Q1. (a) $$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{rx} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{(r-ik)x}}{r-ik} \right|_{-\pi}^{\pi}$$

$$\boxed{\begin{aligned} &= \frac{(-1)^k}{\sqrt{2\pi}} \frac{1}{r-ik} \left(e^{r\pi} - e^{-r\pi}\right) \\ &= (-1)^k \sqrt{\frac{2}{\pi}} \frac{\sinh(r\pi)}{r-ik} \end{aligned}}$$

Thus

$$P_N f(x) = \sum_{|k| \leq N} \sqrt{\frac{2}{\pi}} \sinh(r\pi) \frac{e^{ikx} (-1)^k}{r-ik}$$

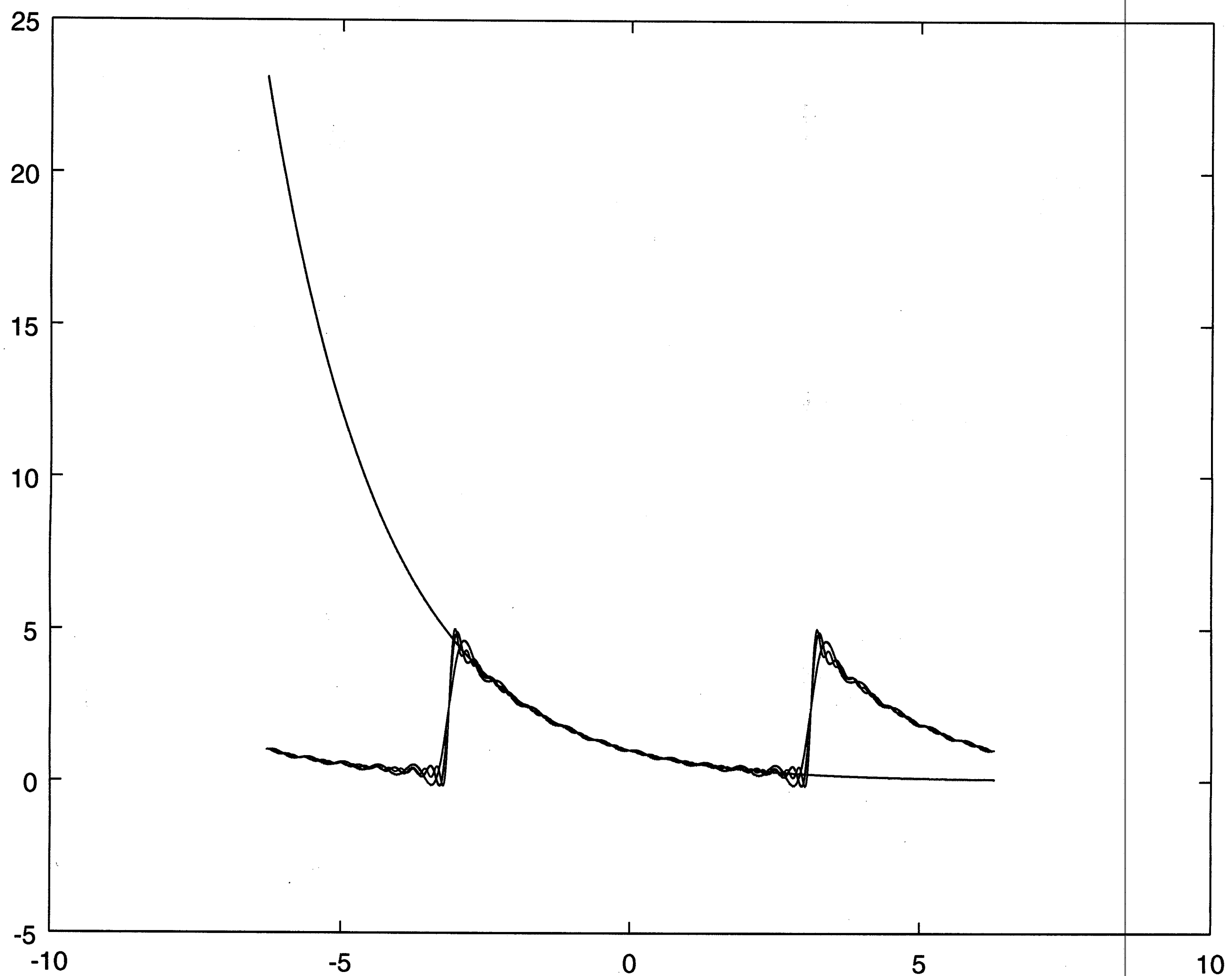
$$= \sqrt{\frac{2}{\pi}} \sinh(r\pi) \sum_{|k| \leq N} \frac{(\cos kx + i \sin kx)(r+ik)}{r^2+k^2} (-1)^k$$

