BACHELOR OF COMPUTER APPLICATIONS

SEMESTER 3

DCA2101

COMPUTER ORIENTED NUMERICAL METHODS

Unit 4

Solution of Algebraic and Transcendental Equations

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1. INTRODUCTION

One of the basic problems in science and engineering is the computation of roots of an equation in the form, $f(x) = 0$. The equation $f(x) = 0$ is called an algebraic equation, if it is purely a polynomial in x; it is called a transcendental equation if f(x) contains trigonometric, exponential or logarithmic functions. For example,

$x^4 + 5x^3 - 6x^2 + 3x + 5 = 0$, is an algebraic equation, whereas

$M = E - e \sin E$ and $ax^2 + \log(x - 3) + e^x \sin x = 0$

are transcendental equations.

To find the solution of an equation $f(x) = 0$, we find those values of x for which $f(x) = 0$ is satisfied. Such values of x are called the **roots** of $f(x) = 0$. Thus *a* is a root of an equation $f(x) = 0$, if and only if, $f(a) = 0$.

Consider $f(x) = x^2 - 1$. Now

$$f(1) = 1^2 - 1 = 0$$

$$f(-1) = (-1)^2 - 1 = 0$$

$$f(2) = 2^2 - 1 = 3$$

Here $x = 1$, and $x = -1$ are the roots of $x^2 - 1 = 0$ (i.e. $a = 1$, $a = -1$) but $f(2) \neq 0$, and so $x = 2$ is not the root of $x^2 - 1 = 0$. Geometrically, a root of the equation $f(x) = 0$ is the value of x at which the graph of $y = f(x)$ intersects the x -axis.

We shall list below some of the basic properties of an algebraic equation:

(i) Every algebraic equation of n^{th} degree, where n is a positive integer, has n and only n roots.

(ii) Complex roots occur in pairs. That is, if $(a + ib)$ is a root of $f(x) = 0$, then $(a - ib)$ is also a root of this equation.

(iii) If $x = a$ is a root of $f(x) = 0$, a polynomial of degree n, then $(x - a)$ is a factor of f(x). On dividing f(x) by $(x - a)$ we obtain a polynomial of degree $(n - 1)$.

(iv) Descarte's rule of signs: The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficients cannot exceed the number of changes in the sign of the coefficients in the polynomial f(x) = 0. Similarly, the number of negative roots of f(x) = 0 cannot exceed the number of changes in the sign of the coefficients of f(– x) = 0.

For example, consider an equation $x^3 - 3x^2 + 4x - 5 = 0$. As there are three changes in sign, also, the degree of the equation is three, and hence the given equation will have all the three positive roots.

If f(x) is a quadratic, cubic, or a biquadratic expression, then algebraic formulae are available for expressing the roots in terms of the coefficients. On the other hand when f(x) is a polynomial of a higher degree or an expression involving transcendental functions.

Solving either a transcendental equation or an algebraic equation can be classified into two types: direct methods and iterative methods. Direct methods require no knowledge of the initial approximation of a root of the equation f(x) = 0, while iterative methods do require a first approximation to initiate iteration.

For example, we can approximate the roots of $1 + \cos x - 5x$, $xe^x - 1$,

$x \tan x - \cosh x$, $x - e^{\frac{1}{x}}$, $e^{-x} - \sin x + 1$ etc., by numerical techniques.

In this unit, we discuss a few numerical techniques for the approximation of the solution of equations of the form f(x) = 0, where f(x) is algebraic or transcendental or a combination of both.

1.1 Objectives:

At the end of this unit the student should be able to:

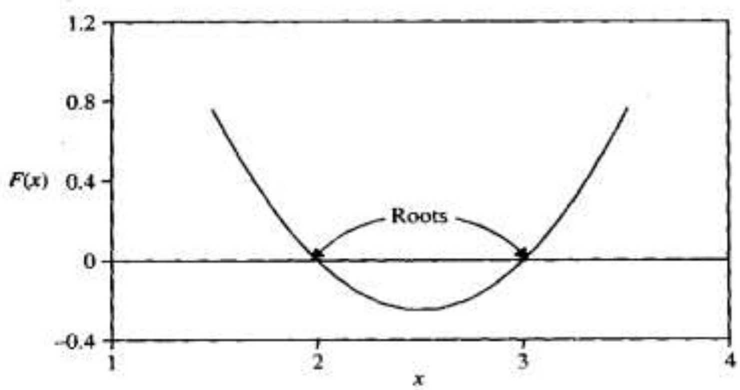
- *Know the types of nonlinear equations.*
- *To know the analytical methods for finding the roots of nonlinear equations.*
- *Approximation methods for finding the roots of algebraic and transcendental equations.*
- *Know the geometrical interpretations of methods.*
- *Analyze the convergence in the numerical methods.*

2. GRAPHICAL AND ANALYTICAL METHODS

Now we present two methods namely the Graphical method and Analytical method, to explain the first approximation root of f(x) = 0.

2.1 Graphical method

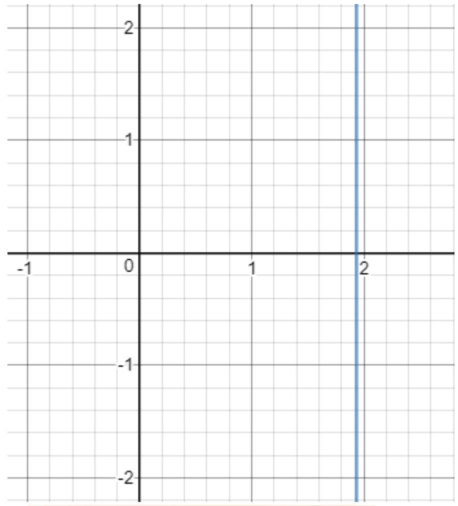
One of the basic methods to find the roots of a nonlinear equation is to plot the equation on a traditional graph paper or, even better, plot it using a graph-plotting software. Each time the curve passes through f(x) = 0, a root of the equation is indicated. To illustrate, let us find the roots of the equation: $f(x) = x^2 - 5x + 6 = 0$, using the graphical method. Once we plot f(x) against x, for $1 \le x \le 4$, it is observed that the curve crosses the f(x) = 0 line twice, at x = 2 and at x = 3. The plot is presented in the following figure. Therefore, the roots of this quadratic equation are 2 and 3. Since we know the analytical solution of this equation, it is easy to verify the results.



