17+0.3725]=2.65.$$

Example: Find $\frac{dy}{dx}$ at x = 0.4 from the following table:

x	0.1	0.2	0.3	0.4
y	1.10517	1.22140	1.34986	1.49182

Solution: Since x = 0.4 lies at the end of the table so we use Newton's backward formula

Here h = 0.1, x_n = 0.4, p = 0

$$\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left[\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots\right]$$

$$=\frac{1}{0.1}\left[0.14196+\frac{1}{2}(0.0135)+\frac{1}{3}(0.00127)\right]$$

$$=\frac{1}{0.1}\left[0.14196+0.00675+0.000423\right]$$

= 1.49133

Example: The distance covered as a function of time by an athlete during his/her run for the 50 meters race is given below

Time (sec)	0	1	2	3	4	5
Distance	0	2.5	8.5	15.5	24.5	36.5

Determine the speed of the athlete at t= 5 sec.

Solution: Since t = 5 lies at the end of table so we will apply Newton's backward formula

t	s	∇	∇^2	∇^3	∇^4	∇^5
0	0					
		2.5				
1	2.5		3.5			
		6.0		-2.5		
2	8.5		1.0		3.5	
		7.0		1.0		3.5
3	15.5		2.0		0.0	
		9.0		1.0		
4	24.5		3.0			
		12.0				
5	36.5					

Here h = 1, $x_n = 5$

By Newton's backward formula, we have

$$\left(\frac{ds}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla s_n + \frac{1}{2}\nabla^2 s_n + \frac{1}{3}\nabla^3 s_n + \frac{1}{4}\nabla^4 s_n + ...\right]$$

$$= \left[12 + \frac{1}{2}(3) + \frac{1}{3}(1) + \frac{1}{4}(0) + \frac{1}{5}(-3.5)\right]$$

$$= \left[12 + 1.5 + 0.33 - 0.7\right]$$

= 13.13

Example: Find $\frac{dy}{dx}$ at x = 0.6 of the following function y = f(x) where

x	0.4	0.5	0.6	0.7	0.8
y	1.5836494	1.7974426	2.0442376	2.3275054	2.6510818

Solution: Since x = 0.6 lies in the middle of the table so we use Striling's formula for differentiation.

Here $x_0 = 0.6$, $x_{-1} = 0.5$, $x_{-2} = 0.4$, $x_1 = 0.7$, $x_2 = 0.8$, $h = 0.1$

p	x	$10^7 y$	$10^7 \Delta y$	$10^7 \Delta^2 y$	$10^7 \Delta^3 y$	$10^7 \Delta^4 y$
-2	0.4	15836494				
			2137932			
-1	0.5	17974426		330018		
			2467950		34710	
0	0.6	20442376		364728		3648
			2832678		38358	
1	0.7	23275054		403086		
			3235764			
2	0.8	26510818				

By Striling's formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2}+\Delta^5 y_{-3}}{3}\right) + \cdots\right\} 10^7 \frac{dy}{dx} =$$

$$\frac{1}{0.1}\left\{\left(\frac{2832678+2467950}{2}\right) - \frac{1}{12}\left(34710 + 38358\right) + \cdots\right\}$$

$$= 26442250$$

$$\frac{dy}{dx} = 2.6442250$$

Example: Find $f'(93)$ from the following table:

x	60	75	90	105	120

f(x)	28.2	38.2	43.2	40.9	37.2

Solution: Since 93 lies near the central point of the table. So we will use striling's formula.

Here $x_0 = 90$, $x = 93$, $h = 15$, $\boldsymbol{p} = \frac{x-x_0}{h} = \frac{93-90}{15} = \frac{3}{15} = \mathbf{0.2}$

p	x	f(x)	Δ	Δ^2	Δ^3	Δ^4
-2	60	28.2				
			10			
-1	75	38.2		-5		
			5		-2.3	
0	90	43.2		-7.3		8.2
			-2.3		5.9	
1	105	40.9		-1.4		
			-3.7			
2	120	37.2				

Putting these values in Striling formula, we get

$$\frac{dy}{dx} = \frac{1}{h}\left\{\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right) + \frac{2P}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \cdots\right\}$$

$$= \frac{1}{15}\left\{\left(\frac{5-2.3}{2}\right) + (0.2)(-7.3) + \frac{3(0.2)^2-1}{3!}\left(\frac{5.9-2.3}{2}\right) + \frac{4(0.2)^3-2(0.2)}{4!}(8.2)\right\}$$

$$= \frac{1}{15}\{1.35 - 1.46 - 0.26406 - 0.1257\}$$

=-0.03331

SAQ 1: A rod is rotating in a plane about one of its ends. If the following table gives the angle "x" in radian through which the rod has turned for different values of time t seconds. Find its angular velocity and acceleration at t = 0.7 sec. & 0.6 sec.

t (sec)	0	0.2	0.4	0.6	0.8	1.0
x (in radian)	0	0.12	0.48	0.10	2.0	3.20

SAQ 2: Find $\mathbf{f}'(0.04)$ from the following table:

x	0.01	0.02	0.03	0.04	0.05	0.06
f(x)	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

SAQ 3: Find $\mathbf{f}'(7.50)$ from the following table:

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
f (x)	0.193	0.195	0.198	0.201	0.203	0.206	0.208

10.3 Derivatives using Newton's General Interpolation (or divided difference) Formula

Consider the Newton's general interpolation formula

$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2)$

$+ (x-x_0)(x-x_1)\,(x-x_2)f(x_0, x_1, x_2, x_3) + \ldots$

Differentiate w r t 'x' successively thrice.

$f'(x) = f(x_0, x_1) + [(x-x_0) + (x-x_1)]\, f(x_0, x_1, x_2) + [(x-x_0)\,(x-x_1) + (x-x_1)\,(x-x_2) + [(x-x_2)(x-x_0)]f(x_0, x_1, x_2, x_3) + \ldots.$

$f''(x) = 2f(x_0, x_1, x_2) + 2[(x-x_0) + (x-x_1) + (x-x_2)] \times f(x_0, x_1, x_2)$

$f'''(x) = 2[3f(x_0, x_1, x_2, x_3)].$

We denote $f(x_0, x_1)$ *as* $[x_0, x_1]$ and the similarly the higher order differences.

