Unit 10 Matrices and Determinants

Structure:

10.1 Introduction

The theory of matrices, introduced by French mathematician Cayley in 1957, is presently a powerful tool in the study of different branches of Mathematics, Physical sciences, biological sciences and business applications. The concept was initially developed for solving equations.

Objectives:

At the end of the unit you would be able to

- solve determinant using their properties
- find the solution of equations using matrices and determinant

10.2 Definition of a Matrix

Definition: A matrix A is a rectangular array of numbers arranged as m horizontal lists, called rows, each list having n elements; the vertical lists called columns.

It is written as

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ . & & \\ . & & \\ . & & \\ . & & \\ a_{m1} & \ldots\ldots\ldots\ldots & a_{m} \end{bmatrix}$$

Note: The element in the i^{th} row and j^{th} column is a_{ij}. So A is also written as $[a_{ij}],\ 1 \le i \le m, 1 \le j \le n$ or simply $[a_{ij}]$. A is called an m $\times$ n matrix. We also write the$(i, j)^{th}$ entry as a_{ij}. When m = n, A is called a square matrix (also called an n – square matrix A)

Example $A = \begin{bmatrix} 2 & -3 & 4 & -1 \\ 1 & 0 & 6 & 5 \\ 4 & -6 & 8 & -3 \end{bmatrix}$**,**

$$B = [2 \quad 1 \quad 3],\quad C = \begin{bmatrix} 5 \\ -7 \\ -8 \\ -1 \end{bmatrix},\quad D = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -4 \\ 5 & 0 & -6 \end{bmatrix}$$

are 3 $\times$ 4,1 $\times$ 3, 4 $\times$ 1 and 3 $\times$ 3 matrices respectively.

Definition: Two matrices $A = [a_{ij}]$ *and* $B = [b_{ij}]$ are equal if (i) the number of rows of A and B are the same (ii) the number of columns of A and B are the same (iii) $a_{ij} = b_{ij}$ for all i, j.

Example: Find the values of a, b, c, d if

$$\begin{bmatrix} 2a+2b & 2a-2b \\ 2c+d & c-d \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 14 & 10 \end{bmatrix}$$

Solution: Equating the corresponding entries of the matrix, we get

2a + 2b = 6 2a – 2b = 2

2c + d = 14 c – d = 10

By adding the first two equations, we get 4a = 8. so a = 2.

2b = 6 – 2a = 6 – 4 = 2. So b = 1

Similarly,
3c = 24. So c = 8, d = 14 – 2c = 14 – 16 = –2

S.A.Q.1: How many entries are there in an m $\times$ n matrix ?

10.3 Operations on Matrices

In this section we define sum of two matrices, difference of two matrices, the scalar multiple of a matrix A by a scalar (real number) k.

Definition: If $A = [a_{ij}]$ *and* $B = [b_{ij}]$ are two m $\times$ n matrices then A + B is defined as an m $\times$ n matrix as follows:

$$A + B = \left[a_{ij} + b_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Note: For getting the sum of A and B, we add to an entry in A, the entry in B in the same place. We can add only two matrices of the same size.

Definition: If $A = [a_{ij}]$ *and* $B = [b_{ij}]$ are two m $\times$ n matrices then A – B is defined as an m $\times$ n matrix as follows:

$$A - B = \left[a_{ij} - b_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Definition: If $A = [a_{ij}]$ and k is any scalar then kA is an m$\times$ n matrix defined as follows:

$$kA = [ka_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Example: If $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$ find A + B, A – B, 2A + 3B

Solution: $A + B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2+7 & 3+8 & 4+9 \\ 5+1 & 6+2 & 7+3 \end{bmatrix} = \begin{bmatrix} 9 & 11 & 13 \\ 6 & 8 & 10 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2-7 & 3-8 & 4-9 \\ 5-1 & 6-2 & 7-3 \end{bmatrix} = \begin{bmatrix} -5 & -5 & -5 \\ 4 & 4 & 4 \end{bmatrix}$$

$$2A + 3B = 2\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} + 3\begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2) & 2(3) & 2(4) \\ 2(5) & 2(6) & 2(7) \end{bmatrix} + \begin{bmatrix} 3(7) & 3(8) & 3(9) \\ 3(1) & 3(2) & 3(3) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 8 \\ 10 & 12 & 14 \end{bmatrix} + \begin{bmatrix} 21 & 24 & 27 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+21 & 6+24 & 8+27 \\ 10+3 & 12+6 & 14+9 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 30 & 35 \\ 13 & 18 & 23 \end{bmatrix}$$

We list some special matrices in the next example.

Example:

a) $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix}$ is called the zero matrix.

We can write it as O. We can have zero matrix of any order.

b) $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is called then n – square unit matrix.

(**Note:** The number of rows is equal to the number of columns in the unit matrix. It is denoted by I_n or simply I when n is understood.)

c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is called a diagonal matrix. A square matrix is diagonal matrix if only the entries on the diagonal are nonzero and other entries are $0'$s.

d) A diagonal matrix having the same number along the diagonal is called a scalar matrix. A scalar matrix is simply kI_n for some scalar k and some positive integer n.

