Unit 14 Boundary Value Problems

Structure:

14.1 Introduction

Consider a second order differential equation

$F(x, y, y', y'') = 0.$

Its general solution contains two arbitrary constants. To determine these constants, we need to prescribe two conditions. These conditions are called initial conditions

i) <u>if y and y' are specified at a certain value of x</u>: The differential equation together with the initial conditions is called the initial value problem (IVP).

ii) <u>if y and y' or their combination is prescribed at two different values of x</u>, then the conditions are called boundary conditions, and differential equation together with the boundary conditions is called BVP.

Note

i) The solution of an initial value problem, in general exists and is unique

ii) The solution of a boundary value problem may exist or may not.

Examples

i) $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 1$ (BVP)

ii) $y'' + y = 0$, $y(0) = 0$, $y'(0) = 0$ (IVP)

iii) $y'' + y = 0$, $y(0) = 0$, $y'(\pi) = 1$ (BVP)

iv) $y'' + y = 0$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$ (BVP).

Objectives:

At the end of this unit the student should be able to:

- Learn the finite difference method and illustrations.
- Distinguish between the initial value problem and boundary value problem.
- Applications of Laplace and Poisson Equations.

14.2 Method of Finite Differences solving BVP

Consider a second order BVP.

$p(x)\, y''(x) + q(x)\, y'(x) + r(x)\, y(x) = f(x)$

with boundary conditions: $y(x_0) = y_0$, $y(x_n) = y_n$ … (1)

Divide $[x_0, x_n]$ into 'n' subintervals each of

length $h = \frac{x_n - x_0}{n}$

Let $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$ and we use the following notation:

$y(x_i) = y_i$, $y'(x_i) = y_i'$, $y''(x_i) = y_i''$

$p(x_i) = p_i$, $q(x_i) = q_i$, $r(x_i) = r_i$, $f(x_i) = f_i$

By Taylor's series expansion:

$$y_{i+1} = y(x_{i+1}) = y(x_0 + (i+1)h) = y(x_0 + ih + h)$$

$$= y(x_i) + hy'(x_i) + \frac{h^2}{2!} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \ldots.$$

Therefore $$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots \qquad \ldots (2)$$

Similarly, $$y_{i-1} = y(x_i - h)$$

$$= y_i - hy_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \ldots \qquad \ldots (3)$$

(2) – (3) gives:

$$y_{i+1} - y_{i-1} = 2hy_i' + 2\frac{h^3}{3!}y_i''' + \ldots$$

Therefore $$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \ldots (4)$$

Error = O (h^2) (higher order of h neglected)

Adding (2) and (3):

$$y_{i+1} + y_{i-1} = 2y_i + 2\frac{h^2}{2!}y_i'' + 2\frac{h^4}{4!}y_i^{iv} + \ldots$$

Therefore $$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \ldots (5)$$

Error = o (h^2)

In finite difference method y ′ and y ″ are replaced by the finite difference using (4) and (5).

<u>The boundary Conditions do not involve y ′ :</u>

To solve the boundary value problem

$$\left.\begin{array}{l} p(x)\, y_i'' + q(x)\, y_i' + r(x)\, y = f(x), \\ \quad y(a) = y_a, \qquad y(b) = y_b \end{array}\right\} \qquad \ldots\ldots (*)$$

Take x = x_i and replace y_i' and y_i'' by the approximations (4) and (5), to get the system of equations:

$$p_i\,\frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} + q_i\,\frac{(y_{i+1} - y_{i-1})}{2h} + r_i\, y_i = f_i$$

with i = 1, 2, 3, ….. , n –1 for the n –1 unknowns:

$y_1, y_2, \ldots , y_{n-1}$

we get y_0 for i = 1,

y_n for i = n –1.

Therefore $y_0 = y(a) = y_a$, $y_n = y(b) = y_b$.

Example

Solve x y″ + y = 0,

y(1) = 1, y(2) = 2 with h = 0.5 and h = 0.25. (BVP)

Solution: Consider h = 0.5; $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$

$$n = \frac{x_n - x_0}{h} = \frac{x_2 - x_0}{0.5} = \frac{2-1}{0.5} = 2$$

$y_0 = y(x_0) = y(1) = 1$

$y_1 = y(x_1) = y(x_0 + 1.\ h) = y(1+0.5) = y(1.5) = ?$

$y_2 = y(x_2) = y(2) = 2.$

Replacing y ″ by the finite difference:

$$x_i\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + y_i = 0 \qquad \dots (1)\ (i = 1)$$

Therefore $$\frac{x_1(y_2 - 2y_1 + y_0)}{(0.5)^2} + y_1 = 0$$

$$\Rightarrow \frac{(1.5)(2 - 2y_1 + 1)}{0.25} + y_1 = 0$$

$\Rightarrow$ $4.5 - 3y_1 + 0.25y_1 = 0$

$\Rightarrow$ $-2.75\ y_1 = -4.5$

$$\Rightarrow y_1 = \frac{-4.5}{-2.75} = 1.6364$$

Therefore $y_1 = y(1.5) = 1.6364$

Consider h = 0.25

$$n=\frac{2-1}{0.25}=4\,; \quad x_0 = 1,\ x_1 = 1.25,\ x_2 = 1.5,\ x_3 = 1.75,\ x_4 = 2.$$

