An Introduction to Stochastic Modeling

Fourth Edition

Student Solutions Manual

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2.1 $E[\mathbf{1}\{A_1\}] = Pr\{A_1\} = \frac{1}{13}$. Similarly, $E[\mathbf{1}\{A_1\}] = Pr\{A_k\} = \frac{1}{13}$ for $k = 1, \ldots, 13$. Then, because the expected value of a sum is always the sum of the expected values, $E[N] = E[\mathbf{1}\{A_1\}] + \cdots + E[\mathbf{1}\{A_{13}\}] = \frac{1}{13} + \cdots + \frac{1}{13} = 1$.

2.3 Write $S_r = \xi_1 + \cdots + \xi_r$ where ξ_k is the number of additional samples needed to observe k distinct elements, assuming that $k-1$ district elements have already been observed. Then, defining $p_k = Pr[\xi_k = 1\} = 1 - \frac{k-1}{N}$ we have $Pr[\xi_k = n] = p_k(1-p_k)^{n-1}$ for $n = 1, 2, \ldots$ and $E[\xi_k] = \frac{1}{p_k}$. Finally, $E[S_r] = E[\xi_1] + \cdots + E[\xi_r] = \frac{1}{p_1} + \cdots + \frac{1}{p_r}$ will verify the given formula.

2.5 Using an obvious notation, the probability that A wins on the $2n+1$ trial is $Pr\left\{\overbrace{A^cB^c \ldots A^cB^c}^{2n\text{ losses}}A\right\} = [(1-p)(1-q)]^n p, n = 0, 1, \ldots$ $Pr\{\text{A wins}\} = \sum_{n=0}^{\infty}[(1-p)(1-q)]^n p = \dfrac{p}{1-(1-p)(1-q)}$. $Pr\{A \text{ wins on } 2n+1 \text{ play} | \text{A wins}\} = (1-\pi)\pi^n$ where $\pi = (1-p)(1-q)$.

$$[E^{\#}\text{trials} \mid \text{A wins}] = \sum_{n=0}^{\infty}(2n+1)(1-\pi)\pi^n = 1 + \frac{2\pi}{1-\pi} = \frac{1+(1-p)(1-q)}{1-(1-p)(1-q)} = \frac{2}{1-(1-p)(1-q)} - 1.$$

2.7 We are given that (*) $Pr\{U > \mathrm{u}, \mathrm{W} > \mathrm{w}\} = [1 - F_u(u)][1 - F_w(w)]$ for all u, w. According to the definition for independence we wish to show that $Pr\{U \le u, W \le w\} = F_u(u)F_w(w)$ for all u, w. Taking complements and using the addition law

$$\begin{aligned} Pr\{U \le u, W \le w\} &= 1 - Pr\{U > u \text{ or } W > w\} \\ &= 1 - [Pr\{U > u\} + Pr\{W > w\} - Pr\{U > u, W > w\}] \\ &= 1 - [(1 - F_U(u)) + (1 - F_W(w)) - (1 - F_u(u))(1 - F_w(w))] \\ &= F_U(u)F_W(w) \text{ after simplification.} \end{aligned}$$

2.9 Use the usual sums of numbers formula (See I, 6 if necessary) to establish

$$\sum_{k=1}^{n} k(n-k) = \frac{1}{6}n(n+1)(n-1); \text{ and}$$

$$\sum_{k=1}^{n} k^2(n-k) = n\sum k^2 - \sum k^3 = \frac{1}{12}n^2(n+1)(n-1), \text{ so}$$

$$E[X] = \frac{2}{n(n-1)}\sum k(n-k) = \frac{1}{3}(n+1)$$

$$E[X^2] = \frac{3}{n(n-1)}\sum k^2(n-k) = \frac{1}{6}n(n+1), \text{ and}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{18}(n+1)(n-2).$$

2.11 Observe, for example, $Pr\{W=z\}=Pr\{U=0,V=2\}+Pr\{U=1,V=1\}=\frac{1}{6}+\frac{1}{6}+\frac{1}{3}$. Continuing in this manner, arrive at

w	1	2	3	4
$Pr\{W=w\}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

2.13

$$\begin{aligned}
Pr\{v<V,U\le u\} &= Pr\{v<X\le u, v<Y\le u\}\\
&= Pr\{v<X\le u\}\,Pr\{v<Y\le u\}\ \text{(by independence)}\\
&= (u-v)^2\\
&= \iint\limits_{(u',v')v<v'\le u'\le u} f_{u,v}(u',v')\,du'dv'\\
&= \int_v^u\left\{\int_{v'}^u f_{u,v}(u',v')\,du'\right\}dv'.
\end{aligned}$$

The integrals are removed from the last expression by successive differentiation, first w.r.t. v (changing sign because v is a lower limit) than w.r.t. u. This tells us

$$f_{u,v}(u,v)=-\frac{d}{du}\frac{d}{dv}(u-v)^2=2 \text{ for } 0<v\le u\le 1.$$

3.1 z has a discrete uniform distribution on $0,1,\dots,9$.

3.3 Recall that $e^{\lambda}=1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots$ and $e^{-\lambda}=1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\cdots$ so that sinh $\lambda\equiv\frac{1}{2}(e^{\lambda}-e^{-\lambda})=\lambda+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\cdots$ Then $Pr\{X \text{ is odd}\}=\sum_{k=1,3,5,\dots}\frac{\lambda^k e^{-\lambda}}{k!}=e^{-\lambda}\sinh(\lambda)=\frac{1}{2}\left(1-e^{-2\lambda}\right)$.

$$\begin{aligned}
E[XY] &= E[X(N-X)]=NE[X]-E\left[X^2\right]\\
&= N^2p-\left[Np(1-p)+N^2p^2\right]=N^2p(1-p)-Np(1-p)
\end{aligned}$$

3.5 $Cov[X,Y] = E[XY]-E[X]E[Y]=-Np(1-p).$

$$\begin{aligned}
Pr\{Z=n\} &= \sum_{k=0}^{n} Pr\{X=k\}Pr\{Y=n-k\}\\
&= \sum_{k=0}^{n}\frac{\mu^k e^{-\mu}\upsilon^{(n-k)}e^{-\upsilon}}{k!(n-k)!}=e^{-(\mu+\upsilon)}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mu^k\upsilon^{n-k}
\end{aligned}$$

3.7 $$= \frac{e^{-(\mu+\upsilon)}(\mu+\upsilon)^n}{n!}\qquad \text{(Using binomial formula.)}$$

z is Poisson distributed, parameter $\mu+\upsilon$.

$$\begin{aligned} Pr\{X+Y=n\} &= \sum_{k=0}^{n} Pr\{X=k, Y=n-k\} = \sum_{k=0}^{n}(1-\pi)\pi^k(1-\pi)\pi^{n-k} \\ &= (1-\pi)^2\pi^n \sum_{k=0}^{n} 1 = (n+1)(1-\pi)^2\pi^n \text{ for } n \geq 0. \end{aligned}$$

