Unit 9 Complex Numbers

Structure:

9.1 Introduction

We recall that, if x and y are real numbers and $i = \sqrt{-1}$ then $x + iy$ is called a complex number. The complex numbers were first introduced by Cardan*(1501 – 1576).* Two hundred years later Euler *(1707 – 1783)* and John Bernoulli recognized the complex numbers introduced by Cardan and studied their properties in detail. In 1983, Sir William Rowan *Hamilton (1805 – 1865)* an Irish mathematician introduced the complex number as an ordered pair of real numbers. In this chapter, we begin the study of complex numbers as ordered pairs.

Objectives:

At the end of the unit you would be able to

- understand the concept of complex numbers.
- apply De Moivre's Theorem in finding the roots of complex numbers.

9.2 Complex Numbers

Let C denote the set of all ordered pairs of real numbers.

That is, $C = \{(x, y); x, y \in R\}$.

On this set C define addition "+" and multiplication "." by,

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ … (1)

$(x_1, y_1) . (x_2, y_2) = (x_1x_2 – y_1y_2, x_1y_2 + x_2y_1)$ … (2)

Then the elements of C which satisfy the above rules of addition and multiplication are called complex numbers. If $z = (x, y)$ is a complex number then x is called the real part and y is called the imaginary part of the complex number z and they are denoted by $x = Re\ z$ and $y = Im\ z$. If (x_1, y_1) and (x_2, y_2) are two complex numbers then $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

(a) Properties of addition

1. **Closure law:** If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ then from (1)
   $z_1 + z_2 = (x_1, y_2) + (x_2, y_2),= (x_1 + x_2, y_1 + y_2)$, which is also an ordered pair of real numbers. Hence $z_1 + z_2 \in C$. Therefore for every $z_1, z_2 \in C$, $z_1 + z_2 \in C$.

2. **Commutative law:** $z_1 + z_2 = z_2 + z_1$ for every $z_1, z_2 \in C$
   Consider$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
   $= (x_2 + x_1, y_2 + y_1) = (x_2, y_2) + (x_1, y_1) = z_2 + z_1$.

3. **Associative law:** $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for every $z_1, z_2, z_3 \in C$
   Proof of this is similar to *above proof.*

4. **Existence of identity element:** There exists an element $(0, 0) \in C$ such that,
   $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$.
   for every $(x, y) \in C$. Here $(0, 0)$ is called the additive identity element of C.

5. **Existence of inverse:** For every $(x, y) \in C$ there exists $(–x, –y) \in C$ such that
   $(x, y) + (–x, –y) = (x – x, y – y) = (0, 0)$.
   Hence $(–x, –y)$ is the additive inverse of (x, y).
   Thus we have shown that the set C is an abelian group w.r.t. the addition of complex numbers defined by (1).

(b) Properties of multiplication

1. **Closure law:** If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in C$ then from (2)
   $z_1z_2 = (x_1, y_1) (x_2, y_2) = (x_1x_2 – y_1y_2, x_1y_2 + x_2y_1)$, which is also an ordered pair of real numbers. Hence $z_1z_2$is also a complex number.
   Thus, for every $z_1, z_2 \in C$, $z_1z_2 \in C$.

2. **Commutative law:** $z_1z_2 = z_2z_1$ for every $z_1, z_2 \in C$.
   Now $z_1z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ ….. *(i)*
   and $z_2z_1 = (x_2, y_2)(x_1, y_1) = (x_2x_1 - y_2y_1, x_2y_1 + x_1y_2)$
   $= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ ….. *(ii)*
   From (i) and (ii) $z_1z_2 = z_2z_1$.
3. **Associative law:** $z_1(z_2z_3) = (z_1z_2)\,z_3$, for every $z_1, z_2, z_3 \in C$.
   Proof is similar to *above proof.*
4. **Existence of identity element:** There exists $(1, 0) \in C$ such that
   $(x, y)(1, 0) = (x \,.\, 1 - y \,.\, 0, x \,.\, 0 + 1 \,.\, y) = (x, y)$ for every $(x, y) \in C$.
   Here $(1, 0)$ is called the multiplicative identity element.
5. **Existence of inverse:** Let $z = (x, y) \neq (0, 0)$, be a complex number. Let (u, v) be the inverse of (x, y).
   Then $(u, v) \,.\, (x, y) = (1, 0)$, the identity element.
   i.e. $(ux - vy, uy + vx) = (1, 0)$.
   Hence $ux - vy = 1$, and $uy + vx = 0$.
   Solving for u and v, we get, $u = \frac{x}{x^2 + y^2}$, $v = \frac{-y}{x^2 + y^2}$
   Hence $\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \in C$ is the multiplicative inverse of (x, y).