$y_0 = y(x_0) = y(1) = 1,$

$y_1 = y(x_1) = y(1.25) = ?$

$y_2 = y(x_2) = y(1.5) = ?$

$y_3 = y(x_3) = y(1.75) = ?$

$y_4 = y(x_4) = y(2) = 2.$

Put i = 1, 2, 3 in equation (1)

That is, $x_i\,[y_{i+1} - 2\,y_i + y_{i-1}] + y_i\,h^2 = 0$

$h = 0.25 = \frac{1}{4}$

Put i = 1, $x_1\,[y_2 - 2\,y_1 + y_0] + y_1\left(\frac{1}{16}\right) = 0$

Therefore $1.25\,[y_2 - 2\,y_1 + 1] + y_1\left(\frac{1}{16}\right) = 0$

$\Rightarrow 16 \times 1.25\,(y_2 - 2\,y_1 + 1) + y_1 = 0$

$\Rightarrow 20\,(y_2 - 2\,y_1 + 1) + y_1 = 0$

$\Rightarrow 20\,y_2 - 39\,y_1 + 20 = 0$ … (2)

Putting i = 2, we get

$24\,y_3 - 47\,y_2 + 24\,y_1 = 0$ … (3)

Put i = 3, we get

$56 - 55\,y_3 + 28\,y_2 = 0$ … (4)

Therefore we have to solve the system (using Cramers rule):

$$39\, y_1 - 20\, y_2 + 0\cdot y_3 = 20$$

$$24\, y_1 - 47\, y_2 + 24\, y_3 = 0$$

$$0\cdot y_1 - 28\, y_2 + 55\, y_3 = 56$$

Let $$D = \begin{vmatrix} 39 & -20 & 0 \\ 24 & -47 & 24 \\ 0 & -28 & 55 \end{vmatrix} = -48207$$

$$y_1 = \frac{\begin{vmatrix} 20 & -20 & 0 \\ 0 & -47 & 24 \\ 56 & -28 & 55 \end{vmatrix}}{D} = \frac{-65140}{-48207} = 1.3513$$

$$y_2 = \frac{\begin{vmatrix} 39 & 20 & 0 \\ 24 & 0 & 24 \\ 0 & 56 & 55 \end{vmatrix}}{D} = \frac{-78816}{-48207} = 1.6349$$

$$y_3 = \frac{\begin{vmatrix} 39 & -20 & 20 \\ 24 & -47 & 0 \\ 0 & -28 & 56 \end{vmatrix}}{D} = \frac{-89208}{-48207} = 1.8508\ .$$

Problem

Solve $(x^3 + 1)\, y'' + x^2\, y' - 4xy = 2$,

$y(0) = 0$, $y(2) = 4$ with $h = 0.5$.

Solution: Here $x_0 = 0$, $y_0 = 0$, $x_n = 2$, $y_n = 4$

$$n = \frac{x_n - x_0}{0.5} = \frac{2-0}{0.5} = 4$$

Therefore $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$

Replacing y', y'' by finite differences, we get

$$\frac{\left(x_i^{\,3}+1\right)}{h^2}\left(y_{i+1}-2y_i+y_{i-1}\right)+\frac{x_i^{\,2}}{2h}\left(y_{i+1}-y_{i-1}\right)-4x_iy_i=2,\quad i = 1, 2, 3$$

These reduce to

$$\begin{bmatrix} -44 & 19 & 0 \\ 7 & 20 & 9 \\ 0 & -61 & 164 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 8 \\ 2 \\ 308 \end{bmatrix} \quad \text{Here take } y_4 = 4$$

for getting third equation

Solving, we get $y_1 = 0.25$, $y_2 = 1.00$, $y_3 = 2.25$.

Problem:

Solve $x^2 y'' + x y' + (x^2 - 3) y = 0$,

$y(1) = 0$, $y(2) = 2$ with $h = 0.25$.

Solution: Here $n = 4$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$, $y_0 = 0$, $y_4 = 2$,

Replacing y', y'' by finite differences, we get

$16\, x_i^2 (y_{i+1} - 2 y_i + y_{i-1}) + 2 x_i (y_{i+1} - y_{i-1}) + (x_i^2 - 3) y_i = 0$, $i = 1,2,3.$

These reduce to

$$\begin{bmatrix} 823 & -440 & 0 \\ 44 & -97 & 82 \\ 0 & -728 & 1569 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1680 \end{bmatrix}$$

Solving, $y_1 = 0.6044$, $y_2 = 1.1304$, $y_3 = 1.5973$.

