BACHELOR OF COMPUTER APPLICATIONS

SEMESTER 3

DCA2101

COMPUTER ORIENTED NUMERICAL METHODS

Unit 5

Matrices and Solutions of Systems of Linear Equations-Direct Methods

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1. INTRODUCTION

Matrices have originated as mere stores of information but, at present, have found very wide applications. For example, in the solution of linear algebraic systems, solutions of ordinary and partial differential equations, eigenvalue problems, electrical networks, framework in mechanics, curve fitting in statistics and transportation problems, etc.

Matrices are useful because they enable us to consider an array of many numbers as a single object, denote it by a single symbol, and perform calculations with these symbols in a very compact form.

In this unit, we will study different methods of solutions to the linear system of equations by direct and indirect methods.

1.1 Objectives:

At the end of this unit the student should be able to:

- *Learn to solve the system of equations and observe their consistency*
- *Learn the direct methods to solve the linear system.*

2. LINEAR EQUATIONS

Definitions

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots \quad \ldots \quad \ldots \quad$ – (*)

$\ldots \quad \ldots \quad \ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m.$

a_{ij} are constants, which are called the *coefficients* of the system and bi are another constants.

(i) If the b_i are all zero, then (*) is called a *homogeneous system*. If at least one b_i is not zero, then (*) is called a *non-homogeneous system*.

A *solution* of (*) is a set of numbers $x_1, x_2, \ldots, x_n$ which satisfy all the m equations. If the system (*) is homogeneous, it has at least one trivial solution $x_1 = 0, x_2 = 0 \ldots x_n = 0$.

We can write the above system of equation (*) as

$AX = B$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}$ is the coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

Note that X has n elements whereas *B* has m elements.

The matrix $\left[A \mid B\right]$ or *[A:B]* $= \begin{bmatrix} a_{11} & a_{12} & \ldots.. & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots.. & a_{2n} & b_2 \\ \ldots. & \ldots. & \ldots. & \ldots. & \ldots. \\ \ldots. & \ldots. & \ldots. & \ldots. & \ldots. \\ a_{m1} & a_{m2} & \ldots. & a_{mn} & b_m \end{bmatrix}$

is called the ***augmented matrix***.

Note: The matrix equation *AX = B* need not always have a solution. It may have *no solution or a unique solution* or an *infinite* number of solutions.

Definition

A system of equations having no solution is called an ***inconsistent system***. A system of equations having one or more solutions is called a ***consistent system***.

Note:

Consider a system of non-homogeneous linear equations *AX = B.*

i) if rank $A \neq rank \left[A \mid B\right]$, then system is **inconsistent**.

ii) if rank $A = rank \left[A \mid B\right]$ = number of unknowns, then system has a unique solution.

iii) if rank $A = rank \left[A \mid B\right] \leq$ number of unknowns, then system has an infinite number of solutions.

Example

Test for consistency and solve

x + y + z = 6

x – y + 2z = 5

3x + y + z = 8

Solution: The augmented matrix $\left[A \mid B\right] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$

Perform $R_2 \to -R_1 + R_2$ and $R_3 \to -3R_1 + R_3$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

Perform $R_3 \to -R_2 + R_3$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 : 6 \\ 0 & -2 & 1 : -1 \\ 0 & 0 & -3 : -9 \end{bmatrix}$$

Now both A and $[A \mid B]$ have all the three rows non-zero. Therefore the rank*(A)* = 3 = *rank* $[A \mid B]$ = the number of independent variables. Therefore the given system of equations is consistent and will have a unique solution.

Now convert the prevailing form of *[A:B]* to a set of equations we get

$x + y + z = 6$ ………. (i)

$-2y + z = -1$……….. (ii)

$-3z = -9$ ……….. (iii)

Now from (iii), $z = 3$, substituting in (ii) we get $y = 2$. From (i) we get $x = 1$.

Example:

Test for consistency and solve

$x + 2y + 3z = 14$

$4x + 5y + 7z = 35$

$3x + 3y + 4z = 21$

Solution: The augmented matrix $[A \mid B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$

Perform $R_2 \rightarrow -4R_1 + R_2$ and $R_3 \rightarrow -3R_1 + R_3$

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$$

Perform $R_3 \rightarrow -R_2 + R_3$

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Therefore the rank $(A) = 2 = rank\ [A \mid B] < 3 =$ the number of independent variables.

Therefore the given system of equations is consistent and will have an <u>infinite number</u> of solutions. Since $n\text{-}r = 3\text{-}2 = 1$, one variable can take arbitrary constant value.

Now convert the prevailing form of $[A/B]$ to a set of equations, we get

$x + 2y + 3z = 14$ ……….(i)

$-3y - 5z = -21$………..(ii)

Take $z = k$ (arbitrary).

