Unit 12 Numerical Solution of Ordinary Differential Equations – I

Structure:

12.1 Introduction

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types. These methods are of even greater importance when we realize that computing machines are now readily available which reduce numerical work considerably.

The fundamental laws of nature are often described by ordinary differential equations. For example: Newton's law F = ma = m$\frac{d^2x}{dt^2}$ is a second order differential equation expressing the fact that force is proportional to accelerations. As another example, if y(t) is the number of bacteria present in a medium at time the rate of growth of the population of bacteria is proportional to the number y itself, that is, $\frac{dy}{dt}$ = ky.

Thus many problems in science and technology can be formulated into differential equations satisfying certain given conditions. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods.

Definition:

A *differential equation* is an equation which involves independent and dependent variables and the derivatives of the dependent variables.

Definition:

The*order of*a differential equation is the order of highest derivative appearing in it.

Definition:

The *degree* of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions so far as the derivatives are concerned.

Example:

1. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$ is a second order and first degree differential equation.
2. $\left(\frac{dy}{dx}\right)^3 + xy = \sin x$ is a first order and third degree differential equation.

Definition:

A *solution* of a differential equation is a relation between the variables which satisfies the given differential equation.

Definition:

The *general solution* of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

Definition:

A *particular solution*is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

Objectives:

At the end of this unit the student should be able to:

- explain the initial value problem
- Apply the different Numerical Techniques in solving differential equations.

12.2 Initial Value Problems

Definition

A differential equation together with the initial condition is called *initial-value problem.*

Thus $\frac{dy}{dx}$ = y, with the initial condition y(0)= 1 is called an initial value problem.

Example:

Consider a first order and first degree differential equation

$$\frac{dy}{dx} = y \qquad – (1)$$

Its general solution is $y = ce^x$. – (2)

In order to obtain the value of the constant c, we need additional information. For example, consider the general solution $y = ce^x$ to the differential equation (1). If we are given a value of y for some x, the constant c can be determined. Suppose y = 1 at x = 0, then $y(0) = ce^0 = 1$.

Therefore c = 1 and the particular solution is $y = e^x$.

Note: Particular solution of a differential equation are usually determined by imposing conditions.

In this unit, we will discuss how to solve the first order and first-degree initial value problems by numerical methods. A *first order initial value problem*can be expressed in the form

$$\frac{dy}{dx} = f(x, y), \text{with } y(x_0) = y_0.$$

Single-step and Multi - step methods

All numerical techniques for solving differential equations involve a series of estimates of y(x) starting from the given conditions. There are two basic approaches that could be used to estimate the values of y(x). They are known as single-step methods and multi-step methods.

In single-step methods, we use information from only one preceding point, that is, to estimate the value $y_i$we need the conditions at the previous point y_{i-1}only. Multistep methods use information at two or more previous steps to estimate a value.

12.3 Picard's method of Successive Approximation

Consider first order ordinary differential equation

$$y' = \frac{dy}{dx} = f(x, y) \quad \text{....(3)}$$

Subjected to initial condition $y(x_0) = y_0$

Integrate eq (3) between x_0 and x, we obtain

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \text{....(4)}$$

Since $f(x, y)$ appears under integral sign so the eq (4) is an integral equation.

Now instead of finding the solution of eq(3) it is enough to find the solution of eq(4).

We will try to find the solution of eq(4) by approximation method.

The first approximation y_1 of y can be obtained by replacing y by y_0 in $f(x,y)$

That is,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx \qquad \ldots(5)$$

Similarly the second approximation can be obtained by replacing y_0 by y_1 in $f(x, y_0)$ of first approximation eq.(5)

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Likewise the third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

Continuing this process, the $(n+1)^{th}$ approximation will be

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)dx$$

We go on repeating this process until the two values of y becomes same up to same degree of accuracy.

This method is known as Picard's Successive Approximation method.

Example: Use Picard's method of successive approximations to find y_1,y_2, y_3 to the solution of the initial value problem $y' = \frac{dy}{dx} = y$, given that y =2 for x = 0. Use y_3 to estimate the value of y (0.8).