Problem:

Solve $y'' + x\,y' - 2y = 0$ … (1)

with $y(1) = y'(1)$, $y(2) = 5$, $h = 0.5$

Solution: $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$

$y_0 = y(x_0) = y(1) = y'(1) = y'(x_0) = y_0' = ?$

$y_1 = y(x_1) = y(1.5) = ?$

$y_2 = y(x_2) = y(2) = 5.$

Substituting y_i'' and y_i' in (1) we get

Where $y_i'' = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ and $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + x_i \frac{[y_{i+1} - y_{i-1}]}{2h} - 2y_i = 0$$

$\Rightarrow 4\,(y_{i+1} - 2y_i + y_{i-1}) + x_i\,(y_{i+1} - y_{i-1}) - 2y_{i = 0}$ … (2)

Put i = 0 in (2),

$4\,(y_1 - 2y_0 + y_{-1}) + x_0(y_1 - y_{-1}) - 2y_0 = 0$ … (3)

Now $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h} = y_{i+1} - y_{i-1}$

$\Rightarrow y_0' = y_1 - y_{-1} \Rightarrow y_{-1} = y_1 - y_0' = y_1 - y_0$ (since $y_0' = y_0$)

Therefore (3) becomes,

$4\,(y_1 - 2y_0 + y_1 - y_0) + x_0(y_1 - y_1 + y_0) - 2y_{0=0}$

Therefore $8y_1 - 13y_0 = 0$ … (4)

Put i = 1, $-10\,y_1 + 2.5\,y_0 = -27.5$ … (5)

$y_1 = 3.25$ $y_0 = 2$

Problem:

Solve x $y'' + y = 0$,

$y'(1) = 0$, $y(2) = 1$, $h = \frac{1}{2}$

Solution: $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$

$y_0 = y(x_0) = y(1) = ?$

$y_1 = y(x_1) = y(1.5) = ?$

$y_2 = y(2) = 1$

$y_0' = 0 \ (= y'(x_0))$

We have

$$x_i \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + y_i = 0$$

$$4x_i (y_{i+1} - 2y_i + y_{i-1}) + y_i = 0 \qquad \ldots (1)$$

Put i = 0 in (1),

$$4x_0 (y_1 - 2y_0 + y_{-1}) + y_0 = 0$$

Also, $y_i' = \frac{y_{i+1} - y_{i-1}}{2h} = y_{i+1} - y_{i-1}$

Take i = 0, we get, $y_0' = y_1 - y_{-1}$

$\Rightarrow \quad y_{-1} = y_1 - y_0' = y_1 \qquad$ (since $y_0' = 0$)

Therefore $\quad 4(y_1 - 2y_0 + y_1) + y_0 = 0$

$\Rightarrow \quad 8y_1 - 7y_0 = 0 \qquad \ldots (2)$

Put i = 1,

$$4x_1 (y_2 - 2y_1 + y_0) + y_1 = 0$$

$6\,(y_2 - 2y_1 + y_0) + y_1 = 0$

$6y_2 - 11y_1 + 6y_0 = 0$

$-11y_1 + 6y_0 = -6$ … (3)

Solving (2) and (3), $y_0 = \frac{48}{29} = 1.6552$

$y_1 = \frac{42}{29} = 1.4482$.

Problem:

Solve $y'' + (1 + x)\,y' - y = 0$,

with $y(0) = y'(0)$, $y(1) + y'(1) = 1$, $h = 0.5$.

Solution: $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$y(0) = y_0 = y_0' = ?$

$y_1 = y(0.5) = ?$

$y_2 = y(1) = ?$

$y_2 + y_2' = 1$

Now $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{(1 + x_i)(y_{i+1} - y_{i-1})}{2h} - y_i = 0$

Put i = 0,

$4\,(y_1 - 2y_0 + y_{-1}) + (1+x_0)\,(y_1 - y_{-1}) - y_0 = 0$ … $(*)$

Now $y_i' = \frac{y_{i+1} - y_{i-1}}{2h} = y_{i+1} - y_{i-1}$

Therefore $y_0' = y_1 - y_{-1}$

$\Rightarrow$ $y_{-1} = y_1 - y_0' = y_1 - y_0$

Substituting in (∗), we get

$\Rightarrow \quad 4\,(y_1 - 2y_0 + y_1 - y_0) + (y_1 - y_1 + y_0) - y_0 = 0$

$8y_1 - 12y_0 = 0$

$\Rightarrow \quad 2y_1 - 3y_0 = 0 \qquad \ldots (1)$

Put i = 1,

$4\,(y_2 - 2y_1 + y_0) + (1+x_1)\,(y_2 - y_0) - y_1 = 0$

$4\,(y_2 - 2y_1 + y_0) + (1.5) \cdot (y_2 - y_0) - y_1 = 0$

$\Rightarrow \quad 5.5y_2 - 9y_1 + 2.5y_0 = 0 \ldots (2)$

Put i = 2,

$4\,(y_3 - 2y_2 + y_1) + (1+x_2)\,(y_3 - y_1) - y_2 = 0$

Consider $\quad y_i' = y_{i+1} - y_{i-1}$

$y_2' = y_3 - y_1$

$\Rightarrow \quad y_3 = y_2' + y_1$

$= (1 - y_2) + y_1$

$= 1 + y_1 - y_2$

Therefore $\quad 4\,(1 + y_1 - y_2 - 2y_2 + y_1) + (1+x_2)\,(1 + y_1 - y_2 - y_1) - y_2 = 0$

