118, F17 Problem Set 09 Solutions

Q1. (a) By definition,

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$$

and we know $e^{-x^2/2}$ is its own Fourier transform:

$$e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2/2} e^{ikx} dk.$$

Hence

$$e^{-x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2/2} e^{ik\sqrt{2}x} dk$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} e^{2ikx} dk.$$

Hence

$$H_n(x) = (-1)^n e^{x^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (2ik)^n e^{2ikx} dk$$

$$= \frac{(-2)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(k-ix)^2} (ik)^n dk$$

$$= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-k^2} (x-ik)^n dk.$$

(b) $|x+ik|^n \le (|x|+|k|)^n$

$$= \sum_{j=0}^{n} \binom{n}{j} |x|^j |k|^{n-j}$$

$$\le \sum_{j=0}^{n} \binom{n}{j} \max(|x|,|k|)^n$$

$$= 2^n \max(|x|,|k|)^n \le 2^n (|x|+|k|)^n.$$

(c) From PS08 we know

$$\|h_n\|^2 = \sqrt{\pi}\,\frac{2^n}{n!}$$

so by Stirling we have

$$\frac{1}{\|h_n\|} \approx \left(\frac{n}{2e}\right)^{n/2}.$$

Also, $\int_{-\infty}^{\infty} e^{-k^2} |k|^n dk = \Gamma\left(\frac{n+1}{2}\right) \approx \left(\frac{n}{2e}\right)^{n/2}$

so $|h_n(x)| \le \frac{2^n}{n!} e^{-x^2/2}\, 2^n \int_{-\infty}^{\infty} e^{-k^2}(|k|^n + |x|^n)\,dk$

$$\le 2^{2n}\left(\frac{e}{n}\right)^n e^{-x^2/2}\left[\Gamma\left(\frac{n+1}{2}\right) + |x|^n\right]$$

$$\le 2^{2n}\left(\frac{e}{n}\right)^n e^{-x^2/2}\left[\left(\frac{n}{2e}\right)^{n/2} + |x|^n\right].$$

Hence

$$\frac{|h_n(x)|}{\|h_n\|} \le \left(\frac{n}{2e}\right)^{n/2}\left(\frac{4e}{n}\right)^n e^{-x^2/2}\left(\left(\frac{n}{2e}\right)^{n/2} + |x|^n\right)$$

$$= \boxed{2^n e^{-x^2/2} + \left(\frac{8e}{n}\right)^{n/2} e^{-x^2/2} |x|^n.}$$

(d) For $|x| \geq \sqrt{2\pi n}$ the function $e^{-x^2/2}|x|^n$ is decreasing as $|x|$ increases.

Hence

$$e^{-x^2/2}|x|^n \le e^{-\pi n}(2\pi n)^{n/2}$$

and

$$e^{-x^2/2} \le e^{-\pi n}$$

So

$$\frac{|h_n(x)|}{\|h_n\|} \le 2^n e^{-\pi n} + \left(\frac{8e}{n}\right)^{n/2} e^{-\pi n} (2\pi n)^{n/2}$$

$$= \left(\frac{2}{e^{\pi}}\right)^n + \left(\frac{16 e \pi}{e^{2\pi}}\right)^{n/2}.$$

Since $\frac{2}{e^{\pi}} \le 0.1$ and

$$\frac{16 e \pi}{e^{2\pi}} = 0.255$$

we have

$$\boxed{\frac{|h_n(x)|}{\| h_n \|} \lesssim 2^{-n} \quad \text{for } |x| \ge \sqrt{2\pi n}}$$

(e) Since $h_n(x)$ is an eigenfunction of the Fourier transform, approximating

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, h_n \rangle h_n(x)}{\| h_n \|^2}$$

also approximates $\hat{f}$ by

$$\hat{f}(k) = \sum_{n=0}^{\infty} \frac{\langle f, h_n\rangle (-i)^n h_n(x)}{\| h_n \|^2},$$

Since h_n is negligible for $|x| \geq \sqrt{2\pi n}$, $h_0 \ldots h_n$ are well-suited to approximate $f(x) = 0$ for $|x| \geq \sqrt{2\pi n}$. Simultaneously $\hat{f}(k)$ is well approximated on $|k| \leq \sqrt{2\pi n}$, because

$$\| \hat{f} - \hat{f}_n \| = \| f - f_n \|$$

by Parseval's equality. Here

$$f_n(x) = \sum_{j=0}^{n} \frac{\langle f, h_j \rangle h_j}{\| h_j \|^2},$$

Here n Hermite functions are used to approximate f with $KT = 2\pi n$ or $\boxed{n = KT/2\pi}$

Of course more may be necessary, so $O(KT)$ looks plausible.