The solution of quadratic equation using graphical method

Often, the equation f(x) = 0 can be rewritten as $f_1(x) = f_2(x)$ and the first approximation to a root of f(x) = 0 can be taken as the abscissa of the point of intersection of the graphs of y $=f_1(x)$ and y $=f_2(x)$. For example, consider, f(x) = x − sin x − 1 = 0. It can be written as x − 1 = sin x. Now, we shall draw the graphs of y = x − 1 and y = sin x as shown in the figure below.

The approximate value of the root is found to be 1.9.



Demerits: If the roots are to be determined as a part of a larger computational task, the graphical method is quite tedious. Also, the root needs to be located precisely on the graph.

2.2 Analytical method

This method is based on 'intermediate value property'. We shall illustrate it through an example. Let f(x) $= 3x - \sqrt{1+\sin x} = 0$. We can easily verify

f(0) = – 1 and f(1) = $3-\sqrt{1+\sin\left(1\times\frac{180}{\pi}\right)} = 3-\sqrt{1+0.84147} = 1.64299.$

We observe that f(0) and f(l) are of opposite signs. Therefore, using intermediate value property we infer that there is at least one root between

x = 0 and x = 1.

This method is often used to find the first approximation to a root of either a transcendental equation or algebraic equation. Hence, in the analytical method, we must always start with an initial interval (a, b), so that f(a) and f(b) have opposite signs.

Self-Assessment Questions - 1

1. The solution of the equation f(x) = 0 is called ___________ .
2. The nth degree polynomial has ___________ roots.
3. The equation of the form $x - \frac{1}{e^x} = 0$, is called ___________.
4. Analytical method based on ___________ property.
5. The value of x at which f(x) = 0 intersects x-axis is ___________.

3. INTERVAL HALVING METHOD (OR BISECTION METHOD)

This method is developed by Bernard Bolzano, a mathematician from Prague (now part of the Czech Republic). His work consisted of texts on the foundation of mathematics, calculus, theories of infinity and paradoxes, and geometry.

3.1 Method

It is based on repeated application of the Intermediate Value Theorem.

The function changes its sign at the two opposite points around a root. These two points can constitute the lower (x_{low}) and upper (x_{high}) limits of the interval.

Step 1: Consider two trail points which by enclose the roots. Two points *a* and *b* enclose a root if f(a) < 0 (negative) and f(b) > 0 (positive) are of opposite signs.

Step 2: Bisect the interval (a, b) and denote the mid-point by x_1, so that

$x_1 = \frac{a+b}{2}$. If $f(x_1) = 0$, we conclude that x_1 is a root of the equation.

Otherwise

Step 3: The root lies either between x_1 and *b* if $f(x_1) < 0$, or the root lies between x_1 and *a* if $f(x_1) > 0$.

If $f(x_1) > 0$ (positive), then

Step 4: Replace b by x_1 and search for the root in this new interval which is half the previous interval. Then the second approximation to the root is

$x_2 = \frac{a+x_1}{2}$. If $f(x_2) = 0$ then x_2 is a root of $f(x) = 0$.

If $f(x_2) < 0$ (negative), then

Step 5: The root lies between x1 and x2. Then the third approximation to the root is $x_3 = \frac{x_1+x_2}{2}$.

Thus at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in the figure. This process of halving the intervals is continued to determine a smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root. This method is also known as the interval halving method.

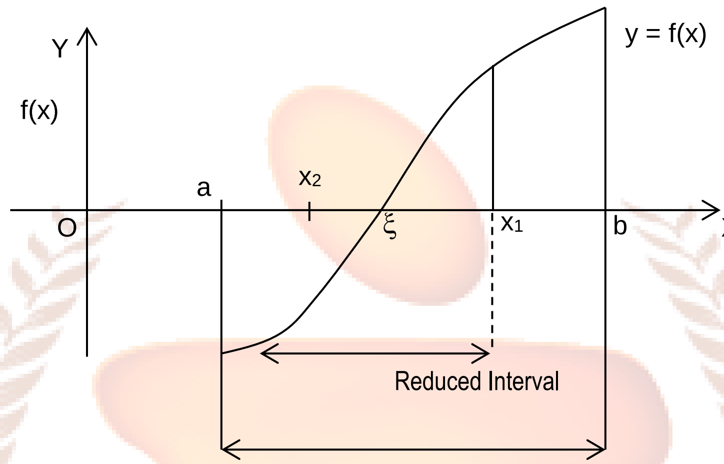


Fig.: The bisection method

Example: Find a real root of the equation $x^3 - 4x - 9 = 0$ using the bisection method.

Solution: Let $f(x) = x^3 - 4x - 9$

$f(1) = 1 - 4 - 9 = -12 < 0$

$f(2) = 8 - 8 - 9 = -9 < 0$

$f(3) = 27 - 12 - 9 = +6 > 0$

Since f(2) is negative and f(3) is positive, a root lies between 2 and 3.

Therefore the first approximation to the root is $x_1 = \frac{2+3}{2} = 2.5$

Then $f(x_1) = f(2.5) = 2.5^3 - 4 \times 2.5 - 9 = -3.375 < 0$.

Hence the root lies between 2.5 and 3. Since $f(2.5) < 0$ and $f(3) > 0$, we have the second approximation to the root is

$$x_2 = \frac{x_1 + 3}{2} = \frac{2.5 + 3}{2} = 2.75$$

Then $f(x_2) = f(2.75) = (2.75)^3 - 4 \times 2.75 - 9 = 0.7969 > 0$.

Therefore conclude that the root lies between 2.5 and 2.75 and so the third approximation to the root is

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

Also f (x_3) = f (2.625) = –1.4121 < 0. Now since f (x_3) < 0 and f(x_2) > 0, we have that root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{2.75 + 2.625}{2} = 2.6875$$

The procedure is repeated and the successive approximations are

x_5 = 2.703125, x_6 = 2.7109375, x_7 = 2.70703125, x_8 = 2.7051 etc.

Therefore $x = x_8$ = 2.7051 is the approximate root.