$\Rightarrow \quad 8y_1 - 15y_2 = -6 \qquad \ldots (3)$

From (1), $\quad y_0 = \frac{2}{3} y_1$

Substituting in (2),

$$\frac{11}{2} y_2 - \frac{22}{3} y_1 = 0$$

$\Rightarrow \quad 33\,y_2 - 44\,y_1 = 0$

$-15y_2 + 8y_1 = -6$ (by (3))

$y_1 = 0.6$

Now $4 - 15y_2 = -6$

$\Rightarrow y_2 = 0.8$

Therefore $y_0 = \frac{2}{3}y_1 = \frac{2}{3}(0.6) = 0.4$.

11.3 Solving of Laplace's and Poisson's Equations

Consider the equation

$A\,u_{xx} + 2\,B\,u_{xy} + C\,u_{yy} + F\,(x, y, u, u_x, u_y) = 0$ … (1)

Equation (1) is said to be

(i) parabolic if $AC - B^2 = 0$

(ii) elliptical if $AC - B^2 > 0$

(iii) hyperbolic if $AC - B^2 < 0$.

Examples: (1) $\frac{du}{dt} = C^2 \frac{d^2u}{dx^2}$

$A = C^2, B = 0, C = 0$

$AC - B^2 = 0$ (parabolic)

(2) $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

$A = 1, C = 1, B = 0$

$AC - B^2 > 0.$ (elliptical)

(3) $u_{tt} = C^2\,u_{xx}$

$A = C^2, C = -1, B = 0$

$AC - B^2 = -C^2 < 0$

Therefore the equation is hyperbolic.

11.3.1 Some Standard Equations:

(1) $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ Two dimensional Laplace equation.

(2) $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=f(x,y)$ Two dimensional Poisson's equation.

(3) $\frac{du}{dt}=C^2\frac{d^2u}{dx^2} \rightarrow$ One dimensional heat equation

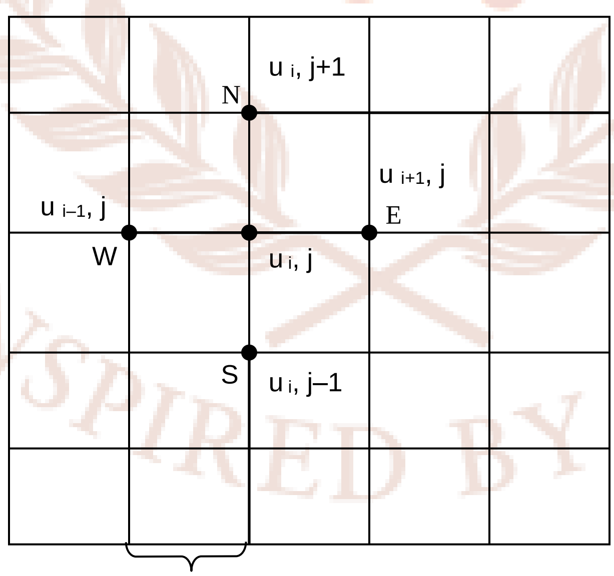
(4) $\frac{d^2u}{dt^2}=C^2\frac{d^2u}{dx^2} \rightarrow$ One dimensional wave equation.

11.3.2 Laplace's and Poisson's equations

To solve Laplace's or Poissons equations in a rectangular region, we divide the region into small squares by drawing lines parallel to the sides of the rectangle. Let h be the side of each square.

Let (x_0, y_0) be an arbitrary interior corner point and $x_i = x_0 + i\,h$, $y_j = y_0 + jh$ … (1)

Define $u_{ij} = u(x_i, y_j)$



$u_{i+1,j} = u(x_{i+1}, y_j) = u(x_i+h, y_j)$

$$= u(x_i, y_j) + h\frac{du}{dx}(x_i, y_j) + \frac{h^2}{2!}\frac{d^2u}{dx^2}(x_i, y_j) + \ldots$$

$$= u_{ij} + h\frac{du_{ij}}{dx} + \frac{h^2}{2!}\frac{d^2u_{ij}}{dx^2} + \ldots \qquad \ldots (1)$$

Similarly,

$$u_{i-1,\,j} = = u_{ij} - h\frac{du_{ij}}{dx} + \frac{h^2}{2!}\frac{d^2u_{ij}}{dx^2} + \ldots \qquad \ldots (2)$$

