Unit 11 Numerical Integration

Structure:

11.1 Introduction

A need for numerical differentiation and integration techniques arises quite extensively and regularly in engineering physics and in other quantitative sciences. This unit tells you how to estimate the derivative or get the integral of a function by Numerical methods. The Interpolation methods of unit 8 & 9 provide the foundation for numerical differentiation and integration.

We need numerical integration techniques in the following situations:

1. Functions do not possess closed form solutions.
2. Closed form solutions exist but these solutions are complex and difficult to use for calculations.
3. Data for variables are available in the form of a table, but no mathematical relationship between them is known, as is often the case with experimental data.

Objectives:

At the end of this unit the student should be able to:

- Use trapezoidal rule to integrate Numerical Functions
- Apply Simpson's Three eight rule in problems.

11.1 Gaussian Quadrature formula

The general problem of numerical integration may be stated as follows: Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots. (x_n, y_n)$ of a function $y = f(x)$ where $f(x)$ is not known explicitly, it is required to compute the value of the definite integral

$$I = \int_a^b f(x)\ dx = \int_a^b y\ dx$$

Let the interval *[a, b]* be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$. Clearly $x_n = x_0 + nh$. Hence the integral becomes

$$I = \int_a^b y\ \ dx = \int_{x_0}^{x_n = x_0 + nh} y\ \ dx$$

Approximating $y = f(x)$ by Newton's forward difference formula, we obtain

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Where $p = \frac{x - x_0}{h}$ $or\ x = x_0 + ph$

$$I = \int_{x_0}^{x_n = x_0 + nh} y\ dx$$

$$= \int_{x_0}^{x_n = x_0 + nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots. \right] dx$$

Since $x = x_0 + ph$, $dx = hdp$

and when $x = x_0$, $p = 0$

$x = x_n = x_0 + nh = x_0 + ph$, $p = n$

Hence the integral becomes

$$I = h \int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2}\Delta^2 y_0 + \frac{(p^3 - 3p^2 + 2p)}{6}\Delta^3 y_0 + \ldots.. \right] dp$$

$$= h \left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{2p^3 - 3p^2}{12}\Delta^2 y_0 + \frac{p^4 - 4p^3 + 4p^2}{24}\Delta^3 y_0 + \ldots \right]_{p=0}^{p=n}$$

$$= h \left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3 - 3n^2}{12}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 4n^2}{24}\Delta^3 y_0 + \ldots \right]$$

which gives on simplification, I =

$$\int_{x_0}^{x_n=x_0+nh} ydx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...\right] \quad ...(1)$$

This is known as***quadrature formula***. From this formula, we can obtain different integration formulae by putting *n = 1, n = 2, n = 3, …* etc.

11.3 Trapezoidal Rule

Setting *n = 1* in the quadrature formula (1), we get only two tabulated points (x_0, y_0) and (x_1, y_1), hence all differences higher than the first will become zero and we obtain

$$\int_{x_0}^{x_1=x_0+h} ydx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}[y_0 + y_1]$$

This gives the integral of y over the interval $[x_0, x_1]$

Similarly $$\int_{x_1=x_0+h}^{x_2=x_0+2h} ydx = \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_2=x_0+2h}^{x_3=x_0+3h} ydx = \frac{h}{2}[y_2 + y_3]$$ and so on.

Lastly $$\int_{x_{n-1}=x_0+(n-1)h}^{x_n=x_0+nh} ydx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding all these results we get

$$\int_{x_0}^{x_1} ydx + \int_{x_1}^{x_2} ydx + \int_{x_2}^{x_3} ydx + ... + \int_{x_{n-1}}^{x_n} ydx$$

$$= \frac{h}{2}\ [(y_0 + y_1) + (y_1+y_2) + (y_2+y_3) + \ldots\ldots..+ (y_{n-1} + y_n)]$$

$$\int_{x_0}^{x_n} y dx = \frac{h}{2}\ [(y_0 + y_n) + 2(y_1+y_2+y_3 +\ldots+y_{n-1})]$$

This is known as *Trapezoidal rule.*

Note: If the tabulated values of *(x,y)* are not given, then divide the interval *(a, b)* in to n subintervals of width h so that $x_0 = a$, $x_1 = x_0 + h, \ldots, x_n = x_0 + nh = b$ and find the corresponding values of *y*, that is $y_0 = y(x_0)$, $y_1 = y(x_1)$, $y_2 = y(x_2)$, . . . , $y_n = y(x_n)$.