Problem

Find a real root of the equation $xe^x - 1 = 0$ correct to four decimal places.

Solution: Let $f(x) = xe^x - 1$.

By trail and error, f(0) = –1 < 0 , f(1) = 1.71870, we have that a root lies between 0 and 1.

Therefore $x_1 = \frac{0+1}{2} = 0.5$.

Now f (x_1) = f (0.5) = (0.5) $e^{0.5}$ – 1 = – 0.1756 < 0.

We minimize the interval by choosing [0.5, 1]. Then $x_2 = \frac{0.5+1}{2} = 0.75$.

Proceeding as above, the following table is constructed where only the sign of the function value is indicated.

i	x_{low}	x_{high}	x_i	$f(x_i)$
1	0	1	0.5	–
2	0.5	1	0.75	+
3	0.5	0.75	0.625	+
4	0.5	0.625	0.5625	–
5	0.5625	0.625	0.59375	+
6	0.5625	0.59375	0.5781	+

7	0.5625	0.5781	0.5703	+
8	0.5625	0.5703	0.5664	–
9	0.5664	0.5703	0.5684	+
10	0.5664	0.5684	0.5674	+
11	0.5664	0.5674	0.5669	–
12	0.5669	0.5674	0.5671	–
13	0.5671	0.5674	0.5672	+
14	0.5671	0.5672	0.56715	+
15	0.5671	0.56715	**0.56713**	+

The required root is 0.5671 which is correct to 4 decimal places.

Problem

Find the real root of $x^3 - 9x + 1 = 0$ using bisection method.

Solution: Given $f(x) = x^3 - 9x + 1 = 0$. Now by trail and error, $f(2) = -9 < 0$ and $f(4) = 29 > 0$. Therefore the root lies between 2 and 4.

Let $x_0 = 2$ and $x_1 = 4$. Now $x_2 = \frac{x_0 + x_1}{2} = \frac{2+4}{2} = 3$ as a first approximation to a root of $f(x) = 0$ and note that $f(3) = 1 > 0$.

Therefore the root lies between 2 and 3. Now $x_3 = \frac{x_0 + x_2}{2} = \frac{2+3}{2} = 2.5$. Also $f(x_3) < 0$ so that $f(2.5)f(3) < 0$. Therefore define the midpoint

$x_4 = \frac{x_3 + x_2}{2} = \frac{2.5+3}{2} = 2.75$.

Similarly we get that $x_5 = 2.875$ and $x_6 = 2.9375$.

Continue this process until the root is obtained to the desired accuracy.

Observation: If the function is continuous and changes its sign between the limits given, the bisection method will surely converge the solution, provided a reasonable tolerance for the error is provided. If there are two or more roots in the interval, this method will find one of them. However, if there is an even number of multiple roots (same root occurring twice), the bisection method will encounter difficulty.

Self-Assessment Questions - 2

6. Bisection method is based on ___________.
7. Bisection method is ___________.
8. The real root of the equation $f(x) = x^3 - 3x - 5=0$ lies between ________.
9. The real root of the equation $f(x) = x^3 + x - 3=0$ lies between ________.
10. Let f(x) be continuous in (a, b) and x_1 be its first approximation root. If $f(x_1)$ is positive then the root lies between ___________.

4. REGULA - FALSI METHOD (OR FALSE POSITION OR METHOD OF

This method resembles the bisection method for finding the real root of a nonlinear equation f(x) = 0.

4.1 Method

Choose two points x_0 and x_1 such that $f(x_1)$ and $f(x_2)$ are of opposite signs. Since the graph of y = f(x) crosses the X-axis between these two points.

This indicates that a root lies between these two points x_1 and x_2.

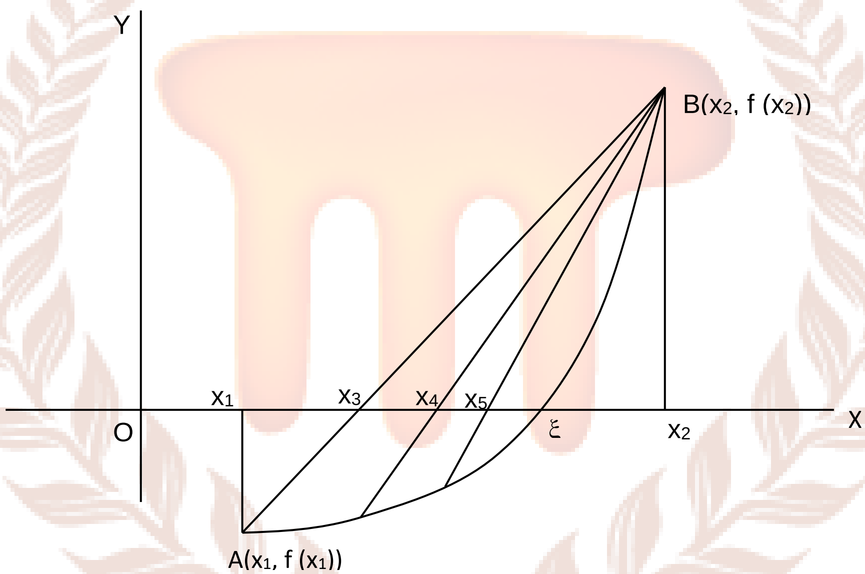


Fig. Regula-Falsi Position

Equation of the chord joining the points A $(x_1, f(x_1))$ and B $(x_2, f(x_2))$ is

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} (x - x_1) \qquad - (i)$$

where $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the slope of the line AB.

The method consists in replacing the curve AB using the chord AB and taking the point of intersection of the chord with the X-axis as an approximation to the root. The point of intersection in the present case is given by putting y = 0 in (i).

Thus we obtain

$$0 - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1). \text{ Solve for } x,$$

we get $$x = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)} \quad \text{– (ii)}$$

Hence the second approximation to the root of $f(x) = 0$ is given by

$$x_3 = x = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)} \quad \text{– (iii)}$$

If $f(x_3)$ and $f(x_1)$ are of opposite signs, then the root lies between x_1 and x_3, and we replace x_2 with x_3 in (iii), and obtain the next approximation. Otherwise, $f(x_3)$ and $f(x_1)$ are of the same sign; we replace x_1 with x_3 and generate the next approximation. The procedure is repeated till the root is obtained to the desired accuracy.

