ECE 650, Winter 2021
A Polynomial-Time Reduction from VERTEX-COVER to CNF-SAT

A *vertex cover* of a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that each edge in E is incident to at least one vertex in C.

VERTEX-COVER is the following problem:

- Input: An undirected graph $G = (V, E)$, and an integer $k \in [0, |V|]$.
- Output: True, if G has a vertex cover of size k, false otherwise.

CNF-SAT is the following problem:

- Input: a propositional logic formula, F, in Conjunctive Normal Form (CNF). That is, $F = c_1 \wedge c_2 \wedge \ldots \wedge c_m$, for some positive integer m. Each such c_i is called a "clause". A clause $c_i = l_{i,1} \vee \ldots \vee l_{i,p}$, for some positive integer p. Each such $l_{i,j}$ is called a "literal." A literal $l_{i,j}$ is either an atom, or the negation of an atom.
- Output: True, if F is satisfiable, false otherwise.

We present a polynomial-time reduction from VERTEX-COVER to CNF-SAT. A polynomial-time reduction is an algorithm that runs in time polynomial in its input. In our case, it takes as input G, k and produces a formula F with the property that G has a vertex cover of size k if and only if F is satisfiable.

The use of such a reduction is that given an instance of VERTEX-COVER that we want to solve, (G, k), we use the reduction to transform it to F, and provide F as input to a SAT solver. The true/false answer from the SAT solver is the answer to the instance of VERTEX-COVER. Assuming the SAT solver works efficiently (for some characterization of "efficient"), we now have an efficient way of solving VERTEX-COVER. Furthermore, the satisfying assignment from the SAT solver can be used to re-construct the solution to VERTEX-COVER.

The reduction

Given a pair (G, k) where $G = (V, E)$, denote $|V| = n$. Assume that the vertices are named $1, \ldots, n$. Construct F as follows.

- The reduction uses $n \times k$ atomic propositions, denoted $x_{i,j}$, where $i \in [1, n]$ and $j \in [1, k]$. A vertex cover of size k is a list of k vertices. An atomic proposition $x_{i,j}$ is true if and only if the vertex i of V is the jth vertex in that list.
- The reduction consists of the following clauses
  - At least one vertex is the ith vertex in the vertex cover:
  $$\forall i \in [1, k], \text{ a clause } (x_{1,i} \vee x_{2,i} \vee \cdots \vee x_{n,i})$$
  - No one vertex can appear twice in a vertex cover.
  $$\forall m \in [1, n], \forall p, q \in [1, k] \text{ with } p < q, \text{a clause } (\neg x_{m,p} \vee \neg x_{m,q})$$
  In other words, it is not the case that vertex m appears both in positions p and q of the vertex cover.
  - No more than one vertex appears in the mth position of the vertex cover.
  $$\forall m \in [1, k], \forall p, q \in [1, n] \text{ with } p < q, \text{ a clause } (\neg x_{p,m} \vee \neg x_{q,m})$$
  - Every edge is incident to at least one vertex in the vertex cover.
  $$\forall \langle i, j \rangle \in E, \text{ a clause } (x_{i,1} \vee x_{i,2} \vee \cdots \vee x_{i,k} \vee x_{j,1} \vee x_{j,2} \vee \cdots \vee x_{j,k})$$

The number of clauses in the reduction is $k + n\binom{k}{2} + k\binom{n}{2} + |E|$.