Thus we have shown that the set of non-zero complex numbers forms an abelian group w.r.t. the multiplication defined by *(2).*

Also we can prove that the multiplication is distributive over addition.

(c) Distributive law: For all $z_1, z_2, z_3 \in C$

i) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (left distributive law)
ii) $(z_2 + z_3)\,z_1 = z_2z_1 + z_3z_1$ (right distributive law)

The complex numbers whose imaginary parts are equal to zero possess the following properties.

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0).$$

and $$(x_1, 0) \,.\, (x_2, 0) = (x_1 x_2, 0).$$

Which are essentially the rules for addition and multiplication of real numbers. We identify the complex number $(x, 0)$ with the real number x. Denote the complex number $(0, 1)$ by i.
Now $i^2 = (0, 1)(0, 1) = (0 \,.\, 0 - 1 \,.\, 1, 0 \,.\, 1 + 1 \,.\, 0)$

$= (-1, 0) = -1.$

Hence $i^2 = -1$.

With this convention we shall show that the ordered pair (x, y) is equal to $x + iy$.

For, $(x, y) = (x, 0) + (0, y)$

$= (x, 0) + (0, 1)\,(y, 0)$

$= x + iy$

Since $(x, 0) = x$, $(y, 0) = y$ and $(0, 1) = i$.

Because of the extreme manipulative convenience we shall continue to use the notation $x + iy$ for the complex number (x, y).

9.3 Conjugate of a Complex Number

Let $z = x + iy$ be a complex number. Then the complex number $x - iy$ is called the complex conjugate or simply, the conjugate of z and is denoted by $\bar{z}$.

Thus, if $z = x + iy$ then $\bar{z} = x - iy$.

For example, if $z = 3+4i$ then $\bar{z} = 3 - 4i$.

Clearly $\overline{(\bar{z})} = z$

$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\, Re\, z,$

and $z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2i\, Im\, z.$

Also, $z \,.\, \bar{z} = (x + iy).\,(x - iy)$

$= x^2 - i^2y^2$

$= x^2 + y^2$, which is a real number.

Thus the product of complex number and its conjugate is a real number.

Theorem: For all $z_1, z_2 \in C$

1. $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$

   i.e., the conjugate of a sum is equal to the sum of the conjugates.
2. $\overline{(z_1 \,.\, z_2)} = \overline{z_1} \,.\, \overline{z_2}$

   i.e., the conjugate of a product is equal to the product of the conjugates.
3. $\overline{\left\{\frac{z_1}{z_2}\right\}} = \frac{\overline{z_1}}{\overline{z_2}},\ z_2 \neq 0$

   i.e., the conjugate of a quotient is equal to the quotient of the conjugates.

9.4 Modulus of a Complex Number

If $z = x + iy$ is a complex number then $\sqrt{x^2 + y^2}$ is called the modulus or absolute value of z and is denoted by | z |.

Thus $|z| = \sqrt{x^2 + y^2}$.

Clearly | z | is a non-negative real number i.e., $|z| \geq 0$.

Let z = 3-i4, then $|z| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5$.

We can easily verify the following:

1. $z \, . \, \bar{z} = |z|^2$ 2. $|z| = |\bar{z}|$ 3. $-|z| \leq Re\, z \leq |z|$.

