BACHELOR OF COMPUTER APPLICATIONS

SEMESTER 3

DCA2101

COMPUTER ORIENTED NUMERICAL METHODS

Unit 6

Solutions of System of Linear Equations – Iterative Methods

Table of Contents

SL No	Topic		Fig No / Table / Graph	SAQ / Activity	Page No
1	Introduction				3
	1.1	Objectives			
2	Gauss - Jacobi Method			1	4 - 9
3	Gauss - Seidel Method			2	10 - 16
4	Eigen Values and Eigen Vectors			3	17 - 27
5	Power Method			4	28 - 31
6	Summary				32
7	Terminal Questions				32 - 33
8	Answers				34

1. INTRODUCTION

The iterative method is applicable when the numerically large coefficients are along the leading diagonal of the coefficient matrix. Such a system is called a *diagonally dominant* system. Sometimes we may have to re-arrange the given system of equations to meet this requirement. This modified procedure is called *partial pivoting*.

1.1 Definition: (Partial Pivoting)

In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the second pivot by interchanging the second equation with the equation having the largest coefficient of y. This process is continued till we arrive at the equation with the single variable.

In the iterative method, we start with a trial solution called initial approximation and improve the solution by iterative processes (repeated application of process) till a solution to the desired level of accuracy is achieved.

1.1 Objectives

At the end of this unit the student should be able to:

- *Learn the Gauss Jacobi's and Gauss Elimination methods*
- *To find the eigen value and the eigen vectors of a matrix*
- *Learn the power method to find the dominant eigenvalue*
- *Apply the Cayley-Hamilton theorem to find the inverse of a matrix*

2. GAUSS - JACOBI'S ITERATION METHOD

This method is an iterative method, where an initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way. In general, when the coefficient matrix of the system of equations is a sparse (many elements are zero), iterative methods have a definite advantage over direct methods in respect of economy in computer memory. Such spare matrices arise in computing the numerical solution of partial differential equations.

To illustrate Jacobi's method, consider a linear system (we take, three equations in 3 unknowns) given by

$a_{11} x + a_{12} y + a_{13} z = b_1$

$a_{21} x + a_{22} y + a_{23} z = b_2$ – (1)

$a_{31} x + a_{32} y + a_{33} z = b_3$,

in which the diagonal elements *a_{11}, a_{22}, a_{33}* do not vanish and they are large as compared to other coefficients. Otherwise the equations should be rearranged so that the conditions are satisfied. Solve for x, y, z respectively and the system can be written as

$$x = \frac{1}{a_{11}} [b1 - a12\, y - a13\, z] \quad \text{-- (i)}$$

$$y = \frac{1}{a_{22}} [b2 - a21\, x - a23\, z] \quad \text{-- (ii)}$$

$$z = \frac{1}{a_{22}} [b3 - a31\, x - a32\, y] \quad \text{-- (iii)}$$

We start with the trial solution $x = x^{(0)}$, $y = y^{(0)}$, $z = z^{(0)}$ (initial approximation) and substitute these in the R.H.S. of (i), (ii) (iii) to obtain the first approximation to the solution denoted by

$$x^{(1)} = x = \frac{1}{a_{11}} \left[b_1 - a_{12} y^{(0)} - a_{13} z^{(0)} \right]$$

$$y^{(1)} = y = \frac{1}{a_{22}}\left[b_2 - a_{21}x^{(0)} - a_{23}z^{(0)}\right] \qquad \text{– (iv)}$$

$$z^{(1)} = z = \frac{1}{a_{33}}\left[b_3 - a_{31}x^{(0)} - a_{32}y^{0)}\right]$$

The second approximation is obtained by substituting the set of values as in (iv) into the equations (i), (ii), (iii). These are denoted by $x^{(2)}, y^{(2)}, z^{(2)}$,

Similarly if $x^{(j)}, y^{(j)}, z^{(j)}$ are a system of j[th] approximations, then the next approximation is given by the formula

$$x^{(j+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}y^{(j)} - a_{13}z^{(j)}\right]$$

$$y^{(j+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x^{(j)} - a_{23}z^{(j)}\right]$$

$$z^{(j+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x^{(j)} - a_{32}y^{(j)}\right], j = 0, 1, 2, 3, 4 \ldots.$$

The procedure is continued till the difference between two consecutive approximations is negligible.

Note

In the absence of any better estimates as $x^{(0)} = y^{(0)} = z^{(0)}$ these are assumed as $x^{(0)} = y^{(0)} = z^{(0)} = 0.$ This method is also called the method of simultaneous displacements.