$$= \sqrt{\frac{2}{\pi}} \sinh(r\pi) \left[\frac{1}{r} + \sum_{k=1}^{N} \frac{(-1)^k}{r^2+k^2} \cdot \right.$$

$$\cdot \left(r\cos kx - k \sin kx \right)$$

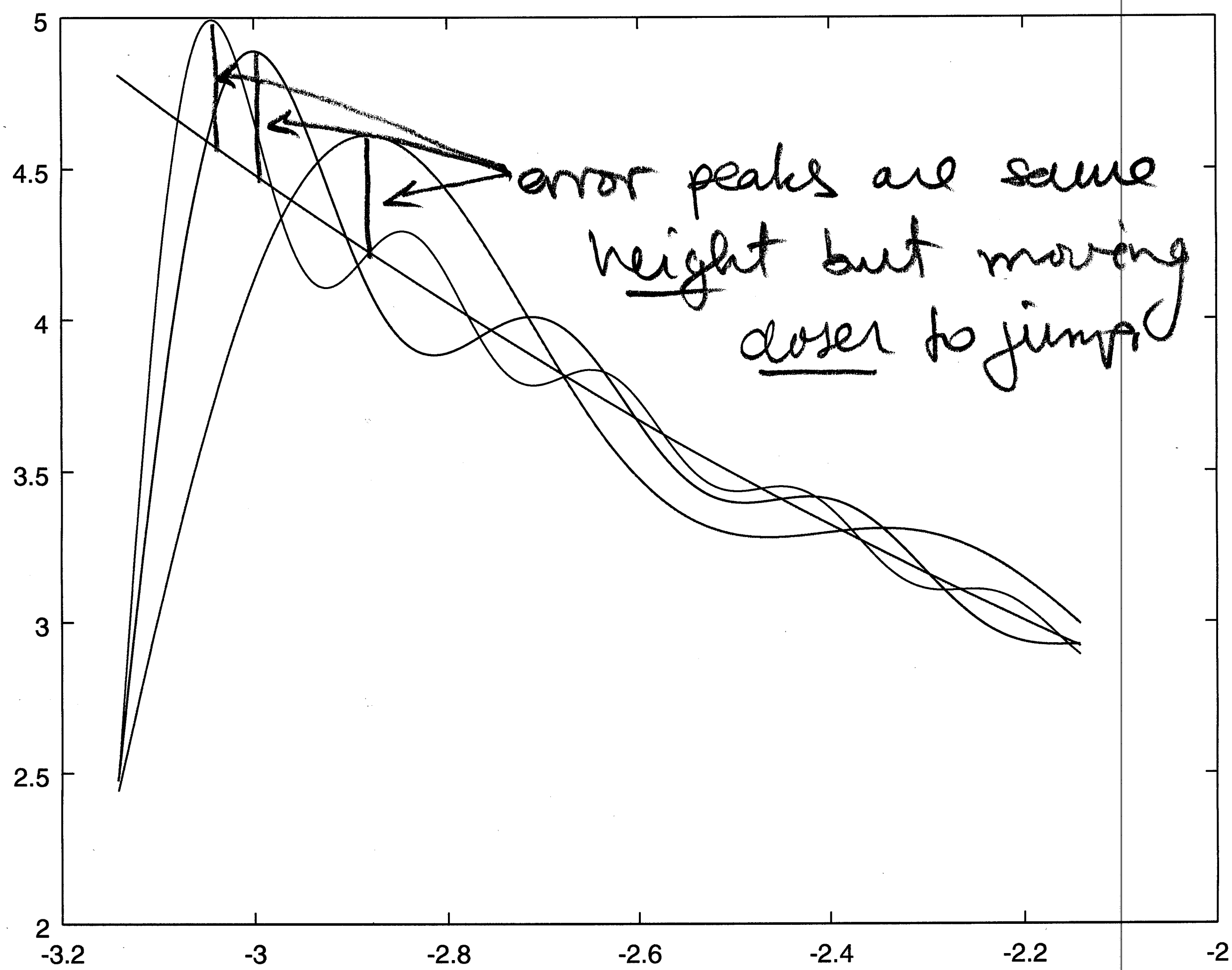
(b)