Theorem: For all $z_1, z_2 \in C$

1. $|z_1 \, . \, z_2| = |z_1| \, . \, |z_2|$

   i.e., modulus of a product is equal to the product of their moduli.

2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

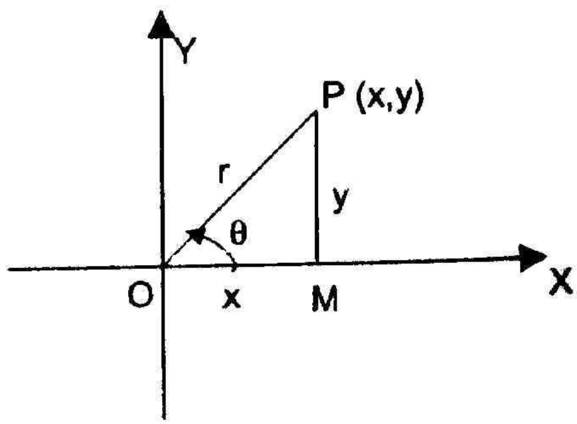
   i.e., modulus of a quotient is equal to the quotient of the moduli.

3. $|z_1 + z_2| \leq |z_1| + |z_2|$

4. $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

9.5 Geometrical Representation of Complex Number

A complex number x + iy can be represented by a point P(x, y) in the Cartesian plane with x as the abscissa and y as the ordinate. Thus every point on the x-axis corresponds to a real number and every point on the y-axis corresponds to a pure imaginary number (iy) and vice versa. Hence x-axis is called the real axis and y-axis, the imaginary axis. And the plane whose points are represented by complex numbers is called the complex plane or Argand plane named after the French mathematician J.R. Argand (1768 – 1822). Although the geometric representation of complex numbers is usually attributed to J.R. Argand but it was Casper Wessel of Norway (1745 – 1818) who first gave the geometric representation of complex numbers.



Now draw PM perpendicular to the x-axis. Let $\angle XOP = \theta$ and OP = r. Clearly OM = x and MP = y.

$$\text{Now} \quad \left.\begin{array}{ll} \cos\theta = \dfrac{OM}{OP} = \dfrac{x}{r} & \therefore x = r\cos\theta \\ \sin\theta = \dfrac{MP}{OP} = \dfrac{y}{r} & \therefore y = r\sin\theta \end{array}\right\} \quad ...(1)$$

Hence, $x + iy = r(\cos\theta + i\sin\theta)$

Thus every complex number $z = x + iy$ can be represented in the form $r(\cos\theta + i\sin\theta)$. This form of a complex number is called the polar form or the trigonometric form.

Squaring and adding the equations given in (1), we get

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$.

$\therefore r = \sqrt{x^2 + y^2}$, which is the modulus of the complex number $z = x + iy$.

Thus $|z|$ represents the distance of the point z from the origin.

The angle θ is called the argument or the amplitude of z and is denoted by $\theta = \arg z$ or $\theta = \text{amp } z$.

Since $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, when n is any integer, θ is not unique. The value of θ satisfying $-\pi < \theta \le \pi$ is called the principal value of the argument.

Note:

1. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

   Then $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

$\therefore |\, z_1 - z_2 \,| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_1)^2}$, which is the distance between the points z_1 and z_2.

2. cos θ + i sin θ is briefly denoted by *cis* θ

Theorem: 1. $cis\theta_1 \; cis\theta_2 = cis(\theta_1 + \theta_2)$

2. $\dfrac{cis\theta_1}{cis\theta_2} = cis(\theta_1 - \theta_2)$

Theorem:

1. amp(z_1z_2)= amp z_1 + amp z_2
2. $amp(\frac{z_1}{z_1}) = amp\ z_1 - amp\ z_2$

Remark:

1. To find the amplitude of a complex number we use the following rule:

sin	cos	θ
+	+	α (say)
+	–	$\pi - \alpha$
–	+	$-\alpha$
–	–	$-(\pi - \alpha)$

For example,

i) if $sin\theta = \frac{\sqrt{3}}{2}, cos\,\theta = \frac{1}{2}$ then $\theta = \frac{\pi}{3}$

ii) If $\sin\theta = \frac{\sqrt{3}}{2}, \cos\theta = -\frac{1}{2}$ then $\theta = \pi - \frac{\pi}{3}$

iii) $\sin\theta = -\frac{\sqrt{3}}{2}, \cos\theta = \frac{1}{2}$ then $\theta = -\frac{\pi}{3}$

iv) if $sin\theta = -\frac{\sqrt{3}}{2}, cos\,\theta = -\frac{1}{2}$ then $\theta = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$.