3.9

$$\begin{aligned} Pr\{U=u, W=0\} &= Pr\{X=u, Y=u\} = (1-\pi)^2\pi^{2u}, u \geq 0. \\ Pr\{U=u, W=w>0\} &= Pr\{X=u, Y=u+w\} + Pr\{Y=u, X=u+w\} = 2(1-\pi)^2\pi^{2u+w} \\ Pr\{U=u\} &= \sum_{w=0}^{\infty} Pr\{U=u, W=w\} = \pi^{2u}\left(1-\pi^2\right). \\ Pr\{W=0\} &= \sum_{w=0}^{\infty} Pr\{U=u, W=0\} = (1-\pi)^2\Big/\left(1-\pi^2\right). \\ Pr\{W=w>0\} &= 2\left[(1-\pi)^2\Big/(1-\pi)^2\left(1-\pi^2\right)\right]\pi^w, \text{ and} \end{aligned}$$

3.11 $$Pr\{U=u, W=w\} = Pr\{U=u\}Pr\{W=w\} \text{ for all } u, w.$$

3.13 Assume that inspected items are independently defective or good. Let $X = \#$ of defects in sample.

$$\begin{aligned} &Pr\{X=0\} = (.95)^{10} = .599 \\ &Pr\{X=1\} = 10(.95)^9(.05) = .315 \\ &Pr\{X \geqq 2\} = 1-(.599+.315) = .086. \end{aligned}$$

3.15 $Pr\{X \leq 2\} = \left(1+2+\frac{2^2}{2}\right)e^{-2} = 5e^{-2} = .677$.

4.1 $$E\left[e^{\lambda Z}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{1}{2}z^2+\lambda z}dz = e^{\frac{1}{2}\lambda^2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\frac{1}{2}(z-\lambda)^2}dz\right\} = e^{\frac{1}{2}\lambda^2}.$$

4.3 $X-\theta$ and $Y-\theta$ are both uniform over $\left[-\frac{1}{2},\frac{1}{2}\right]$, independent of θ, and $W = X-Y = (X-\theta)-(Y-\theta)$. Therefore the distribution of W is independent of θ and we may determine it assuming $\theta = 0$. Also, the density of W is symmetric since that of both X and Y are.

$$Pr\{W>w\} = Pr\{X>Y+w\} = \frac{1}{2}(1-w)^2, \quad w>0$$

So $f_W(w) = 1-w \quad \text{for} \quad 0 \leq w \leq 1$ and $f_W(w) = 1-|w| \quad \text{for} \quad -1 \leq w \leq +1$

4.5 $Pr\{Z<Y\} = \int_0^\infty\left\{\int_x^\infty 3e^{-3y}dy\right\}2e^{-2x}dx = \frac{2}{5}$.

$$\begin{aligned} Pr\{N>k\} &= Pr\{X_1 \leq \xi, \ldots, X_k \leq \xi\} = [F(\xi)]^k, k=0,1,\ldots \end{aligned}$$

5.1 $$Pr\{N=k\} = Pr\{N>k-1\} - Pr\{N>k\} = [1-F(\xi)]F(\xi)^{k-1}, k=1,2,\ldots$$

$$\begin{aligned} Pr\{X>k\} &= \sum_{l=k+1}^{\infty} p(1-p)^l = p(1-p)^{k+1}, k=0,1,\ldots \\ E[X] &= \sum_{k=0}^{\infty} Pr\{X>k\} = \tfrac{1-p}{p}. \end{aligned}$$

5.3

5.5 $E\left[W^2\right]=\int_0^{\infty} P\left\{W^2>t\right\} dt=\int_0^{\infty}\left[1-F_w\left(\sqrt{t}\right)\right] dt=\int_0^{\infty} 2y\left[1-F_w(y)\right] dy$ by letting $y=\sqrt{t}$.

5.7
$$\begin{aligned} Pr\{V>v\} &= Pr\{X_1>v,\ldots,X_n>v\}=Pr\{X_1>v\}\cdot\cdots\cdot Pr\{X_n>v\} \\ &= e^{-\lambda_1 v}\cdot\cdots\cdot e^{-\lambda_n v}=e^{-(\lambda_1+\cdots+\lambda_n)v}, v>0. \end{aligned}$$

v is exponentially distributed with parameter Σ_i^{λ}.

Chapter 2

1.1

(a)

$$\begin{aligned} Pr\{X=k\} &= \sum_{m=k}^{N} \frac{N!}{m!(N-m)!} p^m (1-p)^{N-m} \frac{m!}{k!(m-k)!} \pi^k (1-\pi)^{m-k} \\ &= \binom{N}{k} (\pi p)^k (1-\pi p)^{N-k} \quad \text{for } k = 0, 1, \ldots, N. \end{aligned}$$

(b)

$$\begin{aligned} &E[XY] = E[MX] - E[X]^2; E[M] = Np; \\ &E\left[M^2\right] = N^2 p^2 + Np(1-p); E[X|M] = M\pi \\ &E[X] = N\pi p; E\left[X^2|M\right] = M^2\pi^2 + M\pi(1-\pi); \\ &E\left[X^2\right] = (N\pi p)^2 + N\pi^2 p(1-p) + N\pi p(1-\pi), \\ &E[MX] = E\{ME[X|M]\} = \pi\left[N^2p^2 + Np(1-p)\right] \\ &Cov[X,Y] = E[X(M-X)] - E[X]E[M-X] = -Np^2\pi(1-\pi). \end{aligned}$$

1.3

(a)

$$\begin{aligned} Pr\{U=u|V=v\} &= \frac{2}{2v-1} \quad \text{for } 1 \le u < v \le 6 \\ &= \frac{2}{2v-1} \quad \text{for } 1 \le u = v \le 6. \end{aligned}$$

(b)

$Pr\{S=s,T=t\}$

t/s	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$	
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			
4					$\frac{2}{36}$		$\frac{2}{36}$				
5						$\frac{2}{36}$					

1.5 $Pr\{X=0\} = \left(\frac{3}{4}\right)^{20} = .00317$.

1.7 $Pr\{\text{True S.F.}|\text{Diag. S.F.}\} = \frac{.30(.85)}{.30(.85)+.70(.35)} = .51$

1.9

$$\begin{aligned} Pr\{X=k\} &= \sum_{n=k-1}^{\infty} Pr\{X=k|N=n\}Pr\{N=n\} \\ &= \sum_{n=k-1}^{\infty} \frac{1}{n+2}\frac{e^{-1}}{n!} = \sum_{n=k-1}^{\infty} e^{-1}\left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right] = \frac{e^{-1}}{k!} \text{ for } k=0,1\ldots \quad (\text{Poisson, mean} = 1). \end{aligned}$$

2.1

(a) The bid $X_1 = x$ is accepted if $x > A$; Otherwise, if $0 \le x \le A$, one finds where one started due to the independent and identically distributed nature of the bids.

$$E[X_N|X_1=x] = \begin{cases} x & \text{if } x > A; \\ M & \text{if } 0 \le x \le A \end{cases}$$

(b) The given equation results from the law of total probability:

$$E[X_N] = \int_0^{\infty} E[X_N|X_1=x]\,dF(x).$$

(c) When X is exponentially distributed the conditional distribution of X, given that $X > A$, is the same as the distribution of $A+X$.

(d) When $dF(x) = \lambda e^{-\lambda x}dx$, then

$$\alpha = e^{-\lambda A} \text{ and } \int_A^\infty x\lambda e^{-\lambda x}dx = A\alpha + \alpha\int_0^\infty y\lambda e^{-\lambda y}dy = \alpha\left(A+\frac{1}{\lambda}\right) \text{ and } M = A+\frac{1}{\lambda}.$$

3.1

(a) $\nu = \tau^2 = \lambda,\ \mu = p \quad \sigma^2 = p(1-p)$

$E[Z] = \lambda p; \quad Var[Z] = \lambda p(1-p) + \lambda p^2 = \lambda p.$

(b) z has the Poisson distribution.