Since $e^{-x/2}$ is smooth except at $|x| = \pm\pi$, periodic Fourier series converges well inside $|x| < \pi$.



Breaks down near jumps b/c of Gibbs phenomenon, and of course periodic so poor approx for $|x| > \pi$.

Zoomed in on $[-\pi, -\pi+1]$



Q2(a) $f(x) = x^2$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 e^{-ikx}\, dx$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{d^2}{dk^2} \int_{-\pi}^{\pi} e^{-ikx}\, dx \qquad k \notin \mathbb{Z}!$$

$$= \frac{-1}{\sqrt{2\pi}} \cdot 2 \cdot \frac{d^2}{dk^2} \frac{\sin k\pi}{k}$$

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{1}{k^3} \left[2\pi^2 k^2 \sin k\pi + 2\pi k \cos k\pi - 2 \sin k\pi \right]$$

and since $k \in \mathbb{Z}$

$$\Rightarrow \begin{cases} \sin k\pi = 0 \\ \cos k\pi = (-1)^k \end{cases}$$

$$\boxed{\hat{f}(k) = 2\sqrt{2\pi} \frac{(-1)^k}{k^2}} \quad \text{for } k \neq 0$$

and $\hat{f}(0) = \sqrt{2\pi} \frac{\pi^2}{3}$.

(b) Since $|\hat{f}(k)| \leq \frac{c}{k^2}$ for a constant c, we have

$$\left| f(x) - \frac{1}{\sqrt{2\pi}} \sum_{|k| \leq N} \hat{f}(k) e^{+ikx} \right| =$$

$$\leq \frac{1}{\sqrt{2\pi}} \sum_{|k|>N} |\hat{f}(k)|$$

$$\leq \frac{c}{\sqrt{2\pi}} \sum_{|k|>N} \frac{1}{k^2}$$

$$\leq \frac{2c}{\sqrt{2\pi}} \int_N^\infty \frac{1}{k^2} dk = \frac{2c}{\sqrt{2\pi}} \frac{1}{N}$$

$\to 0$ uniformly as $N \to \infty$.

(c) At $x=\pi$ the Fourier series gives

$$f(x) = \pi^2 = \frac{1}{\sqrt{2\pi}} \sum_k \hat{f}(k)\, e^{ik\pi}$$

$$= \frac{1}{2\pi}\frac{2\pi^3}{3} + \frac{1}{2\pi}\sum_{k\neq 0} \frac{4\pi(-1)^k}{k^2}(-1)^k$$

$$= \frac{\pi^2}{3} + 4\sum_1^\infty \frac{1}{k^2}$$

$$\Rightarrow \quad \boxed{\sum_1^\infty \frac{1}{k^2} = \frac{\pi^2}{6}}$$

(d) By Parseval's theorem

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |\hat{f}(k)|^2$$

$$\shortparallel$$

$$\int_{-\pi}^{\pi} x^4 \, dx = \frac{x^5}{5} \Big|_{-\pi}^{\pi} = \frac{2}{5}\pi^5 ,$$