2.1 Convergence

A sufficient condition of the iterative solution to the exact solution is $\left|a_{ij}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n}\left|a_{ij}\right|, i = 1, 2, \ldots, n$

. When this condition (diagonally dominance) is true, Jacobi's method converges.

Example

We consider the equations.

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

$20x + y - 2z = 17$

After partial pivoting we can rewrite the system as follows:

$20x + y - 2z = 17$

$3x + 20y - z = -18 \qquad -(1)$

$2x - 3y + 20z = 25$

Solve for x, y, z respectively

$x = \frac{1}{20}\left[17 - y + 2z \right]$

$y = \frac{1}{20}\left[-18 - 3x + z \right] \qquad -(2)$

$z = \frac{1}{20}\left[25 - 2x + 3y \right]$

we start from an initial approximation

$x = x^{(0)} = 0, \; y = y^{(0)} = 0, \; z = z^{(0)} = 0.$

Substituting these on the right hand sides of the equations (2), we get

$x = x^{(1)} = \frac{17}{20} = 0.85$

$y = y^{(1)} = \frac{-18}{20} = -0.9$

$z = z^{(1)} = \frac{25}{20} = 1.25,$

where $(x^{(1)}, y^{(1)}, z^{(1)}) = (0.85, -0.9, 1.25)$ are the first approximated values.

Putting these values on the right hand sides of the equations (2) we obtain

$x = x^{(2)} = \frac{1}{20} \quad [17 - (-0.9) + 2 \times 1.25] \quad = \quad 1.0200$

$y = y^{(2)} = \frac{1}{20}$ [−18 − 3 × 0.85 + 1.25] = − 0.965

$z = z^{(2)} = \frac{1}{20}$ [25 − 2 × 0.85 + 3 (−0.9)] = 1.03

The second approximated values of x, y, z are

$(x^{(2)}, y^{(2)}, z^{(2)}) = (1.0200, -0.965, 1.03)$.

Substituting these values on the right hand sides of the equations (2), we have

$x = x^{(3)} = 1.00125$

$y = y^{(3)} = -1.0015$

$z = z^{(3)} = 1.0032$

Continuing this process, we get

The values in the 5th and 6th iterations being practically the same, we can stop the iterations. Hence the solution is

Therefore $x = 1, y = -1, z = 1$.

Example

Find the solution to the following system of equations

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$

using Jacobi's iterative method for the first five iterations.

Solution: The given system is diagonally dominant.

Rewrite the given system as

$$\left.\begin{aligned} x &= \frac{95}{83} - \frac{11}{83}y + \frac{4}{83}z \\ y &= \frac{104}{52} - \frac{7}{52}x - \frac{13}{52}z \\ z &= \frac{71}{29} - \frac{3}{29}x - \frac{8}{29}y \end{aligned}\right\rangle \qquad \ldots\ldots\ldots(1)$$

Take the initial approximation vector as $x = x^{(0)} = 0, \; y = y^{(0)} = 0, \; z = z^{(0)} = 0$.

Then the first approximation is

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} 1.1446 \\ 2.0000 \\ 2.4483 \end{pmatrix} \qquad \ldots\ldots\ldots(2)$$

Now using equation (1), the second approximation is computed from the equations as

$$\left.\begin{aligned} x^{(2)} &= 1.1446 - 0.1325y^{(1)} + 0.0482z^{(1)} \\ y^{(2)} &= 2.0 - 0.1346x^{(1)} - 0.25z^{(1)} \\ z^{(2)} &= 2.4483 - 0.1035x^{(1)} - 0.2759y^{(1)} \end{aligned}\right\rangle \qquad \ldots\ldots\ldots(3)$$

Substituting equation (2) into equation (3), we get the second approximation as

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 0.9976 \\ 1.2339 \\ 1.7424 \end{pmatrix} \qquad \ldots\ldots\ldots(4)$$

Proceed in a similar way, we get the third, fourth and fifth approximations to the required solution and they are tabulated as below.

Iteration number (r) Variables

Iteration number (r)	**Variables**		
	x	**y**	**z**
1	1.1446	2.0000	2.4483
2	0.9976	1.2339	1.7424
3	1.0616	1.4484	1.8844
4	1.8609	1.386	1.8185
5	1.0486	1.2949	1.8733

Self-Assessment Questions -1

1. Jacobi's method is an ________.
2. Jacobi's method is also called as ________ .
3. To apply Jacobi's method, the coefficient matrix must be ________

3. GAUSS – SEIDEL ITERATION METHOD

This is a modification of Jacobi's method. As before the system of equations

$a_{11} x + a_{12} y + a_{13} z = b_1$

$a_{21} x + a_{22} y + a_{23} z = b_2$ – (1)