3.3

(a) $\mu = 0, \sigma^2 = 1, \nu = \frac{1-\alpha}{\alpha}, \tau^2 = \frac{1-\alpha}{\alpha^2}; \quad E[Z] = 0 \quad Var[Z] = \frac{1-\alpha}{\alpha}.$

(b)
$$\begin{aligned} E[z^3] &= 0 \quad E[z^4] = E[N^2 + 2N(N-1)] \\ &= 3\left(\frac{(1-\alpha)^2}{\alpha^2} + \frac{1-\alpha}{\alpha^2}\right) - 2\left(\frac{1-\alpha}{\alpha}\right) = \frac{(1-\alpha)(6-5\alpha)}{\alpha^2} \end{aligned}$$

3.5 $E[Z] = (1+\nu)\mu; \quad Var[Z] = \nu\sigma^2 + \mu^2\tau^2.$

4.1 $E[X_1] = E[X_2] = \int_0^1 p\,dp = \frac{1}{2}$

$$E[X_1X_2] = \int_0^1 p^2 dp = \frac{1}{3}$$

$$Cov[X_1, X_2] = E[X_1X_2] - E[X_1]E[X_2] = \frac{1}{12}.$$

4.3

(a) $Pr\{X = i\} = \int_\infty^0 \frac{\lambda^i e^{-\lambda}}{i!}\theta e^{-\theta\lambda} d\lambda = \frac{\theta}{i!}\int_\infty^0 \lambda^i e^{-(1+\theta)\lambda} d\lambda = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^i, i = 0, 1, \ldots$ (Geometric distribution).

(b) $f(\lambda|X = k) = \frac{(1+\theta)^{k+1}\lambda^k e^{-(1+\theta)\lambda}}{k!}$ (Gamma).

$Pr\{X = 0, Y = 1\} = 0 \neq Pr\{X = 0\}Pr\{Y = 1\}$ so X and Y are *NOT* independent.

4.5 $E[X] = E[Y] = 0, \quad Cov[X, Y] = E[XY] = \frac{1}{9} + \frac{1}{9} - \frac{2}{9} = 0.$

4.7 $f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z}$ for $0 \le x \le z$.

$$f_{X|Z}(x|z) = \frac{f_{x,z}(x,z)}{f_z(z)} = \frac{\alpha^2 e^{-\alpha z}}{\alpha^2 z e^{-\alpha z}} = \frac{1}{z}, \quad 0 < x < z.$$

x is, conditional on $Z = z$, uniformly distributed over the interval [0, z].

5.1 Following the suggestion

$$\begin{aligned} E[X_{n+2}|X_0,\ldots,X_n] &= E[E\{X_{n+2}|X_0,\ldots,X_{n+1}\}|X_0,\ldots,X_n] \\ &= E[X_{n+1}|X_0,\ldots,X_n] = X_n \end{aligned}$$

5.3

$$\begin{aligned} E[X_{n+1}|X_0,\ldots,X_n] &= E\left[2e^{-\varepsilon_{n+1}}X_n|X_n\right] \\ &= X_n E\left[2e^{-\varepsilon_{n+1}}\right] = X_n. \end{aligned}$$

5.5

(b) $Pr\{X_n \ge N \text{ for some } n \ge 0 | X_0 = i\} \le \frac{E(X_0)}{N} = \frac{i}{N}$. (In fact, equality holds. See III, 5.3)

Chapter 3

1.1

$$P_{55} = 1, P_{K,K+1} = \alpha \frac{\binom{k}{1}\binom{5-k}{1}}{\binom{5}{2}}, \quad k = 1,2,3,4$$

$$P_{ij} = 0, \text{otherwise.}$$

1.3 $Pr\{X_2 = G, X_3 = G, X_4 = G, X_5 = D \mid X_1 = G\} = P_{GG}^3 P_{GD} = \alpha^3(1-\alpha).$

2.1 Observe that the columns of P sum to one. Then, b $p_i^{(0)} = 1/4$ for $i = 0,1,2,3$ and by induction $p_i^{(n+1)} = \Sigma_{k=0}^{3} p_k^{(n)} P_{ki} = \frac{1}{4}\Sigma P_{ki} = \frac{1}{4}.$

2.3 $P_{11}^{(3)} = .684772$

2.5 $Pr\{X_3 = 0 \mid X_0 = 0, T > 3\} = P_{00}^{(3)} \Big/ \left(P_{00}^{(3)} + P_{01}^{(3)}\right) = .6652$.

3.1

$$P = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \frac{1}{15} & \frac{14}{15} & 0 & 0 \\ 2 & 0 & \frac{8}{15} & \frac{7}{15} & 0 \\ 3 & 0 & 0 & \frac{3}{5} & \frac{2}{5} \end{array}$$

3.3

(a)

$$P = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & .1 & .4 & .5 \\ 1 & .5 & .5 & 0 \\ 2 & .1 & .4 & .5 \end{array}$$

(b) Long run lost sales per period = (.1) π_1 = .0444….

3.5

$$P = \begin{array}{c|cccc} & 0 & H & HH & HHT \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ H & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ HH & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ HHT & 0 & 0 & 0 & 1 \end{array}$$

$P_{0,k} = \alpha_k$ for $k = 1, 2, \ldots$

$P_{0,0} = 0$

3.7 $P_{k,k-1} = 1 \quad h \geq 1$

3.9

$$P_{k,k-1} = P_{k,k+1} = \left(\frac{k}{N}\right)\left(\frac{N-k}{N}\right)$$
$$P_{k,k} = \left(\frac{k}{N}\right)^2 + \left(\frac{N-k}{N}\right)^2$$

4.1 $0 = 0 \quad 1 = H \quad 2 = HH \quad 3 = HHT$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left.\begin{aligned} v_0 &= 1 + \frac{1}{2}v_0 + \frac{1}{2}v_1 \\ v_1 &= 1 + \frac{1}{2}v_0 + \frac{1}{2}v_2 \\ v_2 &= 1 + \frac{1}{2}v_2 \end{aligned}\right\} \begin{aligned} &\boxed{v_0 = 8} \\ &v_1 = 6 \\ &v_2 = 2 \end{aligned}$$

$0 = 0 \quad 1 = H \quad 2 = HT \quad 3 = HTH$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left.\begin{aligned} v_0 &= 1+\frac{1}{2}v_0+\frac{1}{2}v_1 \\ v_1 &= 1+\frac{1}{2}v_1+\frac{1}{2}v_2 \\ v_2 &= 1+\frac{1}{2}v_0 \end{aligned}\right\} \begin{aligned} &\boxed{v_0=10} \\ &v_1=8 \\ &v_2=6 \end{aligned}$$

4.3 We will verify that $v_m = 2\left(\frac{m+1}{m}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)-3$ solves $v_m = 1+\sum_{j=1}^{m-1}\frac{2j}{m^2}v_j$. Change variables to $V_k = kv_k$. To show: $V_k = 2(k+1)\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)-3k$ solves $V_m = m+\frac{2}{m}(V_1+\cdots+V_{m-1})$. Using the given V_k, use sums of numbers and interchange order as follows:

$$\begin{aligned} \sum_{k=1}^{m-1} V_k &= \sum_{k=1}^{m-1}\sum_{l=1}^{k} 2(k+1)\frac{1}{l} - 3\sum_{k=1}^{m-1} k \\ &= \sum_{l=1}^{m-1}\frac{2}{l}\sum_{k=l}^{m-1}(k+1) - \frac{3m(m-1)}{2} \\ &= \sum_{l=1}^{m-1}\frac{2}{l}\left[\frac{m(m+1)}{2}-\frac{l(l+1)}{2}\right] - \frac{3m(m-1)}{2}. \end{aligned}$$

Then,

$$m+\frac{2}{m}\sum_{k=1}^{m-1}V_k = 2(m+1)\sum_{l=1}^{m-1}\frac{1}{l} - 3m + \frac{2(m+1)}{m} = V_m$$

as was to be shown.