From (ii), $-3y - 5k = -21$ or $y = 7 - \frac{5k}{3}$. Substituting in (i) we get $x = \frac{k}{3}$.

Example

Investigate the values of λ and μ such that the system of equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$, may have

(i) Unique solution
(ii) Infinite number of solutions
(iii) No solution.

Solution: = $\left[A \mid B\right] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

Perform, $R_2 \rightarrow -R_1 + R_2$, and $R_3 \rightarrow -R_1 + R_3$

$$\left[A \mid B\right] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

Perform, $R_3 \rightarrow -R_2 + R_3$

$$\left[A \mid B\right] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

(i) Unique solution: For unique solution, we must have rank A = rank $\left[A \mid B\right]$ = 3. The rank *(A)* will be *3* if *(λ- 3) ≠ 0*, since the other two entries in the last row, are zero. If *(λ- 3) ≠ 0*or *λ ≠3* irrespective of the values of *μ*, the rank $\left[A \mid B\right]$ will also be 3. Therefore the system will have a unique solution if *λ ≠ 3*.

(ii) Infinite solutions: The number of unknown n = 3, we need *rank (A)* = rank $\left[A \mid B\right]$= r < 3. Since first row and second row are non zero, we have that *r = 2*. Therefore rank *(A) = rank* $\left[A \mid B\right]$= 2 only when the last row of $\left[A \mid B\right]$ is completely zero. This is possible if *λ - 3 = 0, μ -10 = 0*. *Therefore the system will have infinite solutions if λ = 3 and μ = 10.*

(iii) No solution: In this case, we must have rank *(A) ≠ rank*$\left[A \mid B\right]$, by case (i) rank*(A) = 3* if *λ* ≠ 3 and hence if *λ = 3* we obtain rank *(A) = 2*. If we impose *(μ - 10) ≠0* other than $\left[A \mid B\right]$ will be *3*.

Therefore the system has no solution if *λ = 3* and *μ ≠10*.

Self-Assessment Questions -1

SAQ 1: Test for consistency and solve

$x + y + z = 0$

$2x + 3y + z = 0$

$3x + 6y + 5z = 0$

SAQ 2: Solve the system of non-homogeneous linear equation

$x - 2y + z = 1$

$2x - 5y + 2z = 2$

$x + y - z = -1$

SAQ 3: Determine the value of a for which the following system of equations

(i) has unique solution

(ii) has no solution

(iii) has infinitely many solutions

$x + y + z = 2$

$x + y + z = -2$

$x + y + (a-5) z = a$

3. SOLUTION OF SYSTEMS OF LINEAR EQUATIONS-DIRECT METHODS

For homogeneous linear equations, $AX = 0$ is always consistent, since a homogeneous linear system has either trivial or an infinite number of solutions.

A system of linear equations of the form $AX = B$, can be solved numerically by two methods. (i) *Direct method* (These methods produce the exact solution after a finite number of steps, disregarding the round-off error) and (ii*) Iterative method* (These methods give a sequence of approximate solutions, which converge when the number of steps tends to infinity). However, for a large system of equations, the direct methods are very tedious and the cost (in terms of time) on the computer is more. For such systems iterative methods are preferable. In this unit, we study some important direct methods.

3.1 Matrix inversion method:

The given equation is $AX = B$.

If A is a non-singular matrix, pre multiplying both sides by A^{-1} (inverse of A), we get

$A^{-1} A X = A^{-1} B$

$\Rightarrow I X = A^{-1}B$ *where I in the identity matrix*

$\Rightarrow X = A^{-1} B$, *since* $IX = X$.

Observations:

(i) This approach becomes useful when we need to solve $AX = B$ for different sets of B values while A remains the same.

(ii) If A is non-singular matrix, then the matrix equation $AX = B$ has a unique solution.

Example: Solve the system of equation by matrix inversion method

$x + y + z = 1$

$x + 2y + 3z = 6$

$x + 3y + 4z = 6$

Solution: The given equation can be written as

$$AX = B$$

$\Rightarrow X = A^{-1}B$ ...(*)

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$

Det A = $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(8-9) - 1(4-3) + 1(3-2) = -1 \neq 0$

$$\therefore \quad A^{-1}\ exists$$

$$\therefore \quad A^{-1} = \frac{1}{\det A} adj(A)$$

First, let us find adj(A)

Cofactor of $a_{11} = 1 = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$

Cofactor of $a_{12} = 1 = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$

Cofactor of $a_{13} = 1 = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

Cofactor of $a_{21} = 1 = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = -1$

Cofactor of $a_{22} = 2 = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 3$

Cofactor of $a_{23} = 3 = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$

Cofactor of $a_{31} = 1 = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$

Cofactor of $a_{32} = 3 = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$

Cofactor of $a_{33} = 4 = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

Cofactor of $A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Adjoint of $A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A} adj\ A = \frac{1}{-1} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

= $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

So from (*), we have

X = $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$

Thus x=1, y=-5, z=5

Example: Solve the following linear simultaneous system of equations by matrix inversion method.