Example

Find $f'(3)$, $f''(7)$ *and* $f'''(12)$ from the following data.

x	2	4	5	6	8	10
y	10	96	196	350	868	1746

Solution: Since the values of x are not equidistant we shall construct the divided difference table.

x	y = f(x)	First divided difference	Second divided difference	Third divided difference
2	10			
4	96	$[x_0, x_1] = 43$		
5	196	$[x_1, x_2] = 100$	$[x_0, x_1, x_2] = 19$	
6	350	$[x_2, x_3] = 154$	$[x_1, x_2, x_3] = 27$	$[x_0, x_1, x_2, x_3]=2$
8	868	$[x_3, x_4] = 259$	$[x_2, x_3, x_4] = 35$	$[x_1, x_2, x_3, x_4] = 2$
10	1746	$[x_4, x_5] = 439$	$[x_3, x_4, x_5] = 45$	$[x_2, x_3, x_4, x_5] = 2$

Substituting the values in the derivatives, we get

$f'(3) = 43 + [1 + (-1)]19 + [(1)(-1)+ (-1)(-2) + (-2)(1)]2 = 41$.

$f''(7) = 2 (19) + 2[5 + 3 + 2](2) = 78$ and

$f'''(12) = 6(2) = 12$.

Example: Given the following pairs of x and y

x	1	2	4	8	10
y= f(x)	0	1	5	21	27

Determine $\frac{dy}{dx}$ at x = 4.

Solution: Since the values of x are not equally spaced so by Newton's divided difference formula, we have

$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 8, x_4 = 10, x = 4$

$f'(x) = f(x_0, x_1) + [(x-x_0) + (x-x_1)] f(x_0, x_1, x_2) + [(x-x_0)(x-x_1) + (x-x_1)(x-x_2) + [(x-x_2)(x-x_0)]f(x_0, x_1, x_2, x_3) + [(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]f(x_0,x_1,x_2,x_3,x_4)+\ldots$

Where $f(x_0,x_1) = 1$, $f(x_0,x_1,x_2) = 0.33$, $f(x_0,x_1,x_2,x_3) = 0$, $f(x_0,x_1,x_2,x_3,x_4) = -0.0069$

x	f(x)	I divided difference	II divided difference	III divided difference	IV divided difference
1	0				

		1			
2	1		0.33		
		2		0	
4	5		0.33		0.0069
		4		0.0625	
8	21		-0.1667		
		3			
10	27				

$f'(4)$ = 1 + [(4-1)+(4-2)](0.33)+ [(4-1)(4-2)+(4-1)(4-4)+(4-2)(4-4)](0)

+ [(4-1)(4-2)(4-4)+(4-1)(4-2)(4-8)+(4-1)(4-4)(4-8)+ (4-2)(4-4)(4-8)](-0.0069)

= 1+ (3+2)(0.33)+ [(3)(2)(-4)](-0.0069)

= 1+1.65 + 0.1656

= 2.8321

Example: Find $f'(6)$ from the following table

x	0	1	3	4	5	7	9
f(x)	150	108	0	-54	-100	-144	-84

Solution:

Here x = 6, x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 4, x_5 = 5, x_6 = 7, x_7 = 9.

By Newton's divided difference formula, we have

$f'(x) = f(x_0, x_1) + [(x-x_0) + (x-x_1)]\, f(x_0, x_1, x_2) + [(x-x_0)(x-x_1) + (x-x_1)(x-x_2) + (x-x_2)(x-x_0)]f(x_0, x_1, x_2, x_3) + [(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]f(x_0,x_1,x_2,x_3,x_4) + \ldots$

x	f(x)	I Divided Difference	II Divided Difference	III Divided Difference	Iv Divided Difference
0	150				

		-42			
1	108		-4		
		-54		1	
3	0		0		0
		-54		1	
4	-54		4		0
		-46		1	
5	-100		8		0
		-22		1	
7	-144		13		
		30			
9	-84				

$f'(6)$ = 1 + [(6-1)+(6-0)](-4)+ [(6-0)(6-1)+(6-0)(6-3)+(6-1)(6-3)](1)

= - 42- 44 + 63

= -23

Example: From the following table, find the value of $f'(10)$

x	3	5	11	27	34
f(x)	-13	23	899	17315	35606

Solution:

x	f(x)	I Divided Difference	II Divided difference	III Divided Difference	IV divided Difference
3	-13				
		18			
5	23		16		
		146		1	

11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

Here $x = 10, x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27, x_4 = 34$

By Newton's divided difference formula, we have

$f'(x) = f(x_0, x_1) + [(x-x_0) + (x-x_1)]\, f(x_0, x_1, x_2) + [(x-x_0)(x-x_1) + (x-x_1)(x-x_2) + [(x-x_2)(x-x_0)]f(x_0, x_1, x_2, x_3) + [(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)]f(x_0,x_1,x_2,x_3,x_4) + \ldots$

$= 18 + [(10-3)+ (10-5)](16) + [(10-3)(10-5)+ (10-5)(10-11)+ (10-11)(10-3)](1)$

$= 18 + [(7)+(5)](16) + [(7)(5) + (5)(-1) + (-1)(7)](1)$

$= 18 + 192 + 23$

$= 233$

SAQ 4: Find $f'(5)$form the following table:

x	0	2	3	4	7	9
f(x)	4	26	58	112	466	922

10.3.1 Maxima and minima of the interpolating Polynomial

The derivative of a function y = f(x) given by a table of values is defined to be the derivatives of the interpolating polynomial, the maxima and minima of f(x) can be obtained by equating the first derivative to zero.

Newton's forward interpolation formula is

$$y = \{y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\}$$ where $x = x_0 + ph$.

$$\frac{dy}{dp} = \Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6} + \Delta^3 y_n + \ldots$$

For y to be a maximum or minimum $\frac{dy}{dp} = 0$.