11.3.1 Error in Trapezoidal rule:

The error of the trapezoidal rule can be obtained in the following way,

The error in the interval $[x_0, x_1]$ is

$$\text{Error} = \int_{x_0}^{x_1} y dx - \frac{h}{2}[y_0 + y_1]$$

$$= y(x_1) - y(x_0) - \frac{h}{2}\ [y_0+y_1]$$

$$= y_1 - y_0 - \frac{h}{2}\ [y_0 + y_1]$$

Expanding y_1 in a Taylor series around $x = x_0$

That is., $y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \ldots$

we get

$$\text{Error} = \left(y_0 + hy'_0 \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \ldots \right) - y_0$$

$$- \frac{h}{2}\left(y_0 + y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \ldots \right)$$

Error $=\left(\frac{h^3}{6}y_0''+....\right)-\left(\frac{h^3}{4}y_0''+....\right)=\frac{-h^3}{12}y_0''+....$

Which is the error in the interval $[x_0, x_1]$.

Therefore, the principal part of error in the interval (x_0, x_1) = $-\frac{h^3}{12}y_0''$.

Similarly, the principal error in the interval (x_1, x_2) = $-\frac{h^3}{12}y_1''$. and so on

Lastly, the principal error in the interval (x_{n-1}, x_n) = $-\frac{h^3}{12}y_{n-1}''$..

Therefore the total error is given by

$$\text{Error} = \frac{-h^3}{12}(y_0''+y_1''+y_2''+....+y_{n-1}'') \qquad (2)$$

$$= \frac{-h^3}{12}ny''(\xi) \qquad \text{where } x_0<\xi<x_n$$

Assuming $y''(\xi)$ is the largest value of the n quantities in (2).

$$\text{Error} = -\frac{(b-a)}{12}h^2y''(\xi)\left(\sin ce\ h=\frac{b-a}{n}\right)$$

Hence the error in the Trapezoidal rule is of order h^2 and it is denoted by $O(h^2)$.

That is, Error = $-\frac{(b-a)}{12}h^2y''(\xi)+O(h^3)$ where $x_0<\xi<x_n$.

Example

Evaluate $\int_0^1 2x\ dx$ using trapezoidal rule with $h = 0.1$.

Solution: Here $a = 0, b = 1, h = 0.1$ and $n = \frac{b-a}{h} = \frac{1-0}{0.1} = 10$

The values of x and y are tabulated as

X	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y = 2x	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
	0	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80	2.0

By trapezoidal rule with n = 10

$$\int_{x_0}^{x_{10}} y dx = \frac{h}{2} \; [(y_0+y_{10}) +2 \, (y_1 + y_2+y_3+y_4+y_5 + y_6 + y_1 + y_8+y_9)]$$

$$\int_0^1 2\,x\,dx = \frac{0.1}{2}[(0+2)+2(0.20+0.40+0.60+0.80+1.00+1.20+1.40+1.60+1.80)]$$

$$= 0.05\,[2+18] = 1.00$$

Therefore $\int_0^1 2\,x\;dx = 1.00$

By actual integration $\int_0^1 2x\,dx = x^2\Big]_0^1 = 1-0=1.$

Example

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h = 0.2. Hence determine the value of π.

Solution: Let y = $\frac{1}{1+x^2}$ be the integrand function.

Here a = 0, b = 1, h = 0.2 and n = $\frac{b-a}{h} = \frac{1-0}{0.2}$ = 5.

Therefore $x_0 = a = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$

The values of y are tabulated as

X	x_0	x_1	x_2	x_3	x_4	x_5
	0	0.2	0.4	0.6	0.8	1.0
y = $\frac{1}{1+x^2}$	y_0	y_1	y_2	y_3	y_4	y_5
	1	0.9615	0.8621	0.7353	0.6098	0.50

By trapezoidal rule (Here $n = 5$)

$$\int_{x_0}^{x_5} y\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}[(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)]$$

$$= 0.1[1.5+6.3374]$$

$$= 0.7837$$

Therefore $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = 0.7837$ (1)

To find the value of π.

By actual integration

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\,\Big]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}. \quad (2)$$

From (1) and (2) we get

$$\frac{\pi}{4} = 0.7837$$

Therefore $\pi = 3.1348$.