Adding (1) and (2)

$$u_{i+1,\,j} + u_{i-1,\,j} = 2u_{ij} + h^2\frac{d^2u_{ij}}{dx^2} + \ldots \qquad \ldots (*)$$

$$\Rightarrow \quad \frac{d^2u_{ij}}{dx^2} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} \qquad \ldots (3)$$

Also, $\dfrac{du_{ij}}{dx} = \dfrac{u_{i+1,j} - u_{ij}}{h} = \dfrac{u_{ij} - u_{i-1,j}}{h}$ (observation)

Similarly, $$\frac{d^2u_{ij}}{dy^2} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} \qquad \ldots (4)$$

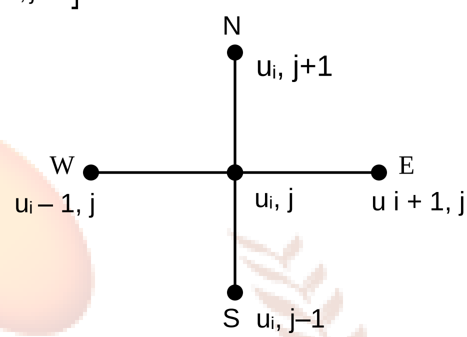
To solve the Laplace equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

Take $x = x_i$ and $y = y_i$ in above equation and use (3) and (4) we get

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} = 0$$

Therefore, $u_{ij} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right]$



is known as standard five point formula.

This can be remembered by

$u_{ij} = \frac{1}{4}$ [N + E + W + S]

Poission's Equations:

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f(x, y)$$

Use (3) and (4) to get $u_{ij} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{ij}\right]$

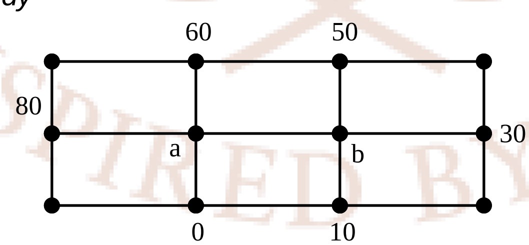
where $f_{ij} = f(x_i, y_j)$.

Self Assessment Questions

1. The one dimensional heat equation $u_t = c^2 u_{xx}$ is ____________
2. The equation $u_{xx} + 2u_{xy} + u_{yy} = 0$ is ____________.
3. The equation $(1+x^2)\ u_{xx} + (5+2x^2)\ u_{xt} + (u+x^2)\ u_{tt} = 0$ is ____________

Problems

Solve $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$ in the given region with indicated boundary conditions:



Sol: The unknown values of *u* at the two interior points are *a* and *b*.

$$a = \frac{1}{4}(b + 80 + 60 + 0)$$

$$= \frac{1}{4}(140 + b)$$

$$b = \frac{1}{4}(30 + a + 50 + 10)$$

$$= \frac{1}{4}(90 + a)$$

Therefore $4a - b = 140$

$-a + 4b = 90$

Solving $15a = 470 \Rightarrow a = 43.33 \quad b = 33.3.$

Problem:

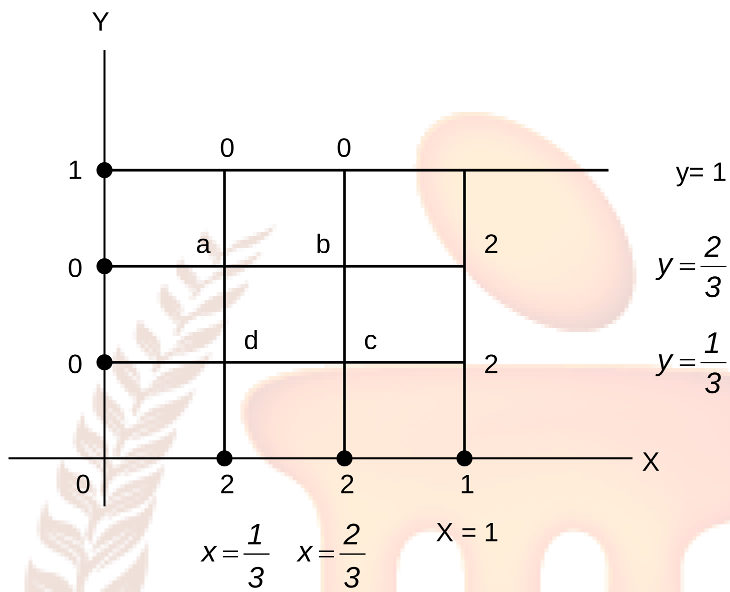
With the step size $h = \frac{1}{3}$, solve

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \qquad \text{in } 0 < x < 1,\ 0 < y < 1$$

$u(x, 1) = u(0, y) = 0$, $u(1, y) = 9(y - y^2)$

$u(x, 0) = 9(x - x^2)$

Sol:



$$u\left(\frac{1}{3},1\right)=0,\quad u\left(\frac{2}{3},1\right)=0,\quad u\left(0,\frac{1}{3}\right)=0, u\left(0,\frac{2}{3}\right)=0$$

$$u\left(\frac{1}{3},0\right)=9\left(\frac{1}{3}-\frac{1}{9}\right)=3-1=2,\qquad u\left(\frac{2}{3},0\right)=9\left(\frac{2}{3}-\left(\frac{2}{3}\right)^2\right)=2$$

u(1, 1) = 0

$$u\left(1,\frac{1}{3}\right)=9\left(\frac{1}{3}-\frac{1}{9}\right)=3-1=2$$

$$u\left(1,\frac{2}{3}\right)=9\left(\frac{2}{3}-\frac{4}{9}\right)=6-4=2.$$

The unknown values of u at the grid points are a, b, c and d.