Therefore, $\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \ldots = 0$

(neglecting the higher order differences).

Or, $\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$

The value of p can be obtained by solving the above quadratic equation

Then if for the above obtained p values $\frac{d^2y}{dp^2} = -ve$, so p is maximum value

And if $\frac{d^2y}{dp^2} = +ve$, so p is minimum value

Note: If the interval is not equally spaced, then we will use divided difference formula or Lagrange's interpolation formula

Example

Find the maximum and minimum value of y from the following table.

x	0	1	2	3	4	5

y	0	$\frac{1}{4}$	0	$\frac{9}{4}$	16	$\frac{225}{4}$

Solution: Here *h =1*. Newton's forward difference formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots, \text{ where } x = x_0 + ph.$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y_0 + \ldots\right]$$

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
0	0					
		0.25				
1	0.25		-0.50			
		-0.25		3		
2	0		2.50		6	
		2.25		9		0
3	2.25		11.50		6	
		13.75		15		
4	16		26.50			
		40.25				
5	56.25					

Choosing the origin at $x_0 = 0$, p = $\frac{x-0}{1} = x$

$$\frac{dy}{dx} = \frac{1}{1}\left[0.25 + \frac{(2p-1)}{2}(-0.50) + \frac{(3p^2-6p+2)}{6}\times 3 + \frac{(4p^3-18p^2+22p-6)}{24}\times 6\right]$$

$= p^3 - 3p^2 + 2p.$

Now $\frac{dy}{dx} = 0 \Rightarrow 4p^3 - 12p^2 + 8p = 0 \Rightarrow 4p(p-2)(p-1) = 0 \Rightarrow p = 0, 1, 2.$

$\frac{d^2y}{dx^2}$ =12p²-24p+8 = 3p²-6p+2

At p = 0, $\frac{d^2y}{dx^2} = 2$ which is positive.

At p = 1, $\frac{d^2y}{dx^2} = -4$ which is negative.

At p = 2, $\frac{d^2y}{dx^2} = 8$ which is positive.

Therefore *y* is maximum at *p = 1* and minimum at *p = 0* and *p = 2*.

The maximum value *y* at *p = 1* (that is, *x =1*) is
y(1) = 0.25.

Example: For what value of x, y is maximum / minimum using the data given below:

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

Solution: Since the values are equally spaced so by Newton's forward difference formula, we have

x	y	Δ	Δ^2	Δ^3	Δ^4
3	0.205				
		0.035			
4	0.240		-0.016		
		0.019		0	
5	0.259		-0.016		0.001
		0.003		0.001	
6	0.262		-0.015		0
		-0.012		0.001	
7	0.250		-0.014		
		-0.026			
8	0.224				

Here p = $\frac{x-x_0}{h}$ = x-3

For finding maximum or minimum we require $\frac{dy}{dx} = 0$. So it is enough to show $\frac{dy}{dp} = 0$

i.e., $\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$

(neglecting higher order differences)

$$\left(\frac{1}{2}(0)\right)p^2 + ((-0.016) - 0.000)p + \left(0.035 - \frac{1}{2}(-0.016) + \frac{1}{3}(0.000)\right) = 0$$

$-0.016\,p + 0.043 = 0$ => p = 0.043 / 0.016 = 2.6875

But p = x-3, => x-3 = 2.6875

x = 2.6875 + 3 = 5.6875

Also, $\frac{d^2y}{dp^2} = \left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{(12p^2 - 36p + 22)}{24}\Delta^4 y_0 + \ldots\right]$

$$= \left[-0.016 + (p-1)(0) + \frac{(6p^2 - 13p + 11)}{12}(0.001) + \ldots\right]$$

$$= \left[\frac{-0.192 + 0.006p^2 - 0.013p + 0.011)}{12}\right]$$

$$= \left[\frac{-0.192 + 0.006(x-3)^2 - 0.013(x-3) + 0.011)}{12}\right]$$

$$= \left[\frac{-0.192 + 0.006x^2 - 0.049x - 0.106}{12}\right]$$

$$= \left[\frac{-0.192 + 0.006(5.6878)^2 - 0.049(5.6878) - 0.106}{12}\right]$$

= -ve

So x = 55.6878 is maximum point

Thus, the max. value of y is 0.25425

Example: Find x for which y is maximum from the following table. Also find the value of y

x	y	Δ	Δ^2	Δ^3	Δ^4

1.0	0				
		0.128			
1.2	0.128		0.288		
		0.416		0.05	
1.4	0.544		0.738		0
		0.754		0.05	
1.6	1.298		0.388		
		1.142			
1.8	2.44				

For finding maximum or minimum we require $\frac{dy}{dx} = 0$. So it is enough to show $\frac{dy}{dp} = 0$

i.e.,$\left(\frac{1}{2}\Delta^3 y_0\right) p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$

$$\left(\frac{1}{2}(0.05)\right) p^2 + (0.288 - 0.05)p + \left(0.128 - \frac{1}{2}(0.288) + \frac{1}{3}(0.05)\right) = 0$$

$$0.025p^2 + 0.238p + \left(0.128 - \frac{1}{2}(0.288) + \frac{1}{3}(0.05)\right) = 0$$

$$0.025p^2 + 0.238p = 0$$

p(0.025p + 0.238)= 0

⇨ p = 0 & p = -0.238/0.025 = -9.52

Also, p = $\frac{x - x_0}{h}$ = $\frac{x-1}{0.2}$

implies,

For p = 0 , x = 1

For p = -9.52, $\frac{x-1}{0.2}$ = -9.52

x = -18

Now,

$$\frac{d^2y}{dp^2} = \left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{(12p^2-36p+22)}{24}\Delta^4 y_0 + \ldots\right]$$

$$= \left[0.288 + (p-1)\ (0.05)\ \right]$$

= 0.288 -0.05, when p = 0

= 2.38

which is positive

so, p = 0 is minimum value

when p = -9.52,

$$\frac{d^2y}{dp^2} = \left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{(12p^2-36p+22)}{24}\Delta^4 y_0 + \ldots\right]$$

$$= \left[0.288 + (p-1)\ (0.05)\ \right]$$

= 0.288 – 0.526

= -0.238

Which is negative so p = -9.52, x = -18 gives the max value

And the maximum value of y is 3.5887

SAQ 5: Find x for which y is maximum and find this value of y

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

SAQ 6: Consider the data

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Then (i) the value $y'(0)$ = __________

(ii) the value $y''(0)$ = ————————.