(i) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$

Solution: (i) We have

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Det. A = -1-12 = -13

Cofactor of $a_{11} = 1 = (-1)^{1+1}(-1) = -1$

Cofactor of $a_{12} = 3 = (-1)^{1+2}4 = -4$

Cofactor of $a_{21} = 4 = (-1)^{2+1}3 = -3$

Cofactor of $a_{22} = -1 = (-1)^{2+2}1 = 1$

Hence, cofactor of A= $\begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}$

So, Adjoint of A= $A^T = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A}A^T = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$$

So, X = $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 12 \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -41 \\ -8 \end{bmatrix}$

Therefore, x = $\frac{-41}{-13} = 3.15$ y =$\frac{-8}{-13} = 0.62$

(ii) We have

A= $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$

DetA $= \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} = 1(2+3) + 1(4+1) + 3(12-2) = 40 \neq 0$

$$\therefore \quad A^{-1} \; exists$$

$$\therefore \quad A^{-1} = \frac{1}{\det A} adj\ A$$

First let us find adj.A

Cofactor of $a_{11} = 1 = (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 5$

Cofactor of $a_{12} = -1 = (-1)^{1+2} \begin{vmatrix} 4 & -1 \\ 1 & 1 \end{vmatrix} = -5$

Cofactor of $a_{13} = 3 = (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10$

Cofactor of $a_{21} = 4 = (-1)^{2+1} \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = 10$

Cofactor of $a_{22} = 2 = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$

Cofactor of $a_{23} = -1 = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = -4$

Cofactor of $a_{31} = 1 = (-1)^{3+1} \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = -5$

Cofactor of$a_{32} = 3 = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} = 13$

Cofactor of $a_{33} = 1 = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} = 6$

Cofactor of $A = \begin{bmatrix} 5 & -5 & 10 \\ 10 & -2 & -4 \\ -5 & 13 & 6 \end{bmatrix}$

Adjoint of $A = \begin{bmatrix} 5 & -5 & 10 \\ 10 & -2 & -4 \\ -5 & 13 & 6 \end{bmatrix}^T = \begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A} adj\, A = \frac{1}{40}\begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}$$

So , we have

$$X = A^{-1}B = \frac{1}{40}\begin{bmatrix} 5 & 10 & -5 \\ -5 & -2 & 13 \\ 10 & -4 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

x= 0, y = 1, z = 2

Self-Assessment Questions -2

SAQ 4: Solve the following system of equations by matrix inversion method.

(i) 2x - 3y - 5z = 11

5x +2y -7z = -12

-4x +3 y +z = 5

(ii) x +y +z = 7

x +2y + 3z = 16

x + 3y +4z = 22

3.2 Gauss elimination method

This is the elementary elimination method and it reduces the system of equations to an equivalent upper triangular system which can be solved by back substitution. The method is quite general and is well-adapted for computer operations.

Here we shall explain it by considering a system of three equations for sake of simplicity.

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)} \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Step I: To eliminate x from second and third equations.

We eliminate *x* from the second equation by subtracting $\left(\frac{a_2}{a_1}\right)$ times the first equation from the second equation. Similarly we eliminate x from the third equation by subtracting $\left(\frac{a_3}{a_1}\right)$ times the first equation from the third equation.

We thus, get the new system

$$a_1 x + b_1 y + c_1 z = d_1$$

$$b'_2 y + c'_2 z = d'_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$b'_3 y + c'_3 z = d'_3$$

Here the first equation is called the pivotal equation and a1 is called the first pivot element.

Step II: To eliminate y from third equation in (ii).

We eliminate y from the third equation of (ii) by subtracting $\left(\frac{b'_3}{b'_2}\right)$ times the second equation from the third equation. We thus, get the new system

$$a_1 x + b_1 y + c_1 z = d_1$$

$$b'_2 y + c'_2 z = d'_2 \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

$$c''_3 z = d''_3$$

Here the second equation is the pivotal equation and b'_2 is the new pivot element.

Now it becomes upper triangular system of equation.

Step III: To evaluate the unknowns

The values of *x, y, z* are found from the reduced system (iii) by back substitution.