Definition: If $A = [a_{ij}]$ is an m × n matrix then the transpose of A denoted by A^T,is defined as $A^T = [a_{ji}]_{\substack{1 \le j \le n \\ 1 \le i \le m}}$

Note: The transpose of an m × n matrix is an n × m matrix.

Just as addition of real numbers is commutative, the addition of matrices is also commutative. As far as addition is concerned matrices behave like numbers. The following theorem lists properties of addition of matrices.

Theorem: If A, B, C are m × n matrices and k and I are scalars.

a) A + (B + C) = (A + B) + C
b) A + 0 = 0 + A where 0 is the m × n zero matrix
c) A + (–A) = (–A) + A = 0 (Here –A denotes (–1) A)
d) A + B = B + A
e) k(A + B) = kA + kB
f) (k + I) A = kA + IA
g) (kI) A = k (IA) = I(kA)
h) IA = A
i) $(A + B)^T = A^T + B^T$
j) $\left(A^T\right)^T = A$
k) $(I_n)^T = I_n$

The theorem can be proved by using the definition of sum of two matrices and scalar multiple of a matrix.

If A and B are two matrices then AB is defined only when the number of columns of A = number of rows of B.

Definition: If A is an m × n matrix and B is an n × p matrix then the product of A and B, denoted by AB, is an m × p matrix and is defined by

$$AB = (ab)_{ik} = a_{i1}\, b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk}$$

for $1 \le i \le m,\ 1 \le k \le p,$

Note: $(ab)_{ik}$ can be understood as follows.

$[a_{i1},\ a_{i2},\ ....\ a_{in}]$ is the ith row of A, $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$ Is the kth column of B and both these have n elements. For calculating $(ab)_{ik}$, multiply the respective elements of ith row of A and kth column of B and add them. The resulting number is $(ab)_{ik}$.

Example: Find AB when

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \text{and } B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 0 & 1 \end{bmatrix}$$

Solution: A is a 2 × 3 matrix and B is a 3 × 2 matrix. So AB is a 2 × 2 matrix.

$$AB = \begin{bmatrix} 2(1)+0(4)+1(0) & 2(2)+0(6)+1(1) \\ -1(1)+0(4)+1(0) & -1(2)+0(6)+1(1) \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & -1 \end{bmatrix}$$

S.A.Q.2: Find BA for matrices A and B given in above example

We have seen that A + B = B + A when A and B are matrices of the same size. But AB ≠ BA in general. It can happen that one of the products is defined whereas the other product is not defined. Let us illustrate this with an example.

Example: Find two matrices A and B such that

a) AB is defined but BA is not
b) BA is defined but AB is not
c) Both are defined but AB ≠ BA
d) Both are defined and AB = BA

Solution:

a) Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

Then AB is defined as the number of columns of A = 3 = number of rows of B.

Number of columns of B = 3 ≠ number of rows of A. Hence BA is not defined.

b) If $A = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ then BA is defined, as number of columns of B = 3 = number of rows of A.

Number of columns of A = 1 ≠ number of rows of B. Hence AB is not defined.

c) Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

$$AB = \begin{bmatrix} 1(-4)+2(3)+3(0) & 1(2)+2(5)+3(1) \\ -2(-4)+0(3)+1(0) & -2(2)+0(5)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 15 \\ 8 & -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4(1)+2(-2) & -4(2)+2(0) & -4(3)+2(1) \\ 3(1)+5(-2) & 3(2)+5(0) & 3(3)+5(1) \\ 0(1)+1(-2) & 0(2)+1(0) & 0(3)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -8 & -10 \\ -7 & 6 & 14 \\ -2 & 0 & 1 \end{bmatrix}$$

Hence AB ≠ BA

d) Consider $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0(1)+1(0)+2(0) & 0(0)+1(1)+2(0) & 0(0)+1(0)+2(1) \\ 2(1)-1(0)+3(0) & 2(0)-1(1)+3(0) & 2(0)-1(0)+3(1) \\ 3(1)+4(0)+0(0) & 3(0)+4(1)+0(0) & 3(0)+4(0)+0(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & 4 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(0)+0(2)+0(3) & 1(1)+0(-1)+0(4) & 1(2)+0(3)+0(0) \\ 0(0)+1(2)+0(3) & 0(1)+1(-1)+0(4) & 0(2)+1(3)+0(0) \\ 0(0)+0(2)+1(3) & 0(1)+0(-1)+1(4) & 0(2)+0(3)+1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & 4 & 0 \end{bmatrix}$$

Thus AB = BA

Theorem: If A is an m $\times$ n matrix and B is an n $\times$ p matrix and k is any scalar, then

a) $(AB)^T = B^T A^T$

b) $AI_n = A \text{ and } I_m A = A$

c) k(AB) = (kA)B = A(kB)

d) OA = O, BO = O where the four zero matrices are k $\times$ m, k $\times$ n, p $\times$ t and n $\times$ t matrices respectively (for some k and t).