As in Problem 4.2, $v_m \sim \log m$.

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{Vmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix} \end{array}$$

4.5 $u_{13} = \frac{7}{12} \qquad u_{53} = \frac{2}{3} \qquad \boxed{u_{43} = \frac{5}{12}}$

$u_{23} = \frac{3}{4} \qquad u_{63} = \frac{5}{6}$

4.7 The stationary Markov transitions imply that $E[\sum_{n=1}^{\infty} \beta^n c(X_n) | X_0 = i, X_1 = j] = \beta h_j$ while $E[\beta^0 c(X_0) | X_0 = i] = c(i)$. Now use the law of total probability.

4.9

$$\begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{8} & \frac{6}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{8} & \frac{4}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{8} & \frac{3}{8} \end{Vmatrix} \end{array}$$

$X_n = \#$ of red balls in use

$$v_5 = \frac{8}{1} + \frac{8}{2} + \frac{8}{3} + \frac{8}{4} + \frac{8}{5} = 18\frac{4}{15} = 18.266\ldots$$

4.11 Label the states (x, y) where $x = \#$ of red balls and $y = \#$ of green balls, "win" $= (1,0)$, "lose" $= \{(0,2),\ (2,0)\}$.

$$\begin{array}{c} \\ (2,2) \\ (1,2) \\ (2,1) \\ (1,1) \\ win \\ lose \end{array}
\begin{array}{c} \begin{array}{cccccc} (2,2) & (1,2) & (2,1) & (1,1) & win & lose \end{array} \\
\left\| \begin{array}{cccccc}
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array} \right\| \end{array}$$

$U_{(2,2),\, win} = \frac{1}{3}$.

4.13 Use the matrix

$$\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 0' & 1' & 2' \\
\hline
0 & 0 & 0 & 0 & .3 & .2 & .5 \\
1 & 0 & 0 & 0 & .5 & .1 & .4 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0' & .3 & .2 & .5 & 0 & 0 & 0 \\
1' & .5 & .1 & .4 & 0 & 0 & 0 \\
2' & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$U_{0,2'} = .67669\ldots$

4.15

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & .96 & .04 & 0 & 0 & 0 \\
2 & 0 & .94 & .06 & 0 & 0 \\
3 & 0 & 0 & .94 & .06 & 0 \\
4 & 0 & 0 & 0 & .96 & .04 \\
5 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$v_1 = 133\frac{1}{3}$

4.17 Let $\varphi_i(s) = E\left[s^T \mid X_0 = i\right]$ for $i = 0, 1$

Then

$$\varphi_0(s) = s[.7\varphi_0(s) + .3\varphi_1(s)]$$
$$\varphi_1(s) = s[.6\varphi_1(s) + .4]$$

which solves to give

$$\varphi_1(s) = \frac{.4s}{1-.6s}$$
$$\varphi_0(s) = \left(\frac{.4s}{1-.6s}\right)\left(\frac{.3s}{1-.7s}\right).$$

4.19

$$P_N = \frac{N}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left[1-\left(\frac{1}{2}\right)^n\right]^{N-1}.$$

As a function of N, this does NOT converge but oscillates very slowly (cycles $\propto \log N$) and very slightly about the (Cesaro limit) $\frac{1}{2\log 2}$.

5.1

$$\begin{aligned}
v_0 &= 1 + a_0v_0 + a_1v_1 + a_2v_2 \\
v_1 &= 1 + (a_0 + a_1)v_1 + a_2v_2 \\
v_2 &= 1 + (a_0 + a_1 + a_2)v_2 \\
v_2 &= \tfrac{1}{1-(a_0+a_1+a_2)} = \tfrac{1}{a^3} = v_0 = v_1.
\end{aligned}$$

5.3

$$P = \begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & \alpha & 1-\alpha & 0 \\
1 & 0 & \alpha & 1-\alpha \\
2 & 1-\alpha & 0 & \alpha
\end{array}$$

5.5 If $X_n = 0$, then $X_{n+1} = 0$ and $E[X_{n+1}|X_n = 0] = 0 = X_n$. If $X_n = i > 0$ then $X_{n+1} = i \pm 1$, each *w. pr.* $\frac{1}{2}$ so $E[X_{n+1}|X_n = i] = i = X_n$. So $\{X_n\}$ is a martingale. If $X_0 = i$, then $Pr\{\max X_n \geq N\} \leq \frac{E[X_0]}{N} = \frac{i}{N}$ by the maximal inequality. Of course, (5.13) asserts $Pr\{\max X_n \geq N\} = \frac{i}{N}$.

6.1

$$\left.\begin{aligned} v_1 &= 1 + .7v_2 \\ v_2 &= 1 + .1v_1 \end{aligned}\right\} v_1 = \tfrac{170}{93} = 1.827957$$
$$\tilde{\pi}_1 = \frac{3}{7}, \quad \tilde{\pi}_2 = \frac{1}{21}, \quad \Phi_1 = \frac{10}{7}, \quad \Phi_2 = \frac{80}{63}$$
$$v_1 = \frac{\Phi_1 + \Phi_2}{1 + \tilde{\pi}_1 + \tilde{\pi}_2} = 1.827957.$$

6.3 For three urns, $v(a,b,c) = E[T] = \frac{3abc}{a+b+c}$. The answer is unknown for four urns.

7.1 Observe that each state is visited at most a single time so that $W_{ij} = Pr\{\text{Ever visit } j|X_0 = i\}$. Clearly then $W_{i,j} = 1$ and $W_{i,j} = 0$ for $j > i$.

$$\begin{aligned} W_{i,i-1} &= P_{i,i-1} = \frac{1}{i} \\ W_{i,i-2} &= P_{i,i-2} + P_{i,i-1}W_{i-1,i-2} = \frac{1}{i} + \frac{1}{i}\left(\frac{1}{i-1}\right) = \frac{1}{i-1}. \\ W_{i,i-3} &= P_{i,i-3} + P_{i,i-2}W_{i-2,i-3} + P_{i,i-1}\,W_{i-1,i-3} \\ &= \frac{1}{i}\left(1 + \frac{1}{i-2} + \frac{1}{i-2}\right) = \frac{1}{i-2}. \end{aligned}$$

Continuing in the manner, we deduce that $W_{i,j} = \frac{1}{j+1}$ for $1 \leq j < i$. 12

7.3 Observe that for j transient and k absorbing, the event $\{X_{n-1} = j, X_n = k\}$ is the same as the event $\{T = n, X_{T-1} = j,\, X_T = k\}$, whence

$$Pr\{X_{T-1} = j, X_T = k|X_0 = i\} = \sum_{n=1}^{\infty} Pr\{X_{n-1} = j, X_n = k|X_0 = i\} = \sum_{n=1}^{\infty} P_{i,j}^{(n-1)} P_{jk} = W_{i,j}P_{jk}.$$

7.5

$$v_k = \tfrac{\pi}{2} B_{N,k}\left[(N+1)^2 - (N-2k)^2\right]$$

where

$$B_{N,k} = 2^{-2N}\binom{2(N-k)}{N-k}\binom{2k}{k}.$$

8.1 No adults survive if a local catastrophe occurs, which has probability $1-\beta$, and if independently, all N dispersed offspring fail to survive, which has probability $(1-\alpha)^N$. Another example in which looking at mean values alone is misleading.