Example

Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$, if the root lies between 2 and 4.

Solution: Let $f(x) = x^3 - 9x + 1$. Now $f(2) = -9$ and $f(4) = 29$.

Since f(2) and f(4) are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4.

Take $x_1 = 2$ and $x_2 = 4$. By the Regula-Falsi method, the first approximation is $x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} = 4 - \frac{29 \times 2}{38} = 2.47368$ and $f(x_3) = -6.12644$.

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as $x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)} = 2.733989$ and $f(x4) = -3.090707$.

Now since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from $x_5 = x_4 - \frac{f(x_4)(x_4 - x_2)}{f(x_4) - f(x_2)} = 2.86125$ and $f(x_5) = -1.32686$.

Continue this procedure till we get the desired degree of accuracy.

Example

Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of Regula- Falsi position, correct to three decimal places.

Let $f(x) = x^3 - 2x - 5$

Then $f(1) = 1 - 2 - 5 = -6 < 0$

$f(2) = 8 - 4 - 5 = -1 < 0$

$f(3) = 27 - 6 - 5 = 16 > 0$

Hence a root lies between 2 and 3.

Take $x_1 = 2$, $x_2 = 3$, $f(x_1) = -1$, $f(x_2) = 16$, in the method of false position, we get

$$x_3 = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)} = 2 - = \frac{(-1)(3-2)}{16-(-1)} = = 2 + \frac{1}{17} = 2.0588$$

Now $f(x_3) = f(2.0588) = (2.0588)^3 - 2 \times 2.0588 - 5 = -0.3908 < 0$.

Since $f(2.0588) < 0$ we have that root lies between 2.0588 and 3.0.

Replace x_1 by x_3, and generate the next approximation using the formula,

$$x4 = x3 - \frac{f(x_3)\,(x_2 - x_3)}{f(x_2) - f(x_3)} = 2.0588 - \frac{(-0.3908)(3 - 2.0588)}{16 - (-0.3908)} = 2.0813.$$

Repeating this process, the successive approximations are $x_5 = 2.0862$,

$x_6 = 2.0915$, $x_7 = 2.0934$, $x_8 = 2.0941$, $x_9 = 2.0943$ etc.

Hence the root $x = 2.094$ correct to three decimal places.

Observation: The regula-falsi method is intended to produce faster convergence to the solution. However, it does not always do so. Sometimes, the values of x_{new} do not improve quickly. One reason for the slow convergence can be the departure from the basic premises on which the false position method is built.

Note: Formula for Regula Falsi method can also be written as,

$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$ where, $f(a)<0$ and $f(b)>0$

Self-Assessment Questions -3

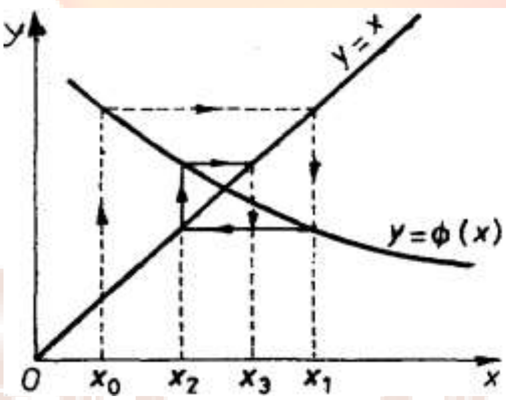
11. What is the first approximation root in Regula-Falsi method?
12. The false position method can be much faster than the bisection method (True/False).
13. Convergence in False position method is ___________.

5. SUCCESSIVE APPROXIMATION METHOD OR PICARD ITERATION

5.1 Initial approximations:

Initial approximations to the root are often known from the physical considerations of the problem. Otherwise graphical methods are generally used to obtain initial approximations of the root. The value of x for which the graph of the equation y = f(x) intersects the X-axis, given the root of f(x) = 0, any value in the neighborhood of this point may be taken as an *initial approximation to the root*.

If the equation f (x) = 0 can be written in the form $f_1(x) = f_2(x)$ then the point of intersection of graphs of $y = f_1(x)$ and $y = f_2(x)$ gives the root of f(x) = 0 and so any value in the neighbourhood of this point can be taken as an *initial approximation* to the root.



Iteration Method (Fixed Point)

Another method of finding initial approximation is by intermediate value theorem, which states that, if f(x) is a continuous function in [a, b] and f(a) f(b) < 0, then the equation f(x) = 0 has at least one real root or an odd number of roots in [a, b]. We can set up a table of values of f(x) for various values of x and obtain a suitable *initial approximation* to the root.

To find the roots of the equation

$$f(x) = 0 \quad \text{– (i)}$$

we rewrite this equation in the form

$$x = \phi(x) \quad \text{– (ii)}$$

For example, f(x) = cos x – 2x + 3 = 0 can be expressed as

$x = \frac{1}{2}(\cos x + 3) = \phi(x)$.

Similarly, the equation $x^3 + x^2 - 1 = 0$ can be expressed as

$$x = (1 - x^3)^{\frac{1}{2}} = \phi(x) \text{ or}$$

$$x = (1 - x^2)^{\frac{1}{3}} = \phi(x).$$

5.2 Method: The roots of $f(x) = 0$ are the same as the point of the intersection of the line $y = x$ and the curve $y = \phi(x)$. This point is known as the fixed point of $\phi(x)$.

Step1: Let $x = x_0$ be an initial approximation of the desired root ξ. Substituting it for x on the right side of $x = \phi(x)$, we obtain the first approximation

$$x_1 = \phi(x_0)$$

Step 2: The second approximation is

$x_2 = \phi(x_1)$ and the successive approximations are given by

$$x_3 = \phi(x_2)$$

$$x_4 = \phi(x_3)$$

- - -

- - -

$x_n = \phi(x_{n-1})$, where xn is the n^{th} iterated root of $f(x) = 0$.

The convergence of an iteration method depends on the suitable choice of the function $\phi(x)$, and x_0, a suitable initial approximation to the root. The following theorem helps in making the choice of x_0 and iteration function $\phi(x)$.