10.4 Difference Equations

Difference calculus also forms the basis of differential equations which are in the theory of probability, in the study of electrical networks, in statistical problems and indeed in all situations in which sequential relation exists at various discrete values of the independent variable. Just as the subject of Differential equations grew out of Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for treatment of discontinuous processes.

10.4.1 Definition:

A difference equation is an equation which involves the relationship between independent variables, dependent variables and the successive difference of the dependent variables.

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$ ….(1)

$\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$ … (2)

$\Delta^2 y_x + 8\Delta y_x + y_x = cos\pi x$ … (3)

are difference equations.

An alternative way of writing a difference equation is as under.

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$

Therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \quad \text{…..(4)}$$

Also since $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$ …..(5)

Therefore (2) takes the form:

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \quad \text{……….(6)}$$

Other ways of writing difference equation is

We know $\Delta = E - 1$

$$\therefore \ \Delta^r = (E-1)^r$$

So from (3), we have

$$\Delta^2 y_x = (E-1)^2 y_x = (E^2 - 2E + 1)y_x = E^2 y_x - 2E y_x + y_x$$

$$= y_{x+2} - 2y_{x+1} + y_x$$

Also, $\Delta y_x = (E-1)y_x = Ey_x - y_x = y_{x+1} - y_x$

$\therefore \quad \Delta^2 y_x + 8\Delta y_x + y_x = y_{x+2} - 2y_{x+1} + y_x + 8(y_{x+1} - y_x) + y_x = cos\pi x$

$\Rightarrow \quad y_{x+2} + 6y_{x+1} + 6y_x = cos\pi x$

Quite often, difference equations are met under the name of recurrence relations.

10.4.2 Order of difference equation

Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus (4) above is the second order, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2$$

and (6) is of the third order, for

$$\frac{(n-2)-(n-1)}{1} = 3$$

10.4.3 Degree of difference equation

The degree of difference equation is defined as the highest power of y.

Example: $y_{x+2} + 6y_{x+1} + 6y_x = 2$ has degree 2 and $5y_{x+1} - x + 2 = 8$ has degree 1.

Note: While finding the order and degree of differential equation, it must always be expressed in a form free of Δs, for the highest power of Δ does not give order or degree of the difference equation.

10.4.4 Definition:

i) **Solution of difference equation** is an expression for $y_{(n)}$ which satisfies the given difference equation.

ii) The **general solution** of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

iii) A **particular solution** (or particular integral) is that solution which is obtained from the general solution by giving particular values to the constants. It is denoted by y_p.

10.4.5 Formation of difference equations:

The following examples illustrate the way in which difference equations arise and are formed.

Example:

Form the difference equation corresponding to the family of curves

$y = ax + bx^2$ …(i)

We have $\Delta y_x = a\,(\Delta x) + b\Delta(x^2)$

$= a(x + 1 - x) + b[(x + 1)^2 - x^2]$

= a+b(2x+1) ….(ii)

And $\Delta^2 y_x = 2b\Delta x = 2b(x + 1 - x) = 2b$ ….(iii)

To eliminate a and b, we have from (iii)

$$b = \frac{1}{2}\Delta^2 y$$

and from (ii) $a = \Delta y - b(2x+1) = \Delta y - \frac{1}{2}\Delta^2 y(2x+1)$

Substituting these values of a and b in (i), we get

$$y = [\Delta y - \frac{1}{2}\Delta^2 y.(2x+1)]x + \frac{1}{2}\Delta^2 y.x^2$$

or $(x^2 + x)\Delta^2 y - 2x\ \Delta y + 2y = 0$

This is the desired difference equation which may be written in terms of E as

$$(x^2 + x)E^2 y_x - (2x)E y_x + 2y = 0$$

$$\left(x^2 + x\right)y_{x+2} - \left(2x^2 + 4x\right)y_{x+1} + \left(x^2 + 3x + 2\right)y_x = 0$$

Example: Find the difference equation satisfied by $y_x = ax^2 + bx + 7$

Solution: We have $y_x = ax^2 + bx + 7$ (i)

$$\Delta y_x = a\Delta x^2 + b\Delta x$$
$$= a((x+1)^2 - x^2) + b\,(x+1-x)$$
$$= a(x^2+1+2x-x^2) + b = (2x+1)a + b$$
$$\Delta^2 y_x = 2a(\Delta x) = 2a(x+1-x) = a$$

⇨ $a = \frac{\Delta y_x}{2}$

⇨ $b = \Delta y_x - (2x+1)\frac{\Delta y_x}{2} = (1-2x)\frac{\Delta y_x}{2}$

Now, put the values of a, b in (i), we get

$$y_x = ax^2 + bx + 7 = \frac{\Delta y_x}{2}x^2 + (1-2x)\frac{\Delta y_x}{2} + 7$$
$$= \frac{(x^2-2x+1)\Delta y_x + 14}{2}$$

Example: From $y_n = A2^n + B(-3)^n$, derive a differential equation by eliminating the constants.

We have $y_n = A.2^n + B(-3)^n$

$y_{n+1} = 2A.\,2^n - 3B(-3)^n$

and $y_{n+2} = 4A.\,2^n + 9B(-3)^n$

Eliminating A and B, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

or $y_{n+2} + y_{n+1} - 6y_n = 0$

10.4.6 Definition: (Linear difference equation)

A linear difference equation is that in which y_{n+1}, y_{n+2} etc. occur to the first degree only and are not multiplied together.