By Symmetry about diagonal, b = d.

$$a=\frac{1}{4}[b+0+0+d]=\frac{1}{4}[2d]=\frac{1}{2}d$$

$$d=\frac{1}{4}[c+0+a+d]=\frac{1}{4}[c+a+2]$$

$$c=\frac{1}{4}\left[b+d+2+2\right]=\frac{1}{4}\left[2d+4\right]=1+\frac{d}{2}$$

$$d=\frac{1}{4}\left[2+\left(1+\frac{d}{2}\right)+\left(\frac{d}{2}\right)\right]=\frac{1}{4}\left[3+d\right]\Rightarrow d=1.$$

Therefore

$a=\frac{1}{2}$

b = 1

$c=\frac{3}{2}$.

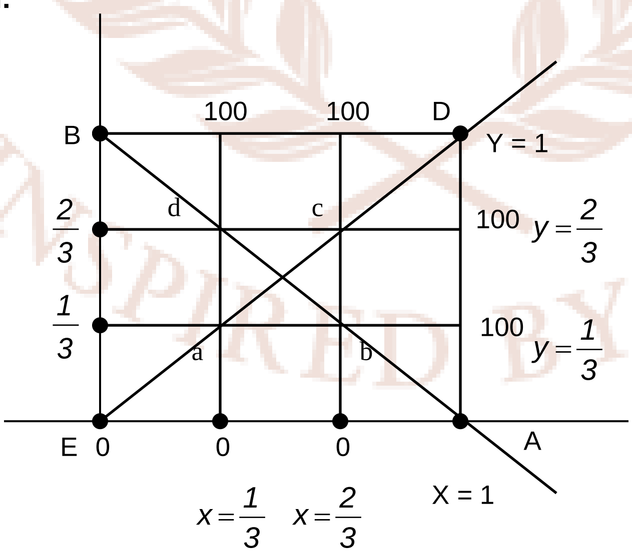
Problem: With h = $\frac{1}{3}$, solve

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=-81xy, \quad 0 < x < 1, \quad 0 < y < 1$$

u(0, y) = u(x, 0) = 0

u(1, y) = u(x, 1) = 100

Solution:



$u(0, y) = 0 \rightarrow$ y – axis

$u(x, 0) = 0 \rightarrow$ x – axis

$u(1, y) = 100$, $u(x, 1) = 100$

By Symmetry, b = d.

Therefore the only three unknowns are a, b, c

We have

$$u_{ij} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{ij}\right]$$ where $f_{ij} = f(x_i, y_j)$

$$a = \frac{1}{4}\left[b + 0 + d + 0 - \frac{1}{9} f\left(\frac{1}{3}, \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[2b - \frac{1}{9}\left(-81 \times \frac{1}{3} \times \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}[2b + 1]$$

$$b = \frac{1}{4}\left[100 + a + c + 0 - \frac{1}{9} f\left(\frac{2}{3}, \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[100 + a + c - \frac{1}{9}\left(-81 \times \frac{2}{3} \times \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}[a + c + 102]$$

$$c = \frac{1}{4}\left[100 + d + 100 + b - \frac{1}{9} f\left(\frac{2}{3}, \frac{2}{3}\right)\right]$$

$$= \frac{1}{4}\left[200 + d + b - \frac{1}{9}\left(-81 \times \frac{2}{3} \times \frac{2}{3}\right)\right]$$

