Unit 13 Numerical Solution of Ordinary Differential Equations – II

Structure:

13.1 Introduction

In the previous unit we studied Taylor's Series method, Eulers method, modified Euler's method. The Taylor's series method of solving differential equations is restricted by the labour involved in the determination of higher order derivatives. Euler's method is less efficient in practical problems since it requires the increment "h" to be small for obtaining reasonable accuracy. In this unit we will study two more methods- Runge-Kutta method and Predicator Corrector method. These methods does not require the calculations of higher order derivatives and theyare designed to give greater accuracy with the advantage of requiring only the function values at some selected points on the sub-interval.

Objectives:

At the end of this unit the student should be able to:

- explain the Runge-Kutta method and Predictor Corrector method.
- Apply these methods in solving differential equations.

13.2 Runge-Kutta method

Runge-kutta method was developed by two German mathematicians- Carl Runge and Wilhelm Kutta. It is a one step method.

13.2.1 Runge-Kutta Second order method

Consider $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

The second order Runge-Kutta formula is derived as

$$y_{i+1} = y_i + \frac{1}{2}[K_1 + K_2] \qquad \text{…………(1)}$$

where

$K_1 = hf(x_i, y_i)$

$K_2 = hf(x_i + h, y_i + K_1)$

where $i = 0, 1, 2, 3, 4$ …..

Example: Given $\frac{dy}{dx} = y - x$ where $y(0) = 2$ find $y(0.1)$ and $y(0.2)$.

Solution: Here $x_0 = 0$, $y_0 = y(x_0) = y(0) = 2$

Let $h = 0.1$

Therefore $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

and $f(x, y) = y - x$.

Put $i = 0$, in Runge-Kutta formula (1), we get

$$y_1 = y_0 + \frac{1}{2}[K_1 + K_2] \qquad \text{……………. ..(2)}$$

where

$K_1 = hf(x_0, y_0) = h(y_0 - x_0) = 0.1\,(2 - 0) = 0.2$

$K_2 = hf(x_0 + h, y_0 + K_1) = h\,[(y_0 + K_1) - (x_0 + h)]$

$= 0.1\,[\,(2 + 0.2) - (0 + 0.1)] = 0.1\,[2.1] = 0.21$

Put $y_0 = 2$, $K_1 = 0.2$ and $K_2 = 0.21$ in (2), we get

$$y_1 = 2 + \frac{1}{2}[0.2 + 0.21]$$

$$= 2 + \frac{1}{2}[0.41] = 2.2050.$$

Therefore $y_1 = y(x_1) = y(0.1) = 2.2050$

To find y (0.2):

Put $i = 1$ in (1) we get

$$y_2 = y_1 + \frac{1}{2}[K_1 + K_2]$$

where $K_1 = hf(x_1, y_1)$, $K_2 = hf(x_1 + h, y_1 + K_1)$

Now $x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$

$K_1 = h f(x_1, y_1) = h(y_1 - x_1) = 0.1 (2.2050 - 0.1) = 0.1(2.1050) = 0.2105.$

$K_2 = hf(x_1 + h, y_1 + K_1) = h [(y_1 + K_1) - (x_1 + h)] = 0.1 [(2.2050 + 0.2105) - (0.1 + 0.1)] = 0.22155.$

Therefore $y_2 = y(x_2) = y(0.2) = 2.2050 + \frac{1}{2}[0.2105 + 0.22153] = 2.421025.$

Hence the required solutions are

$$y(0.1) = 2.2050 \text{ and } y(0.2) = 2.421025.$$

Example: Use Runge-Kutta method of order two to find the solution of $\frac{dy}{dx} = x + y^2$, given that y = 1 when x = 0, at x = 0.2.

Solution: Here $x_0 = 0$, $y_0 = y(x_0) = y(0) = 1$

Let h = 0.2

Therefore $x_1 = x_0 + h = 0 + 0.2 = 0.2$

and $f(x, y) = x + y^2$

Put i = 0, in Runge-Kutta formula (1), we get

$$y_1 = y_0 + \frac{1}{2}\left[K_1 + K_2\right]$$

where

$K_1 = hf(x_0, y_0) = h(x_0 + y_0^2) = 0.2 (0 + 1) = 0.2$

$K_2 = hf(x_0 + h, y_0 + K_1) = h [(x_0 + h) + (y_0 + K_1)^2] = 0.2[(0+0.2) + (1 + (0.2)^2)]$

$= 0.2 [0.2 + 1.04] = 0.248$

Put $y_0 = 1$, $K_1 = 0.2$ and $K_2 = 0.248$ in (2), we get

$$y_1 = 1 + \frac{1}{2}[0.2 + 0.248]$$

$$= 1 + \frac{1}{2}[0.448] = 1.224.$$

Therefore $y_1 = y(x_1) = y(0.1) = 1.224$

13.2.2 Runge - Kutta fourth order method

The fourth order Runge-Kutta formula is defined by

$$y_{i+1} = y_i + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \text{ where } i = 0, 1, 2, 3, \ldots$$

where

$$K_1 = hf(x_i, y_i)$$

$$K_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_i + h, y_i + K_3)$$

This method is most commonly used and is often referred to as Runge-Kutta method only. The error of the method is of order $O(h^5)$.