The proof of above theorem follows from the definition of product of two matrices.

We can define the product of 3 matrices A, B, C when

$$\left.\begin{array}{l} \text{Number of columns of A} = \text{Number of rows of B} \\ \text{And Number of columns of B} = \text{Number of rows of C} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (10.1)$$

The following theorem describes the properties of product of three matrices.
Theorems: Let A, B, C be 3 matrices. Then the following hold good whenever the sums and products of matrices appearing below are defined.

a) (AB) C = A(BC) (Associative law)
b) A(B + C) = AB + AC (Left distributive law)
c) (B + C) A = BA + CA (Right distributive law)

Proof: follows from definition

S.A.Q. 3: If $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ -1 & 0 \end{bmatrix}$ find A + B, 2A – 3B, 3B – 2A, $(A – B)^T$ and $(B – A)^T$.

S.A.Q. 4: If $A = \begin{bmatrix} 4 & 6 & 0 \\ 0 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ -1 & -1 & 1 \end{bmatrix}$

Verify that $(A + B)^T = A^T + B^T$

S.A.Q. 5: Find a matrix A such that

$$3A + \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix}$$

S.A.Q. 6: If $X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ find X and Y.

S.A.Q. 7: A matrix A is said to be symmetric if $A = A^T$. Show that $A + A^T$ is symmetric for a 3×3 matrix A

S.A.Q. 8: If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 0 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ -2 & 1 \end{bmatrix}$

Show that $AB = \begin{bmatrix} -5 & 10 \\ -5 & 5 \\ 0 & 10 \end{bmatrix}$ Does BA exist ?

S.A.Q.9: If $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

show that (A – I) (A + 2I) = $\begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix}$

10.4 Square Matrix and its Inverse

We know that a square matrix is an n × n matrix for some integer n. The set of n × n square matrices satisfy some additional properties.

We know that $AI_n = I_nA = A$ for any n × n square matrix A. We can multiply two n × n matrices and the product is an n × n matrix. In general we can define AA, AAA etc.

We define powers of a square matrix as follows. We define $A^0 = I_n$,

$$A^2 = AA,\ A^3 = AAA, \ldots\ldots, A^n = \underbrace{AA\ldots..A}_{n\ times} \quad \ldots..\ (10.2)$$

The set of all n × n matrices satisfy the properties of indices (powers) of numbers.

Note $A^m A^n = A^{m+n} = A^n . A^m$ .... (10.3)

If a is a non zero real number then we know that $a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1.$

A similar property holds good for some square matrices. In the case of numbers $\frac{1}{a}$ is called the reciprocal of a. But in the case of matrices it is called the inverse of a square matrix.

Definition: A square matrix A is invertible (or non singular) if there exists a square matrix B such that $AB = BA = I_n$ ………….. (10.4)

B is called the inverse of A and is denoted by A^{-1}.

Note: If A has an inverse then A is called as invertible matrix.

Example: Let $A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ If $B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then $AB = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3(2)-5(1) & 3(5)-5(3) \\ -1(2)+2(1) & -1(5)+2(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is the inverse of A.

Theorem: If B is the inverse of a square matrix A then A is the inverse of the matrix B.

Proof: As B is the inverse of A, then by definition,

AB = BA = I_n ………. (10.5)
So BA = AB = I_n ………………………….. (10.6)
From (10.6) we see that A satisfies the condition for the inverse of B. Hence A is the inverse of B.

Now we are going to see a method for finding the inverse of a matrix. However you will have a formula for the inverse of a 2 × 2 matrix

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ………….. (10.7)

For the present, you can verify that

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = \frac{ad-bc}{ad-bc}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

S.A.Q. 10: If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that $A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

S.A.Q. 11: Verify that $\begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$

10.5 Determinants

The determinant of an n – square matrix A is a unique number associated with A and is denoted by det (A) or | A|. | A | is called a determinant of order n.

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ then | A | is denoted by $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

As the definition of | A | is complex for a general n – square matrix A, we define determinant of orders 1, 2, 3 and then extend it for a general n – square matrix A.

Evaluation of determinants

Definition: The determinants of orders 1, 2, 3 are defined as follows

a) $\left| a_{11} \right| = a_{11}$

b) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{12}\ a_{21}$

(We can understand the determinant in the following way).
We (i) multiply the elements in the diagonal from left to right (ii) multiply the elements in the diagonal from right to left (iii) subtract product got in (ii) from the product got in (i)

c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Note: We can calculate the value of a determinant of order 3 as follows:

1. Consider the first element a_{11} in the first row. Attach the sign + (plus)
2. Delete the row and column in which a_{11} appears; that is first row and first column. We get $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
3. Multiply + a_{11} and the value of $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
4. Consider the second element a_{12} in first row. Attach the sign – (minus)
5. Delete the row and second column in which a_{12} appears; that is the first row and second column. We get $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
6. Multiply – a_{12} and the value of $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
7. Consider the third element a_{13} in the first row. Attach the sign + (plus)
8. Delete the row and column in which a_{13} appears; that is the first row and third column. We get $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
9. Multiply + a_{13} and the value of $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Add the values got in steps 3, 6 and 9. This is the value of the given determinant.