$a_{31} x + a_{32} y + a_{33} z = b_3$

is written as

$x = \frac{1}{a_{11}}$ [b1 – a12 y – a13 z]–– (i)

$y = \frac{1}{a_{22}}$ [b2 – a21 x – a23 z] –– (ii)

$z = \frac{1}{a_{33}}$ [b3 – a31 x – a32 y] –– (iii)

Here also we start with the initial approximations $x^{(0)}, y^{(0)}, z^{(0)}$ for x, y, z respectively. Substituting $y = y^{(0)}, z = z^{(0)}$ in (i), we get

$$x^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right]$$

Then putting x = x (1), z = z (0) in (ii), we have

$$y^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x^{(1)} - a_{23}z^{(0)}\right]$$

Next substituting x = in (iii), we obtain

$$z^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right]$$

and so on, that is, as soon as a new approximation for an unknown is found, it is immediately used in the next step.

In general, if $x_1^{(j)}, x_2^{(j)}, x_3^{(j)}$ are a system of jth approximations, then the next approximation is given by the formula

$$x_1^{(j+1)} = \frac{1}{a_{11}}\left[\, b_1 - a_{12}\, x_2^{(j)} - a_{13}\, x_3^{(j)} \right]$$

$$x_2^{(j+1)} = \frac{1}{a_{22}}\left[\, b_2 - a_{21}\, x_1^{(j+1)} - a_{23}\, x_3^{(j)} \right]$$

$$x_3^{(j+1)} = \frac{1}{a_{33}}\left[\, b_3 - a_{31}\, x_1^{(j+1)} - a_{32}\, x_2^{(j+1)} \right], \; j = 0, 1, 2, 3, \ldots.$$

The process of iteration is repeated till the values of x_1, x_2, and x_3 are obtained to the desired degree of accuracy.

Note

i) Convergence in the Gauss-Seidel method is twice as fast as in Gauss-Jacobi's method.

ii) The Gauss-Jacobi and Gauss-Seidel methods converge, for any choice of the first approximation $x_i^{(0)}\ (i = 1, 2, 3\ldots..\, n)$, if every equation of the system satisfies the condition that the sum of the absolute values of the coefficients $\frac{a_{ij}}{a_{ii}}$ is almost equal to, or in at least one equation less than unity, that is, provided that $\sum_{j=1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \le 1,\ (j \ne i)$ where < (less than) sign should be valid in the case of at least one equation.

Problem: Apply Gauss – Seidal iteration method to solve the equations

$3x_1 + 20x_2 - x_3 = -18$

$2x_1 - 3x_2 + 20x_3 = 25$

$20x_1 + x_2 - 2x_3 = 17$

We write the equations in the form (after partial pivoting),

$20x_1 + x_2 - 2x_3 = 17$

$3x_1 + 20x_2 - x_3 = -18$

$2x_1 - 3x_2 + 20x_3 = 25$

and $x_1 = \frac{1}{20}\ [17 - x_2 + 2x_3]$ – (i)

$$x_2 = \frac{1}{20} \ [-18 - 3x_1 + x_3] \qquad -\text{(ii)}$$

$$x_3 = \frac{1}{20} \ [25 - 2x_1 + 3x_2] \qquad -\text{(iii)}$$

Initial approximation: $x_1 = x_1^{(0)} = 0$, $x_2 = x_2^{(0)} = 0$, $x_3 = x_3^{(0)} = 0$.

First iteration:

Taking $x_2 = x_2^{(0)} = 0$, $x_3 = x_3^{(0)} = 0$ in (i), we get $x_1^{(1)} = 0.85$

Taking $x_1 = x_1^{(1)} = 0.85$, $x_3 = x_3^{(0)} = 0$ in (ii), we have

$$x_2^{(1)} = \frac{1}{20} \ [-18 - 3 \times 0.85 + 0] = -1.0275$$

Taking $x_1 = x_1^{(1)} = 0.85$, $x_2 = x_2^{(1)} = -1.0275$ in (iii), we obtain

$$x_3^{(1)} = \frac{1}{20} \ [25 - 2 \times 0.85 + 3 \times (-1.0275)] = 1.0109$$

Therefore $x_1^{(1)} = 0.85$, $x_2^{(1)} = -1.0275$, $x_3^{(1)} = 1.0109$.

Second iteration:

$$x_1^{(2)} = \frac{1}{20} \ [17 - 3 (-1.0275) + 2 \times 1.0109] = 1.0025$$

$$x_2^{(2)} = \frac{1}{20} \ [-18 - 3 \times (-1.0025) + 1.0109] = -0.9998$$

$$x_3^{(2)} = \frac{1}{20} \ [25 - 2 \times 1.0025 + 3 \times (-0.9998)] = 0.9998$$

Similarly in third iteration, we get $x_1^{(3)} = 1.0000$, $x_2^{(3)} = -1.0000$, $x_3^{(3)} = 1.0000$

The values in the 2nd and 3rd iterations being practically the same, we can stop the iterations.