Example

A plane area is bounded by a curve, the x-axis and two extreme ordinates. The area is divided into seven figures by equidistant ordinates *2* inches apart, the heights of the ordinates being *21.65, 21.04, 20.35, 19.61, 18.75, 17.80, 16.75* and *15.90* respectively. Find the approximate value of the area by numerical integration.

Solution: Here $n = 7$, $h = 2$ and the values of y(x) are $y_0 = 21.65$, $y_1 = 21.04$, $y_2 = 20.35$, $y_3 = 19.61$, $y_4 = 18.75$, $y_5 = 17.80$, $y_6 = 16.75$ and $y_7 = 15.90$.

Since $n = 7$, we can apply trapezoidal rule,

$$\int_a^b y\,dx = \frac{h}{2}\left[(y_0+y_7)+2(y_1+y_2+y_3+y_4+y_5+y_6)\right]$$

Area = $\frac{2}{2}\left[(21.65+15.90)+2(21.04 + 20.35 + 19.61 + 18.75 + 17.80 + 16.75)\right]$

= 266.15 square inches.

Example: Evaluate $\int_0^{1.2} e^x\,dx$, taking six intervals using Trapezoidal rule up to three significant figures.

Solution: Here a = 0, b = 1.2, n = 6, $h = \frac{b-a}{n} = \frac{1.2-0}{6} = 0.2$

x	0	0.2	0.4	0.6	0.8	1.0	1.2
y= f(x)	$y_0 = 1$	1.221	1.492	1.822	2.226	2.718	3.320

By trapezoidal rule (Here *n = 5*)

$$\int_{x_0}^{x6} ydx = \frac{h}{2}\ [(y_0+y_6)+2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.2}{2}[(1 + 3.320) + 2(1.221 + 1.492 + 1.822 + 2.226 + 2.718)]$$

$$= 0.1[4.320 + 18.958]$$

$$= 2.3278$$

The exact value of $\int_0^{1.2} e^x\,dx$ = 2.320

SAQ 1: Explain geometrical meaning of Trapezoidal rule.

SAQ 2: Use Trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals.

SAQ 3: Calculate the value of $\int_0^1 \frac{x\,dx}{(1+x)}\,dx$ correct upto three significant figures taking six intervals by trapezoidal rule.

11.4 Simpson's one third rule

Putting *n = 2* in the quadrature formula *(1),* the forward differences of order greater than two do not exist as we have a set of only three values y_0, y_1, y_2 corresponding to x_0, x_1 and x_2. (that is, all the differences of third and higher order become zero).

Therefore

$$\int_{x_0}^{x_2=x_0+2h} ydx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + y_1 - y_0 + \frac{1}{6}y_2 - 2y_1 + y_0\right]$$

(using $\Delta y_0 = y_1 - y_0$, $\Delta^2 y_0 = y_2 - 2y_1 + y_0$)

$$\int_{x_0}^{x_2=x_0+2h} ydx = \frac{h}{3}\ [y_0 + 4y_1 + y_2\ ]$$

Similarly $\int_{x_2}^{x_4=x_0+2h_1} ydx \ = \frac{h}{3}\ [y_2 + 4y_3 + y_4]$ and so on.

Finally $\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}\ [y_{n-2} + 4y_{n-1} + y_n]$ (assuming n to be even).

Adding all these integrals, we have

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}\ [(y_0+4y_1+y_2) + (y_2+4y_3+y_4) + \ \dots + (y_{n-2}+4y_{n-1}+y_n)]$$

Simplifying, we get

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}\ [(y_0 + y_n) + 2(y_2+y_4+y_6 + \dots + y_{n-2}) + 4\ (y_1 + y_3 + y_5 + \dots + y_{n-1})]$$

$$= \frac{h}{3}\ [(y_0 + y_n) + 2\ even + 4\ add]$$

This is called Simpson’s one third rule. Here n must be a multiple of 2.

Note: 1. The error in Simpson’s one third rule is

$$\frac{-(h-a)}{180} h^4 y^{iv}(\xi) \ \ where\ x_0 < \xi < x_n.$$

Where $y^{iv}(\xi)$ is the largest value of the fourth derivatives.

2. Simpson's one-third rule can be applied only when the given interval [a,b] is sub-divided into even number of subintervals each of width h.

Example: Use Simpson's $\frac{1}{3}$ rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking 6 sub-intervals.