Note: We usually denote a determinant by the symbol Δ (read as Delta).

Example: Evaluate the determinant $\Delta = \begin{vmatrix} 0 & 2 & 3 \\ 1 & 4 & 7 \\ 2 & 0 & 4 \end{vmatrix}$

Solution:

$$\Delta = 0\begin{vmatrix} 4 & 7 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 7 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix}$$

$= 0 – 2[1(4) – 2(7)] + 3[1(0) – 2(4)]$
$= 0 – 2 (4 – 14) + 3(0 – 8)$
$= –2(–10) – 24$
$= 20 – 24$
$= –4$

Thus $\Delta = -4$

Evaluation of a determinant in term of any row or column

Recall the definition of determinant of 3-square matrix, we obtained

$$|\, A \,| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The 2 nd– order determinants $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ are called the minors of a_{11}, a_{12}, a_{13}. We can denote the minors by M_{11}, M_{12}, and M_{13}. If we attach the signs, these are called cofactors of a_{11}, a_{12}, a_{13}. We denote them by A_{11}, A_{12}, A_{13}. Then above determinant can also be written as

$$|\, A \,| = a_{11}A_{11}, a_{12}A_{12}, a_{13}A_{13}.$$

We can also define minors of a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} in a similar manner. For example.

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a32 \end{vmatrix}$$

(M_{23} is got by deleting the second row and third column of A)

The cofactors can be defined in a similar manner using the rule of signs given by (10.8)

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \text{………………………….. (10.8)}$$

Any cofactor is got by multiplying the minor and its sign given in (10.8). For example cofactor for a_{32} is – M_{32}.

So $A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Thus we can expand Δ in term of any row or column in a similar way. (10.9) gives the expansion in term of various rows and columns.

$$\left.\begin{aligned} \Delta &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \\ &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\ &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \end{aligned}\right\} \text{………………………………… (10.9)}$$

You may wonder why so many expansions given in (10.9) are necessary. If a row or column has many zeros then evaluating by the elements of that row or column makes the evaluation simpler.

Example:: Evaluate $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 4 & 0 & 0 \\ 1 & 4 & 5 \end{vmatrix}$

Solution: As the second row has two zeros, we expandby the elements of the second row.

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$= 4A_{21} + 0A_{22} + 0A_{23}$$

$$= 4(-1)\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix}$$

(Note: For (2, 1) position, the sign is –. See (4.8)).

= – 4[2(5) – 4(4)]

= – 4(10 – 16)

= – 4(–6)

= 24

The evaluation of a determinant of order n is similar.

For example, if $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$

Then $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14}$

The signs of cofactors can be defined by

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10.10)}$$

Example: Evaluate $\begin{vmatrix} 1 & 4 & 7 & 2 \\ 2 & 4 & 8 & 4 \\ 4 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{vmatrix}$

Solution As the third row has two zeros we expand by the elements of third row. The signs of cofactors of A is determined by using (10.10)

$|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34}$

$= 4A_{31} + 0A_{32} + 0A_{33} + 1A_{34}$

$$= 4(1)\begin{vmatrix} 4 & 7 & 2 \\ 4 & 8 & 4 \\ 2 & 3 & 0 \end{vmatrix} + 1(-1)\begin{vmatrix} 1 & 4 & 7 \\ 2 & 4 & 8 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 4\left(4\begin{vmatrix} 8 & 4 \\ 3 & 0 \end{vmatrix} - 7\begin{vmatrix} 4 & 4 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 4 & 8 \\ 2 & 3 \end{vmatrix}\right) - 1\left(1\begin{vmatrix} 4 & 8 \\ 2 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 1 & 3 \end{vmatrix} + 7\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}\right)$$

$= 4[4(0 - 12) - 7(0 - 8) + 2(12 - 16)] - 1[1(12 - 16) - 4(6 - 8) + 7(4 - 4)]$

$= 4[4(-12) - 7(-8) + 2(-4)] - 1[1(-4) - 4(-2) + 7(0)]$

$= 4(-48 + 56 - 8) - 1(-4 + 8)$

$= 4(-56 + 56) - 1(4)$

$= 4(0) - 4$

$= -4$

Thus $|A| = -4$

S.A.Q. 12: Evaluate the following determinants

a) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix}$ b) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ c) $\begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$

S.A.Q. 13: Evaluate the following determinants

a) $\begin{vmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{vmatrix}$ b) $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 5 & -6 & 4 \\ 4 & 7 & 4 & -3 \\ 7 & 2 & 1 & 6 \end{vmatrix}$

10.6 Properties of Determinants

In this section we list some properties of determinants. These properties enable us to evaluate a determinant in an easier way.

Property 1: If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $m\Delta = \begin{vmatrix} ma_{11} & ma_{12} & ma_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Let A_{11}, A_{12}, A_{13} denote the cofactors of a_{11}, a_{12}, a_{13} in Δ. These are also the cofactors of ma_{11}, ma_{12}, ma_{13}, in $m\Delta$.