$$= \frac{1}{4}\left[200 + d + b + 4\right]$$

$$= \frac{1}{4}\left[2b + 204\right]$$

Therefore $\quad 4a - 2b = 1 \qquad \dots (1)$

$a - 4b + c = -102 \qquad \dots (2)$

$2b - 4c = -204 \qquad \dots (3)$

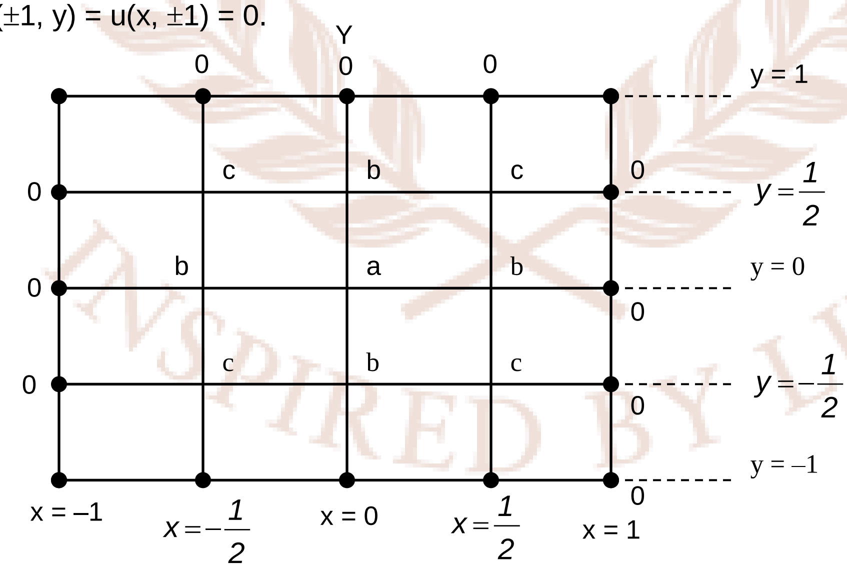
Solving we get a = 26, b = 51 c = 76.

Problem:

With step size $h = \frac{1}{2}$, solve

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = -1, \qquad |x| < 1, \qquad |y| < 1$$

$u(\pm 1, y) = u(x, \pm 1) = 0.$



Given u(1, y) = 0, u(–1, y) = 0

u(x, 1) = 0, u(x, –1) = 0

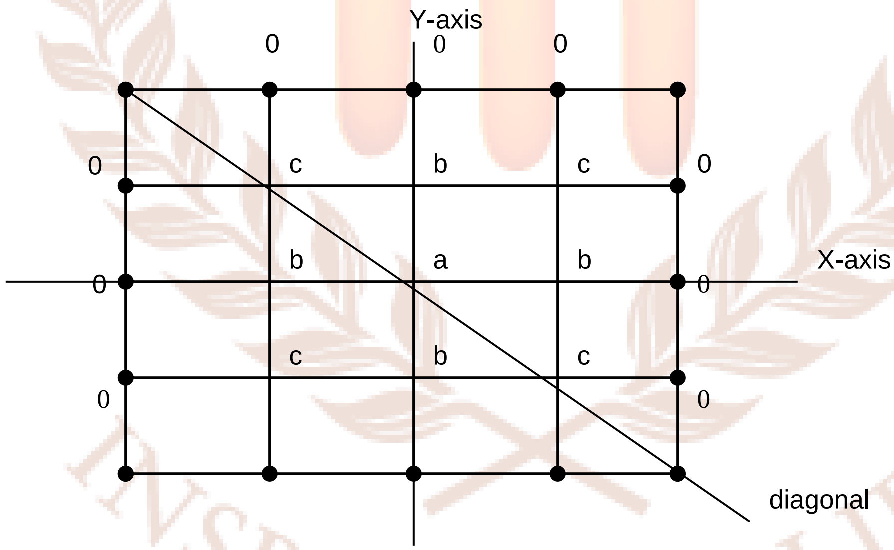
By Symmetry there are only three unknowns a, b, c

$$a=\frac{1}{4}\left[b+b+b+b-\frac{1}{4}f\left(0,0\right)\right]=\frac{1}{4}\left[4b+\frac{1}{4}\right]$$

$$b=\frac{1}{4}\left[0+a+2c+\frac{1}{4}\right]=\frac{1}{4}\left[a+2c+\frac{1}{4}\right]$$

$$c=\frac{1}{4}\left[0+b+b+0+\frac{1}{4}\right]=\frac{1}{4}\left[2b+\frac{1}{4}\right]$$

Solving: $a=\frac{9}{32}$, $b=\frac{7}{32}$, $c=\frac{11}{64}$

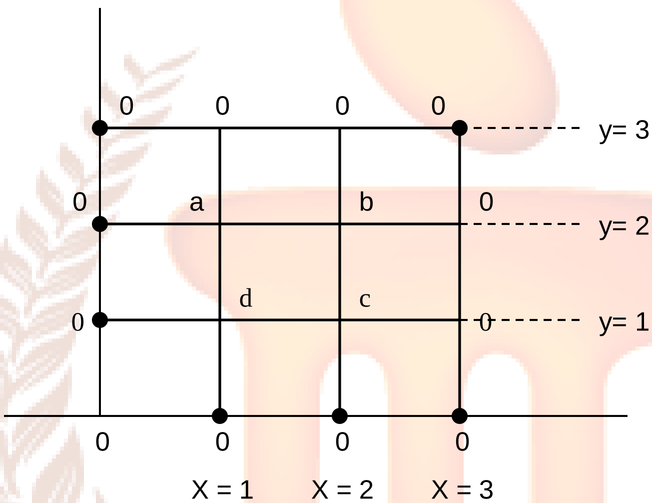


Note

The use of Gauss-Seidal iteration method to solve the system of equations obtained by finite difference method is called Liebmann’s method.