This also can also be derived from the augmented matrix of the system (i),

$$[A|B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Using elementary row transformation, reduce the coefficient matrix A to an upper triangular matrix.

$$\text{That is., } [A|B] \simeq \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

From (v) the system of linear equations is equivalent to

$$a_1 x + b_1 y + c_1 z = d_1$$

$$b'_2 y + c'_2 z = d'_2$$

$$c_3'' z = d_3''$$

By back substitution, we get *z, y,* and *x* constituting the exact solution of the system (i).

Note: The method will fail if one of the pivot elements a_1, b'_2 or c_3'' vanishes. In such cases, the method can be modified by rearranging the rows so that the pivots are non-zero. If this is impossible, then the matrix is singular and the equations have no solution.

Example

Solve by Gauss elimination method.

$2x + y + 4z = 12$

$4x + 11y - z = 33$

$8x - 3y + 2z = 20$

Solution: The augmented matrix of the system is

$$[A/B] = \begin{array}{cc} \begin{bmatrix} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

Perform $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 4R_1$

$$[A/B] = \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{bmatrix}$$

Perform $R_2 \rightarrow \frac{1}{9}R_2$ and $R_3 \rightarrow R_3 + \frac{7}{9}R_2$

$$[A/B] = \begin{matrix} 2 & 1 & 4 & : 12 \\ 0 & 1 & -1 & : 1 \\ 0 & 0 & -21 & : -21 \end{matrix}$$

Now we have

$2x + 1y + 4z = 12$

$1y - 1z = 1$

$-21z = -21$

Therefore $z = 1$

Now by the back substitution we get $y = 2$ and $x = 3$.

Example

Solve the following system of equations by Gauss elimination method.

$5x + y + z + w = 4$

$x + 7y + z + w = 12$

$x + y + 6z + w = -5$

$x + y + z + 4w = -6$

Solution: It is convenient to perform row transformation if the leading entry is *1.* We shall write the augmented matrix by interchanging the first equation with the fourth equation.

$$[A\,|\,B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right]$$

Perform: $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 5R_1$

$$[A\,|\,B] = \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{bmatrix}$$

Perform: $R_4 \to R_4 + \frac{2}{3} R_2$

$$[A\,|\,B] = \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{bmatrix}$$

Perform: $R_4 \to R_4 + \frac{4}{5} R_3$

$$[A\,|\,B] = \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -\frac{117}{5} & \frac{234}{5} \end{bmatrix}$$

Perform $R_4 \to 5R_4$

$$[A\,|\,B] = \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -117 & 234 \end{bmatrix}$$

Now we have

$$1x + 1y + 1z \quad + 4w \quad = -6$$

$$6y + \quad 0z - 3w = 18$$

$$5z \ - \ 3w = 1$$

$$-117w = 234$$

Therefore $w = -2$.

By back substitution we get, $z= -1, \ y= 2, \ x= 1.$

Therefore $(x, y, z, w) = (1, 2, -1, -2)$ is the required solution.

Self-Assessment Questions -3

SAQ 5: Solve by Gauss elimination method.

$x + y + 2z = 4$

$3x + y - 3z = -4$

$2x - 3y - 5z = -5$

SAQ 6: Solve by Gauss elimination method.

$2x + 4y + z = 3$

$3x + 2y - 2z = -2$

$x - y + z = 6$

3.3 Gauss–jordan method

This method is an extension of the Gauss- Elimination method. In the Gauss Elimination method, we transform the given matrix into the upper triangular matrix by row transformation. But here in the Gauss-Jordan method, we transform the given matrix into a diagonal matrix. In this method, we first write the augmented matrix of the given system of equations. Then by row transformation, we try to obtain the left side of the augmented matrix as a diagonal matrix and by doing this all the off-diagonal elements will become zero and diagonal elements will be non-zero. In this way, we will get the unknowns whose value is equal to the values on the right side of the augmented matrix.

Example: Solve the following system of equation by Gauss –Jordan method

x - 2y = -4

-5y +z = -9

4x - 3z = -10

Solution: We have

$$A=\begin{bmatrix}1 & -2 & 0\\ 0 & -5 & 1\\ 4 & 0 & -3\end{bmatrix}, X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}-4\\ -9\\ -10\end{bmatrix}$$

The augmented matrix of above system of equation is

$$[A|B] = \begin{bmatrix}1 & -2 & 0 : -4\\ 0 & -5 & 1 : -9\\ 4 & 0 & -3 : -10\end{bmatrix}$$

Here we will eliminate x from the third row by multiplying the first row successively by -4 and then adding it to third row (i.e., $R_3 \to -4R_1 + R_3$). So the augmented matrix becomes

$$\begin{bmatrix}1 & -2 & 0 : -4\\ 0 & -5 & 1 : -9\\ 0 & 8 & -3 : 6\end{bmatrix} \quad (1)$$