Solution: On comparing $y' = \frac{dy}{dx} = y$ to eq(1) , we get $f(x, y) = y$ and $y_0 = 2, x_0 = 0$

So first approximation will be

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 2 + \int_0^x y_0 \, dx$$

$$= 2 + \int_0^x 2 \, dx$$

$$= 2 + (2x)_0^x$$

$$= 2 + 2x$$

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$= 2 + \int_0^x y_1 \, dx$$

$$= 2 + \int_0^x (2 + 2x) \, dx$$

$$= 2 + (2x + x^2)_0^x$$

$$= 2 + 2x + x^2$$

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx$$

$$= 2 + \int_0^x (2 + 2x + x^2) dx$$

$$= 2 + 2x + x^2 + \frac{x^3}{3}$$

At x = 0.8

$$y_3 = 2 + 2(0.8) + (0.8)^2 + \frac{(0.8)^3}{3} = 4.41$$

Example: Use Picard's methodto solve$y' = \frac{dy}{dx} = x - y$for x = 0.1 and 0.2 given that y = 1 when x = 0.

Solution: We have $f(x, y) = x - y$ and $x_0 = 0, y_0 = 1$

First approximation

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$= 1 + \int_0^x (x - 1) dx$$

$$=\frac{x^2}{2}-x+1$$

When x = 0.1, $y_1 = 0.905$, x = 0.2, $y_1 = 0.82$

Second Approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$= 1 + \int_0^x [x - (\frac{x^2}{2} - x + 1)]dx$$

$$= 1 + \int_0^x (2x - \frac{x^2}{2} - 1)dx$$

$$= 1 + [x^2 - \frac{x^3}{6} - x]_0^x$$

$$= 1 + x^2 - \frac{x^3}{6} - x = -\frac{x^3}{6} + x^2 - x + 1$$

When x = 0.1, $y_2 = 0.909$, x = 0.2, $y_2 = 0.8386$

Third Approximation

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$= 1 + \int_0^x (x + \frac{x^3}{6} - x^2 + x - 1)dx$$

$$= 1 + (x^2 + \frac{x^4}{24} - \frac{x^3}{3} - x)_0^x$$

$$= \frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1$$

When x = 0.1, $y_3 = 0.90967$, x = 0.2, $y_3 = 0.8374$

Example: Use Picard's method to solve $y' = \frac{dy}{dx} = 3x + y^2$ for x = 0.1 given that y = 1 when x = 0.

Solution: Here $\frac{dy}{dx} = f(x,y) = 3x + y^2,\ x_0 = 0, y_0 = 1$

First Approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 1 + \int_0^x (3x + {y_0}^2)dx$$

$$= 1 + \int_0^x (3x + 1)dx$$

$$= \frac{3}{2}x^2 + x + 1$$

When x = 0.1, $y_1 = 1.115$

Second Approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$= 1 + \int_0^x (3x + {y_1}^2)dx$$

$$= 1 + \int_0^x (3x + (\frac{3}{2}x^2 + x + 1)^2)dx$$

$$= 1 + \int_0^x (\frac{9}{4}x^4 + 3x^3 + 4x^2 + 5x + 1)dx$$

$$= \frac{9}{20}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{5}{2}x^2 + x + 1$$

When x = 0.1, $y_2 = 1.1272$

SAQ 1: Using Picard's method of successive approximation find y for $\frac{dy}{dt} = 2y,\ y(0) = 1$

SAQ 2: Using Picard's method of successive approximation find y for $\frac{dy}{dt} = x + y^2,\ y(0) = 0$, upto third order of approximation.

12.4 Taylor's Series Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0. \qquad -(6)$$

Assume that f(x,y) is differentiable sufficient number of times. If y (x) is the solution of (6), then the Taylor's series for y(x) around $x = x_0$ in the terms of $(x - x_0)$ is given by

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' +$$

$$+ \frac{(x-x_0)^4}{4!} y_0'''' + \cdots \qquad -(7)$$

Since the solution is just an assumption so the derivative is not known exactly. But we have the assumption that the function is differentiable sufficient number of times, so the first few derivatives in terms of f and its partial derivatives are

$$y' = \frac{dy}{dx} = f(x, y)$$

$$y'' = f'(x,y) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$$

that is $y'' = f_x + f_y y'$

$$y''' = f''(x,y) = f_{xx} + f_{xy} f + f_{yx} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

$$= f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

Differentiating successively we can obtain y^{iv}, ….