Hence $\frac{2}{5}\pi^5 = 2\pi \cdot \frac{\pi^4}{9} +$

$$+ \; 4 \cdot 2\pi \cdot \sum_{k \neq 0} \frac{1}{k^4}$$

so $$\sum_{1}^{\infty} \frac{1}{k^4} = \frac{1}{16\pi}\left(\frac{2}{5}\pi^5 - \frac{2}{9}\pi^5\right)$$

$$= \boxed{\frac{\pi^4}{90}} = 1.0823 \ldots$$

Q3(a) Use Fourier sine series

$$\hat{f}(k) = \sqrt{2}\int_0^1 f(x)\sin(k\pi x)\,dx$$

$$f(x) = \sqrt{2}\sum_{k=1}^{\infty} \hat{f}(k)\sin(k\pi x).$$

Multiply & integrate by parts to get

$$\hat{u}(k,t)_t = -k^2\pi^2\hat{u}(k,t)$$

Solve ODE

$$\hat{u}(k,t) = e^{-\pi^2k^2t}\,\hat{u}(k,0)$$

Apply ICs

$$= e^{-\pi^2k^2t}\,\hat{f}(k).$$

Sum:

$$u(x,t) = \sqrt{2}\sum_{k=1}^{\infty} e^{-\pi^2k^2t}\,\hat{f}(k)\sin(k\pi x),$$

Evaluate $\hat{f}(k)$:

$$\hat{f}(k) = \sqrt{2}\int_0^1 x(1-x)\sin(k\pi x)\,dx$$

$$= \sqrt{2}\int_0^1 x(1-x)\frac{d}{dx}\left(-\frac{\cos(k\pi x)}{k\pi}\right)dx$$

$$= -\sqrt{2}\int_0^1 (1-2x)\,\frac{\cos k\pi x}{k\pi}\,dx$$

$$= -\sqrt{2}\int_0^1 (1-2x)\frac{d}{dx}\frac{\sin k\pi x}{k^2\pi^2}\,dx$$

$$= \sqrt{2}\int_0^1 -2\,\frac{\sin k\pi x}{k^2\pi^2}\,dx$$

$$= \sqrt{2}\cdot 2\,\frac{1}{k^2\pi^2}\;\frac{-\cos k\pi x}{k\pi}\Bigg|_0^1$$

$$= \frac{2\sqrt{2}}{k^3\pi^3}\left(1-(-1)^k\right)$$

$$= \begin{cases} \dfrac{4\sqrt{2}}{k^3\pi^3} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}$$

(Symmetry about $x = 1/2$.)

Summing up,

$$\boxed{u(x,t) = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{e^{-\pi^2(2k+1)^2 t}}{(2k+1)^3} \cdot \sin(2k+1)\pi x}$$

$$e^{-k^2\pi^2 t}$$

(b) $$u(x,t) = \sqrt{2} \sum_{1}^{\infty} \hat{f}(k) \sin(k\pi x)$$

$$= 2 \sum_{1}^{\infty} \int_0^1 f(y) \sin(k\pi y)\, dy \sin k\pi x$$

$$= \int_0^1 \left(2 \sum_{1}^{\infty} e^{-k^2\pi^2 t} \sin k\pi x \sin k\pi y \right) \cdot f(y)\, dy$$

$$= \int_0^1 K_t(x,y) f(y)\, dy \quad \text{where}$$

$$\boxed{K_t(x,y) = 2 \sum_{1}^{\infty} e^{-k^2\pi^2 t} \sin k\pi x \sin k\pi y}$$

Q4. $\hat{f}(k) = \int_a^b \frac{1}{\sqrt{2\pi}} Q(x) e^{-ikx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b Q(x) \frac{d}{dx}\left(e^{-ikx}/-ik\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[Q(x)\left(e^{-ikx}/-ik\right) \Big|_a^b - \frac{1}{-ik} \int_a^b Q'(x)\, e^{-ikx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} \left[Q(x)\, e^{-ikx} \Big|_a^b - \int_a^b Q'(x)\, e^{-ikx} dx \right]$$