So $\begin{vmatrix} ma_{11} & ma_{12} & ma_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ma_{11} A_{11} + ma_{12}A_{12} + ma_{13}A_{13}$

$= m (a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13})$

$= m\Delta$.

Note: This property holds good when any row or column of Δ is multiplied by m. This property essentially means that any common factor of a row or column can be taken outside the determinant.

Remark: If A is a matrix then mA is got by multiplying each entry of A by m. In the case of determinant $m\Delta$ is got by multiplying the entries of a single row or column by m.

Example:

$$\begin{vmatrix} 12 & 18 & 30 \\ 2 & 4 & 1 \\ 1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2(6) & 3(6) & 5(6) \\ 2 & 4 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 6\begin{vmatrix} 2 & 3 & 5 \\ 2 & 4 & 1 \\ 1 & 3 & 5 \end{vmatrix}$$

Property 2: If |A| = det (A), then det (A^T) = |A|

Proof: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

So | A | = ad – bc = |A^T|

So the second order minors of A and A^T have the same value. As the sign of a cofactor is the same in both A and A^T, the value det (A^T) through expanding along the first column is equal to det (A) through expanding along the first row.

Hence det (A^T) = |A|

Property 3: If two rows or columns of a determinant Δ are interchanged then the value of the Δ is unchanged but the sign is changed.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ If the second and third rows are interchanged we get $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$|\,B\,| = a_{11}\begin{vmatrix} a_{32} & a_{33} \\ a_{22} & a_{23} \end{vmatrix} - a_{12}\begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix}$$

$= a_{11}(a_{32}a_{23} - a_{22}a_{33}) - a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{31}a_{22} - a_{32}a_{21})$

$$= -\left\lfloor a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})\right\rfloor$$

$$= -\left[a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \right]$$

$= -\,|\,A\,|$

Property 4: If a determinant Δ has two identical rows then the value of the determinant is 0.

Proof: If we interchange two identical rows then the value of the new determinant is $-\Delta$. (By property 3). But the new determinant is the same as Δ. So $-\Delta = \Delta$ or $\Delta = 0$.

Example: Evaluate $\Delta = \begin{vmatrix} 1 & 4 & 7 & 9 \\ 2 & 5 & 4 & 7 \\ 9 & 8 & 7 & 9 \\ 1 & 4 & 7 & 9 \end{vmatrix}$

Solution: The first and fourth rows of Δ are identical. By property 4, $\Delta = 0$.

Property: The value of a determinant remains the same when multiple of some rows are added to a particular row. The same is true for columns.

Note: For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kd+lg & b+ke+lh & c+kf+li \\ d & e & f \\ g & h & i \end{vmatrix}$$

Since we add k times the second row and l times the third row to the first row.

Example: Evaluate $\Delta = \begin{vmatrix} 4 & 7 & 10 \\ 2 & 4 & 6 \\ 1 & 2 & 5 \end{vmatrix}$

Solution The first entry in the first row (R_1) is 4. To make it 0, we subtract 4 times the third row (R_3). Thus R_1 of Δ is replaced by $R_1 - 4R_3$ (This is indicated on the right of the determinant). Similarly if we subtract $2R_3$ from R_2 we get 0, as the first element in R_2. In the resulting determinant the first column has two zeros and a one. This makes the evaluation (along the first column) easier.

$$\Delta = \begin{vmatrix} 4 & 7 & 10 \\ 2 & 4 & 6 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4-4(1) & 7-4(2) & 10-4(5) \\ 2-2(1) & 4-2(2) & 6-2(5) \\ 1 & 2 & 5 \end{vmatrix} \begin{matrix} R_1 - 4R_3 \\ R_2 - 2R_3 \\ R_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & -1 & -10 \\ 0 & 0 & -4 \\ 1 & 2 & 5 \end{vmatrix}$$

$= 0A_{11} + 0A_{21} + 1\begin{vmatrix} -1 & -10 \\ 0 & -4 \end{vmatrix}$ (by expanding along the first column)

= 0 + 0 +1 (4 – 0)

= 4

Example: Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$

Solution $\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$= 1\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} + 0A_{21} + 0A_{31} + 0A_{41}$$

$$= a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} + 0\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix}$$

= a (bc – 0)

= abc.

Example: Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b+c & a-b & a-c \\ b^2+c^2 & a^2-b^2 & a^2-c^2 \end{vmatrix} \begin{matrix} C_1 \\ C_2-C_1 \\ C_3-C_1 \end{matrix}$$

$$= 1\begin{vmatrix} a-b & a-c \\ a^2-b^2 & a^2-c^2 \end{vmatrix} + 0A_{12} + 0A_{13}$$

$$= (a-b)\left(a^2-c^2\right) - (a-c)\left(a^2-b^2\right)$$

= (a – b) (a – c) (a + c) – (a – c) (a – b) (a + b)

= (a – b) (a – c) [a + c – a – b]

= (a – b) (a – c) (c – b)

= (a – b) [– (c – a)] [–(b – c)]

= (a – b) (b – c) (c – a)

S.A.Q. 14: Evaluate $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

S.A.Q. 15: Prove that $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

S.A.Q. 16: Evaluate $\Delta = \begin{vmatrix} (x+1) & (x+2) & 1 \\ (x+2) & (x+3) & 1 \\ (x+3) & (x+4) & 1 \end{vmatrix}$

S.A.Q. 17: Evaluate the following determinants

a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix}$ b) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

10.7 The Inverse of a Matrix

In this section we give a method of finding the inverse of a matrix.