Putting $x = x_0$ and $y = y_0$ we get $y_0', y_0'', y_0''', y_0^{iv}$, ….

Substituting the values of y_0, y_0', y_0'', y_0''', y_0^{iv}, in (7) we get y(x) for all values of x.

or

Put $x = x_1 = x_0 + h$ in (7) we get

$$y(x_1) = y_1 = y_0 + hy_0' + \frac{h^2}{2} y'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots.. \qquad -(8)$$

Once y_1 is known, we can compute y_1', y_1'', y_1''' …. from (6) and we have

$$y(x_2) = y(x_1 + h) = y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \qquad -(9)$$

Continuing in this way we get

$$y(x_{n+1}) = y(x_n + h) = y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Remark

Equation (8) can also be written as

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3)$$

where $O(h^3)$ represents all the terms involving third and higher powers of h. If we calculate the value of y_1 by omitting $\mathbf{O(h)^3}$ and higher powers of h, the ***truncation error*** will be kh^3 where k is a constant and the corresponding Taylor series is said to be of *second* order.

In general if $O(h^{n+1})$ and higher powers of h are omitted in a Taylor series expansion then the error will be proportional to $(n+1)^{th}$ power of the step size, namely h and the corresponding Taylor series is said to be of *order n.*

Note:

$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + O(h^5)$ is a Taylor series method of order four.

Example

Use Taylor's series method to solve the initial value problem

$\frac{dy}{dx} = x^2 + y^2$ with y(0) = 1 for x = 0.25 and x = 0.5

Solution: We have

$$\frac{dy}{dx} = y' = x^2 + y^2$$

$$\frac{d^2y}{dx^2} = y'' = 2x + 2yy'$$

$$\frac{d^3y}{dx^3} = y''' = 2 + 2yy'' + 2(y')^2$$

at x = 0, y (0) = y_0 = 1 and therefore,

$$y'(0) = y_0' = 1$$

$$y''(0) = y_0'' = 2$$

$$y'''(0) = y_0''' = 2 + 2 \times 1 \times 2 + 2\,(1)^3 = 8$$

Substituting these values, the Taylor series becomes

$$y(x) = 1 + (x - 0)\,1 + \frac{(x-0)^2}{2\,!}\,2 + \frac{(x-0)^3}{3\,!}\,8 + \ldots..$$

$$y\,(x) = 1 + x + x^2 + \frac{8x^3}{6} + \ldots..$$

The number of terms to be used depends on the accuracy of the solution needed.

Put x = 0.25, we get

$$y\,(0.25) = 1 + (0.25) + (0.25)^2 + \frac{8}{6}\,(0.25)^3 + \ldots.$$

$$= 1.33333$$

Similarly, $y\,(0.5) = 1 + (0.5) + (0.5)^2 + \frac{8}{6}\,(0.5)^3 + \ldots.$

$$= 1.91667$$

Example

Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 to five places of decimals from $\frac{dy}{dx} = x^2 y - 1$, y(0) = 1

Solution: Here $x_0 = 0$, $y(x_0) = y(0) = y_0 = 1$

$$\frac{dy}{dx} = y' = x^2 y - 1$$

Differentiating successively and substituting, we get

$$y' = x^2 y - 1,\ y'_0 = -1$$

$$y'' = 2xy + x^2 y',\ y''_0 = 0$$

$$y''' = 2y + 4xy' + x^2 y'',\ y'''_0 = 2$$

$$y^{iv} = 6y' + 6xy'' + x^2 y''',\ y'''_0 = -6, \text{ etc.}$$

Putting these values in Taylor series, we have

$$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

Hence $y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots$

$= 0.9003$

Similarly y (0.2) = 0.80227.

Example

Apply Taylor's series method to obtain approximate value of y at x = 0.2 for the differential equation $y' = 2y + 3e^x$, y(0) = 0.

