CSE 150 – Honors Foundations of Computer Science – Fall 2025

Homework 5: Counting and Probability

- This homework is worth **119 points**.
- Show your work. It is important to include sufficient explanation/justification in your answers, but you should be concise as well. Do simplify complex expressions when you can, but there is no need to convert everything into a single number at the end. Formulas containing expressions such as $\binom{n}{m}$ are much easier to interpret than their numeric values.
- Do not submit solutions to starred problems. These are additional practice problems designed to help you prepare for the exams.

A. Counting (59 points)

1. In a standard 52-card deck (13 ranks and 4 suits), a hand is a 5-card subset of the set of 52 cards. Express the answer to each part as a formula using factorial, binomial, or multinomial notation.

   (a) Let H be the set of all hands. What is $|H|$?

   (b) Let H_{NP} be the set of all hands that include no pairs; that is, no two cards in the hand have the same rank. What is $|H_{NP}|$?

   (c)* Let H_S be the set of all hands that are straights, that is, the ranks of the five cards are consecutive. What is $|H_S|$?

   (d)* Let H_F be the set of all hands that are flushes, that is, the suits of the five cards are identical. What is $|H_F|$?

   (e)* Let H_{SF} be the set of all straight flush hands, that is, the hand is both a straight and a flush. What is $|H_{SF}|$?

   (f) Let H_{HC} be the set of all high-card hands; that is, hands that do not include pairs, are not straights, and are not flushes. Write a formula for $|H_{HC}|$ in terms of H_{NP}, H_S, H_F and H_{SF}.

2. *Simple Questions on Counting*

   (a) What is the probability that a randomly chosen number between 1 to $1M$ contains the digit 3?

   (b) How many different ways are there to select three dozen colored roses if red, yellow, pink, white, purple and orange roses are available?

   (c) How many different permutations are there of the sequence of letters in "MISSISSIPPI"?

   (d) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their three children for Christmas so that each child gets at least one piece?

   (e) 6 women and 9 men are on the faculty of a CS department. The individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?

3. Below is a list of properties that a group of people might possess. For each property, either give the minimum number of people that must be in a group to ensure that the property holds, or else indicate that the property need not hold even for arbitrarily large groups of people. (Assume that every year has exactly 365 days; ignore leap years.)

   (a) At least 2 people were born on the same day of the year (ignore year of birth).

   (b) At least 2 people were born on January 1.

   (c) At least 3 people were born on the same day of the week.

   (d) At least 4 people were born in the same month.

   (e) At least 2 people were born exactly one week apart.

4. There are 10 students $A, B, \ldots, J$ who will be lined up left to right according to the some rules below.

Rule I: Student A must be rightmost.

Rule II: Student B must be adjacent to C (directly to the left or right of C).

Rule III: Student D is always second. You may answer the following questions with a numerical formula that may involve factorials.

(a) How many possible lineups are there that satisfy all three of these rules?

(b) How many possible lineups are there that satisfy at least one of these rules?

5. Suppose n books are lined up on a shelf. The number of selections of m of the books so that selected books are separated by at least three unselected books is the same as the number of all length-k binary strings with exactly m ones.

   (a) What is the value of k?

   (b) Describe a bijection between between the set of all length k binary strings with exactly m ones and such book selections.

6.* Let X and Y be finite sets.

   (a) How many binary relations from X to Y are there? State your answer in terms of the cardinalities of X and Y.

   (b) How many total functions are there from X to Y?

   (c) How many functions, not necessarily total, are there from X to Y?

   (d) Let $|X| = n$. How many bijections are there from X to X?

7.* Consider the subsets of the set $\{1, ..., 8\}$.

   (a) How many subsets contain 4?

   (b) How many subsets contain 2 and 3 but not 6?

8.* Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

9.* A pizza house's promotional sale commercial reads: "We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's 22,369,621 different ways to choose your pizzas!" The ad writer was a former CSE 150 student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to 22,369,621. Unfortunately, $(2^9)^3/3!$ can't be an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Correct the formula.

B. Probability (60 points)

10. What is the probability that a random n-digit number does *not* contain a 9? Is there a value of n such that in numbers with n or more digits, you are more likely to find numbers containing 9 than those *not* containing a 9? If so, what is it?

11. Let B be the number of heads that come up on $2n$ independent tosses of a fair coin. What is $Pr[B = n]$? For full credit, use Stirling's approximation to simplify your answer to arrive at a compact formula.

12. A box contains 90 good and 10 defective screws. What's the probability that if we pick 10 screws from the box, none will be defective?

13. There is an unpleasant, degenerative disease called Beaver Fever which causes people to tell math jokes unrelentingly in social settings, believing other people will think they're funny. Fortunately, Beaver Fever is rare, afflicting only about 1 in 1000 people. Doctor Meyer has a fairly reliable diagnostic test to determine who is going to suffer from this disease:

- If a person will suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.99.
- If a person will not suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.97.

Let B be the event that a randomly chosen person will suffer Beaver Fever, and Y be the event that Dr. Meyer's diagnosis is "Yes, this person will suffer from Beaver Fever," with $\overline{B}$ and $\overline{Y}$ being the complements of these events.