Hence the solution is $x_1 = 1$, $x_2 = -1$, $x_3 = 1$

Example

Find the solution of the following system of equations.

$$x - \frac{1}{4}y - \frac{1}{4}z = \frac{1}{2}$$
$$-\frac{1}{4}x + y - \frac{1}{4}w = \frac{1}{2}$$
$$-\frac{1}{4}x + z - \frac{1}{4}w = \frac{1}{4}$$
$$-\frac{1}{4}y - \frac{1}{4}z + w = \frac{1}{4}$$

using the Gauss-Seidel method and perform the first five iterations.

Solution: The given system of equations can be rewritten as

$$\left.\begin{aligned} x &= 0.5 + 0.25y + 0.25z \\ y &= 0.5 + 0.25x + 0.25w \\ z &= 0.25 + 0.25x + 0.25w \\ w &= 0.25 + 0.25y + 0.25z \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Taking the initial approximation as w=x=y = z = 0 on the right side of the equation (1) we get $x^{(1)} = 0.5$.

Taking z = 0 and w = 0 and the current value of x, we get

$y^{(1)} = 0.5 + (0.25)(0.5) + 0 = 0.625$ from the second equation of (1).

Now taking w = 0 and the current value of x, we get

$z^{(1)} = 0.25 + (0.25)(0.5) + 0 = 0.375$ from the third equation of (1).

Lastly, using the current values of y and z, the fourth equation of (1) gives

$w^{(1)} = 0.25 + (0.25)(0.625) + (0.25)(0.375) = 0.5$.

The Gauss-Seidal iterations for the given set of equations can be written as

$$\left.\begin{aligned} x^{(r+1)} &= 0.5 + 0.25y^{(r)} + 0.25z^{(r)} \\ y^{(r+1)} &= 0.5 + 0.25x^{(r+1)} + 0.25w^{(r)} \\ z^{(r+1)} &= 0.25 + 0.25x^{(r+1)} + 0.25w^{(r)} \\ w^{(r+1)} &= 0.25 + 0.25y^{(r+1)} + 0.25z^{(r+1)} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Now, by Gauss-Seidel procedure, the second and the subsequent approximations can be obtained and the sequence of the first five approximations is tabulated below.

Iteration number (r)	**Variables**			
	x	**y**	**z**	**w**
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.59375
3	0.84375	0.85938	0.60938	0.61719
4	0.86719	0.87110	0.62110	0.62305
5	0.87305	0.87402	0.62402	0.62451

Example

Solve the system of equations

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

by applying (a) Jacobi's iterative method, (b) Gauss-Seidel iterative method

Solution: The given system of equations are diagonally dominant and the equations be put in the form.

$$\begin{aligned} x_1 &= 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \\ x_2 &= 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \\ x_3 &= 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \\ x_4 &= -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \end{aligned}$$

It can be verified that these equations satisfy the condition

$$\sum_{j=1}^{4} \left| \frac{a_{ij}}{a_{ii}} \right| < 1, (i = 1, 2, 3, 4)$$

Initial approximation:

$x_1 = x_1^{(0)} = 0,\ \ x_2 = x_2^{(0)} = 0,\ \ x_3 = x_3^{(0)} = 0,\ \ x_4 = x_4^{(0)} = 0$.

These results are given in the tables below:

Gauss-Jacobi's method:

Iterations (j)	$x_1^{(j+1)}$	$x_2^{(j+1)}$	$x_3^{(j+1)}$	$x_4^{(j+1)}$
0	0.3000	1.5000	2.7000	– 0.9000
1	0.7800	1.7400	2.7000	– 0.1800
2	0.9000	1.9080	2.9160	– 0.1080
3	0.9624	1.9608	2.9592	– 0.0360
4	0.9845	1.9848	2.9851	– 0.0158
5	0.9939	1.9938	2.9938	– 0.0060
6	0.9975	1.9975	2.9976	– 0.0025
7	0.9990	1.9990	2.9990	– 0.0010
8	0.9996	1.9996	2.9996	– 0.0004
9	0.9998	1.9998	2.9998	– 0.0002
10	0.9999	1.9999	2.9999	– 0.0001
11	1.0000	2.0000	3.0000	0.0000

Gauss – Seidel method

Iteration j	$x_1^{(j+1)}$	$x_2^{(j+1)}$	$x_3^{(j+1)}$	$x_4^{(j+1)}$
0	0.3000	1.5600	2.8860	– 0.1368
1	0.8869	1.9523	2.9566	– 0.0248
2	0.9836	1.9899	2.9924	– 0.0042
3	0.9968	1.9982	2.9987	– 0.0008
4	0.9994	1.9997	2.9998	– 0.0001
5	0.9999	1.9999	3.0000	0.0000
6	1.0000	2.0000	3.0000	0.0000

From the above two tables, it is clear that 12 iterations are required by Jacobi's method to achieve the same accuracy as seven Gauss-Seidel iterations.