Example

Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.1$ given that

$$\frac{dy}{dx} = x^2 - y \text{ and } y(0) = 1.$$

Solution: Here $x_0 = 0$, $x_1 = 0.1 = x_0 + h$

Therefore $h = 0.1$.

Put $i = 0$ in Runge-Kutta fourth order method, we get

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \ldots\ldots(*)$$

where $K_1 = hf(x_0, y_0)$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

Here $f(x, y) = x^2 - y$.

Therefore $K_1 = hf(x_0, y_0) = h\left(x_0^2 - y_0\right) = 0.1(0-1) = -0.1$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = h\left[(x_0 + \frac{h}{2})^2 - (y_0 + \frac{K_1}{2})\right]$$

$$= 0.1\left[(0+0.05)^2 - (1 + (-0.05))\right] = -0.0948$$

Similarly,

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.1\left[(0+0.05)^2 - (1 + (\frac{-0.0948}{2}))\right] = -0.095$$

$$K_4 = hf(x_0 + h_1, y_0 + K_3) = 0.1[(0+0.1)^2 - (1-0.095)] = -0.0895.$$

Substituting these values in (*) ,we get

$$y_1 = 1 + \frac{1}{6}[-0.1 + 2(-0.0948) + 2(-0.095) - 0.0895]$$

$y_1 = 0.9052$

$y_1 = y(x_1) = y(0.1) = 0.9052$

Example: Use fourth order Runge-Kutta method to find y at x = 0.1, given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$.

Solution: Given $f(x, y) = 3e^x + 2y$, $x_0 = 0$, $y_0 = 0$ and $h = 0.1$.

$K_1 = hf(x_0, y_0) = (0.1) f(0, 0) = (0.1) [3e^0 + 2 \times 0] = 0.3.$

K_2 =hf $(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})$ = (0.1)f(0.05, 0.15)= (0.1)[$3e^{0.05}$+2(0.15)] = 0.3454.

K_3 =hf $(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})$ = (0.1) f(0.05, 0.1727)= (0.1)[$3e^{0.05}$+2(0.1727)]

= 0.3499.

K_4 = hf (x_0+ h, y_0+ K_3) = (0.1) f(0.1, 0.3499)= (0.1)[$3e^{0.1}$+2(0.3499)]

= 0.4015.

Substituting these values in y(x_0+h)= $y_0+\frac{1}{6}$ [K_1 + 2K_2 + 2K_3 + K_4], we get

$$y(0.1)= 0 + \frac{1}{6}[0.3 + 2(0.3454) + 2 (0.3499) + 0.4015]$$

Therefore, y(0.1)= 0.3487.

Example:

Given $\frac{dy}{dx}$ = 1 + y^2 where y(0) = 0, find y (0.2), y (0.4)and y (0.6).

Solution: Put *i* = *0* in Runge-Kutta fourth order method, we obtain

$y_1 = y_0+\frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$, where K_1, K_2, K_3, K_4 defined above.

We take $h = 0.2$, $f(x, y) = 1 + y^2$, with $x_0 = 0$, $y_0 = 0$.

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

To find $y(x_1) = y(0.2)$ use the above formula

we obtain

K_1 = hf(x_0, y_0) = (0.2) f(0, 0) = (0.2) [1+0] = 0.2

K_2= hf $(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})$ = (0.2)f(0.1, 0.1)

= (0.2) [1+$(0.1)^2$] = 0.2 (1.01)

= 0.202

K_3 =hf $(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})$ = (0.2) f(0.1, 0.101)

$= 0.2\,(1 + 0.010201) = 0.20204$

$K_4 = hf\,(x_0 + h,\; y_0 + K_3) = (0.2)\, f(0.1,\, 0.20204)$

$= 0.2\,(1 + 0.04082) = 0.20816$

and $y\,(0.2) = y_1 = 0 + \frac{1}{6}\,[(0.2 + 2\,(0.202) + 2\,(0.2024) + 0.20816)]$

$y\,(0.2) = 0.2027.$

To compute y (0.4), we take $i = 1$ in Runge-Kutta formula, we get

$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$ and we have $x_2 = x_1 + h$

$= 0.2 + 0.2$

$= 0.4,$

$y_1 = 0.2027$ and $h = 0.2$.

$K_1 = hf(x_1, y_1) = 0.2\,[1 + (0.2027)^2] = 0.2082$

$K_2 = hf\,(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = 0.2\,[\,1 + (0.3068)^2] = 0.2188$

$K_3 = hf\,(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = 0.2\,[1 + (0.3121)^2] = 0.2195$

$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2\,[1 + (0.4222)^2] = 0.2356.$

Therefore $y_2 = y(x_2) = y(0.4) = 0.2027 + \frac{1}{6}\,[0.2082 + 2\,(0.2188) + 2\,(0.2195)$

$+ 0.2356] = 0.4228$

Proceeding as above we obtain $y\,(0.6) = 0.6841.$

Note: The exact solution of $\frac{dy}{dx} = 1 + y^2$ is obtained from $\frac{dy}{1+y^2} = dx$

Therefore $\tan^{-1} y = x + c$

Putting $x = 0$ and $y = 0$, we get $c = 0$

The particular solution is $\tan^{-1} y = x$ or $y = \tan x$.