For functions with $K \neq T \neq \sqrt{2\pi n}$
the parameter c trades K for T.
$h_n(cx)$ lives on $|cx| \leq \sqrt{2\pi n}$

or

$$|x| \leq \frac{1}{c}\sqrt{2\pi n} = T$$

and b/c $\widehat{f(cx)}(k) = \frac{1}{c}\hat{f}\left(\frac{k}{c}\right)$,
$\hat{f}$ lives on $|k| \leq c\sqrt{2\pi n} = K$.
Thus $KT = 2\pi n$ and

$$c^2 = \frac{K}{T} \quad \text{or} \quad \boxed{c = \sqrt{\frac{K}{T}}}$$

Q2 (a)

$$\mathcal{F}Df(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} \, dx$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, (-ik) \, e^{-ikx} \, dx$$

$$= ik \hat{f}(k)$$

and

$$\mathcal{F}(xf)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, i \frac{d}{dk} e^{-ikx} \, dx$$

$$= i\hat{f}'(k),$$

(b) $$\widehat{D_{ab}f}\,(k) = ik \,\widehat{(a^2-x^2) f'(x)} + b^2 \hat{f}''(k)$$

$$= ik \left(a^2 + \frac{d^2}{dk^2}\right) \hat{f'}(k) + b^2 \hat{f}''(k)$$

$$= -k \left(a^2 + \frac{d^2}{dk^2}\right) \left(k \hat{f}(k)\right) + b^2 \hat{f}''(k)$$

$$= -k\left(a^2 k\hat{f}(k) + 2\hat{f}'(k) + k\hat{f}''(k)\right) + b^2\hat{f}''(k)$$

$$= \left((b^2 - k^2)\hat{f}'\right)' - a^2k^2\hat{f}(k)$$

$$= D_{ba}\hat{f}.$$

So

$$F D_{ab} = D_{ba} F.$$

Note: Take $*$ so $D_{ab} F^* = F^* D_{ba}$.

(c) $\langle g, P_a D_{ab} f(t) \rangle$

$$= \int_{-a}^{a} g(t)\left[\left((a^2 - t^2) f'\right)' - b^2 t^2 f\right] dt$$

$$= \int_{-a}^{a} -g'(t)(a^2 - t^2) f'(t) - b^2 t^2 g f \, dt$$

Symmetric in g and f so

$$P_a D_{ab} = (P_a D_{ab})^*$$

$$= D_{ab}^* P_a^*$$

$$= D_{ab} P_a .$$

Hence D_{ab} and P_a commute.

(d) $P_a Q_b P_a D_{ab} =$

$$= P_a Q_b D_{ab} P_a \qquad \text{(by (c))}$$

$$= P_a F^* P_b F D_{ab} P_a \qquad \text{(def. of } Q_b\text{)}$$

$$= P_a F^* P_b D_{ba} F P_a \qquad \text{(by (b))}$$

$$= P_a F^* D_{ba} P_b F P_a \qquad \text{(by (c))}$$

$$= P_a D_{ab} F^* P_b F P_a \qquad \text{(by (b))}$$

$$= D_{ab} P_a Q_b P_a . \qquad \text{(by (c))}$$

(e) Since the Hermitian operators S and D_{ab} commute, they can be simultaneously diagonalized. Since the eigenvalues of D_{ab} are simple its eigenvectors diagonalize S. Hence those eigenfunctions φ of D_{ab} which satisfy

$$S\varphi = \gamma\varphi$$

with γ close to 1 are local maxima of concentration $\alpha^2(a)$ in time among band limited functions with bandlimit b. They form a basis for bandlimited functions which identifies the functions concentrated on $[-a, a]$.