Therefore the solutions are $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$. In both the methods, we have taken the initial approximations .

Self-Assessment Questions -2

4. Gauss-Seidel method is also called _______.
5. The convergence in the Gauss-Seidel method is ______ than the Jacobi's method.

4. EIGEN VALUES AND EIGEN VECTORS

4.1 Definition

Given a square matrix A of order n, if there exists a scalar λ(real or complex) and a non-zero column matrix X such that AX = λ X, then λ is called an *eigenvalue* of A, and X is called an *eigenvector* of A corresponding to an *eigenvalue* λ.

AX = λ X

= λ IX where I is the unit matrix of the same order as that of A.

$\Rightarrow$AX – λ I X = O, where O is the null matrix

$\Rightarrow$ (A – λ I)X = O.

This is a homogeneous system of n equations with n unknowns of X. A non-trivial solution exists only when the coefficient determinant $|A - \lambda I| = 0$.

On expanding of $|A - \lambda I| = 0$, we get a nth degree equation in λ, called the *characteristic equation* of the matrix A and is of the form

$$(-1)^n \lambda^n + k1\, \lambda^{n-1} + k2\, \lambda^{n-2} + \ldots\ldots + k_n = 0.$$

Solving the equation, we get n roots, say λ 1, λ 2, λ 3 , …. λ n. These are the *eigenvalues* of A, which are also called *eigen roots or characteristic roots* or *latent roots*. For each value of λ there will be an *eigenvector* X ($\neq$ 0.)

4.2 Properties

Let A be a square matrix.

1. The sum of the eigenvalues of A is the sum of the elements of its principal diagonal, that is, trace A = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots\ldots\ \ldots. +. \lambda_n$., where,, $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ \ldots \lambda_n$ are the eigenvalues of A
2. The product of the eigenvalues of A is the determinant of the matrix A, that is, det (A) = $\lambda_1 \lambda_2 \lambda_3 \ldots\ldots\ \ldots \lambda_n$.
3. If λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

4. If λ is the eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigenvalue.
5. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then A^m has the eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer).
6. A and its transpose A^T will have the same eigenvalues.
7. The inverse A^{-1} exists if and only if $\lambda_i \neq 0$ (i = 1, 2, 3, …, n)
8. The constant term in the characteristic equation equals determinant of A.

Example

Find the eigenvalues and eigenvectors of the matrix A = $\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

Solution: $A - \lambda I = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = 0 \Longrightarrow (2-\lambda)(5-\lambda) - 4(1) = 0$$

$\Longrightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Longrightarrow$ $\lambda = 1$, $\lambda = 6$ are the roots of the characteristic equation $\lambda^2 - 7\lambda + 6 = 0$.

If $\lambda = 1$, or $\lambda = 6$, the determinant $|A - \lambda I|$ vanishes, we get non-trivial solutions for the system of equations $(A - \lambda I)X = O$, where O is the null matrix and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$.

Therefore $(A - \lambda I)X = O$ becomes

$$\begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)x_1 + 1x_2 = 0$

$4x_1 + (5-\lambda)x_2 = 0$

Case I: When $\lambda = 1$ the system of equation becomes

$$1x_1 + 1x_2 = 0$$

$$4x_1 + 4x_2 = 0$$

Solve for x_1 and x_2 we get $x_2 = -x_1$.

Put $x_1 = k_1$ we get $x_2 = -k_1$.

Therefore $X = \begin{pmatrix} k_1 \\ -k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $k1 \neq 0$

Case II: When $\lambda = 6$ the system of equation becomes

$$-4x_1 + x_2 = 0$$

$$4x_1 - x_2 = 0$$

Solve for x_1 and x_2, we get $x_2 = 4x_1$

Put $x_1 = k_2$, we get $x_2 = 4k_2$

Therefore $X = \begin{pmatrix} k_2 \\ 4k_2 \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ where $k_2 \neq 0$.

Therefore the eigen values are 1 and 6 and the corresponding eigen vectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ if $k_1 = k_2 = 1$.

Example

Find all eigen values and the corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$ where $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.