Definition: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the adjoint matrix of A (denoted by Adj A) is given by $\text{Adj}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Example: Find the adjoint of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Solution

$$A_{11} = (+1)\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 3(5) - 4(4) = -1$$

$$A_{12} = (-1)\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = (-1)(10 - 12) = 2$$

$$A_{12} = (+1)\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$$

$$A_{21} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = (-1)(10 - 12) = 2$$

$$A_{22} = 1\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 5 - 9 = -4$$

$$A_{23} = (-1)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-1)(4 - 6) = 2$$

$$A_{31} = 1\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$$

$$A_{32} = (-1)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)(4 - 6) = 2$$

$$A_{33} = 1\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$\therefore Adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

We use Adj A for evaluating the inverse of a matrix.

A square matrix is invertible if and only if $|A| \neq 0.$ When $|A| \neq 0,$, the inverse of a matrix A is given by

$$A^{-1} = \frac{1}{|A|} Adj\ A \qquad \text{………………………. (10.11)}$$

If |A| = 0, the matrix a is called singular; otherwise it is non singular. So a matrix is invertible if and only if it is nonsingular.

Example: Find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution A_{11} = d, A_{12} = –c, A_{21} = –b, A_{22} = a.

So Adj $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

|A| = ad – bc. Hence

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{………. (10.12)}$$

Example: Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

Solution

$$A_{11} = (1)\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0; \quad A_{12} = (-1)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = (-1)(-1-2) = 3;$$

$$A_{13} = (1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3; \quad A_{21} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1)(-1-1) = 2;$$

$$A_{22} = 1\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3; \quad A_{23} = (-1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (-1)(1-2) = 1;$$

$$A_{31} = 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1+1 = 2; \quad A_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(1-1) = 0;$$

$$A_{33} = 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2$$

$$\text{Adj}\, A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1 (0) - 1(-3) + 1(3)$

$= 6$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

S.A.Q.18: Find the inverse of $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$

S.A.Q. 19: Test whether A^{-1} exists when $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

S.A.Q. 20: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

10.8 Solution of Equations using Matrices and Determinants

Matrices are useful in representation of data. For example if we want to classify the students of a class in terms of gender and their grades then we can use a matrix for representing the information. Suppose we have three grades A, B, C. Then the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ represents.

The classified data:

a_{11} denotes the number of male students who got grade A.

a_{12} denotes the number of male students who got grade B

a_{13} denotes the number of male students who got grade C.

a_{21} denotes the number of female students who got grade A.

a_{22} denotes the number of female students who got grade B.

a_{23} denotes the number of female students who got grade C.

Solving linear equations using matrices

We can also use matrices for solving n equations in n variables. The ideas is to represent n equations as a single matrix equation and then solve the matrix equation. The next example illustrates this.

Example: Solve x + y = 3

2x + 3y = 8

Solution The given system of equations is equivalent to the single matrix equation AX = B where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Multiplying both side of AX = B by A^{-1}. we get $X = A^{-1}B$.

By definition of A^{-1}, we have

$$A^{-1} = \frac{1}{1(3)-2(1)}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 3(3)-1(8) \\ -2(3)+1(8) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence x = 1, y = 2

Example: Solve the equations

x + y + z = 6

x + 2y + 3z = 14

–x + y – z = –2

Solution: The given systems of equations is equivalent to A X = B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

$$A_{11} = (1)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5$$

$$A_{12} = (-1)\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} = (-1)(-1+3) = -2$$

$$A_{13} = (1)\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3;$$

$$A_{21} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1)(-1-1) = 2;$$

$$A_{22} = (1)\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -1 + 1 = 0;$$

$$A_{23} = (-1)\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = (-1)(1+1) = -2;$$

$$A_{31} = 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1;$$

$$A_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (-1)(3-1) = -2;$$

$$A_{33} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

= 1(–5) + 1(–2) + 1(3)

= –5 – 2 + 3

= –7 + 3

= –4.

$$A^{-1} = \frac{Adj\ A}{|A|} = -\frac{1}{4}\begin{bmatrix} -5 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}^T$$

$$A^{-1} = -\frac{1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$X = A^{-1}\ B = -\frac{1}{4}\begin{bmatrix} -5 & 2 & 1 \\ -2 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ -2 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -5(6)+2(14)+1(-2) \\ -2(6)+0(14)-2(-2) \\ 3(6)-2(14)+1(-2) \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -30+28-2 \\ -12+0+4 \\ 18-28-2 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence x = 1, y = 2, z = 3

10.9 Solving equations using determinants

We can also solve a system of n linear equations in n variables using determinants. The method is provided by Cramer's rule. Cramer's rule for three equations in three variables.