Q3. (a) Fourier transform in x gives

$$\hat{u}_{tt} = k^2 \hat{u}$$

so

$$\hat{u}(k,t) = A e^{-|k|t} + B e^{|k|t}.$$

A bounded solution must have $B = 0$ and then initial conditions imply

$$A = \hat{g}(k),$$

so

$$\hat{u}(k,t) = e^{-|k|t} \hat{g}(k).$$

Inverting the Fourier transform gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} dy \, e^{ikx} dk$$

$$= \int_{-\infty}^{\infty} g(y) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|t} e^{ik(x-y)} dk \right] dy$$

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} g(y) \frac{1}{\pi} \frac{t}{t^2 + (x-y)^2} dy.}$$

(b) Let $y = x + ts$ so

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x-ts)\frac{1}{1+s^2}\,ds.$$

As $t \downarrow 0$, $\left|g(x-ts)\frac{1}{1+s^2}\right| \le \frac{M}{1+s^2}$
which is integrable and

$$g(x-ts) \to g(x)$$

since g is continuous and bounded
by $|g(x)| \le M$ for all $x \in \mathbb{R}$.
Hence by Dominated Convergence

$$u(x,t) \to g(x) \text{ as } t \downarrow 0.$$

(c) $u_{tt} = -u_{xx}$.
Since $u(x,0) = g(x)$ we must have

$$u_{xx}(x,0) = g''(x)$$

and therefore

$$u_{tt}(x,0) = -g''(x).$$

Hence

$$\boxed{\Lambda^2 g(x) = -g''(x).}$$

(d) From (b),

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x-ts)\frac{1}{1+s^2}\,ds$$

so

$$u_t(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g'(x-ts)\frac{-s}{1+s^2}\,ds.$$

Changing variables back to $y = x - ts$,

$$u_t(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g'(y)\,\frac{(y-x)/t}{1+(x-y)^2/t^2}\,\frac{dy}{t}$$

$(t \downarrow 0)$ $$\rightarrow \frac{-1}{\pi}\int_{-\infty}^{\infty} g'(y)\frac{1}{x-y}\,dy$$

so

$$\boxed{K(x,y) = \frac{-1}{\pi}\frac{1}{x-y}.}$$

Q4. $\int_0^t \frac{1}{\sqrt{\pi(t-s)}} h(s)\,ds = g(t)$

$$\Rightarrow \int_0^t \frac{1}{\sqrt{\pi(t-s)}} \int_0^s \frac{1}{\sqrt{\pi(s-\sigma)}} h(\sigma)\,d\sigma\,ds =$$

$$= \int_0^t \frac{1}{\sqrt{\pi(t-s)}} g(s)\,ds$$

$$= \iint_{0 \le \sigma \le s \le t} \frac{1}{\pi\sqrt{(t-s)(s-\sigma)}} h(\sigma)\,d\sigma\,ds$$

$$= \int_0^t h(\sigma) \underbrace{\int_\sigma^t \frac{1}{\pi} \frac{ds}{\sqrt{(t-s)(s-\sigma)}}} \,d\sigma$$

$s = \sigma + \theta(t-\sigma) \quad 0 \le \theta \le 1$

$ds = d\theta\,(t-\sigma)$

$t - s = (1-\theta)(t-\sigma)$

$s - \sigma = \theta(t-\sigma)$

$$= \int_0^t h(\sigma) \left[\int_0^1 \frac{1}{\pi} \frac{1}{\sqrt{\theta(1-\theta)}}\,d\theta\right] d\sigma$$

$$= \int_0^t h(\sigma)\,d\sigma.$$

Differentiating, the FTC gives

$$h(t) = \frac{d}{dt}\int_0^t \frac{g(s)}{\sqrt{\pi(t-s)}}\,ds,$$

If $g(0) = 0$ then

$$\int_0^t \left(\frac{d}{ds}\sqrt{t-s}\right) g(s)\,ds = \sqrt{t-s}\,g(s)\Big|_0^t$$

$$- \int_0^t \sqrt{t-s}\, g'(s)\,ds$$

$$= -\int_0^t \sqrt{t-s}\, g'(s)\,ds$$

$$= \int_0^t -\frac{1}{2}\frac{1}{\sqrt{t-s}}\, g(s)\,ds$$

so

$$h(t) = \frac{d}{dt}\frac{2}{\sqrt{\pi}}\int_0^t \sqrt{t-s}\, g'(s)\,ds$$

$$\boxed{h(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\, g'(s)\,ds.}$$

Q5 (a) Try

$$u(x,t) = \int_0^t K_{t-s}(x)\, h(s)\, ds$$

$$= \int_0^t \frac{e^{-x^2/4(t-s)}}{\sqrt{4\pi(t-s)}}\, h(s)\, ds.$$

Clearly

$u_t = u_{xx}$ for $x>0$ and $t>0$.

As $t \to 0$, $u(x,t) \to 0$ as well.