Now, multiply second row of (1) by -1/5, we get

$$\begin{bmatrix}1 & -2 & 0 : -4\\ 0 & 1 & -1/5 : 9/5\\ 0 & 8 & -3 : 6\end{bmatrix} \quad (2)$$

we will eliminate x and y from the first and third row by multiplying the second row successively by 2 and -8 and then adding it to first row and third row respectively (i.e., $R_1 \to R_1 + 2R_2,\ R_3 \to R_3 - 8R_2$). So the augmented matrix (2) becomes

$$\begin{bmatrix}1 & 0 & -2/5 : -2/5\\ 0 & 1 & -1/5 : 9/5\\ 0 & 0 & -7/5 : -42/5\end{bmatrix} \quad (3)$$

Now, multiply the third row of (3) by -5/7, and we get

$$\begin{bmatrix} 1 & 0 & -2/5 : -2/5 \\ 0 & 1 & -1/5 : 9/5 \\ 0 & 0 & 1 : 6 \end{bmatrix} \quad (4)$$

Multiply the third row by 2/5 and add it to the first row also multiply the third row by 1/5 and add it to the second row(i.e., $R_1 \to R_1 + 2/5R_3,\ R_2 \to 1/5R_3 + R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0:2 \\ 0 & 1 & 0:3 \\ 0 & 0 & 1:6 \end{bmatrix}$$

Equating the values, we get z = 6, y = 3, x = 2

Example: Solve the following system of equations by Gauss –Jordan method

2x + y - 3z = 11

4x - 2y + 3z = 8

-2x + 2y - z = -6

Solution: We have

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & -2 & 3 \\ -2 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ 8 \\ -6 \end{bmatrix}$$

The augmented matrix of the above system of equation is

$$[A|B] = \begin{bmatrix} 2 & 1 & -3:11 \\ 4 & -2 & 3:8 \\ -2 & 2 & -1:-6 \end{bmatrix}$$

Divide first row by 2, we get

$$\begin{bmatrix} 1 & 1/2 & -3/2:11/2 \\ 4 & -2 & 3 : 8 \\ -2 & 2 & -1 : -6 \end{bmatrix}$$

Now, $R_2 \to -4R_1 + R_2,\ R_3 \to 2R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 1/2 & -3/2:11/2 \\ 0 & -4 & 9 : -14 \\ 0 & 3 & -4 : 5 \end{bmatrix}$$

Now divide the second row by -4, we get

$$\begin{bmatrix} 1 & 1/2 & -3/2 & : & 11/2 \\ 0 & 1 & -9/4 & : & 7/2 \\ 0 & 3 & -4 & : & 5 \end{bmatrix}$$

Now, $R_1 \rightarrow R_1 - \frac{1}{2}R_2$, $R_3 \rightarrow R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & -3/8 & : & 15/4 \\ 0 & 1 & -9/4 & : & 7/2 \\ 0 & 0 & 11/4 & : & -11/2 \end{bmatrix}$$

Now divide the third row by 4/11, we get

$$\begin{bmatrix} 1 & 0 & -3/8 & : & 15/4 \\ 0 & 1 & -9/4 & : & 7/2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Now, $R_1 \rightarrow R_1 + \frac{3}{8}R_3$, $R_2 \rightarrow R_2 + \frac{9}{4}R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & -1 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Hence, the solution is x = 3, y = -1, z = -2

Example

Using Gauss–Jordan method, find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 1 & 0 \\ 8 & 4 & 5 \end{bmatrix}$$

Solution:

Writing the given matrix side by side with the unit matrix of order 3, we have

$$[A / I] = \begin{array}{c} \begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 8 & 4 & 5 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Perform the row transformations: $R_1 \rightarrow \frac{R_1}{4}$ and $R_3 \rightarrow R_3 - 2R_1$

$$[A \mid I] = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{bmatrix}$$

Perform the row transformations: $R_3 \to R_3 - 2R_2$

$$[A \mid I] = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}$$

Perform the row transformations: $R_1 \to R_1 - \frac{1}{4}R_2$

$$[A \mid I] = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}$$

Perform the row transformations: $R_1 \to R_1 - \frac{1}{2}R_3$

$$[A \mid I] = \begin{bmatrix} 1 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix} = [I \mid B]$$

Hence the inverse of the given matrix is $B = A^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$.

Example

Solve the following system of equations by matrix inversion method.