8.3

Possible families:	GG	GB	BG	BBG	BBBG...
Probability:	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}\cdots$

Let $N =$ Total children, $X =$ Male children

(a)

$$Pr\{N=2\}=\frac{3}{4},\quad Pr\{N=k\}=\left(\frac{1}{2}\right)^k \text{ for } k\geqq 3.$$

(b)

$$Pr\{X=1\}=\frac{1}{2},\quad Pr\{X=0\}=\frac{1}{4},\quad Pr\{X=k\}=\left(\frac{1}{2}\right)^{k+1},\ k\geqq 2.$$

9.1 $\xi = \#$ of male children

$$\begin{array}{ccccc} k= & 0 & 1 & 2 & 3 \\ Pr\{\xi=k\}= & \frac{11}{32} & \frac{9}{32} & \frac{9}{32} & \frac{3}{32} \end{array}$$

$$\varphi(s)=\frac{11}{32}+\frac{9}{32}s+\frac{9}{32}s^2+\frac{3}{32}s^3.$$

$u=\varphi(u)$ has smallest solution $u_\infty = .76887.$

9.3 Following the hint

$$Pr\{X=k\}=\int \pi(k|\lambda)f^{(\lambda)}d\lambda=\frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)}\left(\frac{\theta}{1+\theta}\right)^{\alpha}\left(\frac{1}{1+\theta}\right)^{k},\quad k=0,1,\ldots.$$

9.5

(a)

$$\begin{array}{lcccc} n= & 1 & 2 & 3 & 4 \\ Pr\{\text{All red}\}= & \frac{1}{4} & \frac{1}{4}\left(\frac{1}{4}\right)^2 & \frac{1}{4}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^4 & \frac{1}{4}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^4\left(\frac{1}{4}\right)^8 \end{array}$$

$$Pr\{\text{All red at generations}\}=\left(\frac{1}{4}\right)^{2n-1}.$$

(b) $Pr\{\text{Culture dies out}\} = Pr\{\text{Red cells die out}\}\ \varphi(s)=\frac{1}{12}+\frac{2}{3}s+\frac{1}{4}s^2.$ Smallest solution to $u=\varphi(u)$ is $u_\infty=\frac{1}{3}.$

9.7

Family	*GG*	*GB*	*BG*	*BBG*	*BBBG*
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

Let $X = \#$ of male children.

$$Pr\{X=0\}=\frac{1}{4} \quad Pr\{X=1\}=\frac{1}{2}$$
$$Pr\{X=k\}=\left(\frac{1}{2}\right)^{k+1} \quad \text{for } k \geqq 2.$$
$$\varphi(s)=\frac{1}{4}+\frac{1}{2}s+\sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^{k+1}s^k=\frac{1}{4}+\frac{1}{2}s+\frac{1}{4}\frac{s^2}{2-s}.$$
$$\varphi'(s)=\frac{1}{2}+\frac{1}{4}\frac{2s(2-s)+s^2}{(2-s)^2} \quad E[X]=\varphi'(1)=\frac{5}{4}.$$
$$Pr\{X>0\}=\frac{3}{4} \quad Pr\{X>k\}=\left(\frac{1}{2}\right)^{k+1}$$
$$E[X]=\sum_{k=0}^{\infty}Pr\{X>h\}=\frac{5}{4}$$

9.9

(a) Let $X = \#$ of male children.

$$Pr\{X=0\}=\frac{1}{4}+\left(\frac{3}{4}\right)\frac{1}{2}=\frac{5}{8}$$
$$Pr\{X=k\}=\frac{3}{4}\left(\frac{1}{2}\right)^{k+1} \quad \text{for } k \geqq 1.$$

(b)

$$\varphi(s)=\frac{5}{8}+\frac{3}{8}\sum_{k=1}^{\infty}\left(\frac{s}{2}\right)^k=\frac{5}{8}+\frac{3}{8}\left(\frac{s}{2-s}\right).$$
$$u_0=0 \quad u_n=\varphi(u_{n-1}) \quad u_5=.9414$$

Chapter 4

1.1 Let X_n be the number of balls in Urn A. Then $\{X_n\}$ is a doubly stochastic Markor chain having 6 states, whence $\lim_{n\to\infty} Pr\{X_n = 0|X_0 = i\} = \frac{1}{6}$.

1.3 The equations for the stationary distribution simplify to:

$$\begin{aligned}
\pi_0 &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)\,\pi_0 \quad \left(\textstyle\sum \alpha_i = 1\right)\\
\pi_1 &= (\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)\,\pi_0\\
\pi_2 &= (\alpha_3+\alpha_4+\alpha_5+\alpha_6)\,\pi_0\\
\pi_3 &= (\alpha_4+\alpha_5+\alpha_6)\,\pi_0\\
\pi_4 &= (\alpha_5+\alpha_6)\,\pi_0\\
\pi_5 &= \alpha_6\pi_0
\end{aligned}$$

$$1 = \left(\sum_{k=1}^{6} k\alpha_k\right)\pi_0$$

$$\pi_0 = \frac{1}{\sum_{k=1}^{6} k\alpha_k} = \frac{1}{\text{Mean of } \alpha \text{ distribution}}$$

1.5
$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left\| \begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array} \right\| \end{array}$$

$$\pi_A = \frac{2}{8};\quad \pi_B = \frac{3}{8};\quad \pi_C = \frac{1}{8}\quad \boxed{\pi_D = \frac{2}{8}}$$

Note: $\pi_{\text{NODE}} \propto \#$ arcs at NODE.

1.7 $\pi_0 = \frac{6}{19}; \pi_1 = \frac{3}{19}; \pi_2 = \frac{6}{19}; \pi_3 = \frac{4}{19}$.

1.9 $\pi_0 = \pi_1 = \frac{1}{10} \quad \pi_2 = \pi_3 = \frac{4}{10}$.