Repeating with Q' gives

$$\sqrt{2\pi}\, \hat{f}(k) = \left(\frac{-1}{ik} Q - \frac{1}{(ik)^2} Q' \right) e^{-ikx} \Big|_a^b + \frac{1}{(ik)^2} \int_a^b Q''(x)\, e^{-ikx} dx \, .$$

Repeating until derivatives of Q vanish gives

$$\boxed{\sqrt{2\pi}\,\hat{f}(k) = \left[-\frac{1}{ik}Q - \frac{1}{(ik)^2}Q' - \frac{1}{(ik)^3}Q'' - \dots - \frac{1}{(ik)^n}Q^{(n)}\right]e^{-ikx}\Big|_a^b}$$

where $\deg(Q) = n$.

Q5 (a) Over the interval $[-1,1]$,

$$\hat{f}(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x)\, e^{-ik\pi x}\, dx$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\pi x}.$$

So

$$\varphi_0(x) = 1 \Rightarrow \hat{\varphi}_0(k) = \sqrt{2}\,\delta_{k0}$$

$$\varphi_1(x) = \operatorname{sign}(x) \Rightarrow \hat{\varphi}_1(0) = 0 \text{ and}$$

$$\Rightarrow \hat{\varphi}_1(k) = \frac{1}{\sqrt{2}}\left[\int_{-1}^{0} -1\, e^{-ik\pi x}\, dx + \int_{0}^{1} +1\, e^{-ik\pi x}\, dx\right]$$

$$= \frac{-\sqrt{2}\, i}{k\pi}\left(1-(-1)^k\right)$$

$$\varphi_2(x) = \varphi_1(2x-1)$$

$$\Rightarrow \hat{\varphi}_2(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_1(2x-1) e^{-ik\pi x} dx$$

$y = 2x-1 \quad dy = 2dx$

$x=1 \Rightarrow y=1 \qquad x = (y+1)/2$

$x=-1 \Rightarrow y=-3$

$$= \frac{1}{\sqrt{2}} \int_{-3}^{1} \varphi_1(y) e^{-ik\pi(y+1)/2} 2\, dy$$

But $\varphi_1(y) = 0$ for $|y| > 1$ so

$$\hat{\varphi}_2(k) = \frac{2}{\sqrt{2}} \int_{-1}^{1} \varphi_1(y) e^{-ik\pi y/2} e^{-ik\pi/2} dy$$

$$\boxed{\hat{\varphi}_2(k) = 2(-i)^k \hat{\varphi}_1(k/2)}$$

But watch out! In computing $\hat{\varphi}_1(k)$ we assumed $k \in \mathbb{Z}$. If $2k \in \mathbb{Z}$ instead then

$$\hat{\varphi}_1(k/2) = \frac{-\sqrt{2}i}{k\pi/2}\left(1 - e^{-ik\pi/2}\right)$$

so

$$\hat{\varphi}_2(k) = 2(-i)^k \frac{-\sqrt{2}\,i}{k\pi/2}\left(1 - e^{-ik\pi/2}\right)$$

$$\boxed{\hat{\varphi}_2(k) = -4\sqrt{2}\,i\,\frac{(-i)^k}{k\pi}\left(1 - e^{-ik\pi/2}\right)}$$

Replacing k by $2k$ gives

$$\hat{\varphi}_1(k) = \frac{1}{2}\, i^{2k}\, \hat{\varphi}_2(2k)$$

(b) $$\boxed{\hat{\varphi}_1(k) = \frac{1}{2}(-1)^k\, \hat{\varphi}_2(2k)}$$

The $1/2$ is scaling and the $(-1)^k$ is due to shifting. Check with

$$\hat{\varphi}_3(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} \varphi_1(2x+1)\, e^{-ik\pi x}$$

$$= \sqrt{2} \int_{-1}^{1} \varphi_1(y)\, e^{-ik\pi y/2}\, e^{ik\pi/2}$$

$$= 2(i)^k\, \hat{\varphi}_1(k/2)$$

and we see

$$\hat{\varphi}_3(k) = (-1)^k \hat{\varphi}_2(k)$$

due to shifting by half a period.