Therefore $y\,(0.2) = \tan 0.2 = 0.202710$

$y\,(0.4) = \tan 0.4 = 0.42279$

$y\,(0.6) = \tan 0.6 = 0.68413$

Example: Compute y(0.1) and y(0.2) by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} = xy + y^2$, y(0) = 1.

Solution: The formulae for the fourth order Runge-Kutta method are

$y_{i+1} = y_i + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$, where K_1, K_2, K_3, K_4 are defined above.

Here $f(x, y) = xy + y^2$.

$x_0 = 0,\ y_0 = y(0) = 1$ and $h = 0.1$.

$x_1 = x_0 + h = 0.1$

$x_2 = x_1 + h = 0.2$

To find $y_1 = y(x_1) = y(0.1)$, put i = 0 in R-K method, we get

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

where $K_1 = hf(x_0, y_0) = h(x_0 y_0 + y_0^2) = 0.1(0 + 1^2) = 0.1$

$K_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = h\,f(0.05, 1.05) = 0.1((0.05)(1.05) + (1.05)^2)$

$= 0.1155.$

$K_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = h\,f(0.05, 1.05775)$

$= 0.1((0.05)(1.05775) + (1.05775)^2) = 0.1172.$

$K_4 = hf(x_0 + h, y_0 + K_3) = hf(0.1, 1.1172)$

$= 0.1((0.1)(1.1172) + (1.1172)^2) = 0.1169$

we have $K_1 = 0.1,\ K_2 = 0.1155,\ K_3 = 0.1172,\ K_4 = 0.1359.$

Therefore $y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1155) + 2(0.1172) + 0.1359]$

$= 1.1169$

Therefore $y(0.1) = 1.1169$

To find $y_2 = y(x_2) = y(0.2)$, put $i = 1$ in R-K method of order 4, we have

$$y_2 = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4], \text{ where}$$

$K_1 = h\,f(x_1, y_1) = 0.1\,f(0.1, 1.1169) = 0.1\,(0.1\,(1.1169) + (1.1169)^2) = 0.1359$

$K_2 = h\,f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = 0.1\,f(0.15, 1.1849)$

$= 0.1\,(0.15\,(1.1849) + (1.1849)^2) = 0.1582$

$K_3 = h\,f(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = 0.1\,f(0.15, 1.196) = 0.1(0.15\,(1.196) + (1.196)^2)$

$= 0.1610$

$K_4 = hf\,(x_1 + h, y_1 + K_3) = 0.1\,f(0.2, 1.2779) = 0.1\,(0.2\,(1.2779) + (1.2779)^2)$

$= 0.1889.$

So $K_1 = 0.1359$, $K_2 = 0.1582$, $K_3 = 0.1610$, $K_4 = 0.1889$.

Therefore $y_2 = 1.1169 + \frac{1}{6}[0.1359 + 2\,(0.1582) + 2\,(0.1610) + 0.1889)]$

$= 1.2774.$

Therefore $y(0.2) = 1.2774$.

SAQ 1: Use Runge-Kutta method of order 4 to find the solution of $\frac{dy}{dx} = x + y$, given that y = 1 when x = 0, at x = 0.1.

SAQ 2: Use Runge-Kutta method of order 4 to find the solution of $\frac{dy}{dx} = x + y^2$, given that y = 1 when x = 0, at x = 0.2 taking h = 0.1

SAQ 3: Use Runge-Kutta method of order 4 to find the solution of $\frac{dy}{dx} = xy$, given that y = 2 when x = 1, at x = 1.4 taking h = 0.2

13.3 Predictor Corrector Method

Predictor- Corrector method is a multi-step procedure for solving ordinary differential equations. It consists of two formulae: *predictor* and *corrector* formulae. In this method we require four prior values of $x_n$to find the values of y_n. The predictor formula is used to determine the estimated solution of y_{n+1}. The value of y_{n+1} is calculated from the previous values of (x_n, y_n).

Once the value of y_{n+1} is estimated then the corrector formula is applied. The corrector formula uses this estimated value of y_{n+1} for computing more accurate value of y_{n+1} . We can repeat the corrector formula several times such that the new value of y_{n+1} is substituted back into the corrector formula to obtain the more accurate value of y_{n+1}. This technique of finding accurate value of y_{n+1} is known as predicator- corrector formula.