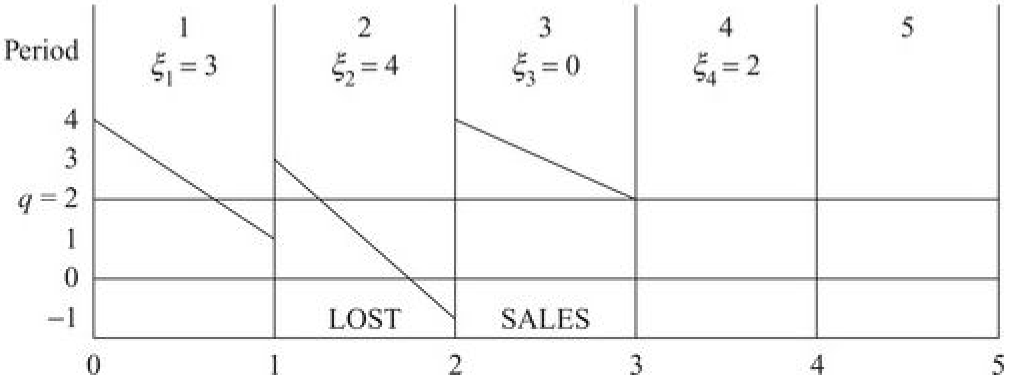
1.11

(a) $\pi_0 = \frac{117}{379} = .3087 \quad \pi_3 = \frac{62}{379} = .1636$

(b) $\pi_2 + \pi_3 = \frac{143}{379} = .3773$.

(c) $\pi_0(P_{02} + P_{03}) + \pi_1(P_{12} + P_{13}) = .1559$

1.13 $\lim_{n\to\infty} Pr\{X_{n-1} = 2 | X_n = 1\} = \frac{\pi_2 P_{21}}{\pi_1} = \frac{6 \times .2}{7} = .1714 \quad \pi_0 = \frac{11}{24} \quad \pi_1 = \frac{7}{24} \quad \pi_2 = \frac{6}{24}$.



(a) $X_0 = 4 \quad X_1 = 1 \quad X_2 = 0 \quad X_3 = 2 \quad X_4 = 2$

(b)
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} .6 & .3 & .1 & 0 & 0 \\ .3 & .3 & .3 & .1 & 0 \\ .1 & .2 & .3 & .3 & .1 \\ .3 & .3 & .3 & .1 & 0 \\ .1 & .2 & .3 & .3 & .1 \end{array} \right\| \end{array}$$

2.1

(c) $\pi_0 + \pi_1 + \pi_2 = .3559 + .2746 + .2288 = .8593$

2.3

(a) $\pi_0 = .2549; \quad \pi_1 = .2353; \quad \pi_2 = .3529; \quad \pi_3 = .1569$

(b) $\pi_2 + \pi_3 = .5098$

(c) $\pi_0 (P_{02} + P_{03}) + \pi_1 (P_{12} + P_{13}) = .2235$.

2.5

(a) $P^{(4)}_{(s,s)(s,s)} + P^{(4)}_{(s,s),(c,s)} = .3421 + .1368 = .4789$.

(b) $\pi_{(s,s)} + \pi_{(c,s)} = .25 + .15 = .40$.

2.7
$$P_{00} = p; \quad P_{0,1} = q = 1 - p; \quad P_{i,i+1} = \alpha q$$
$$P_{ii} = \alpha p + \beta q; \quad P_{i,i-1} = \beta p \text{ for } i \geqq 1.$$

3.1

(a) $f^{(0)}_{00} = 0 \quad f^{(1)}_{0,0} = 1 - a \quad f^{(k)}_{00} = ab(1-b)^{k-2}, \quad k \geqq 2$.

(b) We are asked do show:

$$P^{(n)}_{00} = \sum_{k=0}^{n} f^{(k)}_{00} P^{(n-k)}_{00} \quad \text{where } n \geqq 1 \text{ and}$$
$$P^{(n)}_{00} = \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^n.$$

Some preliminary calculations:

$$(i) \quad \frac{b}{a+b}\sum_{k=2}^{n} f^{(k)}_{00} = \frac{b}{a+b}\sum_{k=2}^{n} ab(1-b)^{k-2} = \frac{ab}{a+b}\left[1-(1-b)^{n-1}\right]$$
$$(ii) \quad \frac{a}{a+b}\sum_{k=2}^{n} f^{(k)}_{00}(1-a-b)^{n-k} = \frac{ab}{a+b}\left[(1-b)^{n-1} - (1-a-b)^{n-1}\right]$$

$$\text{Then} \quad \sum_{k=0}^{n} f^{(k)}_{00} P^{(n-k)}_{00} = f^{(1)}_{00} P^{(n-1)}_{00} + \sum_{k=2}^{n} f^{(k)}_{00} P^{(n-k)}_{00}$$
$$= (1-a)\left[\frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^{n-1}\right] + \frac{ab}{a+b}\left[1-(1-a-b)^{n-1}\right]$$
$$= \frac{b}{a+b} + \left[\frac{(1-a)a}{a+b} - \frac{ab}{a+b}\right](1-a-b)^{n-1}$$
$$= \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^n = P^n_{00}.$$

as was to be shown.

3.3 We first evaluate

$n=1$	2	3	4	5	6
$P^{(n)}_{00} = 0$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{7}{32}$	$\frac{17}{64}$

Because $P^{(n)}_{00} = 1$, (3.2) may be rewritten

$$f_{00}^{(n)} = P_{00}^{(n)} - \sum_{k=1}^{n-1} f_{00}^{(k)} P_{00}^{(n-k)}.$$

Finally $f_{00}^{(1)} = P_{00}^{(1)} = 0$

$$\begin{aligned}
f_{00}^{(2)} &= f_{00}^{(2)} - f_{00}^{(1)} P_{00}^{(1)} = \frac{1}{4} \\
f_{00}^{(3)} &= \frac{1}{8} \\
f_{00}^{(4)} &= \frac{3}{8} - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{5}{16} \\
f_{00}^{(5)} &= \frac{7}{32} - \left(\frac{1}{8}\right)\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{8}\right) = \frac{5}{32}
\end{aligned}$$

4.1

(a) $\pi_0 + \pi_1 = 1$ and $(\beta, \alpha)\mathrm{P} = (\beta, \alpha)$.

(b) A first return to 0 at time n entails leaving 0 on the first step, staying in 1 for $n-2$ transitions, and then returning to 0, whence $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$14 for $n \geqq 2$.

$$\begin{aligned}
m_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{00}^{(n)} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_{00}^{(n)} \\
&= 1 + \sum_{k=2}^{\infty} \alpha\beta \sum_{n=k}^{\infty} (1-\beta)^{n-2} = 1 + \alpha \sum_{k=2}^{\infty} (1-\beta)^{k-2} \\
\text{(c)} \quad &= 1 + \frac{\alpha}{\beta} = \frac{\alpha+\beta}{\beta} = \frac{1}{\pi_0}.
\end{aligned}$$

4.3 The period of the Markov chain is $d = 2$. While there is no limiting distribution, there is a stationary distribution. Set $p_0 = q_N = 1$. Solve.

$$\begin{aligned}
\pi_0 &= \pi_0 & \pi_0 &= \pi_0 \\
\pi_0 &= q_1 \pi_1 & \pi_1 &= \frac{1}{q_1}\pi_0 = \left(\frac{p_0}{q_1}\right)\pi_0 \\
\pi_1 &= p_0 \pi_0 + q_2 \pi_2 & \pi_2 &= \frac{1}{q_2}(\pi_1 - p_0 \pi_0) = \left(\frac{p_0 p_1}{q_1 q_2}\right)\pi_0 \\
\pi_2 &= p_1 \pi_1 + q_3 \pi_3 & \pi_3 &= \left(\frac{p_0 p_1 p_2}{q_1 q_2 q_3}\right)\pi_0 \\
&\vdots \\
\pi_k &= p_{k-1}\pi_{k-1} + q_{k+1}\pi_{k+1} & \pi_{k+1} &= \rho_{k+1}\pi_0
\end{aligned}$$

Where

$$\rho_{k+1} = \frac{p_0 p_1 \cdot\cdots\cdot p_k}{q_1 q_2 \cdot\cdots\cdot q_{k+1}}. \quad \text{Upon adding}$$

$$1 = \pi_0 + \cdots + \pi_N = (\rho_0 + \rho_1 + \cdots\cdots \rho_N)\pi_0$$

$$\pi_0 = \frac{1}{1 + \rho_1 + \rho_2 + \cdots\cdots + \rho_N} = \frac{1}{1 + \sum_{k=1}^{N} \prod_{l=1}^{k} \left(\frac{p_{l-1}}{q_l}\right)}$$

and $\pi_k = \rho_k \pi_0$.