A linear difference equation with constant coefficients is of the form

$y_{n+k} + a_1 y_{n+k-1} + a_2 y_{n+r-2} + \ldots + a_k y_n = f(n)$ …(7)

where $a_1, a_2, \ldots a_r$ are constants.

Elementary Properties:

(i) If $u_1(n), u_2(n), \ldots, u_k(n)$ be k independent solutions of the equation

$y_{n+k} + a_1 y_{n+k-1} \ldots + a_k y_n = 0$ ….(8)

Then its complete solution is

$$U_n = c_1 u_1(n) + c_2 u_2(n) + \ldots + c_k u_k(n)$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

(ii) If V_n is a particular solution of (7), then the complete solution of (7) is

Thus complete solution (denoted as C.S.) of (7) is

$$y_n = \text{C.F.} + \text{P.I.}$$

The part U_n is called Complementary function (C.F.) and the V_n is called the *particular integral* (P.I) of (7).

Rules to find the complementary function:

Consider the homogeneous equation of order k

$y_{n+k} + a_1 y_{n+k-1} \ldots + a_k y_n = 0$ …..(9)

then its auxiliary equation can be obtained by putting $y_n = x^n$, for all n

So eq (8) becomes

$x^{n+k} + a_1 x^{n+k-1} + a_2 x^{n+k-2} + \cdots + a_k x^n = 0$ …..(10)

Or, $x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k = 0$ …..(11)

Eq (11) is characteristic equation of eq.(9)

Let $r_1, r_2, \ldots, r_k$ are roots of eq(11). Then $r_1^n, r_2^n, \ldots, r_k^n$ are all solutions of eq (9)

Case 1: If $r_1, r_2, \ldots, r_k$ are all real and distinct roots eq (11), then the solution of eq (9) is

$$y_n = c_1 r_1^n + c_2 r_2^n + \ldots + c_k r_k^n$$

Case 2: If $r_1 = r_2$, then the solution of (9) will be

$$y_n = (c_1 + c_2 x) r_1^n + c_3 r_2^n + \ldots + c_k r_k^n$$

Case 3: If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = amp(\alpha + i\beta) = tan^{-1}(\frac{\beta}{\alpha})$, then the solution of (9) will be

$$y_n = (c_1\, Cos\theta n + c_2\, Sin\theta n) r^n + c_3 r_2^n + \ldots + c_k r_k^n$$

Case 4: If the roots are repeated complex roots, then

$$y_n = \left((c_1 + c_2 x) Cos\theta + (c_3 + c_4 x)\, Sin\theta\right) r^n + c_5 r_2^n + \ldots + c_k r_k^n$$

Example: Solve $y_{x+1} - 2y_x\, Cos\alpha + y_{x-1} = 0$

Solution: The auxiliary equation of $y_{x+1} - 2y\, Cos\alpha + y_{x-1} = 0$

$$(E^2 - 2E\ Cos\alpha + 1)y_{x-1} = 0$$

i.e.,$x^2 - x2Cos\alpha + 1 = 0$

$$x = \frac{2Cos\alpha\ \pm \sqrt{4cos^2\alpha - 4}}{2} = Cos\alpha\ \pm i\ Sin\alpha$$

Here $r = \sqrt{Cos^2\alpha + sin^2\alpha} = 1$ & $\theta = tan^{-1}\left(\frac{\sin\alpha}{Cos\alpha}\right) = \alpha$

So, the solution is

$$y_{x-1} = (c_1\ Cos\alpha x + c_2\ Sin\alpha x)$$

Example: Solve the difference equation $y_{n+2} - 2y_{n+1} - 8y_n = 0$

Solution: The equation can be written as

$$(E^2 - 2E - 8)y_n = 0$$

The auxiliary equation is $m^2 - 2m - 8 = 0$

So the roots are m = -2,4

The solution is $y_n = c_1(-2)^n + c_2 4^n$

Example: Solve the difference equation $y_{n+3} - y_{n+2} - 8y_{n+1} - 12y_n = 0$

Solution: The equation can be written as

$$(E^3 + E^2 - 8E - 12)y_n = 0$$

The auxiliary equation is $m^3 + m^2 - 8m - 12 = 0$

So the roots are m = -2,-2,3

The solution is $y_n = (c_1 + c_2 n)(-2)^n + c_3 3^n$

SAQ 7: Form the difference equations generated by $y_x = ax + b2^x$.

SAQ 8: Write the difference equation $\Delta^3 y_n + \Delta^2 y_n + \Delta y_n + y_n = 0$ in terms of E

SAQ 9: Form the difference equation from the equation $y = Ax^2 - Bx$

SAQ 10: Solve the difference equation $y_{x+2} - 8y_{x+1} + 15y_x = 0$

SAQ 11: Solve the difference equation $y_{x+2} + y_{x+1} + y_x = 0$

SAQ 12: Solve the difference equation $y_{n+2} - 4y_{n+1} + 13y_n = 0$

Rule to find Particular solution:

An equation of the form

$y_{n+k} + a_1 y_{n+k-1} \ldots + a_k y_n = f(n)$

or$(E^k + a_1 E^{k-1} + \cdots + a_k I)y_n = f(n)$ …(12)

$$\varphi(E)y_n = f(n)$$

Where $a_1, a_2, \ldots, a_k$ are constants, is known as non-homogeneous linear difference equation with constant coefficient.