$3x + y + 2z = 3$

$2x - 3y - z = -3$ ………………………… (i)

$x + 2y + z = 4$

The system (i) can be written as $AX = B$

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

Now $AX = B \Rightarrow X = A^{-1} B$ ……………….. (ii)

By Gauss–Jordan method we can find the inverse of A (solution left to the reader), that is,

$$A^{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\ -\frac{3}{8} & \frac{1}{8} & \frac{7}{8} \\ \frac{7}{8} & -\frac{5}{8} & -\frac{11}{8} \end{bmatrix}$$

Equation (ii) becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\ -\frac{3}{8} & \frac{1}{8} & \frac{7}{8} \\ \frac{7}{8} & -\frac{5}{8} & -\frac{11}{8} \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{8} - \frac{9}{8} + \frac{20}{8} \\ \frac{-9}{8} - \frac{3}{8} + \frac{28}{8} \\ \frac{21}{8} + \frac{15}{8} - \frac{44}{8} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times 1}$$

This implies that $x = 1, y = 2, z = -1$.

Self-Assessment Questions - 4

SAQ 7: Solve the following system of equation by Gauss –Jordan method

$2x + 6y + z = 7$

$x + 2y - z = -1$

$5x + 7y - 4z = 9$

SAQ 8: Solve the following system of equation by Gauss –Jordan method

$2x + 2y + z = 1$

$4x + 2y + 3z = 2$

$x + y + z = 3$

3.4 Crammer's Rule

Steps to apply Crammer's rule:

Step (i): Write the given equations in order so that the constant terms are all on the right side.

equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (i)$$

$$a_3x + b_3y + c_3z = d_3$$

Step (ii): Take Δ = the determinant formed by the coefficients of *x, y, z.*

If the determinant of coefficients be

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

Step (iii): Replace the first column of Δ by constant terms (r.h.s. terms) of the equations and denote as Δx.

$$\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

Step (iv): Replace the second column of Δ by constant terms (r.h.s. terms) of the equations and denote as Δy.

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},$$

Step (v): Replace the third column ofΔby constant terms (r.h.s. terms) of the equations and denote as Δz.

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Step (vi): Write the solution $x = \frac{\Delta x}{\Delta}$, $y = \frac{\Delta y}{\Delta}$, $z = \frac{\Delta z}{\Delta}$.

Example: Apply Crammer's rule to solve the equations

$3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4.$

Solution: Here $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$

$= 3(-3 + 2) - 1(2 + 1) + 2(4 + 3) = 8$

$\Delta x = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 8$. Similarly, $\Delta y = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 16$ and $\Delta z = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -8$

Therefore $x = \frac{\Delta x}{\Delta} = \frac{8}{8} = 1$, $y = \frac{\Delta y}{\Delta} = \frac{16}{8} = 2$, and

$z = \frac{\Delta z}{\Delta} = \frac{-8}{8} = -1$. Hence $(x, y, z) = (1, 2, -1)$

This method is quite general but involves a lot of labor when the number of equations exceeds four. Therefore, Crammer's rule is not suitable for large systems.

Example: Apply Crammer's rule to solve the equations

$3x + 4y = -14$

$-2x - 3y = 11$

Solution: Here $\Delta = \begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix} = -9 + 8 = -1$

$$\Delta x = \begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix} = 42 - 44 = -2$$

$$\Delta y = \begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix} = 33 - 28 = 5$$

Therefore $x = \frac{\Delta x}{\Delta} = \frac{-2}{-1} = 2, \ y = \frac{\Delta y}{\Delta} = \frac{5}{-1} = -5$

Hence (x,y) = (2, -5)

Example: Apply Crammer's rule to solve the equations

x + 2 y + 3z = 17

3x + 2y+ z = 11

x - 5y + z = -5.

Solution: Here $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}$

$$= 1(2+5) - 2(3-1) + 3(-15-2) = -48$$

$$\Delta x = \begin{vmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{vmatrix} = 17(2+5) - 2(11+5) + 3(-55+10) = -48$$

$$\Delta y = \begin{vmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{vmatrix} = 1(11+5) - 17(3-1) + 3(-15-11) = -96$$

$$\Delta z = \begin{vmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & -5 \end{vmatrix} = 1(-10+55) - 2(-15-11) + 17(-15-2) = -192$$

Therefore $x = \frac{\Delta x}{\Delta} = \frac{-48}{-48} = 1, \ y = \frac{\Delta y}{\Delta} = \frac{-96}{-48} = 2\,, z = \frac{\Delta z}{\Delta} = \frac{-192}{-48} = 4$

Self-Assessment Questions -5

SAQ 9: Apply Crammer's rule to solve the equations

$2x - y + 3z = -3$

$-x - y + 3z = -6$

$X - 2y - z = -2.$

SAQ 10: Apply Crammer's rule to solve the equations

$3x - y + 5z = -2$

$-4x + y + 7z = 10$

$2x + 4y - z = 3.$

2.5 LU decomposition

This method is based on the fact that a square matrix *A* can be factorized into the form *LU*, where *L* is unit lower triangular and *U* is upper triangular if all the principal minors of A are nonsingular.