Solution: Here n = 6, a = 0, b = 0.6, therefore length of the interval is,

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

The values of x and $y = e^{-x^2}$ correct to four decimal places are tabulated.

X	x_0	x_1	x_2	x_3	x_4	x_5	x_6
	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y = e^{-x^2}$	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1.0000	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977

Simpson's one third rule for n = 6 is given by

$$\int_{x_0}^{x_6} ydx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\int_0^{0.6} e^{-x^2} = \frac{0.1}{3} [(1.0+0.6977)+4(0.9900+0.9139+0.7788)+2(0.9608+0.8521)]$$

$$= 0.5351$$

Therefore $\int_0^{0.6} e^{-x^2} dx = 0.5351$

Note: $\int e^{-x^2} dx$ is a non-integrable by analytical methods.

Example: Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ by Simpson's $\frac{1}{3}$rd rule by dividing $\left[0, \frac{\pi}{2}\right]$ into 6 equal parts.

Solution: Here a = 0, b = $\frac{\pi}{2}$, n = 6,

Length of each part, h = $\frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$ *or* $15°$

The values of θ and the corresponding values of y = $\sqrt{\cos\theta}$ correct to four decimal places are tabulated.

θ	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$y = \sqrt{\cos\theta}$	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1.0000	0.9828	0.9306	0.8409	0.7071	0.5087	0.0000

By Simpson's $\frac{1}{3}$ rd rule for $n = 6$ is given by

$$\int_0^{\theta_6} y\,d\theta = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta = \frac{\pi}{12\times 3}$ [(1+0)+4(0.9828+0.8409+0.5087)+2(0.9306+0.7071)]

= 1.1873

Therefore $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta$ = 1.1873

Example: The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (minute) as follows :

t	0	2	4	6	8	10	12	14	16	18	20
v	0	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution: If s(kms) be the distance covered in time t (min), then

$$\frac{ds}{dt} = v$$

$$s = \int_0^{20} \frac{ds}{dt}\, dt = \int_0^{20} v\, dt$$

$$\int_0^{20} vdt = \frac{h}{3} \left[(v_0 + v_{10}) + 4\,(v_1 + v_3 + v_5 + v_7 + v_9) + 2\,[v_2 + v_4 + v_6 + v_8) \right]$$

by Simpson's one third rule.

Here h = 2, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$, $v_4 = 29$, $v_5 = 32$, $v_6 = 20$, $v_7 = 11$, $v_8 = 5$, $v_9 = 2$, $v_{10} = 0$ (from table).

Hence the required distance

$$S = \int_0^{20} vdt = \frac{2}{3} \left[(0 + 0) + 4\,(10 + 25 + 32 + 11 + 2) + 2\,(18 + 29 + 20 + 5)\right]$$

$$= \frac{2}{3} [\, 0 + 320 + 144] = 309.33 \text{ kms.}$$

Example: A curve passes through the points as given in the table.

X	1	2	3	4	5	6	7	8	9
Y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3

Find:

i) the area bounded by the curve, the *x-axis, x = 1 and x = 9.*

ii) the volume of the solid generated by revolving this area about the x-axis.

Solution: Here h = 1.

Area A = $\int_{x=1}^{9} ydx$.

Use Simpson's one third rule (since the number of ordinates is 9, odd),

$$\int_{x_0}^{x_6} ydx = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

Therefore $\int_{x=1}^{9} ydx$

= $\frac{1}{3}\left[(0.2+2.3)+4(0.7+1.3+1.7+2.1)+2(1+1.5+1.9)\right]$ = 11.5 sq. units.

iii) The volume V = $\pi\int_{1}^{9} y^2dx$.

= $\frac{1}{3}\left[(0.2)^2+(2.3^2)+4((0.7)^2+(1.3)^2+(1.7)^2+(2.1)^2)+2(1^2+(1.5)^2+(1.9)^2)\right]$

= π18.99 = 59.6588 cubic units.

Example: Find the distance travelled by a train between 8.20 A.M. and 9 A.M. from the following data.

Time	8.20 A.M	8.30 AM	8.40 AM	8.50 AM	9 AM
Speed in miles/hour	24.2	35	41.3	42.8	39.4

Solution: Let s be the distance travelled and v is the velocity, then

s = $\int_{t_1}^{t_2} vdt$.

Since n = 4 and time interval = 10 minutes = $\frac{1}{6}$ hr.