Let $\{x_i\}$ be the sequence obtained by a given method and let $x = \xi$ denotes the root of the equation $f(x) = 0$. Then the method is said to be convergent if and only if $\underset{n \to \infty}{Lt}\, xn -| x_n - \xi| = 0$.

We now state an important theorem; the proof is out of the scope of the book.

Theorem:

If i) ξ is a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$.

ii) I, be any interval containing the point $x = \xi$.

iii) $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2 \ldots\ldots x_n$ will converge to the root ξ provided the initial approximation x0 is chosen in I.

This is a sufficient condition for convergence of iterations.

Example

Find a real root of the transcendental equation cos x – 3x+1 = 0, correct to four decimal places using iteration method.

Let $f(x) = \cos x - 3x + 1$.

Now $f(0) = \cos 0 - 0 + 1 = 2 > 0$ and $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{3\pi}{2} + 1 = -\frac{3\pi}{2} + 1 < 0$. Therefore a root lies between 0 and $\frac{\pi}{2}$.

Rewriting the given equation $\cos x - 3x + 1 = 0$ as

$x = \frac{1}{3}(1 + \cos x) = \phi(x)$ – (i)

$\Rightarrow \phi(x) = \frac{1}{3}(1 + \cos x)$ (say)

Differentiate with respect to x on both the sides,

$\phi'(x) = \frac{-\sin x}{3}$ and

Since $|\sin x| \leq 1$, we have $|\phi'(x)| = \left|\frac{-\sin x}{3}\right| \leq \frac{1}{3} < 1$.

Therefore $|\phi'(x)| < 1$ for all x in $(0, \frac{\pi}{2})$.

Hence the iteration method can be applied to the equation (i) and we start with $x_0 = 0$

$\left[\text{or any choice of x in the interval } \left(0 \frac{\pi}{2}\right)\right]$

$x_1 = \phi(x_0) = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + \cos 0) = 0.6667$

$x_2 = \phi(x_1) = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos 0.6667) = 0.5953$

$x_3 = \phi(x_2) = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos 0.5953) = 0.6093$

$x_4 = \phi(x_3) = \frac{1}{3}(1 + \cos x_3) = \frac{1}{3}(1 + \cos 0.6093) = 0.6067$

$x_5 = \phi(x_4) = \frac{1}{3}(1 + \cos x_4) = \frac{1}{3}(1 + \cos 0.6067) = 0.6072$

$x_6 = \phi(x_5) = \frac{1}{3}(1 + \cos x_5) = \frac{1}{3}(1 + \cos 0.6072) = 0.6071$

$x_7 = \phi(x_6) = \frac{1}{3}(1 + \cos x_6) = \frac{1}{3}(1 + \cos 0.6071) = 0.6071$

Hence we take the solution as x = 0.6071 correct to 4 decimal place.

Problem

Find a real root of $2x - \log_{10} x = 7$ correct to 3 decimal places using iteration method.

Solution: Let $f(x) = 2x - \log_{10} x - 7$. Now

$f(3) = 6 - \log_{10} 3 - 7 = -1.4471 < 0$ and $f(4) = 8 - \log_{10} 4 - 7 = 0.398 > 0$.

Therefore a root lies between 3 and 4.

Rewriting the given equation as

$$x = \frac{1}{2}[\log_{10} x + 7] = \phi(x) \quad \text{(say)}.$$

Now $\phi'(x) = \frac{1}{2x} \log_{10} e$

Therefore $\left|\phi'(x)\right| = \frac{1}{2}\left|\frac{\log_{10} e}{x}\right| = \frac{1}{2}\left|\frac{0.4343}{x}\right| < 1$ where $3 < x < 4$.

Hence the iteration method can be applied.

Take $x_0 = 3.6$ as initial approximation. Since $\left|f(4)\right| < \left|f(3)\right|$, the root is near to 4.

$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$

$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78863$

$x_3 = \phi(x_2) = \frac{1}{2}(\log_{10} 3.78863 + 7) = 3.78924$

$x_4 = \phi(x_3) = \frac{1}{2}(\log_{10} 3.78924 + 7) = 3.78927$

Hence we take the solution as 3.7892.

Problem

Use the method of successive approximation to determine the real root of equation e−x = 10x, and carry out four iterations.

Solution: Let $f(x) = e^{-x} - 10x = 0$. Clearly $f(0) = 1$ and $f(1) = -9.6321$. Since $\left|f(0)\right| < \left|f(1)\right|$, the root is near to $x = 0$.

Rewrite the given equation as $x = \frac{1}{10^{e-x}} = \phi(x)$.

Therefore $\left|\phi'(x)\right| = \frac{1}{10}e^{-x} = \frac{1}{10e^{x}} < 1$ for all x in (0, 1).

Hence the method of iteration can be applied. Start with the initial approximation $x_0 = 0$, then

$x_1 = \phi(x_0) = \frac{1}{10} = 0.1$, $f(x_1) = -0.09516$. Similarly, the successive approximations are

$x_2 = \phi(x_1) = \frac{1}{10}\, e^{-0.1} = \frac{0.904837}{10} = 0.09048$, $x_3 = \phi(x_2) = 0.091349$,

$x_4 = \phi(x_3) = 0.091274$. Hence the required root is 0.0913.

Self-Assessment Questions - 4

14. The method of successive approximation also called ___________.
15. Rewrite the equation xex =1, in the form of x = ⍰(x).
16. Find a positive root between 0 and 1 of the equation xex =1, by the iteration method.

6. NEWTON – RAPHSON METHOD

This is a very powerful method for finding the real root of an equation in form f(x) = 0.

6.1 Method

Step 1: Let x_0 be an approximate root of the equation f(x) = 0.

If $x_1 = x_0 + h$ (where h is very small) be the exact root, then $f(x_1) = 0$.