Two predictor- corrector methods are:

1. Miline's Method
2. Adams- Moulton's Method

13.3.1 Milne's method

Consider the differential equation

$$\frac{dy}{dx} = y' = f(x, y) \qquad \text{.....(3)}$$

By Newton's forward formula, we have

$$f(x) = f(x_0 + h)$$

$$= f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots$$

Where $p = \frac{x - x_0}{h}, x = x_0 + ph$

Since $y' = f(x)\ \ and\ y_0' = f(x_0)$, so by replacing $f(x_0)$ by y_0' in above equation, we get

$$y' = y_0' + p\Delta y_0' + \frac{p(p-1)}{2!}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0' + \cdots \qquad \text{....(4)}$$

Integrating eq(4) from x_0 to $x_0 + 4h$ i.e., from p = 0 to p = 4 and dx = h dp, we get

$$\int_{x_0}^{x_0+4h} y'\, dx = h\int_0^4 \left(y_0' + p\Delta y_0' + \frac{p(p-1)}{2!}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0' + \cdots\right) dp$$

$$y_{x_0+4h} - y_{x_0} = h[py_0' + \frac{p^2}{2}\Delta y_0' + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0'$$

$$+ \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0' + \cdots]_0^4$$

$$y_4 - y_0 = h[4y_0' + 8\Delta y_0' + \tfrac{20}{3}\Delta^2 y_0' + \tfrac{8}{3}\Delta^3 y_0' + \cdots] \qquad \ldots(5)$$

Also, $\Delta = E - 1$, so eq (5) becomes

$$y_4 - y_0 = h[4y_0' + 8(E-1)y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0' + \cdots]$$

$$= h[y_0' + 8(E-1)y_0' + \tfrac{20}{3}(E-1)^2 y_0' + \tfrac{8}{3}(E-1)^3 y_0' + \cdots$$

$$= h[y_0' + 8Ey_0' - 8y_0' + \tfrac{20}{3}(E^2 + 1 - 2E)y_0' + \tfrac{8}{3}(E^3 - 1 - 3E^2 + 3E)y_0' + \cdots]$$

$$= h[y_0' + 8y_1' - 8y_0' + \tfrac{20}{3}y_2' + \tfrac{20}{3}y_0' - \tfrac{40}{3}y_1' + \tfrac{8}{3}y_3' - \tfrac{8}{3}y_0' - 8y_2' + 8y_1' + \cdots]$$

$$= h[\left(4 - 8 + \tfrac{20}{3} - \tfrac{8}{3}\right)y_0' + \left(8 - \tfrac{40}{3} + 8\right)y_1' + \left(\tfrac{20}{3} - 8\right)y_2' + \tfrac{8}{3}y_3' + \cdots]$$

$$= h[\tfrac{8}{3}y_1' - \tfrac{4}{3}y_2' + \tfrac{8}{3}y_3' + \cdots]$$

$$= \tfrac{4}{3}h[2y_1' - y_2' + 2y_3' + \cdots]$$

$$= \tfrac{4}{3}h[2y_1' - y_2' + 2y_3' + \cdots]$$

Neglecting terms higher than degree three, we have

$$y_4 = y_0 + \tfrac{4}{3}h[2y_1' - y_2' + 2y_3'] \qquad \ldots(6)$$

So eq (6) can be used to predict the values of y_4 when the values of y_0, y_1, y_2 and y_3 are known.

The general **predictor formula** is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_n' + 2y')$$

The corrector formula is obtained by integrating eq(4) from x_0 to $x_0 + 2h$ i.e., from p = 0 to p = 2 and dx = h dp, we get

$$\int_{x_0}^{x_0+2h} y'\,dx = h\int_0^2 \left(y_0' + p\Delta y_0' + \frac{p(p-1)}{2!}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0' + \cdots\right) dp$$

$$y_{x_0+2h} - y_{x_0} = h\Big[py_0' + \frac{p^2}{2}\Delta y_0' + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0'$$

$$+ \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0' + \cdots \Big]_0^2$$

$$y_2 - y_0 = h[2y_0' + 2\Delta y_0' + \tfrac{1}{3}\Delta^2 y_0' + \ldots] \qquad \ldots(7)$$

Also, $\Delta = E - 1$, so eq (7) becomes

$$y_2 - y_0 = h[2y_0' + 2(E-1)y_0' + \frac{1}{3}(E-1)^2 y_0' + \cdots]$$

$$= h[2y_0' + 2Ey_0' - 2y_0' + \tfrac{1}{3}(E^2 + 1 - 2E)y_0' + \cdots]$$

$$= h[2y_0' + 2y_1' - 2y_0' + \tfrac{1}{3}y_2' + \tfrac{1}{3}y_0' - \tfrac{2}{3}y_1' + \cdots]$$

$$= h[\tfrac{1}{3}y_0' + \tfrac{4}{3}y_1' + \tfrac{1}{3}y_2' + \cdots]$$

$$= \frac{1}{3}h[y_0' + 4y_1' + y_2' + \cdots]$$

Neglecting higher terms, we get

$$y_2 = y_0 + \tfrac{1}{3}h[y_0' + 4y_1' + y_2'] \qquad \ldots.(8)$$

So $y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$

This is Milne's **Corrector formula**.

In, general

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f_{n-1} + 4f_n + f_{n+1}]$$

Note: In the formulae obtained above we have considered the differences only up to third order as it will fit up with a polynomial of degree four. So to solve the first order differential equation, we find the first three values of y and x in addition to the initial values and then we find the next value of y by (6). The value of y thus obtained is then substituted in y′ = $f(x,y)$ to get y′. The value is then substituted in (8), to get the corrected value of the new y. If the corrected value of y agrees closely with the predicted value then we proceed to the next interval .