Therefore h = $\frac{1}{6}$.

Let v_0 = 24.2, v_1 = 35, v_2 = 41.3, v_3 = 42.8, v_4 = 39.4.

By Simpson's 1/3rd rule, for n = 4 is given by

s = $\frac{h}{3}\left[\left(v_0+v_4\right)+4(v_1+v_3)+2v_2\right]$

= $\frac{1}{3}.\frac{1}{6}[(24.2+39.4)+4(35+42.8)+2(41.3)]=25.4 miles.$

SAQ 4. Show that $\int_0^1 \frac{\log(1+x^5)}{1+x^2} = 0.1730$, using Simpson's 1/3 rd rule , by dividing the interval into ten equal parts.

SAQ 5. Calculate the approximate value of $\int_{-3}^{3} x^4 dx$ using Simpson's one-third rule.

SAQ 6. Calculate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's one-third rule.

11.5 Simpson's three eighth rule

Put *n = 3* in the quadrature formula. Since *x* takes one of the four values $x_0, x_1, x_2,$ and x_3, all differences of fourth and higher order become zero and we obtain

$$\int_{x_0}^{x_3} y dx = 3h\left[y_0+\frac{3}{2}\Delta y_0+\frac{3}{4}\Delta^2 y_0+\frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0+\frac{3}{2}(y_1-y_0)+\frac{3}{4}(y_2-2y_1+y_0)+\frac{1}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}[y_0+3y_1+3y_2+y_3]$$

Similarly

$$\int_{x_3}^{x_6} y dx = \frac{3h}{8}[y_3+3y_4+3y_5+y_6]$$

and so on.

Assuming that n is a multiple of three, and adding all these integrals, we get

$$\int_{x_0}^{x_n} ydx = \frac{3h}{8} [(y_0 + 3y_1 + 3\, y_2 + y_3) + (y_3+3y_4+3y_5+y_6) + \ldots$$

$$\ldots + [(y_{n-3})+3y_{n-2} + 3y_{n-1} + y_b)]$$

Simplifying, we get

$$\int_{x_0}^{x_n} ydx = \frac{3h}{8} [(y_0 + 3(y_1 + y_2 + y_4 + \ldots + y_{n-1})+2\,(y_3 + y_6 + \ldots + y_{n-3})+y_n)$$

This rule, called Simpson's three eight rule.

Observations:

1: The error in Simpson's three eight rule is – $\frac{3}{80}(h-a)h^4 y^{iv}(\xi)$.

2: Simpson's three eight rule is not so accurate as Simpson's one third rule, since the error in Simpson's three eight rule is large compared to Simpson's one third rule.

Example: Evaluate $\int_0^1 \frac{dx}{1+x}$ by using Simpson's three eight rule by dividing the interval *[0, 1]* into *6* equal parts. Hence deduce the value of $\log_e 2$.

Solution: Here $a = 0$, $b = 1$, $n = 6$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

Therefore $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{2}{6}$, $x_3 = \frac{3}{6}$, $x_4 = \frac{4}{6}$, $x_5 = \frac{5}{6}$ *and* $x_6 = \frac{6}{6} = 1$

The values of $y = \frac{1}{1+x}$ are tabulated as

X	x_0	x_1	x_2	x_3	x_4	x_5	x_6
	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
y =	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1.0	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$

Simpson's three eight rule for n = 6 is given by

$$\int_{x_0}^{x_6} ydx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$\int_0^1 \frac{1}{1+x}dx = \frac{\frac{3}{6}}{8}\left[\left(1+\frac{1}{2}\right)+3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)+2\left(\frac{2}{3}\right)\right]$$

$$= 0.693195$$

Therefore $\int_0^1 \frac{dx}{1+x} = 0.693195$

Now to deduce the value of $\log_e 2$.

we have $\int_0^1 \frac{dx}{1+x} = \log(1+x)\Big]_0^1$

$$= \log_e 2 - \log_e 1 \text{(since } \log_e 1 = 0)$$

$$\int_0^1 \frac{dx}{1+x^2} = \log_e 2$$

Equating this with the above numerical value, we have

$\log_e 2 = 0.6932$

Note: The actual value of $\log_e 2 = 0.6931472$.