Step 2: Expand $f(x_1) = f(x_0 + h) = 0$ by Taylor's series, we obtain

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

Since h is small, we neglect terms containing h^2 and its higher powers, then

$$f(x_0) + h f'(x_0) = 0 \text{ or } h = -\frac{f(x_0)}{f'(x_0)} .$$

Step 3: Write a better approximation to the root

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

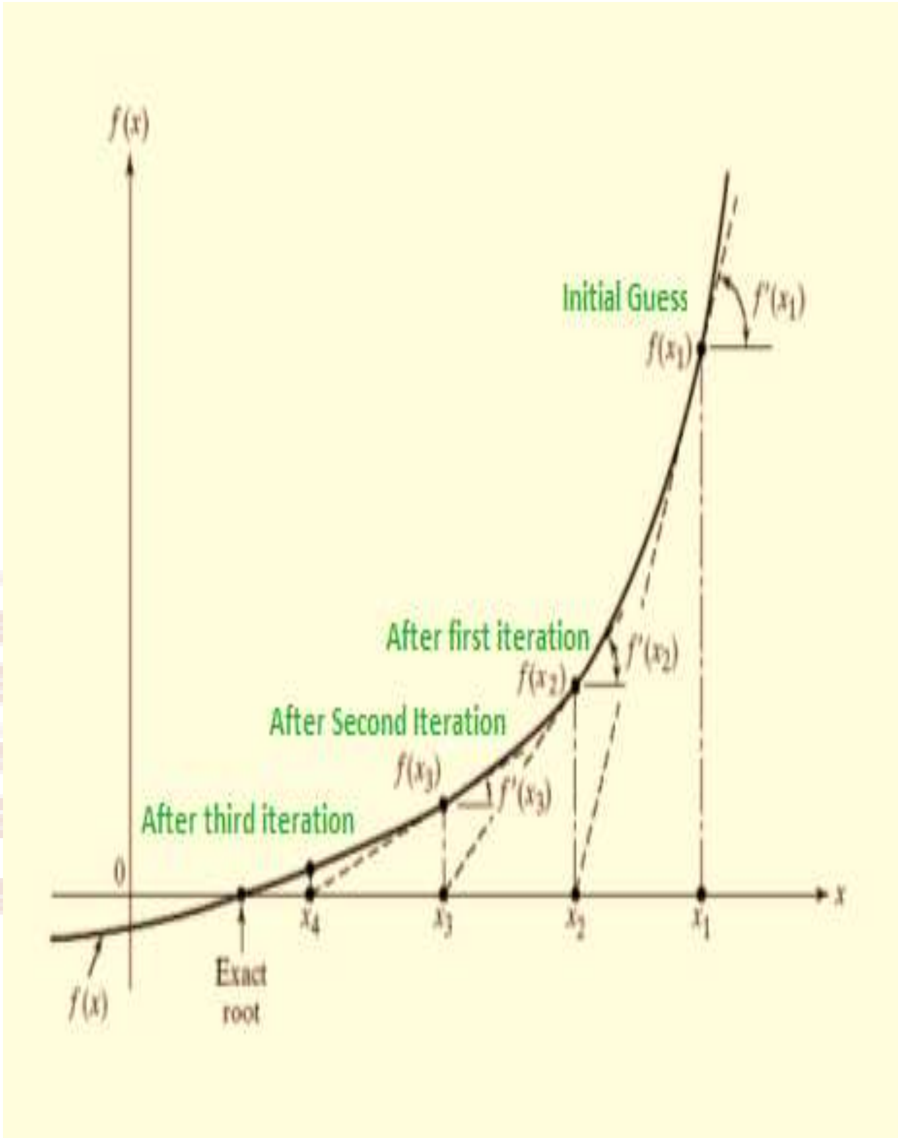
Step 4: Repeat step 2, starting with x_1, write a still better approximation x_2 as $x_2 = x_1 - \frac{f(x_0)}{f'(x_0)}$.

Step 5: Continue, to get $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, n = 0, 1, 2, 3, …,

which is known as the **Newton – Raphson formula.**

6.2 Geometrical interpretation

Suppose, the graph of the function y = f(x) crosses the X-axis at α (see in the Fig), then $x = \alpha$ is the root of the equation f(x) = 0.



Newton-Raphson's Method

Let x_0 be a point closer to the root α, then the equation of the tangent at $P_0(x_0,f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$

This tangent cuts the x-axis at $R_0(x_1, 0)$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

which is a first approximation to the root α. If P_1 is a point on the curve corresponding to x_1, then the tangent at P_1 cuts the x-axis at $R_1(x_2, 0)$, which is still closer to a, than x_1. Therefore,

x_2 is a second approximation to the root. Continuing this process, we arrive at the root α, very rapidly, which is evident from Fig. Thus, in this method, we have replaced the part of the curve between the point P_0 and X-axis with a tangent to the curve at P0, and so on.

Consider the following example.

Example

Find a real root of the equation $x^3 - 2x - 5 = 0$ by Newton– Raphson method.

Solution: Let $f(x) = x^3 - 2x - 5$. Differentiate with respect to x, we get

$$f'(x) = 3x^2 - 2.$$

Now observe that $f(2) = -1 < 0$ and $f(3) = 16 > 0$ and hence a root lies between 2 and 3. Since $|f(2)| < |f(3)|$, a root is near to x = 2.

Let us take the initial approximate root $x_0 = 2$.

Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3; \ldots \qquad -(i)$$

$$= x_n - \frac{(x_n^3 - 2x_n - 5)}{3x_n^2 - 2}$$

Put n = 0 in (i), the first approximation is

$$x_1 = x_0 - \frac{(x_0^3 - 2x_0 - 5)}{3x_0^2 - 2} = 2 - \frac{(8-4-5)}{3 \times 4 - 2} = 2 + \frac{1}{10} = 2.1$$

Now $f(x_1) = f(2.1) = (2.1)^3 - 2(2.1) - 5 = 0.061$ and

$$f'(x_1) = f'(2.1) = 3 \times (2.1)^2 - 2 = 11.23$$

Put n = 1 in (i), the second approximation is

$$x_2 = x_1 - \frac{(x_1^3 - 2x_1 - 5)}{3x_1^2 - 2} = 2.1 - \frac{[(2.1)^3 - 2(2.1) - 5]}{3(2.1)^2 - 2} = 2.094568$$

Similarly $x_3 = 2.094568 - \dfrac{0.00018437}{11.16164532} = 2.09455.$

Hence the required root is 2.0945.

Example

Find the Newton's scheme of iteration for obtaining the square root of a positive number N.

Solution: The square root of N can be carried out as a root of the equation $x^2 - N = 0$. Let $f(x) = x^2 - N$.