4.5 Recall that a return time is always at least 1.

(a) Straight forward (b) To simplify, use $\sum \pi_i = 1$ and

$$\sum_i \pi_i P_{ik} = \pi_k \text{ to get } \sum_i \pi_i m_{ij} = 1 + \sum_{k \neq j} \pi_k m_{kj} = 1 + \sum_{i \neq j} \pi_i m_{ij}$$

and subtract $\sum_{i \neq j} \pi_i m_{ij}$ from both sides to get $\pi_j m_{jj} = 1$.

4.7 Measure time in *trips*, so there are two trips each day. Let $X_n = 1$ if car and person are at the same location prior to the n*th* trip; $= 0$, if not. The transition probability matrix is

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \boldsymbol{P} = \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} 0 & 1 \\ 1-p & p \end{array} \right\| \end{array}$$

In the long run he is *not* with car for $\pi_0 = \frac{1-p}{2-p}$ fraction of trips, and walks in rain $\pi_0 p = \frac{p(1-p)}{2-p}$ fraction of trips. The fraction of *days* he/she walks in rain is $\frac{2p(1-p)}{2-p}$.

With two case, let X_n be the number of cars at the location of the person.

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \boldsymbol{P} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{array} \right\| \end{array}$$

$\pi_0 = \frac{1-p}{3-p}$ and fraction of days walk in rain is $2p\pi_0 = \frac{2p(1-p)}{3-p}$. Note that the person never gets wet if $p = 0$ *or* $p = 1$.

5.1 The stationary distributions for P_B and P_c are respectively, $(\pi_3, \pi_4) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(\pi_5, \pi_6, \pi_7) = (.4227\ .2371\ .3402)$. The hitting probabilities from the transient states to the recurrent classes are

$$
\boldsymbol{U} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c} \begin{array}{cc} 3-4 & 5-7 \end{array} \\
\left\|\begin{array}{cc} .4569 & .5431 \\ .1638 & .8362 \\ .4741 & .5259 \end{array}\right\|
\end{array}
$$

This gives

$$
\boldsymbol{P}^{\infty} = \begin{array}{c|ccc|cc|ccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & & & & .2284 & .2284 & .2296 & .1288 & .1848 \\
1 & & & & .0819 & .0819 & .3534 & .1983 & .2845 \\
2 & & & & .2371 & .2371 & .2223 & .1247 & .1789 \\
\hline
3 & & \mathbf{O} & & \tfrac{1}{2} & \tfrac{1}{2} & & \mathbf{O} & \\
4 & & & & \tfrac{1}{2} & \tfrac{1}{2} & & & \\
\hline
5 & & & & & & .4227 & .2371 & .3402 \\
6 & & & & & \mathbf{O} & .4227 & .2371 & .3402 \\
7 & & & & & & .4227 & .2371 & .3402 \\
\hline
\end{array}
$$

Chapter 5

1.1 Following the hint:

$$Pr\{X=k\} = \int_0^1 e^{-\lambda(1-x)} \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} dx = \frac{\lambda^k e^{-\lambda}}{(k-1)!} \int_0^1 x^{k-1} dx = \frac{\lambda^k e^{-\lambda}}{k!}$$

1.3 $g_x(s) = \sum_{k=0}^{\infty} \frac{\mu^k e^{-\mu}}{k!} s^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu s)^k}{k!} = e^{-\mu} e^{\mu s} = e^{-\mu(1-s)}, |s| < 1.$

1.5

(a)
$$\begin{aligned} \frac{1-p_0(h)}{h} &= \frac{1-e^{-\lambda h}}{h} = \frac{\lambda h - \frac{1}{2}\lambda^2 h^2 + \frac{1}{3!}\lambda^3 h^3}{h} \\ &= \lambda - \frac{1}{2}\lambda^2 h + \frac{1}{3!}\lambda^2 h^2 - \cdots \to \lambda \text{ as } h \to 0. \end{aligned}$$

(b) $\frac{p_1(h)}{h} = \frac{\lambda h e^{-\lambda h}}{h} = \lambda e^{-\lambda h} \to \lambda \text{ as } h \to 0.$

(c) $\frac{p_2(h)}{h} = \frac{\frac{1}{2}\lambda^2 h^2 e^{-\lambda h}}{h} = \frac{1}{2}\lambda^2 H e^{-\lambda h} \to 0 \text{ as } h \to 0.$

1.7 $Pr\{\text{Survive at time } t\} = \sum_{k=0}^{\infty} Pr\{\text{Survive } k \text{ shocks}\}\, Pr\{k \text{ shocks}\} = \sum_{k=0}^{\infty} \alpha^k \frac{e^{-\lambda t}(\lambda t)^k}{k!} = e^{-\lambda t(1-\alpha)}.$

1.9

(a) $E[X(T)|T=t] = \lambda t \quad E\left[X(T)^2|T=t\right] = \lambda t + \lambda^2 t^2.$

(b)
$$\begin{aligned} E[X(T)] &= \int_0^1 \lambda t\, dt = \frac{1}{2}\lambda = 1 \quad \text{when } \lambda = 2. \\ E\left[X(T)^2\right] &= \int_0^1 \left(\lambda t + \lambda^2 t^2\right) dt = \frac{1}{2}\lambda + \frac{1}{3}\lambda^2. \\ Var[X(T)] &= E\left[X(T)^2\right] - E[X(T)]^2 = \frac{1}{2}\lambda + \frac{1}{3}\lambda^2 - \frac{1}{4}\lambda^2 \\ &= \frac{1}{2}\lambda + \frac{1}{12}\lambda^2 = \frac{4}{3} \quad \text{when } \lambda = 2. \end{aligned}$$

1.11 The gamma density

$$f_k(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \quad x > 0$$

$$Pr\{X(n,p)=0\} = (1-p)^n = \left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}, n \to \infty$$