That is, $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0.$

Expanding we get

$(8 - \lambda\ [(7-\lambda)(3-\lambda) - 16] + 6\,[-6\,(3-\lambda + 8] + 2\,[24 - 2\,(7-\lambda\ ] = 0$

$\Longrightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

That is, $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$on simplification.

$\Longrightarrow \lambda(\lambda - 3)(\lambda - 15) = 0$

Therefore $\lambda = 0, 3, 15$ are the eigen values of A.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, then we have

$$(A - \lambda I)X = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, $(8 - \lambda)x - 6y + 2z = 0$

$-6x + (7-\lambda)y - 4z = 0$

$2x - 4y + (3 - \lambda)z = 0$

Case I: Let $\lambda = 0$ we have

$8x - 6y + 2z = 0$ (i)

$-6x + 7y - 4z = 0$ (ii)

$2x - 4y + 3z = 0$ (iii)

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{24-14} = \frac{-y}{-32+12} = \frac{z}{56-36}$$

$$\frac{x}{10} = \frac{-y}{-20} = \frac{z}{20} \quad \text{or} \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Therefore (x, y, z) are proportional to (1, 2, 2) and we can write x = 1k, y = 2k, z = 2k where k (≠0) is an arbitrary constant. However, it is enough to keep the values of (x, y, z) in the simplest form x = 1, y = 2, z = 2 (putting k = 1). These values satisfy all the equations simultaneously.

Thus the eigen vector X1 corresponding to the eigen value $\lambda = 0$ is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Case II:Let $\lambda = 3$ and the corresponding equations are

$$5x - 6y + 2z = 0 \quad -\text{(iv)}$$

$$-6x + 4y - 4z = 0 \quad -\text{(v)}$$

$$2x - 4y = 0 \quad -\text{(vi)}$$

From (iv) and (v) we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

That is, $\frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16}$

That is, $\frac{x}{2} = \frac{-y}{1} = \frac{z}{-2}$

Thus $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

Case III: Let $\lambda = 15$ and the associated equations are

$$-7x - 6y + 2z = 0 \quad \text{(vii)}$$

$$-6x - 8y - 4z = 0 \qquad \text{(viii)}$$

$$2x - 4y - 12z = 0 \qquad \text{(ix)}$$

From (vii) and (viii) we have

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

That is, $\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$

Thus $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 15$.

Therefore $\lambda = 0, 3, 15$ are the eigen values of A and

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ are the corresponding eigen vectors.

Example

Find all the eigen roots and the eigen vector corresponding to the least eigen root of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$.

This implies $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$

Expanding we obtain, $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$. ....(1)

By inspection, $\lambda = 2$ is a root. Then $(\lambda - 2)$ is a factor of the (1).

Therefore dividing equation (1) with $(\lambda - 2)$.

We get that $\lambda^2 - 10\lambda + 16 = 0$.

This means $(\lambda - 2)(\lambda - 8) = 0$.

Therefore the eigen values are $\lambda = 2, 2, 8$.

We find the eigen vector corresponding the eigen value $\lambda = 2$. When $\lambda = 2$, the set of equations are

$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

The above set of equations represents a single independent equation,

$2x - y + z = 0$ and hence we can choose two variables $z = k_1$ and $y = k_2$.

Therefore $X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{k_2 - k_1}{2} \\ k_2 \\ k_1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 2$

where k_1, k_2 are not simultaneously equal to zero.

We find the eigen vector corresponding the eigen value $\lambda = 2$

$-2x - 2y + 2z = 0$

$-2x - 5y - z = 0$

$2x - y - 5z = 0$

Applying the rule of cross multiplication for

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

That is, $\frac{x}{12} = \frac{-y}{6} = \frac{z}{6}$

That is, $\frac{x}{2} = \frac{-y}{1} = \frac{z}{1}$

Thus $X_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 15$

4.3 Cayley – hamilton theorem

Every square matrix satisfies its own characteristic equation.

(That is., if the characteristic equation for the nth order square matrix A is

$| A - \lambda I | = (-1)^n \lambda^n + k1\, \lambda^{n-1} + k2\, \lambda^{n-2} + \ldots + k_n = 0$, then

$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \ldots + k_n I = O$ where I is the identity matrix of order n and O is the null matrix of order n.

Example

If $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ is the characteristic equation of the matrix A =

$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then $A^3 - 18A^2 + 45A = O$ where O is the null matrix of order 3.

4.4 Inverse of a square matrix

(using Cayley – Hamilton theorem):

If A is a square matrix of order 3 (for convenience) then its characteristic equation

$| A - \lambda I | = 0$ can be put in the form $\lambda^3 + k_1 \lambda^2 + k_2 \lambda + k3 = 0$.