Solution: Differentiating successively and substituting x = 0, y = 0, we get

$$y' = 2y + 3e^x,\ y'(0) = 2(0) + 3e^0 = 3$$

$y'' = 2y' + 3e^x$, $y''(0) = 2y'(0) + 3 = 9$

$y''' = 2y'' + 3e^x$, $y'''(0) = 2y''(0) + 3 = 21$.

$y^{iv} = 2y''' + 3e^x$, $y^{iv}(0) = 2y'''(0) + 3 = 45$.

Substituting these values in the Taylor's series, we have

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}\,y''(0) + \frac{x^3}{3!}\,y'''(0) + \frac{x^4}{4!}\,y^{iv}(0) + \ldots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots.$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots.$$

Therefore $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots = 0.8110$.

SAQ 3: Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ *and* $y(0) = 1$. Find the value of y(0.1) and y(0.2), using Taylor series method.

SAQ 4: Given $\frac{dy}{dx} = \frac{1}{x^2+y}$ *and* $y(4) = 4, h = 0.2$. Find the value of y(4.2) and y(4.4), using Taylor series method

Merits:

i) The method of numerical solution by using Taylor series is of the single-step untruncated type.

ii) The method is very powerful if we can calculate the successive derivatives of y in an easy manner.

iii) If there is a simple expression for the higher derivatives in terms of the previous derivatives of y, Taylor's method will work very well.

Demerits:

The differential equation $\frac{dy}{dx}$ = f(x, y), the function f(x, y) may have a complicated algebraic structure. Then the evaluation of higher order derivatives may become tedious and so this method has little application for computer programmes.

12.5 Euler's method

Consider the initial value problem,

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0 .$$

We have Taylor's series expansion,

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots\ldots\ldots\ldots$$

Put $x = x_1 = x_0 + h$, we get

$$y(x_1) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots$$

Approximating Taylor's expansion up to the first degree term of h only since $h^2, h^3, \ldots$ are very small,

we have

$$y(x_1) = y(x_0) + hy'(x_0)$$

$$= y(x_0) + hf(x_0, y_0)$$

(Since $y'(x_0) = \left(\frac{dy}{dx}\right)_{at x = x_0} = f(x_0, y_0)$)

or

$$y_1 = y_0 + hf(x_0, y_0)$$

Similarly, if $x_2 = x_1 + h$ we can write

$$y_2 = y_1 + h f(x_1, y_1)$$

In general, if $x_{i+1} = x_i + h$ we have

$$y_{i+1} = y_i + hf(x_i, y_i) \text{ where } i = 0, 1, 2, 3, 4 \ldots\ldots\ldots$$

This is known as Euler's formula or Euler's method.

Notation: $y_i = y(x_i)$. Therefore $y_0 = y(x_0)$, $y_1 = y(x_1) = y(x_0 + h)$

Observation: In Euler's method, we approximate the curve of solution by the tangent in each interval. This is the simplest single-step method. The accuracy in this method is very poor and to obtain the value of y to a reasonable degree of accuracy we must take h sufficiently small with the result the process becomes very slow.

Example

Find the solution for x = 0.1 and x = 0.2 by Euler's method for the equation

$$\frac{dy}{dx} = 1 - y,\ y(0) = 0.$$

Solution: The given differential equation is

$$\frac{dy}{dx} = 1 - y$$

Therefore $f(x, y) = 1 - y$

Also we have $x_0 = 0$, $y_0 = 0$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

Putting i = 0 in Euler's formula $y_{i+1} = y_i + hf(x_i, y_i)$ – (10)

we get

$y(x_1) = y_1 = y_0 + h\,f(x_0, y_0)$

$y(x_1) = y_1 = y_0 + h(1 - y_0)$

$= 0 + 0.1(1 - 0) = 0.10$

Therefore $y(0.1) = y_1 = 0.10$

Put i = 1 in (10) we get

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y_2 = y(x_2) = y_1 + hf(x_1, y_1) = y_1 + h(1 - y_1) = 0.1 + 0.1(1 - 0.1)$

$= 0.19$

Hence $y(0.1) = 0.10$ and $y(0.2) = 0.19$

Example

Use Euler's method to solve $\frac{dy}{dx} + 2xy^2 = 0, 0 \le x \le 1$, taking h = $\frac{1}{4}$, given that y =1 at x = 0 initially.