Example:

Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides x = 0, y = 0, x = 3, y = 3 with u = 0 on the boundary and mesh length = 1



We have

$$u_{ij} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{ij}\right] \quad \text{where } f_{ij} = f(x_i, y_j)$$

By Symmetry a = c

$$a = \frac{1}{4}\left[0 + d + b + 0 - f(1,2)\right] \qquad (1)$$

$$= \frac{1}{4}\left[d + b + 150\right]$$

$$b = \frac{1}{4}\left[0 + a + 0 + c - \left(-10\left(4 + 4 + 10\right)\right)\right]$$

$$= \frac{1}{4}\left[2a + 180\right]$$

$$= \frac{1}{2}\left[a + 90\right] \qquad (2)$$

$$d = \frac{1}{2}\left[a + 60\right] \qquad (3)$$

$$c = \frac{1}{4}\left[b + d + 150\right]$$

Therefore a = c.

Solving: (1), (2), (3) by Gauss-Seidal iteration method, at the 4th iteration, we get

a = 75, b= 82.5, c = 75, d = 67.5

11.4 Summary

Partial differential equations occur in many branches like hydrodynamics, elasticity, etc. Several numerical methods have been proposed for the solution of partial differential equations, but only the finite difference methods have became more gainfully employed than others. In this unit, we discussed five point formula, Laplace and Poisson equations with necessary illustrations.

11.5 Terminal Questions

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given u = 0 on the four boundaries dividing the square into 16 subsquares of length 1 unit.

11.6 Answers

Self Assessment Questions

1. Parabolic
2. Parabolic
3. Hyperbolic

Terminal Questions

1. $u_1 = u_3 = u_7 = u_9 = -3$
   $u_2 = u_4 = u_6 = u_8 - 2$ and $u_5 = -2$

List of Algorithms

1. **Bisection Algorithm:**

   To find a solution to f(x) = 0 given the continuous function f on [a, b] where f(a) and f(b) have opposite signs

   INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0,

   OUTPUT approximate solution p or message of failure.

   Step 1: Set i = 1

   Step 2: While i ≤ N_o do steps 3 – 6 .

   Step 3: Set p = a + $\frac{(b-a)}{2}$ (Compute p_i)

   Step 4: If f(p) = 0 or $\frac{(b-a)}{2}$ < TOL then OUTPUT (p); (procedure completed successfully)

   STOP

   Step 5: set i = i +1

   Step 6: If f(a) f(p) > 0 then set a = p (Compute a_i, b_i)

   else set b = p

   Step 7: OUTPUT ("Method failed after No iteration, N_0 = ; N_0); (Procedure completed unsuccessfully)

   STOP

2. **Composite Simpsons Algorithm**

   To approximate the integral I = $\int_a^b f(x)dx$

   INPUT end points a, b; even positive integer n.

   OUTPUT approximation XI to I

   Step 1: Set h = $\frac{(b-a)}{n}$

   Step 2: Let XIO = f(a) + f(b);

X I 1 = 0; (summation of $f(x_{2i-1})$)

X I 2 = 0; (summation of $f(x_{2i})$)

Step 3: For i = 1, 2, …, n –1 do steps 4 and 5

Step 4: Set X = a+ih

Step 5: If i is even then set X I 2 = X I 2 + f(x) else set XI 1 = X I 1 + f(x)

Step 6: Set X 1 = h (X I O + 2. X I 2 + 4. XI 1)/3

Step 7: OUTPUT (XI);

STOP

3. **Algorithm for Newton-Raphson method for a polynomial equation.**

1. Read x_0, e, n, N

   Remarks: x_0 is the initial guess of the root, e the allowed error, n the order of the polynomial and N the total number of iterations, x_0 is guessed using the rules of polynomial equation.
2. for i = 0 to n in steps of 1 do Read a_i end for
3. for i = 0 to n –1 in steps of 1 do Read b_i end for
4. $P \leftarrow a_n$
5. $b_{n-1} \leftarrow a_n$
6. $S \leftarrow b_{n-1}$
7. for k = 1 to N in steps of 1 do
8. for i = 1 to n – 1 in steps of 1 do
9. $b_{n-(i+1)} \leftarrow a_{n-i} + x_0\, b_{n-I}$
10. $S \leftarrow b_{n-(i+1)} + x_0\, s$

    end for
11. $P \leftarrow a_0 + b_0\, x_0$
12. $x_1 \leftarrow x_0 - (P/s)$
13. if $\left| x_1 - \dfrac{x_0}{x_1} \right| \le e$ Go To 18 else

14. $x_0 \leftarrow x_i$

    end if

    end for

15. Write " Root not found in N iterations"
16. Write S, P, x_1, x_0
17. Stop
18. Write "Root found in k iterations"
19. $x_0 \leftarrow x_1$
20. Write x_0, S, P
21. Stop

Reference Books

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