1 Poisson-normal

$$\begin{aligned}
\eta_{ij} &= \log \mu_{ij} + b_j + e_{ij} \\
b_j &\sim N\left(0, \tau^2\right) \\
e_{ij} &\sim N\left(0, \sigma^2\right) \\
\eta_{ij} &= \log \lambda_{ij} \\
\lambda_{ij} &= \exp \eta_{ij} \\
y_{ij}|b_j &\sim \text{Poisson}\left(\lambda_{ij}\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(y_{ij}|b_j\right) &= \lambda_{ij} \\
\mathrm{Var}\left(y_{ij}|b_j\right) &= \lambda_{ij}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(\lambda\right) &= \exp\left(\log \mu + \left(\tau^2 + \sigma^2\right)/2\right) \\
\mathrm{Var}\left(\lambda\right) &= \exp\left(2\log \mu + 2\tau^2 + 2\sigma^2\right) - \exp\left(2\log \mu + \tau^2 + \sigma^2\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(y\right) &= \mathrm{E}\left(\lambda\right) \\
&= \exp\left(\log \mu + \left(\tau^2 + \sigma^2\right)/2\right) \\
&= \mu \exp\left(\tau^2/2\right)\exp\left(\sigma^2/2\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left(y\right) &= \mathrm{Var}\left(\mathrm{E}\left(y_{ij}|b_j\right)\right) + \mathrm{E}\left(\mathrm{Var}\left(y_{ij}|b_j\right)\right) \\
&= \mathrm{Var}\left(\lambda\right) + \mathrm{E}\left(\lambda\right) \\
&= \exp\left(\log \mu + \left(\tau^2 + \sigma^2\right)/2\right) + \exp\left(2\log \mu + 2\tau^2 + 2\sigma^2\right) - \exp\left(2\log \mu + \tau^2 + \sigma^2\right) \\
&= \mu \exp\left(\tau^2/2\right)\exp\left(\sigma^2/2\right) + \mu^2\left(\exp\left(2\tau^2 + 2\sigma^2\right) - \exp\left(\tau^2 + \sigma^2\right)\right) \\
&= \mu^2\left(\exp\left(2\tau^2 + 2\sigma^2\right) - \exp\left(\tau^2 + \sigma^2\right)\right) + \mu \exp\left(\tau^2/2\right)\exp\left(\sigma^2/2\right)
\end{aligned}$$

2 Poisson-gamma

$$\begin{aligned}
\eta_{ij} &= \log \mu_{ij} + b_j \\
b_j &\sim N\left(0, \tau^2\right) \\
u_{ij} &\sim \Gamma\left(\phi^{-1}, \text{rate} = \phi\right) \\
\eta_{ij} &= \log\left(\lambda_{ij} u_{ij}\right) = \log \lambda_{ij} + \log u_{ij} \\
\lambda_{ij} u_{ij} &= \exp \eta_{ij} \\
y_{ij}|b_j, u_{ij} &\sim \text{Poisson}\left(\lambda_{ij} u_{ij}\right) \\
y_{ij}|b_j &\sim \text{Negative Binomial}\left(\lambda_{ij}, \phi\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(y_{ij}|b_j\right) &= \lambda_{ij}\left(\frac{1-\phi}{\phi}\right) \\
\mathrm{Var}\left(y_{ij}|b_j\right) &= \lambda_{ij}\left(\frac{1-\phi}{\phi^2}\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(\lambda\right) &= \exp\left(\log \mu + \tau^2/2\right) \\
\mathrm{Var}\left(\lambda\right) &= \exp\left(2\log \mu + 2\tau^2\right) - \exp\left(2\log \mu + \tau^2\right)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}\left(y\right) &= \mathrm{E}\left(\lambda_{ij}\left(\frac{1-\phi}{\phi}\right)\right) \\
&= \mathrm{E}\left(\lambda_{ij}\right)\left(\frac{1-\phi}{\phi}\right) \\
&= \exp\left(\log\mu + \tau^2/2\right)\left(\frac{1-\phi}{\phi}\right) \\
&= \mu\exp\left(\tau^2/2\right)\left(\frac{1-\phi}{\phi}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(y\right) &= \mathrm{Var}\left(\mathrm{E}\left(y_{ij}|b_j\right)\right) + \mathrm{E}\left(\mathrm{Var}\left(y_{ij}|b_j\right)\right) \\
&= \mathrm{Var}\left(\lambda\left(\frac{1-\phi}{\phi}\right)\right) + \mathrm{E}\left(\lambda\left(\frac{1-\phi}{\phi^2}\right)\right) \\
&= \mathrm{Var}\left(\lambda\right)\left(\frac{1-\phi}{\phi}\right) + \mathrm{E}\left(\lambda\right)\left(\frac{1-\phi}{\phi^2}\right)^2 \\
&= \left(\exp\left(2\log\mu + 2\tau^2\right) - \exp\left(2\log\mu + \tau^2\right)\right)\left(\frac{1-\phi}{\phi}\right) + \left(\exp\left(\log\mu + \tau^2/2\right)\right)\left(\frac{1-\phi}{\phi^2}\right)^2 \\
&= \mu^2\left(\exp 2\tau^2 - \exp\tau^2\right)\left(\frac{1-\phi}{\phi}\right) + \mu\exp\left(\tau^2/2\right)\left(\frac{1-\phi}{\phi^2}\right)^2
\end{aligned}
$$

$$
\mathrm{Var}\left(y\right) = \mu^2\left(\exp\left(2\tau^2 + 2\sigma^2\right) - \exp\left(\tau^2 + \sigma^2\right)\right) + \mu\exp\left(\tau^2/2\right)\exp\left(\sigma^2/2\right)
$$