Applying Trapezoidal rule to the above problem, we get

$$\int_{0}^{x_6} y dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$\int_{0}^{1} \frac{dx}{1+x} = \frac{1}{12}\left[(1+\frac{1}{2}) + 2(\frac{6}{7} + \frac{3}{4} + \frac{2}{3} + \frac{3}{5} + \frac{6}{11})\right] = 0.69487$$

That is., $\int_{0}^{1} \frac{dx}{1+x} = 0.69487$

Applying Simpson's one third rule to the above problem, we obtain

$$\int_{x_0}^{x_6} y dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\int_{0}^{1} \frac{dx}{1+x} = \frac{1}{18}\left[(1+\frac{1}{2}) + 4(\frac{6}{7} + \frac{2}{3} + \frac{6}{11}) + 2(\frac{3}{4} + \frac{3}{5})\right]$$

$$= 0.693169$$

That is., $\int_{0}^{1} \frac{dx}{1+x} = 0.693169$

we have

1) Using Trapezoidal rule,

$$\int_{0}^{1} \frac{dx}{1+x} = 0.69487 \qquad -(1)$$

2) Simpson's one third rule,

$$\int_{0}^{1} \frac{dx}{1+x} = 0.693169 \qquad -(2)$$

3) Simpson's three eight rule,

$$\int_{0}^{1} \frac{dx}{1+x} = 0.693195 \qquad -(3)$$

4) Actual integration,

$$\int_0^1 \frac{dx}{1+x} = 0.693147 \qquad -(4)$$

Comparing (1) and (4) errors in trapezoidal rule is

0.693147 – 0.69487 = – 0.001723

Comparing (2) and (4) error in Simpson's one third rule is

0.693147 – 0.693169 = – 0.000022

Comparing (3) and (4) error in Simpson's three eight rule is

0.693147 – 0.693195 = – 0.000048

Simpson's one third rule is more accurate compare to Trapezoidal rule and Simpson's three eight rule.

Observation: Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following example illustrate the application.

Example: Find the approximate value of y = $\int_0^{\pi} \sin x dx$ using (i) Trapezoidal rule,

(ii) Simpson's 1/3rd rule by dividing the range of integration into six equal parts. Calculate the percentage error form its true value in both the cases.

Solution: We shall first divide the range of integration *(0, π)* into six equal parts so that each part is of width π/6 and write the table:

x	*0*	*π/6*	*π/3*	*π/2*	*2π/3*	*5π/6*	*π*
y = sin x	*0.0*	*0.5*	*0.8660*	*1.0*	*0.8660*	*0.5*	*0.0*

Applying trapezoidal rule, we have

$$\int_0^{\pi} \sin x dx = \frac{h}{2}(y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5))$$

Here, h, the width of the interval is $\pi/6$. Therefore,

$$y = \int_0^{\pi} \sin x dx = \frac{\pi}{12}(0 + 0 + 2(3.732)) = \frac{3.1415}{6} \times 3.732 = 1.9540.$$

Applying Simpson's rule, we have

$$\int_0^{\pi} \sin x dx = \frac{h}{3}(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4))$$

$$= \frac{\pi}{18}\left[0 + 0 + (4 \times 2) + (2)(1.732)\right] = \frac{3.1415}{18} \times 11.464 = 2.0008.$$

But the actual value of the integral is

$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = 2.$$

Hence, in the case of trapezoidal rule,

the percentage error = $\frac{2 - 1.954}{2} \times 100 = 2.3$

In the case of Simpson's rule, the percentage error is

$$\frac{2 - 2.0008}{2} \times 100 = 0.04.$$

Example: A solid of revolution is formed by rotating about the X-axis the area between the X-axis, the lines x = 0 and x = 1, and a curve through the points with the following coordinates:

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed, giving the answer to three decimal places.

Solution: Let V is the volume of the solid formed. Then we know that V = $\pi\int_0^1 y^2 dx$.

Now we find the values of y^2 and these are tabulated below, correct to four decimal places.

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9793	0.9195	0.8261	0.7081

Take h = 0.25. Now by the Simpson's $\frac{1}{3}^{rd}$ rule,

V = $\frac{\pi.(0.25)}{3}[1^2 + 2(.9793^2 + 0.8261^2) + 2(0.9195^2 + 0.7081^2)]$= 2.6858

Example: Evaluate $\int_0^{\frac{\pi}{2}} e^{Sinx} dx$ correct to four decimal places by Simpson's three-eighth rule dividing the interval $(0, \frac{\pi}{2})$ into three equal parts.

