Ch-1
Error & computation

Theorem:

If $f(x)$ is continuous in $a \leq x \leq b$ and if $f(a)$ and $f(b)$ are of opposite sign, then $f(c) = 0$ for at least one number c such that $a < c < b$ means $c \in (a, b)$.

Rolle's Theorem:-

If $f(x)$ is a function on $[a, b]$ and must
(i) $f(x)$ is continuous in $[a, b]$
(ii) $f(x)$ is differential in (a, b)
(iii) $f(a) = f(b)$
then $\exists$ (exist) at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Example:-

(1) verify rolle's theorem in $[-1, 1]$ for the function $f(x) = x^2$

Solution:

We know that $f(x) = x^2$ is differentiable and continous on $[-1, 1]$, and

$f(1) = (1)^2 = 1$

$f(-1) = (-1)^2 = 1$.

So, $f(1) = 1 = f(-1)$

Then, $f'(x) = 2x$ exist in $[-1, 1]$ means, $c \in (-1, 1)$ such that $f'(c) = 0$.

$f'(c) = 2c$.

$0 = 2c$

$c = 0$.

Hence Rolle's theorem is verified.

(2) Verify Rolle's theorem in $[2,4]$ for the function $f(x) = x^2 - 6x + 8$.

Solution :-

$\cdot$ $f(x) = x^2 - 6x + 8$ is polynomial

Hence $f(x)$ is continuous and differentiable in $[2,4]$.

Now, $f(2) = 4 - 12 + 8 = 0$

$f(4) = 16 - 24 + 8 = 0$.

$f(2) = f(4) = 0$.

$f'(x) = 2x + 6(1) + 0$

$= 2x + 6$

$c \in (2,4)$ such that $f'(c) = 0$.

$f'(x) = 2x + 6$

$f'(c) = 2c + 6$

$0 = 2c + 6$

$c = 3 \in (2,4)$

Hence, Rolle's theorem is verified.

* **Intermediate Value Theorem :-**

If $f(x)$ is continous function in $[a,b]$ and K be any number between $f(a)$ and $f(b)$. Then there exists number c in (a,b) such that $f(c) = K$. means at least one $x_0 \in (a,b)$ such that $f(x_0) = C$.

Example :-

(1) $f(x) = x^2 + x - 1$ then verify Intermediate value theorem ~~on~~ on $[0,1]$.

Sol: $f(0) = 0 + 0 - 1 = -1 < 0$

$f(1) = 1 + 1 - 1 = 1 > 0$.

Hence, $f(0)$ and $f(1)$ are different signs, so at least one real root lies between 0 and 1.

* Lagrunge's Mean value theorem :-

If $f(x)$ is function on $[a,b]$. and must
(i) $f(x)$ is continous in $[a,b]$
(ii) $f(x)$ is differentiable in (a,b)
there there $\exists$ (exist) at least one $c \in (a,b)$
such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

Example :-

① check validity of LMVT for $f(x) = x^2 - 1$ in $[-1,5]$.
also find 'c' corresponding to LMVT of exist.

Sol:

$f(x) = x^2 - 1$

$f(x)$ is a polynomial, so $f(x)$ is differentiable and continous in $[-1,5]$.

$f'(x) = 2x$.

$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$2c = \dfrac{f(5) - f(-1)}{5-(-1)}$

$2c = \dfrac{[5^2-1] - [(-1)^2-1]}{6}$

$2c = \dfrac{24-0}{6}$ $\boxed{\therefore c = 2}$

Here $c \in (-1,5)$ such that LMVT is varified.

(2) Verify Lagrange's mean value theorem for function $f(x) = \sqrt{x^2-4}$ in interval $[2,4]$

Sol:- $f(x) = \sqrt{x^2-4}$ is continous & differentiable.
then

$$f'(x) = \frac{2x}{2\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}}$$

$$f'(c) = \frac{f(4)-f(2)}{4-2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\left[\sqrt{(4)^2-4}\right] - \left[\sqrt{(2)^2-4}\right]}{2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12} - \sqrt{0}}{2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{2\sqrt{3}}{2}$$

$$(c)^2 = \left(\sqrt{3}\sqrt{c^2-4}\right)^2$$

$$c^2 = 3(c^2-4)$$

$$c^2 = 3c^2 - 12$$

$$2c^2 = 12$$

$$c^2 = 6$$

$$c = \sqrt{6} \quad \text{and} \quad c \in (2,4)$$

Hence, Lagrang's mean value theorem is verified.