Example: Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2, y(0) = 1, h = 0.1$, for x = 0.4, using Milne's method. Given that

x	0.1	0.2	0.3
y	1.105	1.223	1.355

Solution: Here $f(x,y) = 1 + xy^2$,

For $x_0 = 0,\ y_0 = 1, y_0' = 1 + (0)(1)^2 = 1$

$x_1 = 0.1, \quad y_1 = 1.105, y_1' = 1 + (0.1)(1.105)^2 = 1.1221$

$x_2 = 0.2, \quad y_2 = 1.223, y_2' = 1 + (0.2)(1.223)^2 = 1.2991$

$x_3 = 0.3, \quad y_3 = 1.355, y_3' = 1 + (0.3)(1.355)^2 = 1.5508$

By Predicator formula,

$$y_4 = y_0 + \frac{4}{3}h[\,2y_1' - y_2' + 2y_3']$$

$$y_4 = 1 + \frac{4}{3}(0.1)[\,2(1.1221) - (1.2991) + 2(1.5508)] = 1.5395$$

So, $y_4' = 1 + x_4 y_4' = 1 + (0.4)(1.5395)^2 = 1.9480$

Now, by Corrector formula, we have

$$y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$$

$$y_4 = 1.223 + \frac{1}{3}(0.1)[(1.2991) + 4(1.5508) + (1.9480)] = 1.5348$$

Hence corrected value is $y_4' = 1 + x_4 y_4' = 1 + (0.4)(1.5348)^2 = 1.9422$

Again using corrector formula, we have

$$y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$$

$$y_4 = 1.223 + \frac{1}{3}(0.1)[(1.2991) + 4(1.5508) + (1.9422)] = 1.5346$$

Hence corrected value is

$y_4' = 1 + x_4 y_4' = 1 + (0.4)(1.5346)^2 = 1.94199 \cong 1.942$.

Example: Solve the initial value problem $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2, y(0) = 1,$

y(0.1)=1.06, y(0.2)= 1.12, y(0.3) = 1.21 for x = 0.4, using Milne's method.

Solution: Here $f(x,y) = \frac{1}{2}(1 + x^2)y^2,$

$y_0 = 1, y_1 = 1.06,\ y_2 = 1.12, y_3 = 1.21, h = 0.1$

So, $y_1' = \frac{1}{2}(1 + x_1{}^2)y_1{}^2 = \frac{1}{2}(1 + (0.1)^2)(1.06)^2 = 0.5674$

$$y_2' = \frac{1}{2}(1 + x_2{}^2)y_2{}^2 = \frac{1}{2}(1 + (0.2)^2)(1.12)^2 = 0.6522$$

$$y_3' = \frac{1}{2}(1 + x_3{}^2)y_3{}^2 = \frac{1}{2}(1 + (0.3)^2)(1.21)^2 = 0.7980$$

By Predicator formula,

$$y_4 = y_0 + \frac{4}{3}h[\,2y_1' - y_2' + 2y_3']$$

$$y_4 = 1 + \frac{4}{3}(0.1)[\,2(0.5674) - (0.6522) + 2(0.7980)] = 1.2765$$

So, $y_4' = \frac{1}{2}(1 + x_4{}^2)y_4{}^2 = \frac{1}{2}(1 + (0.4)^2)(1.2765)^2 = 0.9451$

Now by corrector formula, we have

$$y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$$

$$y_4 = 1.12 + \frac{1}{3}(0.1)[(0.6522) + 4(0.7980) + (0.9451)] = 1.2780$$

Hence corrected value is

$$y_4' = \frac{1}{2}(1 + x_4{}^2)y_4{}^2 = \frac{1}{2}(1 + (0.4)^2)(1.2780)^2 = 0.9473$$

Again using corrector formula, we have

$$y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$$

$$y_4 = 1.12 + \frac{1}{3}(0.1)[(0.6522) + 4(0.7980) + (0.9473)] = 1.2781$$

Hence corrected value is

$$y_4' = \frac{1}{2}(1 + x_4{}^2)y_4{}^2 = \frac{1}{2}(1 + (0.4)^2)(1.2781)^2 = 0.9474$$

Example: Given $\frac{dy}{dx} = 0.2x + 0.1y,$ with the following data

x	0.00	0.05	0.10	0.15
y	2.000	2.0103	2.0211	2.0323

Find the next entry in the table using Milne's method.