The general solution of (12) is

$$y_n = C.I + P.I = y_c + y_p$$

Complementary solution y_c is the general solution of the homogeneous equation that is left hand side of eq (12) and the particular integral (P.I.) is obtained from $\frac{1}{\varphi(E)}f(n)$

Case I: when $f(n) = a^n$

$\therefore P.I = \frac{1}{\varphi(E)}f(n) = \frac{1}{\varphi(a)}a^n$ provided $\varphi(a) \neq 0$

If $\varphi(a) = 0$, then $\therefore (E-a)^k y_n = a^n$

$$\therefore P.I = \frac{1}{(E-a)^k}a^n = \frac{n(n-1)(n-2)\ldots(n-(k-1))}{k!}a^{n-k}$$

Case II: when $f(n) = n^k$

$$\therefore P.I = \frac{1}{\varphi(E)}f(n) = \frac{1}{\varphi(1+\Delta)}n^k = [\varphi(1+\Delta)]^{-1}n^k$$

i.e., we expand $[\varphi(1+\Delta)]^{-1}$ in ascending power of Δ

Case III: when $f(n) = Sin\ an\ or\ Cos\ an$

$$\therefore P.I = \frac{1}{\varphi(E)}f(n) = \frac{1}{\varphi(E)}\operatorname{Sin} an$$

$$= \frac{1}{\varphi(E)}\frac{e^{ian} - e^{-ian}}{2i} = \frac{1}{2i}\left(\frac{(e^{ia})^n}{\varphi(E)} - \frac{(e^{-ia})^n}{\varphi(E)}\right)$$

$$= \frac{1}{2i}\left(\frac{e^{ian}}{\varphi(e^{ia})} - \frac{e^{-ian}}{\varphi(e^{-ia})}\right)$$

Similarly, $\therefore P.I = \frac{1}{\varphi(E)}\operatorname{Cos} an = \frac{1}{2i}\left(\frac{e^{ian}}{\varphi(e^{ia})} + \frac{e^{-ian}}{\varphi(e^{-ia})}\right)$

provided $\varphi(e^{ia}) \neq 0, \varphi(e^{-ia}) \neq 0$

Case IV: when $f(n) = a^n R(n)$,where R(n)is a polynomial in n.

$$\therefore P.I = \frac{1}{\varphi(E)} a^n R(n) = a^n \frac{1}{\varphi(aE)} R(n)$$

Example: Solve the difference equation $y_{n+2} - 2y_{n+1} + 5y_n = 2.3^n - 7^n$.

Solution: To find the complimentary solution we have

$$y_{n+2} - 2y_{n+1} + 5y_n = 0$$

It can be written as

$$(E^2 - 2E + 5)y_n = 0$$

Its auxiliary equation is

$$m^2 - 2m + 5 = 0$$

The roots are m = $1 \pm i2$

So, $r = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $\theta = tan^{-1}2$

$$C.I = y_c = 5^{\frac{n}{2}}(c_1\, Cos\, n\theta + c_2 \sin n\theta), where\ \theta = tan^{-1}2$$

Now, the particular integral (P.I.) = $y_p = \frac{1}{E^2-2E+5}(2.3^n - 7^n)$

$$= 2\frac{1}{E^2-2E+5}.3^n - \frac{1}{E^2-2E+5}7^n$$

$$= 2\frac{1}{3^2-2.3+5}.3^n - \frac{1}{7^2-2.7+5}7^n$$

$$= 2\frac{1}{8}.3^n - \frac{1}{40}7^n$$

$$= \frac{1}{4}.3^n - \frac{1}{40}7^n$$

Hence, the solution is $y_n = C.I. + P.I. = y_c + y_p$

$$= 5^{\frac{n}{2}}(c_1\, Cos\, n\theta + c_2 \sin n\theta) + \frac{1}{4}.3^n - \frac{1}{40}7^n$$

Example: Find a particular solution of the equation $y_{n+2} - 7y_{n+1} + 12y_n = 3n^2 + 2n + 2$.

Solution: Particular integral (P.I.) = $y_p = \frac{1}{E^2-7E+12}(3n^2 + 2n + 2)$

We can write $E^2 - 7E + 12 = (2 - \Delta)(3 - \Delta)$

Now, we will write $3n^2 + 2n + 2$ in factorial notation

$$3n^2 + 2n + 2 = An^{(2)} + Bn^{(1)} + C$$

$$= An(n-1) + Bn + C \qquad ....(13)$$

Now, put n = 0 in (13), we get $C = 2$

Put n = 1 in (13), we get

$$B + 2 = 7, \quad => \quad B = 5$$

Put n = 2 in (13), we get

$$2A + 10 + 2 = 18, \quad => \quad A = 3$$

So $3n^2 + 2n + 2 = 3n^{(2)} + 5n^{(1)} + 2$

$$\therefore y_p = \frac{1}{(2-\Delta)(3-\Delta)}(3n^{(2)} + 5n^{(1)} + 2)$$

$=\frac{1}{6}(1-\frac{\Delta}{2})^{-1}(1-\frac{\Delta}{3})^{-1}(3n^{(2)} + 5n^{(1)} + 2)$

$= \frac{1}{6}(1+\frac{\Delta}{2}+\frac{\Delta^2}{4}+\cdots)(1+\frac{\Delta}{3}+\frac{\Delta^2}{9}+\cdots)(3n^{(2)} + 5n^{(1)} + 2)$

$= \frac{1}{6}(1+\frac{5\Delta}{6}+\frac{19\,\Delta^2}{36}+\cdots)(3n^{(2)} + 5n^{(1)} + 2)$

$= \frac{1}{2}n^{(2)} + \frac{5}{6}n^{(1)} + \frac{1}{3} + \frac{5}{6}n^{(1)} + \frac{25}{36} + \frac{19}{36}$

$= \frac{1}{2}n(n-1) + \frac{10}{6}n + \frac{1}{3} + \frac{25}{36} + \frac{19}{36}$

$= \frac{1}{2}n^2 + \frac{7}{6}n + \frac{14}{9}$

Example: Find the particular solution of $8y_{n+2} - 6y_{n+1} + y_n = 5\,\mathrm{Sin}\frac{n\pi}{2}$.