Consider the system of equations

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

This system can be written as $AX = B$.

Let $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Therefore $LUX = B$.

Put $UX = Y$, then $LY = B$, which is equivalent to the system

$y_1 = b_1$

$l_{21}y_1 + y_2 = b_2$

$l_{31}y_1 + l_{32}y_2 + y_3 = b_3.$

We can solve by forward substitution. When Y is known, the system $UX = Y$ becomes,

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$

$u_{22}x_2 + u_{23}x_3 = y_2$

$u_{33}x_3 = y_3$, which can be solved by the back substitution.

Now $A = LU \Longrightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Multiplying the matrices on the left and equating the corresponding elements of both sides, we get

$u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

$l_{21}u_{11} = a_{21}$, $l_{21}u_{12} + u_{22} = a_{22}$, $l_{21}u_{13} + u_{23} = a_{23}$,

$l_{31}u_{11} = a_{31}$, $l_{31}u_{12} + l_{32}u_{22} = a_{32}$, $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}.$

Solving them we get

$l_{21} = \frac{a_{21}}{a_{11}}$; $l_{31} = \frac{a_{31}}{a_{11}}$; $u_{22} = a_{22} - \frac{a_{31}}{a_{11}} a_{12}$; $u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$

$l_{32} = \dfrac{a_{32} - (\frac{a_{31}}{a_{11}})a_{12}}{u_{22}}$ from which u33 can be computed.

Now we have a systematic procedure to evaluate the elements of L and U. First we determine the first row of U and the first column of L; then we determine the second row of U and the second column of L, and finally, we compute the third row of U. This procedure can be obviously generalized. When the factorization is effected, the inverse of A can be computed from the formula

$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$.

Example

Solve the equations

$2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

by LU decomposition method.

Solution: We have $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

$u_{11} = 2, u_{12} = 3, u_{13} = 1$

$l_{21}u_{11} = \frac{a_{21}}{u_{11}} \Rightarrow l_{21} = \frac{1}{2}$ and $l_{31}u_{11} = 3 \Longrightarrow l_{31} = \frac{3}{2}$.

For u_{22} and u_{23} we have the equations,

$l_{21}u_{12} + u_{22} = 2$, and $l_{21}u_{13} + u_{23} = 3$.

This implies that u22 =   and u23= .

Finally, l_{32} and u_{33} are obtained from $l_{31}u_{12} + l_{32}u_{22} = 1$, and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$.

Therefore $l_{32} = -7$ and $u_{33} = 18$.

Hence A = $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$ and hence the given system of equations can be written as

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Take $\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$.

Solving by forward substitution, we get $y_1 = 9$, $y_2 = \frac{3}{2}$ and $y_3 = 5$.

With these values of y_1, y_2, y_3, by back substitution, we obtain

$x = \frac{35}{18}$, $y = \frac{29}{18}$ and $z = \frac{5}{18}$.

Example: Solve the equations

$8x - 3y + 2z = 20$

$4x + 11y - z = 33$

$6x + 3y + 12z = 36$

By *LU* decomposition method.

Solution: We have

A= $\begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$

Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}$

By the formula, we have

$u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

$l_{21}u_{11} = a_{21}$, $l_{21}u_{12} + u_{22} = a_{22}$, $l_{21}u_{13} + u_{23} = a_{23}$,

$l_{31}u_{11} = a_{31}$, $l_{31}u_{12} + l_{32}u_{22} = a_{32}$, $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$.

So,

$\boldsymbol{u}_{11} = 8$, $\boldsymbol{u}_{12} = -3$, $\boldsymbol{u}_{13} = 2$

$l_{21}u_{11} = a_{21} \Rightarrow l_{21}=a_{21}/ u11= 4 / 8 = ½$

$l_{31}u_{11} = 6 \Longrightarrow l_{31}= 6 / 8 = 3 / 4.$

$l_{21}u_{12} + u_{22} = a_{22}$

$\boldsymbol{u}_{22} = 11 - l_{21}u_{12} = 11+ 3 / 2 = 25 / 2$

$l_{21}u_{13} + u_{23} = a_{23} = -1$

$\boldsymbol{u}_{23} = -1 - l_{21}u_{13} = -1 – 1 =-2$

$l_{31}u_{12} + l_{32}u_{22} = a_{32} = 3$

$l_{32} = (3 - l_{31}u_{12}) / u_{22} = 21 / 50$

Also,

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a33$

$u_{33}= 12 – (l_{31}u_{13} + l_{32}u_{23})=567 / 50$

Hence, A = $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 21/50 & 1 \end{bmatrix}\begin{bmatrix} 8 & -3 & 2 \\ 0 & 25/2 & -2 \\ 0 & 0 & 567/50 \end{bmatrix}$