(a) The description above explicitly gives the values of the following quantities. What are their values?

$$Pr[B] \qquad Pr[Y|B] \qquad Pr[\overline{Y}|\overline{B}]$$

(b) Write formulas for $Pr[\overline{B}]$ and $Pr[Y|\overline{B}]$ solely in terms of the explicitly given quantities in part (a). (Use their expressions, not their numeric values.)

(c) Write a formula for the probability that Dr. Meyer says a person will suffer from Beaver Fever solely in terms of $Pr[B]$, $Pr[\overline{B}]$, $Pr[Y|B]$ and $Pr[Y|\overline{B}]$.

(d) Write a formula solely in terms of the expressions given in part (a) for the probability that a person will suffer Beaver Fever given that Doctor Meyer says they will. Then calculate the numerical value of the formula.

(e) Suppose there was a vaccine to prevent Beaver Fever, but the vaccine was expensive or slightly risky itself. If you were sure you were going to suffer from Beaver Fever, getting vaccinated would be worthwhile, but even if Dr. Meyer diagnosed you as a future sufferer of Beaver Fever, the probability you actually will suffer Beaver Fever remains low (about 1/32 by part (d)). In this case, you might sensibly decide not to be vaccinated — after all, Beaver Fever is not that bad an affliction. So the diagnostic test serves no purpose in your case. You may as well not have bothered to get diagnosed. Even so, the test may be useful. Suppose Dr. Meyer had enough vaccine to treat 2% of the population. If he randomly chose people to vaccinate, he could expect to vaccinate only 2% of the people who needed it. But by testing everyone and only vaccinating those diagnosed as future sufferers, he can expect to vaccinate a much larger fraction people who were going to suffer from Beaver Fever. Estimate this fraction.

14. Suppose we change the rules in the Monty Hall problem so that the host can open the door selected by the player with a probability p. If the host does this, then the player wins or loses at this point. Does this change of rules change your answer on whether the switching strategy is better or not? If it does, indicate if this depends on the value of p, and if so, the precise value of p when this happens. Draw a tree diagram with these new rules in order to arrive at the answer to this question.

    Hint: This change of rules adds an additional edge in the "door revealed" column, corresponding to the door selected by the player. The probability on this edge will be p. Rework the probability of the "switch wins" strategy to answer this question.

15. There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

16.* A fair coin is flipped n times. What is the probability that all the heads occur at the end of the sequence? (If no heads occur, then "all the heads are at the end of the sequence" is vacuously true.)

17.* There is a subject in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time, regardless of whether or not a problem has an error. If you ask a lecturer, he will identify whether or not there is an error with only 75% accuracy. We formulate this as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

- E: the problem has an error

- T: TA says the problem has an error
- L: the lecturer says the problem has an error

(a) Write down the sample space $\mathcal{S}$, i.e., the set of all possible outcomes in this experiment.

(b) Express E, T and L in terms of this sample space, i.e., identify the set of outcomes corresponding to each event.

(c) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T and L.

(d) Suppose you have questions about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. What is the probability that there is actually an error in the problem? Assume that the correctness of the lecturer's answer and the TA's answer are independent of each other, regardless of whether there is an error.

18.* Consider a 90% accurate test for a rare condition **bunny-itis**. We define these events:

$A =$ The test comes out positive. $\quad\quad \overline{A} =$ The test comes out negative.

$B =$ You have bunny-itis. $\quad\quad \overline{B} =$ You do not have bunny-itis.

Events A and B satisfy the following:

$Pr[A \mid B] = \frac{9}{10}$. $\quad\quad Pr[A \mid \overline{B}] = \frac{1}{10}$ $\quad\quad Pr[B] = \frac{1}{20}$.

Find the following probabilities:

(a)* $Pr[A \cap B]$

(b)* $Pr[A \cap \overline{B}]$

(c)* $Pr[A]$

(d)* $Pr[B \mid A]$

Suppose you go to the doctor to take the test for bunny-itis. The test comes back positive. Is it more likely that you *actually* have bunny-itis or not? Explain. (No more than a few sentences.)

19.* The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends. Assume that the Red Sox win each game with probability 3/5, regardless of the outcomes of previous games. Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

(c) What is the probability that the Red Sox wins the series?

20.* A 52-card deck is thoroughly shuffled and you are dealt a hand of 13 cards.

(a) If you have one ace, what is the probability that you have a second ace?

(b) If you have the ace of spades, what is the probability that you have a second ace? Remarkably, the answer is different from part (a).

21.* In the game with strange dice, you propose a change to the rules: the pay out will be proportional to the difference in the rolls. Answer this question by computing the expected gain (or loss) for each player. Expected gain is the weighted average $\sum_{\omega\in\mathcal{S}} Gain(\omega)\cdot Pr[\omega]$, where $Gain(\omega)$ denotes the gain of the player in the outcome ω. Gain made by a player will be a negative quantity if he/she loses in that outcome.

(a) Does this change who has an advantage in a single roll game? Answer this question using the tree diagram. ***Hint:*** Note that $\mathcal{S}$ corresponds to the set of leaves in the tree diagram, so $Pr[\omega]$ and $Gain(\omega)$ in the above formula correspond to the probabilities and gains associated with each leaf.

(b) What can you say about multiple roll games? We are expecting an explanation here, and not a tree diagram.