By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3 \ldots$$

Since $f'(x) = 2x$, we have $x_{n+1} = x_n - \dfrac{{x_n}^2 - N}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{N}{x_n}\right)$.

Problem

Evaluate $\sqrt{12}$ to four decimal places by Newton's-Raphson formula.

Solution: Let $x = \sqrt{12}$ so that $x^2 - 12 = 0$

Take $f(x) = x^2 - 12$. Now $f(3) = 9 - 12 = -3 < 0$, $f(4) = 16 - 12 = 4 > 0$.

Therefore a root of f(x) lies between 3 and 4.

$f'(x) = 2x$.

Let $x_0 = 3.5$ be the initial approximated value.

By Newton's – Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3 \ldots$$

$$= x_n - \frac{(x_n^2 - 12)}{2\,x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right) \qquad \text{– (i)}$$

Put n = 0, we get

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643$$

Similarly putting n = 1, 2, 3 . . . . in (i), we get

$$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{12}{x_2}\right) = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Since the approximations x_2 and x_3 are the same, we have the value corrected up to 4 decimal places, which is 3.4641.

Problem

Find a real root of the equation x sin x + cos x = 0 by Newton's-Raphson method, by taking the initial approximation as $x = \pi$

Solution: Let f(x) = x sin x + cos x. Differentiating with respect to x

f'(x) = x cos x + sin x – sin x = x cos x

By Newton's formula,

$$x_{n+1} = x_n - \frac{(x_n \sin x_n + \cos x_n)}{x_n \cos x_n}, \quad n = 0, 1, 2, 3 \ldots$$

with $x_0 = \pi$ we have

$$x_1 = x_0 - \frac{(x_0 \sin x_0 + \cos x_0)}{x_0 \cos x_0} = \frac{(\pi \sin \pi + \cos \pi)}{\pi \cos \pi} = 2.82328$$

Similarly, we obtain

$x_2 = 2.7986$

$x_3 = 2.7984$

$x_4 = 2.7984$

Therefore 2.7984 is the required root.

Observations:

1) Newton – Raphson formula converges provided the initial approximation x_0 is chosen sufficiently close to the root ξ.
2) Newton – Raphson method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.
3) The Newton – Raphson method has the fastest convergence. It is an excellent method if one is already near the root. It is useful in the case of large values of $f'(x)$.

Newton – Raphson process has a second order or quadratic convergence.

6.3 Generalised newton's method

If ξ be a root of f (x) = 0 which is repeated m-times, then

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}$$

which is called the generalised Newton's formula. It reduces to Newton – Raphson formula for m = 1.

Since ξ is a root of f (x) = 0 with multiplicity m, then it is also a root of f'(x) = 0 with multiplicity m – 1, of f'' (x) = 0 with multiplicity m – 2 and so on. Hence the expressions

$x_0 - m \frac{f(x_0)}{f'(x_0)}$, $x_0 - (m-1) \frac{f'(x_0)}{f''(x_0)}$, $x_0 - (m-2) \frac{f''(x_0)}{f'''(x_0)}$, …, must have the same value if there is a root with multiplicity m, provided that the initial approximation x_0 is chosen sufficiently close to the root.

Formulae for multiplicity *m*:

We have

$$x_0 - m \frac{f(x_0)}{f'(x_0)} = x_0 - (m-1) \frac{f'(x_0)}{f''(x_0)}$$

$$\Rightarrow m \frac{f(x_0)}{f'(x_0)} = (m-1) \frac{f'(x_0)}{f''(x_0)} \Rightarrow \frac{m-1}{m} = \frac{f(x_0) f''(x_0)}{[f'(x_0)]^2}$$

$$\Rightarrow 1 - \frac{1}{m} = \frac{f(x_0)\, f''(x_0)}{[f'(x_0)]^2} .$$

$$\text{Therefore } m = \frac{[f'(x_0)]^2}{[f'(x_0)]^2 - f(x_0)\, f''(x_0)}$$

Example

Find the double root of the equation $x^3 - x^2 - x + 1 = 0$

Solution: Let $f'(x) = x^3 - x^2 - x + 1$

$$f'(x) = 3x^2 - 2x - 1 \text{ and } f''(x) = 6x - 2$$

Let $x_0 = 0.9$ (taken randomly) be the initial approximated value.

We have $x_0 - m\dfrac{f(x_0)}{f'(x_0)} = x_0 - (m-1)\dfrac{f'(x_0)}{f'(x_0)}$ – (i)

Here $m = 2$, $x_0 = 0.9$, $f(x_0) = f(0.9) = 0.019$, $f'(x_0) = f'(0.9) = -0.37$ and $f''(x_0) = f''(0.9) = 3.4$

Therefore (i) becomes

$$x_0 - \frac{2f(x_0)}{f'(x_0)} = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

we have $x_0 - \dfrac{2f(x_0)}{f'(x_0)} = 0.9 - \dfrac{2 \times 0.019}{-0.37} = 1.003$, and

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.9 - \frac{(-0.37)}{3.4} = 1.009$$

The closeness of these values indicates that there is a double root near

$x = 1$.

For the next approximation we choose $x_1 = 1.01$

we get $x_1 - \dfrac{2f(x_1)}{f'(x_1)} = 1.01 - \dfrac{2 \times 0.0002}{0.0403} = 1.0001$

and $\quad x_1 - \dfrac{f'(x_1)}{f''(x_1)} = 1.01 - \dfrac{0.0403}{4.06} = 1.0001$

This shows that there is a root at x = 1.0001 which is quite near the actual root x = 1.

Observation: For $f(x) = x^3 - x^2 - x + 1 = (x - 1)^2 (x + 1)$. Therefore x = 1 is a double root with multiplicity m = 2.

Self-Assessment Questions - 5

17. The Newton's Raphson method also called as ___________.
18. Newton's Raphson method is ___________ convergence.
19. Newton – Raphson method is generally used for ___________.
20. If ξ be a root of f (x) = 0 which is repeated m-times, then, by Generalized Newton's formula, x_{n+1} = ___________.

7. RAMANUJAN'S METHOD

This method is described by Indian mathematical genius Srinivasa Ramanujan (1887-1920). This is an iterative method which can be used to determine the smallest root of the equation of the form f(x) = 0.