And hence the given system of equation can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 21/50 & 1 \end{bmatrix}\begin{bmatrix} 8 & -3 & 2 \\ 0 & 25/2 & -2 \\ 0 & 0 & 567/50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$$

Take

$$\begin{bmatrix} 8 & -3 & 2 \\ 0 & 25/2 & -2 \\ 0 & 0 & 567/50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Then $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/4 & 21/50 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$

Solving by forward substitution, we get

p = 20, q = 23, r = 567/50

with these values of p, q, r by back substitution in

$$\begin{bmatrix} 8 & -3 & 2 \\ 0 & 25/2 & -2 \\ 0 & 0 & 567/50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 23 \\ 567/50 \end{bmatrix}$$

We obtain,

$\frac{567}{50}z = \frac{567}{50}$,, z = 1, y =2, x = 3

Self-Assessment Questions -6

SAQ 11: Solve the equations

3x - y + 2z = 12

x + 2y +3 z = 11

2x - 2y - 1z = 2

By LU decomposition method.

4. SUMMARY

Matrices are an important branch of linear algebra. We discussed the matrix approach to solving system of linear equations and consistency through the rank of a matrix. These concepts are useful tools for further computations and writing efficient algorithms. We provide a sufficient number of illustrations for each method.

5. TERMINAL QUESTIONS

1. Test for consistency and solve

   $x - 4y + 7z = 14$

   $3x + 8y - 2z = 13$

   $7x - 8y + 26z = 5$

2. Test for consistency and solve

   $x + 2y + 2z = 1$

   $2x + y + z = 2$

   $3x + 2y + 2z = 3$

   $y + z = 0.$

3. Solve the following system of equations by matrix inversion method

   $x + y + z = 6$

   $x + 2y + 3z = 14$

   $-x + y - z = -2$

4. Solve the following system of equations by matrix inversion method

   $2x + y - 3z = 11$

   $4x - 2y + 3z = 8$

   $-2x + 2y - z = -6$

5. Solve by Gauss elimination method.

$$6x + 3y + 2z = 6$$

$$6x + 4y + 3z = 0$$

$$20x + 15y + 12z = 0$$

6. Solve the following system of equations by Gauss elimination method;

$$x + y + z = 9$$

$$x - 2y + 3z = 8$$

$$2x + y - z = 3$$

7. Solve the following system of equation by Gauss –Jordan method

$$2x + y + 3z = 1$$

$$4x - 3y + 5z = -7$$

$$-3x + 2y + 4z = -3$$

8. Solve the following system of equation by Gauss –Jordan method

$$2y + z = 4$$

$$x + y + 2z = 6$$

$$2x + y + z = 7$$

9. Apply Crammer's rule to solve the equations

$$8x - 7y + 10z = 15$$

$$2x + 3y + 8z = 7$$

$$5y + 9 = 4x + 2z$$

10. Solve by LU decomposition method, the following system of equations

$$2x + y + 4z = 12$$

$$4x + 11y - z = 33$$

$$8x - 3y + 2z = 20.$$

6. ANSWERS

Self Assessment Questions

1. Trivial solution i.e. x = y = z = 0
2. x = 0, y = 0, z = 1
3. unique sol. if a ≠ 6, no solution if a = 6 & a ≠ 2
4. (i) x = 1, y = 2, z = 3, (ii) x = 1, y = 2, z = 3
5. x = 1, y = -1, z = 2
6. x = 2, y = -1, z = 3
7. x=10, y = -3, z = 5
8. x = -9/2, y = 5/2, z = 5
9. x = 1, y = 2, z = -1
10. x = -212/187, y = 273/187, z = 107/187
11. x = 3, y = 1, z = 2.

Terminal Questions

1. ρ(A) = 2 ≠ 3 = ρ (A: B). The system is inconsistent.
2. ρ (A) = 2 < 3 = ρ (A : B). System has infinite number of solutions. The solution to the system is x = 1, y = – k, z = k where k is a arbitrary constant.
3. x = 1, y = 2, z = 3.
4. x = 3, y = -1, z = -2
5. x = 9, y = - 36, z = 30
6. The solution to the system x = 2, y = 3, z = 4.
7. x = 1, y = 2, z = -1
8. x= 11/5, y = 7/5, z = 6/5
9. x = 7, y = 3, z = -2
10. The matrix L = $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 9 & 0 \\ 8 & -7 & -21 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The solution to the system is x = 3, y = 2, z = 1.