Consider the system of three linear equation in three variables x, y, z.

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

Let Δ be the co-efficient determinant i.e., the determinant of the coefficients of the variables x, y, z such that $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

(If $\Delta = 0$ the system has no unique solution).

By Cramer's rule we have

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\Delta} \qquad y = \frac{\begin{vmatrix} a_{11} & b_{11} & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\Delta} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\Delta}$$

or

$$x = \frac{\Delta_1}{\Delta} \qquad y = \frac{\Delta_2}{\Delta} \qquad z = \frac{\Delta_3}{\Delta}$$

Note: As in the previous section the system of equations Ax = B. Then |A| = Δ. Now Δ_1 is obtained by replacing the first column of Δ by B. Similarly Δ_2 and Δ_3 are obtained by replacing the second and third columns of Δ by B respectively.

Example: Solve

2x + 3y + 4z = 20

x + y + 2z = 9

3x + 2y + z = 10

Solution: In this example,

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} \quad \Delta_1 = \begin{vmatrix} 20 & 3 & 4 \\ 9 & 1 & 2 \\ 10 & 2 & 1 \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} 2 & 20 & 4 \\ 1 & 9 & 2 \\ 3 & 10 & 1 \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} 2 & 3 & 20 \\ 1 & 1 & 9 \\ 3 & 2 & 10 \end{vmatrix}$$

$\Delta = 2(1 - 4) - 3(1 - 6) + 4(2 - 3) = 5$

$\Delta_1 = 20(1 - 4) - 3(9 - 20) + 4(18 - 10) = 5$

$\Delta_2 = 2(9 - 20) - 20(1 - 6) + 4(10 - 27) = 10$

$\Delta_3 = 2(10 - 18) - 3(10 - 27) + 20(2 - 3) = 15$

Hence

$$x = \frac{\Delta_1}{\Delta} = 1, \quad y = \frac{\Delta_2}{\Delta} = 2, \quad z = \frac{\Delta_3}{\Delta} = 3.$$

S.A.Q. 21: Solve the following system of equations using matrices

a) 2x + 3y – z = 9
x + y + z = 9
3x – y – z = –1

b) 2x – y + 3z = – 9
x + y + z = 6
x – y + z = 2

S.A.Q. 22: Solve the following system of equations using Cramer's rule

a) 5x – 6y + 4z = 15
7x + 4y – 3z = 19
2x + y + 6z = 46

b) x + y + z = 9
2x + 5y + 7z = 52
2x + y – z = 0

10.10 Summary

In this unit, we discuss about the concept of matrices and determinants. The different types matrices is defined, the concept of inverse of matrix is well defined with good examples, studied about determinants and its different properties and solving equations using matrices and determinants is explained with the help of standard examples.

10.11 Terminal Questions

1. Find the values of x, y, z and t satisfy the matrix relationship

$$\begin{bmatrix} x+3 & z+4 & t-2 \\ 2y+5 & 4x+5 & 3t+1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2t+5 \\ -5 & 2x+1 & -20 \end{bmatrix}$$

2. Find the values for x, y, z that satisfy the matrix relationship

$$3\begin{bmatrix} 2 & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -1 & 2z \end{bmatrix} + \begin{bmatrix} 4 & x+2 \\ y+z & 3 \end{bmatrix}$$

3. Find a matrix A satisfying

$$\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} + 2A = \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

   Show that AB = AC. (In the case of real numbers, ab = ac will imply that b = c. But this is not so for matrices as this example shows)

5. If $A=\begin{bmatrix}1 & 0 & -2\\ 0 & -3 & 1\\ 2 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1 & 0\\ -2 & 0 & 3\\ 0 & 2 & -1\end{bmatrix}$ evaluate AB – BA.

6. If $A=\begin{bmatrix}-2 & -4\\ 3 & 6\end{bmatrix}$, show that $A^2=4A$

7. If $A=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$, show that $A^2 - 2A - 5I = 0$

8. If $A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$ find AA^T and A^TA

9. If $A=\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}$ Evaluate A^2 and A^3.

10. If $\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix} A=\begin{bmatrix}4 & -6\\ 2 & 1\end{bmatrix}$ find A.

11. If $A\begin{bmatrix}5 & 3\\ 7 & 4\end{bmatrix}=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ find A

12. $If\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}A\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}-2 & 4\\ 3 & -1\end{bmatrix}$, find A.