Solution: Particular integral (P.I.) = $y_p = \frac{1}{8E^2-6E+1}(5\,\mathrm{Sin}\frac{n\pi}{2})$

$= \frac{1}{8E^2-6E+1}5(\frac{e^{i(\frac{n\pi}{2})} - e^{-i(\frac{n\pi}{2})}}{2i})$

$= \frac{5}{2i}\frac{1}{8E^2-6E+1}((e^{\frac{i\pi}{2}})^n - ((e^{-\frac{i\pi}{2}})^n)$

$= \frac{5}{2i}\frac{1}{8E^2-6E+1}((i)^n - ((-i)^n)$

$= \frac{5}{2i}[\frac{1}{8E^2-6E+1}(i)^n - \frac{1}{8E^2-6E+1}(-i)^n]$

$= \frac{5}{2i}[\frac{1}{8i^2-6i+1}(i)^n - \frac{1}{8(-i)^2-6(-i)+1}(-i)^n]$

$= \frac{5}{2i}[-\frac{1}{7+6i}(i)^n + \frac{1}{7-6i}(-i)^n]$

$= \frac{5}{2i}[\frac{-(7-6i)(\cos\frac{\pi}{2}+i\,Sin\,\frac{\pi}{2})^n+(7+6i)(\cos\frac{\pi}{2}-i\,Sin\,\frac{\pi}{2})^n}{(7-6i)(7+6i)}]$

$$= \frac{5}{2i}[\frac{-(7-6i)(\cos\frac{n\pi}{2}+i\ Sin\ \frac{n\pi}{2})+(7+6i)(\cos\frac{n\pi}{2}-i\ Sin\ \frac{n\pi}{2})}{49+36}]$$

$= \frac{1}{34i}[-7\cos\frac{n\pi}{2} - 7i\ Sin\ \frac{n\pi}{2} + 6i\ Cos\ \frac{n\pi}{2} +$

$$6\ Sin\frac{n\pi}{2} + 7\ Cos\ \frac{n\pi}{2} - 7i\ Sin\ \frac{n\pi}{2} +$$

$6i\ Cos\frac{n\pi}{2} - 6Sin\ \frac{n\pi}{2}]$

$$= -\frac{i}{34}\{-14iSin\frac{n\pi}{2} + 12i\ cos\frac{n\pi}{2}\}$$

$$= \frac{1}{17}\{6\ cos\frac{n\pi}{2} - 7\ Sin\frac{n\pi}{2}\}$$

Example: Solve of $y_{n+2} - 2\ Cos\alpha\ y_{n+1} + y_n = Cos\ \alpha n.$

Solution: The given difference equation can be written as

$$(E^2 - 2Cos\alpha\ E + 1)\ y_n = Cos\ \alpha n$$

The auxiliary equation is

$m^2 - 2Cos\alpha\ m\ + 1 = 0$

The roots are $(Cos\alpha \pm i\ Sin\alpha)$, r = 1, $\theta = \alpha$

Thus, the complimentary solution is $A\cos n\ \alpha + B\sin n\alpha$

The particular solution is given by

Particular integral (P.I.)

$$y_p = \frac{1}{(E^2 - 2Cos\alpha\ E + 1)}(Cos\ \alpha n)$$

$$= \frac{1}{(E^2-2(\frac{e^{i\alpha}+e^{-i\alpha}}{2})\ E+1}(\frac{e^{i\alpha n}+e^{-i\alpha n}}{2})$$

$$=\frac{1}{(E^2-(e^{i\alpha}+e^{-i\alpha})E+1}(\frac{e^{i\alpha n}+e^{-i\alpha n}}{2})$$

$$= \frac{1}{2}\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}((e^{i\alpha})^n + (e^{-i\alpha})^n)$$

$$= \frac{1}{2}\{\frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}\left(e^{i\alpha}\right)^n + \frac{1}{(E-e^{i\alpha})(E-e^{-i\alpha})}\left(e^{-i\alpha}\right)^n\}$$

$$= \frac{1}{2}\{\frac{1}{(E-e^{i\alpha})(e^{i\alpha}-e^{-i\alpha})}\left(e^{i\alpha}\right)^n + \frac{1}{(e^{-i\alpha}-e^{i\alpha})(E-e^{-i\alpha})}\left(e^{-i\alpha}\right)^n\}$$

$$= \frac{1}{2}\{\frac{1}{(E-e^{i\alpha})2i\ Sin\alpha}\left(e^{i\alpha}\right)^n + \frac{1}{(-2i\ Sin\alpha)(E-e^{-i\alpha})}\left(e^{-i\alpha}\right)^n\}$$

$$= \frac{1}{4i\ Sin\alpha}\{\frac{1}{(E-e^{i\alpha})}\left(e^{i\alpha}\right)^n - \frac{1}{(E-e^{-i\alpha})}\left(e^{-i\alpha}\right)^n\}$$

$$= \frac{1}{4i\ Sin\alpha}\{\ n\left(e^{i\alpha}\right)^{n-1} - n\left(e^{-i\alpha}\right)^{n-1}\}$$

$= \frac{n}{4i\ Sin\alpha}\{e^{i(n-1)\alpha} - e^{-i(n-1)\alpha}\}$

$= \frac{n}{4i\ Sin\alpha} 2i\ Sin(n-1)\alpha$

$= \frac{nSin(n-1)\alpha}{2Sin\alpha}$

Hence, $y_n = C.F. + P.I. = A\cos n\,\alpha + B\sin n\alpha + \frac{nSin(n-1)\alpha}{2Sin\alpha}$

Example: Solve $y_{n+2} - 7y_{n+1} - 8y_n = x(x-1)2^x$.

Solution: The equation can be written as

$$(E^2 - 7E - 8)\, y_n = x(x-1)2^x$$

The auxiliary equation is $(m^2 - 7m - 8)$ = 0

The roots are m = 8, -1

The C.F. is $y_c = A8^x + B(-1)^x$

Particular integral (P.I.)

$$y_p = \frac{x(x-1)2^x}{(E^2 - 7E - 8)}$$

$= \ 2^x \frac{x(x-1)}{((2E)^2 - 7(2E) - 8)}$

$= 2^x \frac{x(x-1)}{(4E^2 - 14E - 8)}$

$= 2^x \frac{x(x-1)}{(4(1+\Delta)^2 - 14(1+\Delta) - 8)}$

$= 2^x \frac{x(x-1)}{(4\Delta^2 - 6\Delta - 8)}$

$= 2^{x-1} \frac{x(x-1)}{(2\Delta^2 - 3\Delta - 9)}$

$= -\frac{1}{9} 2^{x-1}[\,1 + \left(\frac{2\Delta^2 - 3\Delta}{9}\right) + \cdots]x^{(2)}$