Solution: We have $f(x, y) = 0.2x + 0.1y$

For $x_0 = 0,\ y_0 = 2, y_0' = 0.2(0) + 0.1(2) = 0.2$

$x_1 = 0.05,\quad y_1 = 2.0103,\ y_1' = 0.2(0.05) + 0.1(2.0103) = 0.21103$

$x_2 = 0.10,\quad y_2 = 2.0211, y_2' = 0.2(0.10) + 0.1(2.0211) = 0.22211$

$x_3 = 0.15,\quad y_3 = 2.0323, y_3' = 0.2(0.15) + 0.1(2.0323) = 0.23323$

By Predicator formula,

$$y_4 = y_0 + \frac{4}{3}h[\ 2y_1' - y_2' + 2y_3']$$

$$y_4 = 2 + \frac{4}{3}(0.05)[\ 2(0.21103) - (0.22211) + 2(0.23323)] = 2.04431$$

So, $y_4' = 0.2x_4 + 0.1y_4 = (0.2)(0.2) + (0.1)(2.04431) = 0.244431$

Now by corrector formula, we have

$$y_4 = y_2 + \frac{1}{3}h[y_2' + 4y_3' + y_4']$$

$$y_4 = 2.0211 + \frac{1}{3}(0.05)[(0.22211) + 4(0.23323) + (0.244431)] = 2.0444245$$

SAQ 4: By using Milne's method find the solution of

$\frac{dy}{dx} = 2e^x - y, y(0) = 2$for x = 0.4, 0.5 with

$y_1 = 2.010, y_2 = 2.040, y_3 = 1.32372.09$ for x = 0.1, 0.2, 0.3.

SAQ 5: Compute y(2) by Milne's method if y(x) is the solution of

$\frac{dy}{dx} = \frac{1}{2}(x + y)$ assuming y(0) = 2, y(0.5)=2.636, y(1) = 3.595, y(1.5)= 4.968.

SAQ 6: Solve the equation $\frac{dy}{dx} = 1 + y^2$ for x = 0.8 and 1.0 by Milne's method given that y(0) = 0, y(0.2) = 0.2027, y(0.4) = 0.4228, y(0.6) = 0.6841

13.3.2 Adams- Moulton's Method

This is another Predictor-corrector method, which is used to find y_{n+1} using the values of $y_n, y_{n-1}, y_{n-2}, y_{n-3}$.

Consider the differential equation

$$\frac{dy}{dx} = f(x,y)$$

Integrating it between the limits x_n to x_{n+1}, we get

$$\int_{x_{n+1}}^{x_n} \frac{dy}{dx} = \int_{x_{n+1}}^{x_n} f(x,y)\,dx \qquad \ldots(9)$$

Or, $y_{n+1} - y_n = \int_{x_{n+1}}^{x_n} f(x,y)dx$ …(10)

By Newton's backward interpolation formula, we have

$$f(x,y) = f_n + p\nabla f_n + \frac{p(p+1)}{2}\nabla^2 f_n + \frac{p(p+1)(p+2)}{6}\nabla^3 f_n + \frac{p(p+1)(p+2)(p+3)}{24}\nabla^4 f_n + \cdots \qquad \ldots(11)$$

Where $p = \frac{x-x_n}{h}, x = x_n + ph$

Substituting (11) in (10), we obtain

$$y_{n+1} - y_n = \int_{x_{n+1}}^{x_n} [f_n + p\nabla f_n + \frac{p(p+1)}{2}\nabla^2 f_n + \frac{p(p+1)(p+2)}{6}\nabla^3 f_n + \frac{p(p+1)(p+2)(p+3)}{2}\nabla^4 f_n + \cdots]dx$$

By changing the variable of integration from x to p, the limits of integration also changes as

When $x = x_n, \quad p = \frac{x-x_n}{h} = \frac{x_n - x_n}{h} = 0$

$x = x_{n+1}, \quad p = \frac{x_{n+1} - x_n}{h} = \frac{(x_n + h) - x_n}{h} = \frac{h}{h} = 1$

And dx = hdp

So, we get,

$y_{n+1} = y_n +$

$$h\int_0^1 [f_n + p\nabla f_n + \frac{p(p+1)}{2}\nabla^2 f_n + \frac{p(p+1)(p+2)}{6}\nabla^3 f_n + \frac{p(p+1)(p+2)(p+3)}{2}\nabla^4 f_n + \cdots]dp$$

$$= y_n + h[pf_n + \frac{p^2}{2}\nabla f_n + \frac{1}{2}(\frac{p^3}{3} + \frac{p^2}{2})\nabla^2 f_n + \frac{1}{6}\left(\frac{p^4}{4} + p^3 + p^2\right)\nabla^3 f_n + \frac{1}{24}(\frac{p^5}{5} + \frac{3p^4}{2} + \frac{11p^4}{3} + 3p^2)\nabla^4 f_n + \cdots]_0^1$$

$$= y_n + h[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \cdots]$$

We know that

$\nabla f_n = f_n - f_{n-1}$

$\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$

$\nabla^3 f_n = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}$

On simplification, we get

$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}$

$y_{n+1} = y_n + \frac{h}{24}[55\, y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$ …(12)

This is known as **Adam's Predictor formula**.

Now, to obtain the corrector formula, we use Newton's backward difference interpolation formula about y_{n+1} instead of y_n

Thus, we have

$$y_{n+1} = y_n + h\int_0^1 [f_{n+1} + p\nabla f_{n+1} + \frac{p(p+1)}{2}\nabla^2 f_{n+1} + \frac{p(p+1)(p+2)}{6}\nabla^3 f_{n+1} + \frac{p(p+1)(p+2)(p+3)}{2}\nabla^4 f_{n+1} + \cdots]dp$$

Which on simplification gives

$\boldsymbol{y_{n+1} = y_n + \frac{h}{24}[9\, y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']}$ …(13)

This is **Adam's Corrector formula**.

