1 Poisson-normal

$$
\begin{aligned}
\eta_{ij} &= \log \mu_{ij} + b_j + e_{ij} \\
b_j &\sim N\left(0, \tau^2\right) \\
e_{ij} &\sim N\left(0, \sigma^2\right) \\
\eta_{ij} &= \log \lambda_{ij} \\
\lambda_{ij} &= \exp \eta_{ij} \\
y_{ij} | b_j &\sim \text{Poisson}\left(\lambda_{ij}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(y_{ij} | b_j\right) &= \lambda_{ij} \\
\mathrm{Var}\left(y_{ij} | b_j\right) &= \lambda_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(\lambda\right) &= \exp\left(\log \mu + \left(\tau^2 + \sigma^2\right)/2\right) \\
\mathrm{Var}\left(\lambda\right) &= \exp\left(2 \log \mu + 2\tau^2 + 2\sigma^2\right) - \exp\left(2 \log \mu + \tau^2 + \sigma^2\right)
\end{aligned}
$$

2 Poisson-gamma

$$
\begin{aligned}
\eta_{ij} &= \log \mu_{ij} + b_j \\
b_j &\sim N\left(0, \tau^2\right) \\
u_{ij} &\sim \Gamma\left(\phi, \text{rate} = \phi\right) \\
\eta_{ij} &= \log\left(\lambda_{ij} u_{ij}\right) = \log \lambda_{ij} + \log u_{ij} \\
\lambda_{ij} u_{ij} &= \exp \eta_{ij} \\
y_{ij} | b_j, u_{ij} &\sim \text{Poisson}\left(\lambda_{ij} u_{ij}\right) \\
y_{ij} | b_j &\sim \text{Negative Binomial}\left(\lambda_{ij}, \phi\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(y_{ij} | b_j\right) &= \lambda_{ij} \\
\mathrm{Var}\left(y_{ij} | b_j\right) &= \lambda_{ij} + \lambda_{ij}^2/\phi
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(\lambda\right) &= \exp\left(\log \mu + \left(\tau^2\right)/2\right) \\
\mathrm{Var}\left(\lambda\right) &= \exp\left(2 \log \mu + 2\tau^2\right) - \exp\left(2 \log \mu + \tau^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left(y\right) &= \exp\left(\log \mu + \left(\tau^2\right)/2\right) \\
\mathrm{Var}\left(y\right) &= \mathrm{E}\left(\mathrm{Var}\left(y_{ij} | b_j\right)\right) + \mathrm{Var}\left(\mathrm{E}\left(y_{ij} | b_j\right)\right) \\
&= \mathrm{E}\left(\lambda_{ij} + \lambda_{ij}^2/\phi\right) + \mathrm{Var}\left(\lambda\right) \\
&= \mathrm{E}\left(\lambda\right) + \phi^{-1}\mathrm{E}\left(\lambda^2\right) + \mathrm{Var}\left(\lambda\right) \\
&= \mathrm{E}\left(\lambda\right) + \phi^{-1}\mathrm{Var}\left(\lambda\right) + \phi^{-1}\mathrm{E}\left(\lambda\right)^2 + \mathrm{Var}\left(\lambda\right) \\
&= \mathrm{E}\left(\lambda\right) + \phi^{-1}\mathrm{E}\left(\lambda\right)^2 + \mathrm{Var}\left(\lambda\right)\left(1 + \phi^{-1}\right)
\end{aligned}
$$