2. The value of the amplitude θ must satisfy the equations $cos\,\theta = \frac{x}{r}$ and $sin\theta = \frac{y}{r}$, where $r = \sqrt{x^2 + y^2}$. Some times we combine these equations dividing one by another. In that case we get $tan\,\theta = \frac{y}{x}$ or $\theta = tan^{-1}\frac{y}{x}$. Because of the difference in principal values of sin^{-1}, cos^{-1} *and* tan^{-1} the value of the argument is not necessarily the principal value of $tan^{-1}\frac{y}{x}$. For example, $-1 + i = \sqrt{2}\left(cos\frac{3\pi}{4} + i\, sin\frac{3\pi}{4}\right)$ and so $amp(-1+i) = \frac{3\pi}{4}$ but $\tan^{-1}(-1) = -\frac{\pi}{4}$.

9.6 Exponential Form of a Complex Number

If *x* is real, it can be proved, in the advanced mathematics that the functions e^x, *sin x, cos x* etc. can be expressed in the form of an infinite series.

i.e. $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \qquad \text{… (1)}$$

$$sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad \text{… (2)}$$

$$cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad \text{… (3)}$$

Assuming that (1) holds good for a complex number also, replacing *x* by i*x* in (1) we get,

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + ...$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)$$

$$= \cos x + i \sin x$$

Thus, $e^{ix} = \cos x + i \sin x$

This is called the Euler's formula.

We know that a complex number $z = x + iy$ can be expressed in the polar form as

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

The complex number can also be written in the form

$z = x + iy = r\, e^{i\theta}, -\pi < \theta \le \pi.$

This is called the exponential form of a complex number.

Example: Express the following complex numbers in the polar form and hence find their modulus and amplitude.

1) $\sqrt{3} + i$ 2) $1 - i$ 3) $-1 + i\sqrt{3}$

Solution:

1) Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$.

On equating the real and imaginary parts, we get

$r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1.$

Squaring and adding, $r^2\cos^2\theta + r^2\sin^2\theta = \left(\sqrt{3}\right)^2 + 1^2$

$\therefore r^2 = 4$ or $r = 2$

Hence $\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$

Hence $\theta = \dfrac{\pi}{6}$.

$\therefore \sqrt{3} + i = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

Therefore modulus = 2 and amp $\left(\sqrt{3} + i\right) = \dfrac{\pi}{6}$

2. Let $1 - i = r(\cos\theta + i\sin\theta)$

$\therefore r\cos\theta = 1,$ $r\sin\theta = -1$

$r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$ i.e. $r^2 = 2,$ or $r = \sqrt{2}$.

$\therefore \cos\theta = \dfrac{1}{\sqrt{2}},\ \sin\theta = -\dfrac{1}{\sqrt{2}}$

Therefore, $\theta = -\dfrac{\pi}{4}$

$$\therefore 1-i=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$

i.e. Modulus = $\sqrt{2}$, $amp(1-i)=-\frac{\pi}{4}$.