Example: Obtain the solution of initial value problem $\frac{dy}{dx} = y - x^2, y(0) = 1$, y(0.2)=1.1218, y(0.4) = 1.4682, y(0.6) = 1.7379. Estimate y(0.8) by Adam's Moulton's Method.

Solution: We are given that

$$\frac{dy}{dx} = f(x,y) = y - x^2$$

Taking

$x_0 = 0, y_0 = 1, x_1 = 0.2, y_1 = 1.1218, x_2 = 0.4, y_2 = 1.4682, x_3 = 0.6, y_3 = 1.7379$

$$y_0' = f(x_0, y_0) = y_0 - x_0{}^2 = 1 - 0 = 1$$

$$y_1' = f(x_1, y_1) = y_1 - x_1{}^2 = 1.1218 - (0.2)^2 = 1.0818$$

$$y_2' = f(x_2, y_2) = y_2 - x_2{}^2 = 1.4682 - (0.4)^2 = 1.3082$$

$$y_3' = f(x_3, y_3) = y_3 - x_3{}^2 = 1.7379 - (0.6)^2 = 1.3779$$

By Adam's Predictor formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[55\, y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

So,

$$y_4 = y_3 + \frac{h}{24}[55\, y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$y_4 = 1.7379 + \frac{0.2}{24}[55\,(1.3779) - 59(1.3082) + 37(1.0818) - 9(1)]$$

$$= 1.7379 + (0.008333)[75.7845 - 77.1838 + 40.0266 - 9]$$

$$= 1.7379 + 0.24688 = 1.98478$$

Hence

$$y_4' = f(x_4, y_4) = y_4 - x_4{}^2 = 1.98478 - (0.8)^2 = 1.34478$$

Now by Adam's corrector formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[9\, y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

$$y_4 = y_3 + \frac{h}{24}[9\, y_4' + 19y_3' - 5y_2' + y_1']$$

$$y_4 = 1.7379 + \frac{0.1}{24}[9\,(1.34478) + 19(1.3779) - 5(1.3082) + 1.0818]$$

$$= 1.7379 + (0.008333)[12.10302 + 26.1801 - 6.541 + 1.0818]$$

$$= 1.73791 + 0.27352 = 2.011431$$

Example: Using Adams-Moultan method, obtain the solution of

$\frac{dy}{dx} = x^2y + x^2$ at x = 1.4, given the values

x	1	1.1	1.2	1.3
y	1	1.233	1.548488	1.978921

Solution: We are given that

$$\frac{dy}{dx} = f(x,y) = x^2y + x^2$$

Taking

$x_0 = 1, y_0 = 1, x_1 = 1.1, y_1 = 1.2.33, x_2 = 1.2, y_2 = 1.548488, x_3 = 1.3, y_3 = 1.978921$

$$y_0' = f(x_0,y_0) = {x_0}^2 y_0 + {x_0}^2 = 1 + 1 = 2$$

$$y_1' = f(x_1,y_1) = {x_1}^2 y_1 + {x_1}^2 = (1.1)^2(1.233) + (1.1)^2 = 2.70193$$

$$y_2' = f(x_2,y_2) = {x_2}^2 y_2 + {x_2}^2 = (1.2)^2(1.548488) + (1.2)^2 = 3.669822$$

$$y_3' = f(x_3,y_3) = {x_3}^2 y_3 + {x_3}^2 = (1.3)^2(1.978921) + (1.3)^2 = 5.034376$$

By Adam's Predictor formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[55\, y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

So,

$$y_4 = y_3 + \frac{h}{24}[55\, y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$y_4 = 1.978921 + \frac{0.1}{24}[55\,(5.034376) - 59(3.669822) + 37(2.70193) - 9(2)]$$

$$= 1.978921 + (0.004167)[276.89068 - 216.519498 + 99.97141 - 18]$$

$$= 1.978921 + 0.5931416 = 2.5720626$$

Hence

$$y_4' = f(x_4,y_4) = {x_4}^2 y_4 + {x_4}^2 = (1.4)^2(2.57206) + (1.4)^2 = 7.0012$$

Now by Adam's corrector formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[9\, y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

$$y_4 = y_3 + \frac{h}{24}[9\, y_4' + 19y_3' - 5y_2' + y_1']$$

$$y_4 = 1.978921 + \frac{0.1}{24}[9\,(7.0012) + 19(5.034376) - 5(3.669822) + 2.70193]$$