At $x=0$, the boundary condition will be satisfied if

$$u(0,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\, h(s)\, ds = g(t)$$

or

$$D^{-1/2} h(t) = 2g(t).$$

From Q4, solution is

$$h(t) = 2\int_0^t \frac{1}{\sqrt{\pi(t-s)}}\, g'(s)\, ds$$

$$= 2D^{1/2} g(t).$$

Thus

$$u(x,t) = \int_0^t \frac{e^{-x^2/4(t-s)}}{\sqrt{4\pi(t-s)}}\, 2\int_0^s \frac{g'(\sigma)}{\sqrt{\pi(s-\sigma)}}\, d\sigma\, ds$$

$$\boxed{= \int_0^t g'(\sigma) \left(\int_\sigma^t \frac{e^{-x^2/4(t-s)}}{\pi\sqrt{(t-s)(s-\sigma)}}\, ds \right) d\sigma.}$$

(b) Since $u_{xx} = u_t$,

and

$$u_t(0,t) = g'(t),$$

we must have

$$u_{xx}(0,t) = \boxed{\Lambda^2 g(t) = g'(t).}$$

Check:

$u(0,t) = \int_0^t g'\, d\sigma$

$= g(t)$ ✓

[Note that $v = u_x$ is the solution of

$v_t = v_{xx}$ $\quad x>0,\ t>0$

$v(x,0) = 0$ $\quad$ (since $u(x,0)=0$)

$v(0,t) = \Lambda g(t)$ $\quad$ (def. of Λ).

Hence $w = u_{xx}$ is the solution of

$w_t = w_{xx}$ $\quad x>0,\ t>0$

$w(x,0) = 0$ $\quad$ (since $v(x,0)=0$)

$w(0,t) = \Lambda^2 g(t)$ $\quad$ (def. of Λ).

(c) $$\frac{d}{dx}\int_\sigma^t \frac{e^{-x^2/4(t-s)}}{\pi\sqrt{(t-s)(s-\sigma)}}\,ds$$

$$= \frac{d}{dx}\int_0^{t-\sigma} \frac{e^{-x^2/4s}}{\pi\sqrt{s(t-\sigma-s)}}\,ds$$

$$= \frac{d}{dx}\int_0^1 \frac{e^{-x^2/4(t-\sigma)s}}{\pi\sqrt{(t-\sigma)s(t-\sigma-(t-\sigma)s)}}\,(t-\sigma)\,ds$$

$$= \frac{d}{dx}\int_0^1 \frac{e^{-x^2/4(t-\sigma)s}}{\pi\sqrt{s(1-s)}}\,ds$$

$$= \frac{-x}{2\pi(t-\sigma)}\int_0^1 \frac{e^{-x^2/4(t-\sigma)s}}{\pi\sqrt{s(1-s)}}\,\frac{ds}{s}$$

$$= \frac{-x}{2\pi(t-\sigma)}\int_0^1 e^{-y}\frac{1}{\sqrt{\frac{x^2}{4(t-\sigma)y}\left(1-\frac{x^2}{4(t-\sigma)y}\right)}}\,\frac{dy}{y}$$

$$= \frac{+1}{\pi\sqrt{t-\sigma}}\int_{x^2/4(t-\sigma)}^{\infty} \frac{e^{-y}}{\sqrt{1-\frac{x^2}{4(t-\sigma)y}}}\,\frac{dy}{\sqrt{y}}$$

By dominated convergence, as $x \downarrow 0$ we get

$$\frac{d}{dx}\bigg|_{x=0} \int_\sigma^t \frac{e^{x^2/4(t-s)}}{\pi\sqrt{(t-s)(s-\sigma)}}\, ds =$$

$$= \frac{1}{\pi\sqrt{t-\sigma}} \int_0^\infty e^{-y} \frac{dy}{\sqrt{y}}$$

$$= \frac{\Gamma(1/2)}{\pi\sqrt{t-\sigma}} = \frac{1}{\sqrt{\pi}\sqrt{t-\sigma}}.$$

Hence

$$u_x(0,t) = \int_0^t g'(\sigma) \frac{1}{\sqrt{\pi(t-\sigma)}}\, d\sigma$$

or

$$u_x(0,t) = D^{1/2} g(t).$$

Hence $$\boxed{\Lambda = D^{1/2}}$$

and $\Lambda^2 = D$ as expected.