3. Let $-1+i\sqrt{3}=r(\cos\theta+i\sin\theta)$.

   Hence, r $\cos\theta=-1$, $r\sin\theta=\sqrt{3}$

   $\therefore r^2\cos^2\theta+r^2\sin^2\theta=1+3$, or $r^2=4$, or $r=2$

   Hence, $\cos\theta=-\frac{1}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}$, $\theta=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

   $$\therefore -1+i\sqrt{3}=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

   $\therefore$Modulus = 2, amp $\left(-1+i\sqrt{3}\right)=\frac{2\pi}{3}$

Example: If $a=\cos\theta+i\sin\theta, 0<\theta<2\pi$ prove that $\frac{1+a}{1-a}=i\cot\frac{\theta}{2}$

Solution:

$$L.H.S.=\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$$

$$=\frac{2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\frac{2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\left[\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}\right]$$

Multiplying and dividing b y i,

$$=i\cot\frac{\theta}{2}\frac{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}$$

$$=i\cot\frac{\theta}{2}=R.H.S.$$

Self Assessment Questions

1. Find the smallest positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1.$

Example: If $x + iy = \sqrt{\frac{a+ib}{c+id}}$ prove that $\left(x^2 + y^2\right)^2 = \frac{a^2 + b^2}{c^2 + d^2}$

Solution:

Now $x + iy = \sqrt{\frac{a+ib}{c+id}}$. Taking the conjugate on both sides

We get, $x - iy = \sqrt{\frac{a-ib}{c-id}}$

Multiplying, $(x+iy)(x-iy) = \sqrt{\frac{(a+ib)(a-ib)}{(c+id)(c-id)}}$

$\therefore x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$ $\quad \therefore \left(x^2 + y^2\right)^2 = \frac{a^2+b^2}{c^2+d^2}$

9.7 De Moivre's* Theorem

If n is any integer, then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ ……... (1)

And if n is a rational fraction say $\frac{p}{q}$ then $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ has q values and one of its values is $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta.$

Proof: Case (i) Let n be a positive integer.

In this case we shall prove (1) by mathematical induction.

If $n = 1$ *then* $(\cos\theta + i\sin\theta)^1 = \cos 1 . \theta + i\sin 1 . \theta.$

$= \cos\theta + i\sin\theta.$

Hence (1) is true for $n = 1.$ Assume that *(1)* is true for $n = m,$

i.e., $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ (Induction hypothesis)...(2)

Multiplying both sides of *(2) by* $\cos\theta + i\sin\theta$ we get

$(\cos\theta + i\sin\theta)^{(m+1)} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$

$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\sin m\theta\cos\theta + \cos m\theta\sin\theta)$

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta)$

$= \cos (m + 1)\ \theta + i \sin (m + 1)\theta$

Hence the theorem is true for $n = m + 1$.

Hence by mathematical induction the theorem is true for all positive integers n.

Case (ii).Let n be a negative integer.

$\therefore n = -m$, where m is a positive integer.

Consider $(\cos \theta + i \sin \theta)^n \quad = (\cos \theta + i \sin \theta)^{-m}$

$$= \frac{1}{(\cos\theta + i \sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i \sin m\theta} \text{ from case } (i)$$

$$= \frac{\cos m\theta - i \sin m\theta}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \quad \because i^2 = -1$$

$$= \cos m\theta - i \sin m\theta$$

$$= \cos (-m)\ \theta + i \sin (-m)\theta$$

$$= \cos n\theta + i \sin n\theta.$$

Case (iii). Let n be a rational fraction i.e., $n = \frac{p}{q}$, where p and q are integers and $q > 0$.

Let $\qquad z = \cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta$

$$\therefore z^q = \left[\cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta\right]^q$$

$$= \cos q\frac{p}{q}\theta + i \sin q\frac{p}{q}\theta$$

$$= \cos p\theta + i \sin p\theta$$

$$z^q = (\cos\theta + i \sin \theta)^p,$$

which is an algebraic equation of degree q. Hence from fundamental theorem of algebra it has q roots. Therefore taking q^{th} root on both sides, we get $z = (\cos\theta + i \sin\theta)^{\frac{P}{Q}}$

Hence $(cos\,\theta + i\,sin\theta)^{\frac{p}{q}}$ has q values and one of them is $z = cos\frac{p}{q}\theta + i\,sin\frac{p}{q}\theta.$

This completes the proof of the theorem.