(c) $$Pf(x) = \int_{-1}^{1} \sum_{j=1}^{4} \varphi_j(x)\varphi_j(y) f(y)\,dy$$

Since $\varphi_j(x) = \sum_k \hat{\varphi}_j(k)\, e_j(x)$

we have

$$Pf(x) = \sum_{j=1}^{4} \varphi_j(x) \langle f, \varphi_j \rangle$$

$$\langle \varphi_j, f \rangle = \sum_{-\infty}^{\infty} \hat{f}_j(k) \overline{\hat{\varphi}_j(k)}$$

so

$$Pf(x) = \sum_{j=1}^{4} \varphi_j(x) \sum_{-\infty}^{\infty} \hat{f}(k) \overline{\hat{\varphi}_j(k)}$$

$$= \sum_{-\infty}^{\infty} \hat{P}(x,k)\, \hat{f}(k)$$

$$\boxed{\hat{P}(x,k) = \sum_{j=1}^{4} \varphi_j(x) \overline{\hat{\varphi}_j(k)}}$$

Q6. (a)

$$\hat{f}(k)\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx \cdot \int_{-\pi}^{\pi} g(y)e^{-iky}\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x)g(y)\, e^{-ik(x+y)}\,dy\,dx.$$

Let's try to use $x+y$ and $x-y$ as new variables of integration.

Set $x+y = \xi$ $\qquad x-\pi \le \xi \le x+\pi$

$dy = d\xi$

$y = \xi - x$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{x-\pi}^{x+\pi} f(x)\, g(\xi - x)\, e^{-ik\xi}\,d\xi\,dx$$

Since g and $e^{-ik\xi}$ are 2π-periodic,

$$\int_{x-\pi}^{x+\pi} g(\xi-x)e^{-ik\xi}\,d\xi = \int_{-\pi}^{\pi} g(\xi-x)e^{-ik\xi}\,d\xi$$

So $\hat{f}(k)\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\xi} \int_{-\pi}^{\pi} g(\xi - x) f(x)\, dx\, d\xi$

$= \hat{h}(k)$

where $\boxed{h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(x-y) f(y)\, dy.}$

By Fourier inversion, we also have

$$h(x) = \frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k) e^{ikx}.$$

(b) Let $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2a} & |x| \le a \\ 0 & |x| > a \end{cases}$

~~so that~~

$$h(x) = f * g(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2a} \int_{x-a}^{x+a} g(y)\,dy$$

$$= \frac{1}{2a} \int_{x-a}^{x+a} g(y)\,dy$$

gives the moving average of g over intervals of length $2a$. Then

$$\hat{f}(k) = \begin{cases} 1 & k = 0 \\ \frac{\sin(ka)}{ka} & k \ne 0. \end{cases}$$

Hence

we also have by (a)

$$h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k) e^{ikx}$$

and at $x = 0$

$$h(0) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k),$$

Since $g(0) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k)$,

we have

$$g(0) = h(0)$$

$$= \frac{1}{2a} \int_{-a}^{a} g(y)\,dy$$

$$\Leftrightarrow \quad \sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k).$$

(c) Since f is smooth near $x=0$ we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} e^{ikx} \qquad |x| < a.$$

In particular

$$\frac{\sqrt{2\pi}}{2a} = f(0) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka}$$

so

$$\sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} = \frac{2\pi}{2a} = \frac{\pi}{a}.$$