13. Evaluate the following determinants

a) $\begin{vmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{vmatrix}$ b) $\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}$ c) $\begin{vmatrix}a & c & b\\ b & a & c\\ c & b & a\end{vmatrix}$

14. Evaluate the following determinants

a) $\begin{vmatrix}2 & 4 & -2 & 3\\ 1 & -2 & 1 & 0\\ -2 & 0 & -1 & 3\\ 2 & 3 & -2 & 3\end{vmatrix}$ b) $\begin{vmatrix}2 & -1 & 1 & 3\\ -1 & 2 & 4 & 2\\ 0 & 3 & -1 & 1\\ 1 & -2 & 5 & 0\end{vmatrix}$ c) $\begin{vmatrix}4 & 2 & -1 & 2\\ -2 & 1 & 2 & -3\\ 3 & 2 & -4 & 1\\ 2 & -3 & 0 & 2\end{vmatrix}$

15. Show that $\begin{vmatrix}bc & b+c & 1\\ ca & c+a & 1\\ ab & a+b & 1\end{vmatrix}=(a-b)(b-c)(c-a)$

16. Evaluate the following determinants

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \end{vmatrix}$

17. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

18. Prove that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

19. Solve the following system of equations using (i) matrices (ii) determinants

a) x + 2y – z = 3
3x – y + 2z = 2
2x – 2y + 3z = 2

b) 2x + 3y – z = 9
x + y + z = 9
3x – y – z = –1

c) a + b + z = 6
a + 2b + 3c = 14
–a + b – z = –2

d) 2a + 3b + c = 8
4a + b + c = 6
a + b + c = 3

10.12 Answers

Self Assessment Questions

1. mn entries
2. $\begin{bmatrix} 0 & 0 & 3 \\ 2 & 0 & 10 \\ -1 & 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 6 \\ 8 & 8 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ -4 & -14 \\ 15 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 8 \\ 4 & -6 \\ 20 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 2 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & -7 \\ -2 & 4 & -2 \end{bmatrix}$
5. $\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$
6. $X = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$

8. BA does not exist since the number of columns of B = 2 $\neq$ 3 = the number of rows of A.
12. a) – 9 b) 2 c) 419
13. a) – 9 b) 2 c) 419
(Expand using the first column, first column and first row respectively.)
14. $\Delta = (a - b)(b - c)(c - a)$
16. The row operations are $R_2 - R_1$ and $R_3 - R_2$. Answer is 0.
17. a) Answer 0; the row operations are R_1, R, $- R_1$, $R_3 - R_2$, R_4. b) –3. The row operations R_1, R_2, $R_3 - R_2$, $R_4 - R_3$ reduce Δ to

$$(-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}.$$ Apply R_1, $R_2 - R_1$, R_3.

18. $\frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$
19. As $|A| = 0$, A^{-1} does not exist.
20. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
21. a) x = 2, y = 3, z = 4 b) x = 1, y = 2, z = 3
22. a) x = 3, y = 4, z = 6 b) x = 1, y = 3, z = 5

Terminal Questions

1. x = –2, y = –5, z = –8, t = –7
2. x = 4, y = 1, z = 3
3. $\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}$
5. $\begin{bmatrix} 0 & -8 & 9 \\ 2 & -1 & -14 \\ 6 & 5 & 1 \end{bmatrix}$
8. $\begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}, \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$

9. $A^2 = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -23 \\ 0 & 8 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

12. Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \; C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}, \; D = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

    Then BAC = D. So A = B^{-1} DC^{-1},,

    $B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}, \; c^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}, \; A = \begin{bmatrix} 24 & 13 \\ -34 & 18 \end{bmatrix}$

13. a) –8b) abc + 2fgh – af^2 – bg^2 – ch^2 c) $a^3 + b^3 + c^3 - 3abc$

14. a) – 15 (Row operations: R_1, R_2, R_3 –R_1, R_4 – R_1)

    b) 102 (Row operations: $R_1 + 2R_4$, $R_2 + R_4$, R_3, R_4

    c) 87 (column operations: $C_1 + 4C_3$, $C_2 + 2C_3$, C_3, $C_4 + 2C_3$).

15. Apply row operations R_1, $R_2 - R_1$, $R_3 - R_1$. Expand using last column.

16. a) and b) answer 1 (Apply row operations R_1, $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$)

17. Apply R_1, $R_2 - R_1$, $R_3 - R_1$, we get $\begin{vmatrix} b-a & a-b \\ c-a & a-c \end{vmatrix}$. This is equal to (b – a) (a – c) – (a – c) (c – a) – (b – a) (a – c) – (–1) (–1) (b – a) (a – c) = 0

18. Multiplying R_1 by a, R_2 by b and R_3 by c and then dividing by abc we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} ab^2c^2 & abc & a(b+c) \\ a^2bc^2 & abc & b(c+a) \\ a^2b^2c & abc & c(a+b) \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc} \begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix} \text{ (By taking out abc from columns 1 and 2)}$$

$$= abc \begin{vmatrix} bc & 1 & ab+ac+bc \\ ca & 1 & bc+ab+ca \\ ab & 1 & ca+bc+ab \end{vmatrix} C_3 + C_1$$

$$= (abc)(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$ (by taking out ab + bc + ca from C_3)

19. a) x = –1, y = 4, z = 4 b) x = 2, y = 3, z = 4
c) a = 1, b = 2, c = 3 c) a = 1, b = 2, c = 0