Solution: By Euler's Formula, $y_{i+1} = y_i + hf(x_i, y_i)$; that is

y(x+h) = y(x) + hf(x, y)

By data, f(x, y) = $-2xy^2$; x = 0, y = 1, h = 0.25.

We need to find y in 0 ≤ x ≤ 1 with h = 0.25.

This means that we have to compute y(0.25), y(0.5), y(0.75) and y(1).

We give the table of computations.

x	y	f(x, y) = $-2xy^2$	y(x + 0.25) = y(x) + 0.25 f(x, y).
0	1	0	y(0.25) = y(0) + (0.25)(0) = 1
0.25	1	-0.5	y(0.5) = y(0.25)+(0.25)(-0.5) = 1- 0.125 = 0.875.
0.5	0.875	-0.7656	y(0.75) = y(0.5)+(0.25)(-0.7656) = 0.875 – 0.1914 = 0.6836.
0.75	0.6836	-0.701	y(1) = y(0.75)+(0.25)(-0.701) = 0.6836 – 0.17525 = 0.50835.

Problem:

Use Euler's method to solve $\frac{dy}{dx} = x + y, y(0) = 0,$ choosing h = 0.2 carryout five steps.

Solution: By Euler's Formula, $y_{i+1} = y_i + hf(x_i, y_i)$; that is

y(x+h) = y(x) + hf(x, y)

By data, f(x, y) = x + y; x = 0, y = 0, h = 0.2.

Step no.	x	y	f(x, y) = x + y.	y(x + 0.2) = y(x) + 0.2f(x, y)
1	0	0	0	y(0.2) = y(0) + (0.2)(0) = 0.
2	0.2	0	0.2	y(0.4) = 0+(0.2)(0.2) = 0.04.
3	0.4	0.04	0.44	y(0.6) = 0.04 + (0.2)(0.44) = 0.128.
4	0.6	0.128	0.728	y(0.8) = 0.128 + (0.2)(0.728) = 0.2736.
5	0.8	0.2736	1.0736	y(1) = 0.2736 + (0.2)(1.0736) = 0.48832.

Example: Use Euler's method to solve the initial value problem $\frac{dy}{dt} = 1 - t + 4y, y(0) = 1,\ in\ the\ interval\ 0 \le t \le 0.5$ with h = 0.1.

Solution: By Euler's Formula, $y_{i+1} = y_i + hf(t_i, y_i)$; that is

y(t+h) = y(t) + hf(t, y)

By data, f(t, y) = $1 - t + 4y$; x = 0, y = 1, h = 0.1.

We need to find y in 0 ≤ t ≤ 0.5 with h = 0.1.

This means that we have to compute y(0.1), y(0.2), y(0.3) , y(0.4) and y(0.5).

Step no.	t	y	f(t, y) = $1 - t + 4y$.	y(t + 0.1) = y(t) + 0.1f(t, y)
1	0	1	5	y(0.1) = y(0) + (0.1)(5) = 1+0.4 =1.5
2	0.1	1.5	6.9	y(0.2) = 1.5 +(0.1)(6.9) = 2.19
3	0.2	2.19	9.56	y(0.3) = 6.9 + (0.1)(9.56) = 7.856.
4	0.3	7.856	32.124	y(0.4) = 7.856 + (0.1)(32.124) = 11.0684.
5	0.4	11.0684	44.8736	y(0.5) = 11.0684 + (0.1)(44.8736) = 15.55576.

SAQ5: Use Euler's method to solve the following differential equation $\frac{dy}{dt} = \frac{1}{2}y, y(0) = 1\ and\ 0 \le x \le 0.2$. Use h = 0.1

SAQ6: Given $\frac{dy}{dt} = \frac{y-t}{y+t},\ y(0) = 1$. Find y approximately at x = 0.1 in five steps using Euler's method.

12.6 Modified Euler's method

This is a modification of Euler's method. Instead of $f(x_0, y_0)$ in Euler's method, if we take the average of f(x, y) at (x_0, y_0) and (x_1, y_1) we get the modifed Euler's method as

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

where y_1 is given by $y_1 = y_0 + h\, f(x_0, y_0)$. $y_1^{(1)}$ is the first modified value of y_1.