Note: Replacing θ *by* – θ *in (1),* we get

$$(cos\ \theta - i\ sin\ \theta)^n = cos\ n\theta - i\ sin\ n\theta$$

Self Assessment Questions

2. Simplify

$$\frac{(cos\,3\theta + i\,sin3\theta)^5\ .\ (cos\,2\theta - i\,sin2\theta)^3}{(cos\,4\theta + i\,sin4\theta)^2\ .\ (cos5\theta - i\,sin5\theta)^4}$$

Example: Prove that $\left(-1 + i\sqrt{3}\right)^{3n} + \left(1 - 1i\sqrt{3}\right)^{3n} = 2^{3n} = 2^{3n+1},$ where n is any integer

Solution:

Expressing $-1 + i\sqrt{3}$ in the polar form, we get

$$-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Taking conjugate on both sides, we get

$$-1 - i\sqrt{3} = 2\left(cos\frac{2\pi}{3} - i\,sin\frac{2\pi}{3}\right)$$

L.H.S. = $\left[2\left(cos\frac{2\pi}{3} + i\,sin\frac{2\pi}{3}\right)\right]^{3n} + \left[2\left(cos\frac{2\pi}{3} - i\,sin\frac{2\pi}{3}\right)\right]^{3n}$

$= 2^{3n}(cos\ 2n\pi + i\,sin\,2n\pi) + 2^{3n}(cos\ 2n\pi - i\,sin\,2n\pi)$

Using De Moivre's theorem

$= 2^{3n}\ .\ 2\ cos\ 2n\pi = 2^{3n+1}$ since $cos\ 2n\pi = 1$

= R.H.S.

Example: Prove that

$$(1 + cos\,\theta + i\,sin\theta)^n + (1 + cos\,\theta - i\,sin\theta)^n = 2^{n+1}\ cos^n\,\frac{\theta}{2}\ .\ cos\left(\frac{n\theta}{2}\right)$$

Where n is any integer.

Solution:

$$L.H.S. = \left(2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n + \left(2\cos^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n + \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)\right]^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) + 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

From De Moivre's theorem.

$$= 2^n\cos^n\frac{\theta}{2}\,.\,2\cos\frac{n\theta}{2} = 2^{n+1}\cos^n\frac{\theta}{2}\cos\left(\frac{n\theta}{2}\right) = R.H.S.$$

9.8 n^th Roots of a Complex Number

If $z^n = a$, where a is a non-zero complex number and n, is a positive integer then z is called the n^{th} root of a. Since the given equation is of degree n, there are n roots of the equation. Hence solving $z^n = a$, we obtain n, n^{th} roots of a.

Example: Find the cube roots of $\sqrt{3} + i$ and represent them on the Argand plane. Also find their continued product.

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\therefore r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$.

Squaring and adding $r^2\cos^2\theta + r^2\sin^2\theta = 3+1$

$\therefore r^2 = 4,\ \therefore r = 2$

Hence;

$$\cos\theta = \frac{\sqrt{3}}{2};\ \sin\theta = \frac{1}{2}$$

$\therefore \theta = \frac{\pi}{6}$ (Principal value)

$$\theta = 2n\pi + \frac{\pi}{6}$$

$$\therefore \sqrt{3} + i = 2\left[\cos\left(2n\pi + \frac{\pi}{6}\right) + i\sin\left(2n\pi + \frac{\pi}{6}\right) = 2.cis\left(2n\pi + \frac{\pi}{6}\right)\right]$$

$\left(\sqrt{3}+i\right)^{\frac{1}{3}} = \left[2cis\left(2n\pi + \frac{\pi}{6}\right)\right]^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}.\ cis\frac{1}{3}\left(2n\pi + \frac{\pi}{6}\right)$