Solution: Here *a = 0, b =*$\frac{\pi}{2}$*, n = 3*

$$h = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}$$

and $y = e^{sinx}$. So, we have

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = e^{sinx}$	1	1.64872	2.45960	2.71828

So, by Simpson's three-eighth rule, we have

$$\int_0^{\frac{\pi}{2}} e^{Sinx} dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3]$$

$$= \frac{3}{8}\frac{\pi}{6}[1 + 3(1.64872 + 2.45960) + 2.71828]$$

$$= \frac{3}{8}\frac{\pi}{6}[16.04324]$$

= 3.1499

Example: Evaluate $\int_0^{12} \frac{dx}{1+x^2}$ by Simpson's three-eighth rule and taking seven ordinates.

Solution: Here *a = 0, b =12 , n+1 = 7 => n = 6*

$$h = \frac{12-0}{6} = 2$$

x	0	2	4	6	8	10	12
y=f(x)	y_0=1	y_1=0.2	y_2=0.05882	y_3=0.02703	y_4=0.01538	y_5=0.0099	y_6=0.0069

So, by Simpson's three-eighth rule, we have

$$\int_0^{12} \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3(2)}{8}[(1 + 0.0069) + 3(0.2 + 0.05882 + 0.01538 + 0.0099) + 2(0.02703)]$$

$$= \frac{6}{8}[(1.0069) + 0.8523 + 0.05406]$$

$$= \frac{6}{8}[1.91326] = 1.4349$$

Example: Calculate the approximate value of $\int_{-3}^{3} x^4 dx$ using Simpson's three-eight rule by dividing the range into six equal parts.

Solution: Here *a = -3, b =3 , n = 6*

$$h = \frac{3-(-3)}{6} = 1$$

x	-3	-2	-1	0	1	2	3
y=f(x)	y_0=81	y_1=16	y_2=1	y_3=0	y_4=1	y_5=16	y_6=81

So, by Simpson's three-eighth rule, we have

$$\int_0^{12} \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}[(81+81)+3(16+1+1+\ 16)+2(0)]$$

$$= \frac{3}{8}[162+102]$$

$$= 99$$

SAQ 7: Calculate the approximate value of $\int_0^{\frac{\pi}{2}} sinx\, dx$ using Simpson's three-eight rule.

SAQ 8: Evaluate $\int_2^6 \log x\, dx$ by using Simpson's three- eight rule, taking n = 6, correct to five decimal places.

11.6 Summary

In numerical integration, we developed the following formulas, trapezoidal rule, Simpson's one third rule and Simpson's three eight formulas. We have also discussed the truncation errors in the above quadrature formulas. It is important to know that to apply Simpson's one third rule, the given interval must be divided into an even number of equal intervals. Simpson's three eight rule, n must be a multiple of 3.

11.7 Terminal Questions

1. Given that

x	1	2	3	4	5	6	7
f (x)	2.105	2.808	3.614	4.604	5.857	7.451	9.467

evaluate $\int_{1}^{7} f(x)\,dx$ using Trapezoidal rule.

2. Evaluate $\int_{4}^{5.2} \log_e x dx$ taking 6 equal strips by applying Simpsons $\frac{3}{8}$ rule.

3. Use Simpson's $\frac{3}{8}$ rule to evaluate $\int_{1}^{4} e^{\frac{1}{x}} dx$ by taking n = 3.

4. Evaluate $\int_{0}^{\frac{\pi}{2}} e^{Sinx} dx$ by using (i) Trapezoidal rule (ii) Simpson's one third rule, (Take h = 1)

5. Evaluate $\int_{0}^{5} \frac{dx}{4x+5}$ by Trapezoidal rule using 11 ordinates.

6. Find the value of log $2^{1/3}$ from $\int_{0}^{1} \frac{x^2 dx}{1+x^3}$ using Simpson's 1/3rd rule with h = 0.25.

11.8 Answers

Self Assessment Questions

1. The area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_0+nh is approximately equal to the sum of the areas of the n trapeziums.
2. 0.26
3. 0.30512

5. 98
6. 1.3662
7. 1.1003
8. 2.32947

Terminal Questions

1. 30.12
2. 1.8279
3. 4.9257
4. 1.4769 (using Trapezoidal rule)

   1.4317 (using Simpson's $\frac{1}{3}$ rd rule)
5. 0.4055
6. 0.2311