Let $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$ be the second approximation to y_1.

The third approximation to y_1 is given as

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Thus the (j + 1)th iteration formula for the approximation of y_1 is

$$y_1^{(J+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(J)})\right], \; j = 0, 1, 2, 3, 4, \ldots$$

We repeat this process till two consecutive values of y agree. Let y_1 be the final value obtained to the desired accuracy. Using this value of y_1, we compute y_2 using Euler's method as

$$y_2 = y_1 + h\, f(x_1, y_1)$$

Now let $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$

where $y_2^{(1)}$ is the first approximation formula for y_2. The second approximation formula for y_2 as

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right].$$

we repeat this process until two consecutive values agrees. Then we proceed to calculate y_3 as above and continue the process till we calculate y_n.

Example:

Given $\frac{dx}{dy}$ = 1 + xy, y(0) = 2. Find y(0.1), y(0.2) and y(0.3) by modified Euler's method.

Solution: Here $x_0 = 0$, $y_0 = y(0) = 2$ and $f(x, y) = 1 + xy$

Let us take h = 0.1

Therefore $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

To find $y_1 = y(x_1) = y(0.1)$

we have Euler's formula for i = 1 in (1),

$y_1 = y_0 + h f(x_0, y_0) = y_0 + h(1 + x_0 y_0) = 2 + 0.1(1 + 0 \times 2) = 2.1$

Using modified Euler's method.

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = y_0 + \frac{h}{2}[(1 + x_0 y_0) + (1 + x_1 y_1)]$$

$$= 2 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times 2.1)] = 2.1105$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = y_0 + \frac{h}{2}\left[(1 + x_0 y_0) + (1 + x_1 y_1^{(1)})\right]$$

$$= 2 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times 2.1105)] = 2.1105$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = y_0 + \frac{h}{2}\left[(1 + x_0 y_0) + (1 + x_1 y_1^{(2)})\right]$$

$$= 2 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times 2.1105)] = 2.1105 .$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] = y_0 + \frac{h}{2}\left[(1 + x_0 y_0) + (1 + x_1 y_1^{(3)})\right]$$

$$= 2 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times 2.1105)] = 2.1105.$$

Therefore final value of $y_1 = 2.1105$.

To find $y_2 = y(x_2) = y(0.2)$

Now starting value of

$y_2 = y_1 + h f(x_1, y_1) = y_1 + h(1 + x_1 y_1) = 2.1105 + 0.1(1 + 0.1 \times 2.1105)$

$= 2.2316$

Using modified Euler's method

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)] = y_1 + \frac{h}{2}[(1 + x_1 y_1) + (1 + x_2 y_2)]$$

$$= 2.1105 + \frac{0.1}{2}\left[(1+0.1\times 2.1105)+(1+0.2\times 2.2316)\right] = 2.2434$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = y_0 + \frac{h}{2}\left[(1+x_1y_1)+(1+x_2y_2^{(1)})\right]$$

$$= 2.1105 + \frac{0.1}{2}\left[(1+0.1\times 2.1105)+(1+0.2\times 2.2434)\right] = 2.2435$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] = y_1 + \frac{h}{2}\left[(1+x_1y_1)+(1+x_2y_2^{(2)})\right]$$

$$= 2.1105 + \frac{0.1}{2}\left[(1+0.1\times 2.1105)+(1+0.2\times 2.2435)\right] = 2.2434$$

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right] = y_1 + \frac{h}{2}\left[(1+x_1y_1)+(1+x_2y_2^{(3)})\right]$$

$$= 2.1105 + \frac{0.1}{2}\left[(1+0.1\times 2.1105)+(1+0.2\times 2.2434)\right] = 2.2434$$

Here $y_2^{(3)}$ and $y_2^{(4)}$ is same. Therefore the value of $y_2 = 2.2434$.