Hence

$$\sum_{-\infty}^{\infty} \frac{\sin(ka)}{k} = \pi$$

and for $a = 1$

$$\sum_{-\infty}^{\infty} \frac{\sin(k)}{k} = \pi.$$

Since $\frac{\sin(k)}{k}$ is even,

$$1 + 2\sum_{1}^{\infty} \frac{\sin(k)}{k} = \pi$$

so

$$\boxed{\sum_{1}^{\infty} \frac{\sin(k)}{k} = \frac{\pi - 1}{2}}$$

By (b), we have

$$\sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \frac{\sin(k)}{k} = \pi$$

if and only if

$$\frac{\sqrt{2\pi}}{2a} = \frac{1}{2a}\int_{-a}^{a} \frac{\sqrt{2\pi}}{2a}\, dx = \frac{\sqrt{2\pi}}{2a}.$$

Specializing to $a = 1$ gives

$$\sum_{-\infty}^{\infty} \frac{\sin^2 k}{k^2} = \pi$$

and since $\frac{\sin k}{k} \to 1$ as $k \to 0$

$$\boxed{\sum_{1}^{\infty} \frac{\sin^2 k}{k^2} = \frac{\pi - 1}{2}.}$$

Question 1 (a) Compute the complex exponential Fourier coefficients $\hat{f}(k)$ of

$$f(x) = \mathrm{e}^{rx}$$

for the interval $|x| \leq \pi$.

(b) For the case $r = -1/2$ plot partial sums versus f for $N = 10$, 20, 30 on the larger interval $|x| \leq 2\pi$. Explain the regions of your plot where convergence appears to be fast versus slow.

Question 2 (a) Compute the complex exponential Fourier coefficients $\hat{f}(k)$ of

$$f(x) = x^2$$

for the interval $|x| \leq \pi$.

(b) Show that the Fourier series converges uniformly for $|x| \leq \pi$.

(c) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(d) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Question 3 (a) Solve the heat equation

$$u_t = u_{xx}$$

for $0 \leq x \leq 1$ with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = x(1-x)$.

(b) Express the solution as an integral operator

$$u(x,t) = \int_0^1 K_t(x,y)u(y,0)\ \mathrm{d}y$$

and find the kernel $K_t(x,y)$.

Question 4 Let $-\pi < a < b < \pi$ and $Q(x)$ be a polynomial of degree d. Evaluate the complex exponential Fourier coefficients of $f(x) = Q(x)$ for $a < x < b$ and $f(x) = 0$ otherwise.

Question 5 (a) Compute the complex exponential Fourier coefficient $\hat{\varphi}_j(k)$ over the interval $[-1,1]$ of the four functions φ_j defined in Question 5 of Problem Set 02.

(b) Explain the relations between the four sequences $\hat{\varphi}_j(k)$ in terms of the scaling and shifting relations between the functions φ_j.

(c) Express the projection P from Question 5 of Problem Set 02 in the form

$$Pf(x) = \sum_{-\infty}^{\infty} \hat{P}(x,k)\hat{f}(k)$$

and find the coefficient functions $\hat{P}(x,k)$.

Question 6 (a) Let f and g be 2π-periodic piecewise smooth functions such that

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k) \mathrm{e}^{\mathrm{i}kx}$$

and

$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{g}(k) \mathrm{e}^{\mathrm{i}kx}.$$

Define $h = f \star g$ by

$$h(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k) \mathrm{e}^{\mathrm{i}kx}.$$

Express $\hat{f}$ and $\hat{g}$ as integrals, combine them, and reverse the order of integration and summation to obtain an integral formula for h in terms of f and g.

(b) Let $g \in L^2(-\pi,\pi)$ have complex exponential Fourier coefficients $\hat{g}(k)$. Show that (cf. `https://arxiv.org/abs/0806.0150`)

$$\sum_{-\infty}^{\infty} \hat{g}(k) = \sum_{-\infty}^{\infty} \frac{\sin(ka)}{ka} \hat{g}(k)$$

if and only if

$$g(0) = \frac{1}{2a} \int_{-a}^{a} g(y)\ \mathrm{d}y.$$

Note that $\frac{\sin(ka)}{ka} \to 1$ as $k \to 0$.

(c) Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = (\pi - 1)/2.$$