Question 1 (a) Show that the Hermite polynomial $H_n(x)$ satisfies

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-k^2} (x - \mathrm{i}k)^n \, \mathrm{d}k.$$

(b) Show that

$$|(x + \mathrm{i}k)^n| \leq 2^n (|x|^n + |k|^n).$$

(c) Use Stirling's approximation $n! \approx (n/e)^n$ to show

$$\frac{|h_n(x)|}{\|h_n\|} \leq 2^n \mathrm{e}^{-x^2/2} + \left(\frac{8e}{n}\right)^{n/2} \mathrm{e}^{-x^2/2} |x|^n.$$

(d) Show that

$$\frac{|h_n(x)|}{\|h_n\|} \leq \left(\frac{16\mathrm{e}\pi}{\mathrm{e}^{2\pi}}\right)^{n/2}$$

for $|x| \geq \sqrt{2\pi n}$.

(e) Explain why scaled Hermite functions $h_0(cx), h_1(cx), \ldots, h_n(cx)$ might form a suitable basis for approximating functions $f \in L^2$ which are approximately band- and time-limited in the sense that

$$\int_{|x|\geq T} |f(x)|^2 \, \mathrm{d}x \leq \epsilon^2 \|f\|^2$$

and

$$\int_{|k|\geq K} |\hat{f}(k)|^2 \, \mathrm{d}k \leq \epsilon^2 \|\hat{f}\|^2.$$

How should n and c relate to K and T?

Question 2 (a) Show that

$$FDf(k) = \widehat{f'}(k) = \mathrm{i}k\hat{f}(k) = \mathrm{i}kFf(k)$$

and

$$F(xf)(k) = \widehat{xf}(k) = \mathrm{i}\hat{f}'(k) = \mathrm{i}DFf(k).$$

(b) Show that the differential operator

$$D_{ab}f(x) = \left((a^2 - x^2)f'(x)\right)' - b^2x^2f(x)$$

satisfies

$$FD_{ab} = D_{ba}F.$$

(c) Show that D_{ab} commutes with the orthogonal projection onto time-limited functions

$$P_a f(t) = f(t)$$

for $|t| \leq a$ and

$$P_a f(t) = 0$$

for $|t| > a$.

(d) Use (b) and (c) to show that D_{ab} commutes with the integral operator

$$S_{ab} f(t) = P_a Q_b P_a f(t) = \frac{1}{\pi} \int_{-a}^{a} \frac{\sin b(t-s)}{t-s} f(s)\ ds$$

where $Q_b = F^* P_b F$ is the orthogonal projection onto bandlimited functions.

(e) Explain why the eigenfunctions of D_{ab} might be useful in representing approximately time- and band-limited functions.

Question 3 (a) Use Fourier transform to find a bounded solution u of

$$u_{xx} + u_{tt} = 0$$

in the upper half plane $x \in R$, $t > 0$, with boundary conditions

$$u(x, 0) = g(x)$$

where $g \in L^2(R)$ is bounded and continuous.

(b) Show that u attains its boundary values in the sense that

$$u(x,t) \to g(x)$$

as $t \to 0$.

(c) Assume that $g' \in L^2(R)$ is also bounded and continuous. Argue directly from the Laplace equation that if the Dirichlet-Neumann operator Λ is defined by

$$u_t(x,t) \to \Lambda g(x)$$

as $t \to 0$, then Λ must satisfy

$$\Lambda^2 g(x) = -g''(x).$$

(d) Find the kernel of the Hilbert transform operator H such that

$$\Lambda g = H(g').$$

Question 4 Solve the integral equation

$$D^{-1/2}h(t) = \int_0^t \frac{1}{\sqrt{\pi(t-s)}} h(s)\mathrm{d}s = g(t)$$

where g is a nice function with $g(0) = 0$. (Hint: Square $D^{-1/2}$.)

Question 5 (a) Solve the initial-boundary value problem for the heat equation

$$u_t = u_{xx}$$

for $x > 0$, $t > 0$, with homogeneous initial conditions

$$u(x, 0) = 0$$

and boundary conditions

$$u(0, t) = g(t)$$

where g is a nice function with $g(0) = 0$. (Hint: Try $u(x,t) = \int_0^t K_{t-s}(x)h(s)\mathrm{d}s$ and solve an integral equation for h.)

(b) Assume that $g' \in L^2(R)$ is also bounded and continuous. Argue directly from the heat equation that if

$$u_x(x, t) \to \Lambda g(t)$$

as $x \to 0$, then the Dirichlet-Neumann operator Λ must satisfy

$$\Lambda^2 g(t) = g'(t).$$

(c) Find the Dirichlet-Neumann operator Λ.