7.1 Method

Step 1: Assume that f(x) is of the form

$f(x) = 1 - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots)$.

Step 2: For smaller values of x, we can write

$[1 - (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots)]^{-1} = b_1 + b_2x + b_3x^2 + \dots$

$\Rightarrow 1 + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots) + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots)^2 + \dots$

$= b_1 + b_2x + b_3x^2 + \dots$ (by binomial expansion on left hand side)

Step 3: Comparing the coefficients of like powers of x on both sides, we get $b_1 = 1$, $b_2 = a_1 = a_1b_1$, $b_3 = a_1^2 + a_2 = a_1b_2 + a_2b_1$, …,

$b_n = a_1b_{n-1} + a_2b_{n-2} + \dots + a_{n-1}b_1$, n = 2, 3, …

Step 4: The successive convergent viz., $\frac{b_n}{b_{n+1}}$, approaches a root of the equation f(x) = 0 (The proof of this statement is out of the scope the book).

Example

Find the smallest root of the equation $f(x) = x^3 - 6x^2 + 11x - 6 = 0$, by Ramanujan's method.

From the equation we have

$$\left(1 - \frac{11x - 6x^2 + x^3}{6}\right)^{-1} = b_1 + b_2x + b_3x^2 + \dots$$

Here $a_1 = \frac{11}{6}$, $a_2 = \frac{1}{6} - 1$, $a_3 =$, $a_4 = a_5 = \dots = 0$.

Hence $b_1 = 1$, $b_2 = a_1 = \frac{11}{6}$, $b_3 = a_1b_2 + a_2b_1 = \frac{121}{36} - 1 = \frac{85}{36}$,

$b_4 = a_1b_3 + a_2b_2 + a_3b_1 = \frac{575}{216}$, $b_5 = a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 = \frac{3661}{1296}$,

$b_6 = a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2 + a_5b_1 = \frac{22631}{7776}$.

Therefore,

$\frac{b_1}{b_2} = \frac{6}{11} = 0.54545$, $\frac{b_2}{b_3} = \frac{66}{85} = 0.7764705$, $\frac{b_3}{b_4} = \frac{102}{115} = 0.8869565$, $\frac{b_4}{b_5} = \frac{3450}{3661} = 0.9423654$,

$\frac{b_5}{b_6} = \frac{3138}{3233} = 0.9706155$.

The smallest root of the given equation is unity and it can be seen that the successive convergence approach is to this root.

Self-Assessment Questions - 6

21. Ramanujan method is ___________ approach.
22. Ramanujan method is used to determine ___________.
23. When can the Ramanujan method be preferred ?

8. SUMMARY

In this unit, the techniques for the solution of nonlinear equations have been presented. Two basic methods for finding the roots of a nonlinear equation are the graphical method and the analytical method. The main drawback of these methods is that the accuracy of the solution may be low. Numerical methods involving the iterative solution of nonlinear equations are more powerful, especially when the equations are complicated. These methods can be divided into two categories: bracketing and open methods. The bracketing methods require the limits between which the root lies, whereas the open methods require the initial estimate of the solution. Bisection and false position methods are two well-known examples of bracketing methods. These methods have been illustrated with examples. Among the open methods, the successive approximation, and Newton-Raphson methods are most frequently used. Further finding the smallest root of a nonlinear equation, namely the Ramanujan method is discussed.

9. TERMINAL QUESTIONS

1. Find a root for each of the following equations correct to four decimal places using the bisection method.
   a) $x^3 - x - 1 = 0$
   b) $x^3 - x - 4 = 0$
   c) $x^3 + x^2 - 1 = 0$
   d) $2x = \cos x + 3$
   e) $x \sin x + \cos x = 0$
2. Find a root for each of the following equations correct to four decimal places using the Regula-Falsi method.
   a) $\cos x - xe^x = 0$
   b) $x^3 - 5x - 7 = 0$
   c) $x - e^{-x} = 0$
   d) $x^3 - 20\,x + 20 = 0$
   e) $x = 1 + 0.3 \cos x$

3. Use the successive approximation method to find a real root of the following equations correct to 4 significant figures:
   a) $\cos x = 3x - 1$
   b) $e^{-x} = 10x$
   c) $x^3 - x^2 - 1 = 0$
   d) $x^3 - 2x - 5 = 0$
   e) $x - e^{-x} = 0$
4. Use Newton–Raphson method to obtain a root correct to 3 decimal places of the following :
   a) $x^3 - 5x + 3 = 0$
   b) $\cos x - 3x + 1 = 0$
   c) $x^3 - 4x - 9 = 0$
   d) $\text{Sin } x = 1 - x$
   e) $x^3 + x^2 + x + 7 = 0$
5. Using Newton–Raphson method, establish the formula

   $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate the square root of N. Hence find the square root of 5 correct to four places of decimals.
6. Find the multiplicity root of the equation $x^3 - x^2 - x + 1 = 0$ near $x_0 = 0.9$.

10. ANSWERS

Self Assessment Questions

1. root of f(x) = 0.
2. at most n roots.
3. transcendental equation.
4. intermediate value property.
5. root of f(x) = 0.
6. Repeated application of Intermediate value property.
7. Iterative method
8. 2 and 3.
9. 1 and 2.
10. a and x_1.
11. The point of intersection of the chord with the X-axis.
12. True.
13. Linear Convergence.
14. Picard iteration
15. The form of $x = \phi(x)$ of the given equation is $x = e^{-x}$.
16. A positive root is 0.5671477.0
17. The method of tangents.
18. Quadratic convergence.
19. Improve the result obtained by other methods.
20. $x_n - m\dfrac{f(x_n)}{f'(x_n)}$.
21. Iterative approach.
22. Smallest root of the function.
23. Function consists of infinite series.

Terminal Questions:

1. a)1.3281

   b)1.7963

c)0.7549

d)1.524

e)2.7984

2. a)0.5169

b)2.7474

c)0.5671

d)1.0594

e)1.1284

3. a)0.6071

b)0.0913

c)1.4660

d)2.0946

e)0.5671

4. a)0.6566

b)0.6071

c)0.5671

d)0.5110

e)– 2.1049

5. 2.2361

6. 1.773 $\simeq$ 2.0.