To find $y_3 = y(x_3) = y(0.3)$

Starting value of (from the Euler's method)

$y_3 = y_2 + h\, f(x_2, y_2) = y_2 + h\,(1 + x_2 y_2) = 2.2434 + 0.1\,(1 + 0.2 \times 2.2434)$
$= 2.38827$

Now $y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3)\right] = y_2 + \frac{h}{2}\left[(1+x_2y_2)+(1+x_3y_3)\right]$

$$= 2.38827 + \frac{0.1}{2}\left[(1+0.2\times 2.2434)+(1+0.3\times 2.38827)\right] = 2.5465$$

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right] = y_2 + \frac{h}{2}\left[(1+x_2y_2)+(1+x_3y_3^{(1)})\right]$$

$$= 2.2434 + \frac{0.1}{2}\left[(1+0.2\times 2.2434)+(1+0.3\times 2.5465)\right] = 2.5488$$

$$y_3^{(3)} = y_2 + = \frac{h}{2}\left[f(x_2, y_2) + f\left(x_3, y_3^{(2)}\right)\right] = y_2 + \frac{h}{2}\left[(1+x_2y_2)(1+x_3y_3^{(2)})\right]$$

$= 2.2434 + \frac{0.1}{2}[(1 + 0.2 \times 2.2434) + (1 + 0.3 \times 2.5488)] = 2.40406$

$y_3^{(4)} = y_2 + = \frac{h}{2}\left[f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right] = y_2 + \frac{h}{2}[(1 + x_2 y_2)(1 + x_3 y_3^{(3)})]$

$= 2.2434 + \frac{0.1}{2}[(1 + 0.2 \times 2.2434) + (1 + 0.3 \times 2.40406)] = 2.40189$

$y_3^{(5)} = y_2 + = \frac{h}{2}\left[f(x_2, y_2) + f\left(x_3, y_3^{(4)}\right)\right] = y_2 + \frac{h}{2}[(1 + x_2 y_2)(1 + x_3 y_3^{(4)})]$

$= 2.2434 + \frac{0.1}{2}[(1 + 0.2 \times 2.2434) + (1 + 0.3 \times 2.40189)] = 2.40186$

Therefore final value $y_3 = 2.4019$.

Hence $y_1 = y(x_1) = y(0.1) = 2.1105$

$y_2 = y(x_2) = y(0.2) = 2.2434$

$y_3 = y(x_3) = y(0.3) = 2.4019$

Note: Modified Euler's method is an implicit method. It improves the result by iteration techniques.

Example: use the modified Euler's method to solve the differential equation $\frac{dy}{dx} = x + y^2, y(0) = 1$. Take the step size h = 0.1

Solution: Here $x_0 = 0$, $y_0 = y(0) = 1$ and $f(x, y) = 1 + xy^2$

Let us take $h = 0.1$

Therefore $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$

To find $y_1 = y(x_1) = y(0.1)$

we have Euler's formula for i = 1 in (1),

$y_1 = y_0 + h f(x_0, y_0) = y_0 + h(1 + x_0 y_0^2) = 1 + 0.1(1 + 0 \times 1) = 1.1$

Using modified Euler's method.

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = y_0 + \frac{h}{2}[(1 + x_0 y_0^2) + (1 + x_1 y_1^2)]$

$= 1 + \frac{0.1}{2}[(1 + 0) + (1 + 0.1 \times (1.1)^2)] = 1.10605$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = y_0 + \frac{h}{2}[(1 + x_0 y_0^2) + \left(1 + x_1 {y_1^{(1)}}^2\right)]$

$= 1 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times (1.10605)^2)] = 1.10612$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)}))] = y_0 + \frac{h}{2}[(1 + x_0 y_0^2) + \left(1 + x_1 {y_1^{(2)}}^2\right)]$

$= 1 + \frac{0.1}{2}[(1+0) + (1 + 0.1 \times (1.10612)^2)] = 1.10612$

Therefore final value of $y_1 = 1.10612$.