Then by Cayley – Hamilton theorem, we have

$$A^3 + k_1 A^2 + k_2 A + k_3 I = O$$

Post multiplying by A^{-1}, we have

$$A^2 + k_1 A + k_2 I + k_3 A^{-1} = O, \text{ since } A^{-1}A = I,\ IA = A$$

Therefore $A^{-1} = -\frac{1}{k_3} [A^2 + k_1 A + k_2 I]$.

Example

Apply Cayley-Hamilton theorem compute the inverse of the matrix

A = $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ and verify the answer.

Solution: The characteristic equation of A is $| A - \lambda I | = 0$.

That is, $\begin{vmatrix} 0-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$ ……………………..(1)

On expanding, we get $\lambda^3 - 3\lambda^2 - 8\lambda + 2 = 0$.

Applying Cayley-Hamilton theorem, $A^3 - 3A^2 - 8A + 2I = O$.

Post multiplying with A^{-1} we have,

$A^2 - 3A - 8I + 2A^{-1} = O$. This implies $A^{-1} = \frac{-1}{2}(A^2 - 3A - 8I)$ ………(2)

Now $A^2 = A.A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix}$

Thus (ii), becomes

$$\frac{-1}{2}\left\{\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\}.$$

Therefore

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}.$$

It can be verified that $AA^{-1} = I$.

Example

Verify Cayley – Hamilton theorem and compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$. Therefore

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 3 & 2-\lambda & 3 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Therefore $\lambda^3 - 5\lambda^2 + \lambda + 1 = 0$, is the characteristic equation of A.

We have to show that $A^3 - 5A^2 + A + I = O$ (replacing λ by A).

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 6 \\ 12 & 10 & 15 \\ 6 & 5 & 8 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 & 6 \\ 12 & 10 & 15 \\ 6 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 23 & 19 & 29 \\ 57 & 47 & 72 \\ 29 & 24 & 37 \end{bmatrix}$$

L.H.S. of $A^3 - 5A^2 + A + I$ becomes,

$$\begin{bmatrix} 23 & 19 & 29 \\ 57 & 47 & 72 \\ 29 & 24 & 37 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 6 \\ 12 & 10 & 15 \\ 6 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 23-25+1+1 & 19-20+1+0 & 29-30+1+0 \\ 57-60+3+0 & 47-50+2+1 & 72-75+3+0 \\ 29-30+1+0 & 24-25+1+0 & 37-40+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Therefore $A^3 - 5A^2 + A + I = 0$

Hence Cayley – Hamilton theorem is verified.

To find the inverse of A

We have $A^3 - 5A^2 + A + I = O$. Post multiplying by A^{-1} we have

$$A^2 - 5A + I + A^{-1} = O$$

Therefore $A^{-1} = -A^2 + 5A - I$

$$\Rightarrow A^{-1} = -\begin{bmatrix} 5 & 4 & 6 \\ 12 & 10 & 15 \\ 6 & 5 & 8 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Self-Assessment Questions -3

6. If A is n × n-matrix, then its characteristic equation $|A - \lambda I| = 0$ gives an ____.
7. Find the dominant eigen value and the corresponding eigen vector of (i) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and (ii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

5. POWER METHOD

In many engineering problems, it is required to compute the numerically largest eigenvalue and the corresponding eigenvector. In such cases, the following power method is quite convenient which is also well suited for solid machine computation.

Method: (To find the largest eigen value and the corresponding eigen vector).

Suppose A is the given square matrix.

Step 1: Choose the initial vector such that the largest element is unity.

(choose initially an eigen vector $X^{(0)} = (1, 0, 0)^t$ or $(0, 1, 0)^t$ or $(0, 0, 1)^t$ etc)

Step 2: This normalized (taking the largest component out as a common factor) vector $X^{(0)}$ is pre-multiplied by the given matrix.

(evaluate the matrix product $AX^{(0)}$ which is written as $\lambda^{(1)}X^{(1)}$ after normalization).

Step 3: This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to the eigen vector.

Step 4: Compute $AX^{(1)}$ and again put in the form $AX^{(1)} = \lambda^{(1)}\ X^{(2)}$ by normalization which gives the second approximation. Similarly, we evaluate $AX^{(2)}$ and put it in the form $AX^{(2)} = \lambda^{(3)}\ X^{(3)}$.

Step 5: Repeat this process till the difference between two successive iterations is negligible. The values so obtained are respectively the largest eigen value and the corresponding eigen vector of the given square matrix A.

Example

Determine the largest eigen value and the corresponding eigen vector of the

matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ by power method taking the initial vector as $[1, 0, 0]^t$.