$= -\frac{1}{9} 2^{x-1}[\,1 - \frac{\Delta}{3} + \frac{2\Delta^2}{9}]x^{(2)}$

$= -\frac{1}{9} 2^{x-1}[\,x^{(2)} - \frac{\Delta}{3}x^{(2)} + \frac{2\Delta^2}{9}x^{(2)}]$

$= \ -\frac{1}{9} 2^{x-1}[\,x^{(2)} - \frac{2}{3}x^{(1)} + \frac{4}{9}]$

$= -\frac{1}{9} 2^{x-1}[\,x(x-1) - \frac{2}{3}x + \frac{4}{9}]$

$= -\frac{1}{9^2} 2^{x-1}[\frac{9x^2 - 15x + 4}{9}]$

Hence the solution is $y_n = C.F. + P.I. = A8^x + B(-1)^x - \frac{1}{9^2} 2^{x-1}[\frac{9x^2 - 15x + 4}{9}]$

SAQ 13: Find the particular solution of the equation

$$y_{n+3} - 5y_{n+2} + 7y_{n+1} - 3y_n = n^2 + 4n + 1$$

SAQ 14: Find the solution of

$$y_{x+1} - ay_x = \cos nx$$

SAQ 15: Solve $y_{x+2} - 8y_{x+1} + 16y_x = 4^x$

10.5 Summary

In this unit, we developed formulas for differentiation from an interpolating polynomial. Numerical differentiation techniques may be used to obtain the derivative of continuous as well as tabulated functions. We have illustrated the derivation with Newton's forward and backward difference interpolation formulas to obtain derivatives, but numerical techniques provide only approximations to derivatives.

10.6 Terminal Questions

1. Calculate the first and second derivatives at the point x = 2.2 and x = 2.0, of the function tabulated in the following

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

2. Find first derivatives of y at x = 0.5 from the following data:

x	0	1	2	3	4
y	1	1	15	40	85

3. From the following data obtain the first and second derivatives of y = $\log_e x$ at x = 500, x = 550, x = 520.

x	500	510	520	530	540	550
y = $\log_e x$	6.2146	6.2344	6.2538	6.2729	6.2916	6.3099

4. Given the following data, find the rate of change of y w.r.t. x at x = 2.

x	2	4	6	8	10
y	105	42.7	25.3	16.7	13

5. Form the difference equation generated by $y_x = a2^x + b3^x + c$

6. Find $f'(0.6)$&$f''(0.6)$from the following table:

x	0.4	0.5	0.6	0.7	0.8
f(x)	1.5836	1.7974	2.0442	2.3275	2.6510

7. Compute the value of $f'(3.1)$ & $f''(3.2)$ from the following table:

x	1	2	3	4	5
f(x)	0	1.4	3.3	5.6	8.1

8. Solve the following difference equations:
   i) $4y_{n+2} + 25y_n = 0$
   ii) $y_{n+3} + y_{n+2} - y_{n+1} - y_n = 0,$
   iii) $(\Delta - 2)^2(\Delta - 5)y_n = 0,$
   iv) $(\Delta^2 - 5\Delta + 4)y_n = 0,$

10.7 Answers

Self Assessment Questions

1. At x = 0.7, velocity = 4.496, acceleration = 7.25 rad/sec
2. At x = 0.6, velocity = 3.816, acceleration = 6.75
3. 2.0.2561
4. 0.223
5. 98
6. 5.x= 1.576, y = 0.9999
7. $y'(0) = -27.9$, $y''(0) = 117.67$
8. $(\mathbf{1 - x})\mathbf{yx + 2} - (\mathbf{3x - 2})\mathbf{yx + 1} + \mathbf{2xyx} = \mathbf{0}$
9. $(E^3 - 2E^2 + 2E)y_n = 0$
10. $(x^2 + x)y_{n+2} - (2x^2 + 4x)y_{n+1} + (x^2 + 3x + 2)y_n = 0$

11. $y_x = c_1 3^x + c_2 5^x$

12. $y_x = c_1 Cos\frac{2\pi x}{3} + c_2 Sin\frac{2\pi x}{3}$

13. $y_x = 13^{\frac{n}{2}}(c_1 Cos\,\theta n + c_2 Sin\,\theta n), \theta = tan^{-1}\left(\frac{3}{2}\right)$

14. $-\frac{1}{2}(\frac{1}{12}n^4 + \frac{1}{2}n^3 - \frac{1}{12}n^2 + \frac{3}{2}n + 1)$

15. $y_n = Aa^x + \frac{Cos\,n\,(x-1) - a\,Cos\,nx}{(1-2a\,Cosn + a^2)}$

16. $y_n = (A + Bx)4^x + 4^x\frac{x(x-1)}{32}$

Terminal Questions

1. $\left(\frac{dy}{dx}\right)_{x=2.2}$ =9.0228, $\left(\frac{dy}{dx}\right)_{x=2.0}$ = 7.3896, $\left(\frac{d^2y}{dx^2}\right)_{x=2.2}$ $=8.992$ = 8.992.

2. $\left(\frac{dy}{dx}\right)_{x=0.5}$.= 0.625

3. $\left(\frac{dy}{dx}\right)_{x=500}$ =0.002014, $\left(\frac{d^2y}{dx^2}\right)_{x=500}$ $=-0.00000518$.

4. Rate of change of y w.r.t. “x” is

   $\left(\frac{dy}{dx}\right)_{x=2}$ = – 52.4

5. Refer to Section 7.4.

6. 2.6445, 3.64833

7. 2.16507, 0.39283

8. (i) $y_n = \left(\frac{5}{2}\right)^n (c_1 Cos\frac{n\pi}{2} + c_2 Sin\frac{n\pi}{2})$

   (ii) $y_n = (c_1 + c_2 n)(-1)^n + c_3$

   (iii) $y_n = (c_1 + c_2 n)(3)^n + c_3 6^n$

   (iv) $y_n = c_1(2)^n + c_2 5^n$