To find $y_2 = y(x_2) = y(0.2)$

Now starting value of

$y_2 = y_1 + h f(x_1, y_1) = y_1 + h (1 + x_1 y_1) = 1.10612 + 0.1 (1 + 0.1 \times 1.10612) = 1.21718$

Using modified Euler's method

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)] = y_1 + \frac{h}{2}[(1 + x_1 y_1^2) + (1 + x_2 y_2^2)]$

$= 1.10612 + \frac{0.1}{2}[(1 + 0.1 \times 1.10612) + (1 + 0.1 \times (1.21718)^2)] = 1.21903$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)}))] = y_1 + \frac{h}{2}[(1 + x_1 y_1^2) + \left(1 + x_2 {y_2^{(1)}}^2\right)]$

$= 1.10612 + \frac{0.1}{2}[(1 + 0.1 \times 1.10612) + (1 + 0.1 \times (1.21903)^2)] = 1.21905$

Therefore the value of $y_2 = 1.21905$.

SAQ 7. Using the modified Euler's method, solve $\frac{dy}{dx} = y + x^2, y(0) = 1$ to find y(0.2) and y(0.4), correct to 3 decimal places.

SAQ 8. Using the modified Euler's method, solve $\frac{dy}{dx} = \frac{2y}{x} + x^3, y(1) = 0.5$ to find y(1.2) and y(1.4), correct to 3 decimal places.

12.7 Summary

In this unit, we considered various methods of numerical solution of first order and first degree differential equations. They include Taylor series method, Euler's method.

12.8 Terminal Questions

1) Use Taylor's series method to find y at x = 0.1 (upto fifth derivative term) given that $\frac{dy}{dx} = x - y^2$ with y (0) = 1.

2) Find yat x = 1.02, given $\frac{dy}{dx} = xy - 1$with y (1) = 2. Apply Taylor's series method.

3) Use Taylors series method to solve the equation $\frac{dy}{dx} = 3x + y^2$ to approximate y when x =0.1, given that y = 1 when x = 0.

4) Use the fourth order Taylor series method with a single integration step to determine y(0.2). Given that $\frac{dy}{dx} + 4y = x^2, y(0) = 1.$

5) Use Picard's methodto solve$y' = \frac{dy}{dx} = 2y - 2x^2 - 3$given that y = 2 when x = 0.

6) Find the third approximation of the solution of the equation $\frac{dy}{dx} = 2 - (\frac{y}{x})$ by Picard's method, where y = 2, when x = 1.

7) Given the equation $\frac{dy}{dx} = 3x^2 + 1$with y (1) = 2, estimate y(2) by Eulers method using (i) h = 0.5 and (ii) h = 0.25.

8) Using modified Euler' method find the value of y at x = 0.2 with h = 0.1 where $\frac{dy}{dx}$= 1 – y with y(0) = 0.

9) Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, y (1) = 1. Evaluate y(1.3) by modified Euler's method.

10) Given $y' = -y \ and \ y(0) = 1$. Determine the value of y at x = 0.01, 0.02, 0.03, 0.04, by Euler's method

12.9 Answers

Self Assessment Questions

1. (a). $y = \sum_{i=0}^{k} \frac{(2)^i}{i!} x = e^2 x$
   (b)
2. $y_1 = \frac{x^2}{2}$, $y_2 = \frac{x^2}{2} + \frac{x^5}{20}$, $y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$
3. 1.0665, 1.1672
4. 4.009998, 4.019234
5. y_1= 1.05, y_2 = 1.1025
6. y_1= 1.02, y_2 = 1.10392, y_3= 1.0577, y_4 = 1.075, y_5 = 1.092
7. y(0.2) = 1.224, y(0.4) = 1.514
8. y(1.2)= 1.0228, y(1.4) = 1.8847

Terminal Questions

1. 0.9138
2. 2.02061
3. 1.12722
4. 0.4539
5. $y_1 = 2 + x - \frac{2x^3}{3}$, $y_2 = 2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}$,$y_3 = 2 + x + x^2 - \frac{x^4}{3} - \frac{2x^5}{15}$
6. $y_1 = 2x - 2logx$, $y_2 = 2 + (logx)^2$,$y_3 = 2x - 2logx - (logx)^3$
7. 7.875 and 9.90626
8. y(0.1) = 0.09524, y(0.2) = 0.181408
9. 0.9662
10. y(0.01) = 0.9900, y (0.02) = 0.9801, y(0.03) = 0.9703, y(0.04) = 0.9606