Solution: Given $X^{(0)} = (1, 0, 0)^t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the initial eigen vector

$$AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

(Since 2 is the largest value in $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$)

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2\cdot 5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(1)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2\cdot 8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2\cdot 93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2\cdot 98 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.994 \end{bmatrix} = 2.997 \begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}.$$

Therefore we can conclude that the largest eigen value is approximately 3 and the corresponding eigen vector is $(1, 0, 1)^t$.

Example

Determine the largest eigen value and the corresponding eigen vector of the

matrix $A = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by power method taking the initial vector as $[1, 1, 1]^t$.

Solution: Given $X^{(0)} = (1, 1, 1)^t = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the initial eigen vector

$$A X^{(0)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

(Since 6 is the largest value in $\begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix}$)

$$AX^{(1)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34\begin{bmatrix} 1.0 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82\begin{bmatrix} 1.0 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.45 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 4.01 \end{bmatrix} = 7.94\begin{bmatrix} 1.0 \\ -0.69 \\ 0.51 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.49 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 6 & 2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

Therefore the largest eigen value is $\lambda = \lambda^{(6)} = 8$ and the corresponding eigen vector

$X = X^{(6)} = (1, -0.5, 0.5)^t$.

Note

Finding the least eigenvalue and the corresponding eigen vector:

In this case, we proceed as follows. We note that $AX = \lambda X$. Pre-multiply by A^{-1}, we get

$A^{-1}AX = A^{-1} \lambda X = \lambda A^{-1}X \Longrightarrow X = \lambda A^{-1}X$. This can be written as

$A^{-1}X = \frac{1}{\lambda} X$. This shows the inverse matrix has a set of eigenvalues which are the reciprocals of the eigenvalues of A. Thus, for finding the eigenvalue of the least magnitude of the matrix A, we have to apply power method to the inverse of A.

Self-Assessment Questions -4

8. Power method is used to find ________.

6. SUMMARY

In this unit, we have discussed the Matrices, elementary transformations, rank of a matrix, and solution of systems of linear equations by the iterative methods and eigen value problems. In particular finding the largest eigen value and the corresponding eigen vector of the matrix.

7. TERMINAL QUESTIONS

1. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ find the values of

   $A - B$, $2A + B$, $A + B$, $3A - 2B$.

2. $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ find AB, BA.

3. Find the rank of the following matrices

   $$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

4. Solve by Gauss-elimination method

   $x_1 + x_2 + x_3 + x_4 = 2$

   $2x_1 - x_2 + 2x_3 - x_4 = -5$

   $3x_1 + 2x_2 + 3x_3 + 4x_4 = 7$

   $x_1 - 2x_2 - 3x_3 + 2x_4 = 5$

5. Solve the following system of equations by

   a) Jacobi's iteration method and b) Gauss – seidel iteration method.

   $2x_1 + x_2 - 3x_3 + 9x_4 = 31$

   $3x_1 - 4x_2 + 10x_3 + x_4 = 29$

   $2x_1 + 12x_2 + x_3 - 4x_4 = 13$

   $13x_1 + 5x_2 - 3x_3 + x_4 = 18$

   carry out 4 iterations.

6. Find all the eigen values and the corresponding eigen vectors for the matrix

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

7. Applying Cayley – Hamilton theorem, compute the inverse of the matrix

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ and verify the answer.

8. Using power method find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$, carry out 5 iterations by taking $X^{(0)} = (1, 0, 0)^t$

9. Find the eigen-value of largest modulus and the associated eigen vector of the matrix

$$\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

10. Find the largest eigen-value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

8. ANSWERS

Self Assessment Questions

1. Iterative method.
2. Simultaneous displacement method.
3. Strictly diagonally dominant
4. Method of successive displacements
5. Faster
6. n^{th} degree polynomial
7. (i) The dominant eigen value = 3.987 and the corresponding eigen vector is $\begin{pmatrix} 0.6677 \\ 1 \end{pmatrix}$.

   (ii) The dominant eigen value = 5.3719 and the corresponding eigen vector is $\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$

   .
8. Largest eigen value and the corresponding eigen vector.

Terminal Questions

1. By rule of matrix addition and subtraction
2. By rule of matrix multiplication
3. Using the concept of rank of a matrix
4. (0, 1, −1, 2)
5. (0.9805, 1.9516, 3.0084, 3.9364)
6. $\lambda = 3, 6, 9$ $X^t = (1, 2, 2), (2, 1, -2), (2, -2, 1)$
7. Ans. $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$
8. Ans: 25.18, $(1, 0.04, 0.07)^t$.
9. Ans.: X = (0.44, 0.76, 1.00)t, λ =11.84
10. Ans.: λ = 6.941, X = $(0.298, 0.063, 1.0)^t$