$= 1.978921 + (0.004167)[63.0108 + 95.653144 - 18.34911 + 2.70193]$

$= 1.978921 + 0.5959509 = 2.57487$

Example: Using Adams-Moultan method, obtain the solution of

$5x\frac{dy}{dx} + y^2 = 2$ at x = 4.4, given that

x	4.0	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143

Solution: We are given that

$$5x\frac{dy}{dx} + y^2 = 2$$

So, $$\frac{dy}{dx} = \frac{2-y^2}{5x}$$

Taking

$x_0 = 4.0, y_0 = 1, x_1 = 4.1, y_1 = 1.0049, x_2 = 4.2, y_2 = 1.0097, x_3 = 4.3, y_3 = 1.0143$

$$y_0' = f(x_0, y_0) = \frac{2 - y_0{}^2}{5x_0} = \frac{2-1}{5(4)} = \frac{1}{20} = 0.05$$

$$y_1' = f(x_1, y_1) = \frac{2 - y_1{}^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = \frac{-0.990176}{20.5} = 0.0483$$

$$y_2' = f(x_2, y_2) = \frac{2 - y_2{}^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = \frac{1}{21} = 0.046691$$

$$y_3' = f(x_3, y_3) = \frac{2 - y_3{}^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = \frac{1}{21.5} = 0.04517$$

By Adam's Predictor formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[55\,y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

So,

$$y_4 = y_3 + \frac{h}{24}[55\,y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$y_4 = 1.0143 + \frac{0.1}{24}[55\,(0.04517) - 59(0.046691) + 37(0.0483) - 9(0.05)]$$

$= 1.0143 + (0.004167)[2.48435 - 2.754769 + 1.7871 - 0.45]$

$= 1.0143 + 0.0044448 = 1.0187$

Hence

$$y_4' = f(x_4, y_4) = \frac{2 - {y_4}^2}{5x_4} = \frac{2 - (1.0187)^2}{5(4.4)} = \frac{0.96225}{22} = 0.04374$$

Now by Adam's corrector formula, we have

$$y_{n+1} = y_n + \frac{h}{24}[9\,y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

$$y_4 = y_3 + \frac{h}{24}[9\,y_4' + 19y_3' - 5y_2' + y_1']$$

$$y_4 = 1.0143 + \frac{0.1}{24}[9\,(0.04374) + 19(0.04517) - 5(0.046691) + 0.0483]$$

$$= 1.0143 + (0.004167)[0.39366 + 0.85823 - 0.233455 + 0.0483]$$

$$= 1.0143 + 0.00432 = 1.01862$$

SAQ 7: Use the Adams-Moulton method on (2,3) with h = 0.1 for the initial value problem

$\frac{dy}{dx} = f(x,y) = -xy^2, y(2) = 1$ and the exact solution is $y(x) = \frac{2}{x^2-2}$

SAQ 8: Approximate the value at x = 0.4 of the following differential equation $\frac{dy}{dx} = f(x,y) = 0.5y, y(0) = 1.0$, using the Adams- Moulton method. (Hint: First find y(0.1), y(0.2), y(0.3) by Euler's method or any other method convenient to you. Then use can make use of Adam's Moulton method)

SAQ 9: Using Adams-Moulton method, determine y(0.4) given the differential equation $\frac{dy}{dx} = f(x,y) = \frac{1}{2}xy$ and the data is

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

13.4 Summary

In this unit, we studied the concept of Runge-Kutta methods and Predictor Corrector method with examples.

13.5 Terminal Questions

1) Use Runge-Kutta method of order two and four to estimate y(0.2) of the equation $\frac{dy}{dx} = 3x + \frac{y}{2}$, y (0) = 1 by taking h = 0.2.
2) Use Runge-Kutta method of order four to solve $\frac{dy}{dx} = \frac{1}{x+y}$, y (0.4) = 1 at x = 0.5.
3) Use Milne's predictor –corrector method to compute the solution at x = 0.4 given that $\frac{dy}{dx} = f(x,y) = xy + y^2$, y(0) = 1. Take h = 0.1 and obtain the starting values for Milne's method using Runge-Kutta method of order 4.
4) Find y(0.8) using Milne's method, if y(x) is the solution of the differential equation $\frac{dy}{dx} = f(x,y) = -xy^2$, y(0)=2 assuming y(0.2) = 1.92308, y(0.4)=1.72414, y(0.6)= 1.47059
5) Solve the initial value problem $\frac{dy}{dx} = x - y^2, y(0) = 1$ to find y(0.4) by Adam's method. Starting solutions required are to be obtained using Runga-Kutta method of order 4, using step value h = 0.1.
6) Use Runge-Kutta method of order 4 to find the solution of $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$,given that y = 1 when x = 0, at x = 0.2, 0.4 taking h = 0.2
7) Solve $\frac{dy}{dx} = f(x,y) = 2 - xy^2$ with y(0) = 1. Show by Milne's method, that y(1) = 1.6505 taking h = 0.2

13.6 Answers

Self Assessment Questions

1. 1.11034
2. 1.2736
3. 2.99485

4. $y_4 = 2.162,\ y_5 = 2.256$
5. $y_2 = 6.8734$
6. $y_{0.8} = 1.0294,\ y_{1.0} = 1.5557$
7. y(2.4) = 0.0001144
8. $y_4^p = 1.1988\,, y_4^c = 1.2171$
9. y(0.4) = 1.1749

Terminal Questions

1. 1.165 and 1.16722
2. 1.0674
3. y(0.4) = 1.8392
4. y(0.8) = 1.21808
5. y(0.4) = 0.779
6. 1.196, 1.3752