* Taylor's Series :-

If $f(x)$ is continous and processes continous derivatives of order n in interval that include $x=a$, then in that interval.

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} f^2(a) + \dots$$
$$\dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n(x).$$

Example :-

① $2x^3 + 7x^2 + x - 1$ in power of $(x-2)$ by taylor's theorem

Sol :-

As Taylor's theorem,

$$f(x) = f(2) + \frac{(x-2)}{1!} f^1(2) + \frac{(x-2)^2}{2!} f^2(2) + \dots\dots$$

Now, $f(x) = 2x^3 + 7x^2 + x - 1 \Rightarrow f(2) = 16 + 28 + 2 - 1 = 45$

$f^1(x) = 6x^2 + 14x + 1 - 0 \Rightarrow f^1(2) = 24 + 28 + 1 = 53.$

$f^2(x) = 12x + 14 + 0 \Rightarrow f^2(2) = 24 + 14 = 38$

$f^3(x) = 12 + 0. \Rightarrow f^3(2) = 12$

$f^4(x) = 0. \Rightarrow f^4(2) = 0$

Now put value in Taylor's theorem,

$$2x^3 + 7x^2 + x - 1 = 45 + \frac{53(x-2)}{1!} + \frac{38}{2!}(x-2)^2 + \frac{12}{3!}(x-3)^3$$

$$\boxed{= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3}.$$

M	T	W	T	F	S	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

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(2) Expand $\sin x$ in power of $(x-\pi/2)$ by taylor's theorem.

Sol:- As per Taylor's theorem

$$f(x) = f(\pi/2) + \frac{(x-\pi/2)}{1!} f^1(\pi/2) + \frac{(x-\pi/2)^2}{2!} f^2\left(\frac{\pi}{2}\right) + \ldots$$

Now,
$$f(x) = \sin x \Rightarrow f(\pi/2) = 1$$
$$f^1(x) = \cos x \Rightarrow f^1(\pi/2) = 0$$
$$f^2(x) = -\sin x \Rightarrow f^2(\pi/2) = -1$$
$$f^3(x) = -\cos x \Rightarrow f^3(\pi/2) = 0$$
$$f^4(x) = \sin x \Rightarrow f^4(\pi/2) = 1$$

$$\sin x = 1 + \frac{(x-\pi/2)}{1!}(0) + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}(0) + \frac{(x-\pi/2)^4}{4!}(1) + \ldots$$

$$\sin x = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{4!} + \ldots$$

Example

The derivative, $f'(x)$ of a function $f(x)$ can be approximated by the equation,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

If $f(x) = 7e^{0.5x}$ and $h = 0.3$

a) Find the approximate value of $f'(2)$

b) True value of $f'(2)$

c) True error for part (a)

Solution

b) The exact value of $f'(2)$ can be found by using our knowledge of differential calculus.

$$f(x) = 7e^{0.5x}$$

$$f'(x) = 7 \times 0.5 \times e^{0.5x}$$

$$= 3.5e^{0.5x}$$

So the true value of $f'(2)$ is

$$f'(2) = 3.5e^{0.5(2)}$$

$$= 9.5140$$

True error is calculated as

$E_t =$ True Value – Approximate Value

$$= 9.5140 - 10.263 = -0.722$$

Example

Following from the previous example for true error, find the relative true error for $f(x)=7e^{0.5x}$ at $f'(2)$ with $h=0.3$

Solution

$$E_t = -0.722$$

Relative True Error is defined as

$$\in_t = \frac{\text{True Error}}{\text{True Value}}$$

$$= \frac{-0.722}{9.5140} = -0.075888$$

as a percentage,

$$\in_t = -0.075888 \times 100\% = -7.5888\%$$

Example

For $f(x) = 7e^{0.5x}$ at $x = 2$ find the following,

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) approximate error for the value of $f'(2)$ for part b)

Solution

a) For $x = 2$ and $h = 0.3$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

Example

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3} = 10.263$$

b) For $x = 2$ and $h = 0.15$

$$f'(2) \approx \frac{f(2 + 0.15) - f(2)}{0.15}$$

$$= \frac{f(2.15) - f(2)}{0.15}$$

Example

$$= \frac{7e^{0.5(2.15)} - 7e^{0.5(2)}}{0.15}$$

$$= \frac{20.50 - 19.028}{0.15} = 9.8800$$

c) So the approximate error, E_a is

$E_a =$ Present Approximation – Previous Approximation

$$= 9.8800 - 10.263$$

$$= -0.38300$$