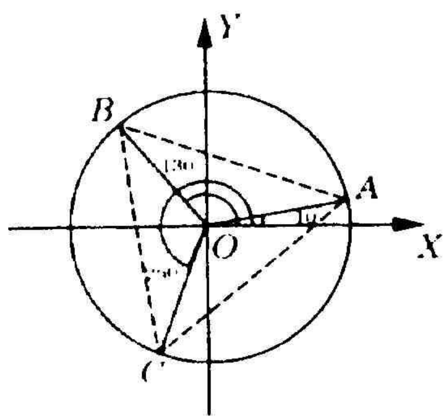
$= 2^{\frac{1}{3}}.\ cis\ \frac{(12n+1)\pi}{18}$

Substituting $n = 0, 1, 2$ (or any three consecutive values of n), we obtain the cube roots of $\sqrt{3}+i$

They are $2^{\frac{1}{3}}\ cis\frac{\pi}{18},\ 2^{\frac{1}{3}}\ cis\frac{13\pi}{18},\quad 2^{\frac{1}{3}}\ cis\frac{25\pi}{18}$

i.e., $2^{\frac{1}{3}}\ cis10°,\ 2^{\frac{1}{3}}\ cis130°,\quad 2^{\frac{1}{3}}\ cis250°.$

To represent these roots on the Argand plane consider a circle whose centre is at the origin and whose radius is $2^{\frac{1}{3}}$. Since modulus of each of these roots is $2^{\frac{1}{3}}$, these roots lie on the circle.



In above figure the points A, B, C represent the cube roots of $\sqrt{3}+i.$

Since $\angle AOB = \angle BOC = \angle COA = 120°,\ A, B, C$ are the vertices of an equilateral triangle

Continued product = $2^{\frac{1}{3}}\ cis\frac{\pi}{18}\ .\ 2^{\frac{1}{3}}\ cis\frac{13\pi}{18}.2^{\frac{1}{3}}cis\frac{25\pi}{18}$

$$= \left(2^{\frac{1}{3}}\right)^3 cis\left(\frac{\pi}{18} + \frac{13\pi}{18} + \frac{25\pi}{18}\right)$$

$$= 2cis\frac{13\pi}{6} = 2cis\left(2\pi + \frac{\pi}{6}\right)$$

$$= 2cis\frac{\pi}{6} = \sqrt{3} + i.$$

Note:

1. The cube roots of unity are $1, \omega, \omega^2$ where $\omega = cis\frac{2\pi}{3}$. Also, $1 + \omega + \omega^2 = 0.$
2. The fourth roots of unity are $1, -1, i, -i.$
3. In general the nth roots of unity are $1, \omega, \omega^2, \ldots\ldots\ldots\omega^{n-1}$ where $\omega = cis\frac{2\pi}{n}.$

9.9 Summary

In this unit, we discuss about the concept of complex numbers in detail. The idea of representing a complex number in Polar Form is explained in a simple manner. The method of finding the roots of a Complex Numbers using De Moivere's Theorem is well illustrated.

9.10 Terminal Questions

1. State and prove De Moivre's Theorem
2. Find the Cube roots of the Complex Numbers 1+i and express it in the Argand Diagram

9.11 Answers

Self Assessment Questions

1. Now $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)}$

$$= \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = i.$$

Therefore $\left(\frac{1+i}{1-i}\right)^n = (\frac{2i}{2})^n = i^n$

Now by inspection $n = 4$ is the smallest positive integer such that $i^n = 1$.

2. Given expression $= \dfrac{[cis3\theta]^5 \,.\, [cis(-2\theta)]^3}{[cis4\theta]^2 \,.\, [cis(-5\theta)]^4}$

$$= \frac{\left[(cis\theta)^3\right]^5 \,.\, \left[(cis\theta)^{-2}\right]^3}{\left[(cis\theta)^4\right]^2 \,.\, \left[(cis\theta)^{-5}\right]^4}$$

$$= \frac{[cis\theta]^{15} \,,\, [cis\theta]^{-6}}{[cis\theta]^8 \,.\, [cis\theta]^{-20}}$$

$$= [cis\theta]^{15-6-8+20} = [cis\theta]^{21}$$

$= cis\ 21\theta$

$= cos\ 21\theta + i\ sin\ 21\theta$