2.1 $$\frac{Pr\{X(n,p)=k+1\}}{Pr\{X(n,p)=k\}} = \frac{(n-k)p}{(k+1)(1-p)} \to \frac{\lambda}{k+1}, n \to \infty, \lambda = np.$$

$$\begin{aligned} Pr\{A\} &= \frac{2N(2N-2)\cdot\dots\cdot(2N-2N+2)}{2N(2N-1)\cdot\dots\cdot(2N-n+1)} = \frac{2^n\binom{N}{n}}{\binom{2N}{n}}. \\ &= .7895 \quad \text{when } n=10, N=100. \end{aligned}$$

$$\begin{aligned} \prod_{i=1}^{n-1}\left(1-\frac{i}{2N-i}\right) &\cong \prod_{i=1}^{n-1} e^{-\frac{i}{2N-i}} \cong \exp\left\{-\sum_{i=1}^{n-1}\left(\frac{i}{2N}\right)\right\}. \\ &= \exp\left\{\frac{-n(n-1)}{4N}\right\} = .7985 \quad \text{when } n=10, N=100. \end{aligned}$$

2.3

2.5 X_r, the number of points within radius 1 of origin is binomially distributed with parameters N and $p = \frac{\pi}{\pi r^2} = \frac{\lambda}{N}$.
$\Pr\{X_r = k\} = \binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k} \to \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \dots.$

2.7 $Pr\{k \text{ bacteria in region of area } a\} = \binom{N}{k}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k} \to \frac{c^k e^{-c}}{k!}$ as $\begin{matrix} N\to\infty \\ a \to o \end{matrix}, \frac{Na}{A} \to c$.

2.9 $p_1 = p_2 = p_3 = .1, p_4 = .2 \quad \sum p_i^2 = .07$

k	$Pr\{S_n = k\}$	$\left(\frac{1}{2}\right)^k e^{\frac{1}{2}}/k!$	$Diff.$
0	.5832	.6065	−.0233
1	.3402	.3033	.0369
2	.0702	.0758	−.0056
3	.0062	.0126	−.0064
4	.0002	.0016	−.0014

2.11 From the hint,

$Pr\{X \text{ in } B\} \le Pr\{Y \text{ in } B\} + Pr\{X \ne Y\}$. Similarly
$Pr\{Y \text{ in } B\} \le Pr\{X \text{ in } B\} + Pr\{X \ne Y\}$.

Together

$$\left|Pr\{X \text{ in } B\} - Pr\{Y \text{ in } B\}\right| \le Pr\{X \ne Y\}.$$

3.1 To justify the differentiation as the correct means to obtain the density, look at

$$\int_{w_1}^{\infty}\int_{w_2}^{\infty} f_{W_1,W_2}\left(w_1', w_2'\right) dw_1' dw_2' = \left[1 + \lambda\left(w_2 - w_1\right)\right] e^{-\lambda w_2}.$$

3.3

$$\begin{aligned} f_{S_0,S_1}\left(s_0, s_1\right) &= f_{W_1,W_2}\left(s_0, s_0 + s_1\right) (\text{Jacobean} = 1) \\ &= \lambda^2 \exp\{-\lambda\left(s_0 + s_1\right)\} = \left(\lambda e^{-\lambda s_0}\right)\left(\lambda e^{-\lambda s_1}\right). \end{aligned}$$

Compare with Theorem 3.2. for another approach.

3.5 One can adapt the solution of Exercise 1.5 for a computational approach. For a different approach,

$$Pr\{X(T) = 0\} = Pr\{T < W_1\} = \frac{\theta}{\lambda + \theta} (\text{Review I}, 5.2).$$

Using the memoryless property and starting afresh at time W_1 17

$$Pr\{X(T) > 1\} = Pr\{W_2 \le T\} = Pr\{W_1 \le T\} Pr\{W_2 \le T | W_1 \le T\} = \left(\frac{1}{\lambda + \theta}\right)^2.$$

Similarly,

$$Pr\{X(T) > k\} = \left(\frac{\lambda}{\lambda + \theta}\right)^{k+1} \text{ and } Pr\{X(T) = k\} = \left(\frac{\theta}{\lambda + \theta}\right)\left(\frac{\lambda}{\lambda + \theta}\right)^k, \quad k \ge 0$$

3.7 The failure rate is $\lambda = 2$ per year. Let $X = \#$ of failures in a year

If Stock (Spares)	**Inoperable unit if**	$Pr\{\text{Inoperable}\}$
0	$X \geqq 1$	.8647
1	$X \geqq 2$	.5940
2	$X \geqq 3$	.3233
3	$X \geqq 4$	.1429
4	$X \geqq 5$	.0527
5*	$X \geqq 6$	.0166

3.9

(a) $Pr\left\{W_1^{(1)} < W_1^{(2)} = \frac{\lambda_1}{\lambda_1 + \lambda_2}\right\}$

$$Pr\left\{W_1^{(1)} < W_2^{(2)}\right\} = Pr\left\{W_2^{(1)} < W_1^{(2)}\right\} + Pr\left\{W_1^{(2)} < W_2^{(1)} < W_2^{(2)}\right\}$$
$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$
(b)
$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2\left[1+2\frac{\lambda_2}{\lambda_1+\lambda_2}\right]$$

$$f_{w_1,\ldots,w_{k-1},w_{k+1},\ldots w_n|X(1)=n,w_k=w}(w_1,\ldots,w_{k-1},w_{k+1},\ldots,w_n)$$
4.1
$$= \frac{n!}{\frac{n!}{(k-1)!(n-k)!}w^{k-1}(1-w)^{n-k}} = (k-1)!\left(\frac{1}{w}\right)^{k-1}(n-k)!\left(\frac{1}{1-w}\right)^{n-k}$$

4.3 The joint distribution of $U = \frac{W_1}{W_2}$

$$V = \left(\frac{1-W_3}{1-W_2}\right) \quad \text{and} \quad W = W_2 \text{ is}$$
$$f_{U,V,W}(u,v,w) = 6w(1-w) \quad \text{for} \quad 0 \le u,v,w \le 1$$

whence

$$f_{U,V}(u,v) = \int_0^1 6w(1-w)\,dw = 1 \quad \text{for} \quad 0 \le u,v \le 1$$

$$Pr\{W_1 > w|N(t)=n\} = Pr\{U_1 > w,\ldots U_n > w\} = \left(1-\frac{w}{t}\right)^n = \left(1-\frac{\beta w}{n}\right)^n$$
4.5 if $n = \beta t \to e^{-\beta w}$ if $t \to \infty, n \to \infty, n = \beta t$.

Under the specified conditions, W is exponentially distributed, but *not* with rate λ.

$$E\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{n} f(W_i)\,|X(t)=n\right] Pr\{X(t)=n\}$$
$$= \lambda t\, E[\,f(U)] \quad U \text{ is unif. } (0,t]$$
4.7
$$= \lambda \int_0^t f(u)\,du.$$

$$E\left[W_1 W_2 \cdots W_{N(t)}\right] = \sum_{n=0}^{\infty} E\left[W_1 \cdots W_n | N(t)=n\right] Pr\{N(t)=n\}$$
$$= \sum_{n=0}^{\infty} E\left[U_1,\ldots U_n\right] Pr\{N(t)=n\}$$
4.9
$$= \sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t\left(1-\frac{1}{2}t\right)}.$$

4.11 The limiting density is

$$
\begin{aligned}
f(x) &= c && \text{for } 0<x<1;\\
&= c(1-\log_e x) && \text{for } 1<x<2;\\
&= c\left[1-\log x+\int_2^x \frac{\log(t-1)}{t}\,dt\right], && \text{for } 2<x<3;\\
&\ \vdots
\end{aligned}
$$

where $c=\exp\{-\text{Euler's constant}\}=.561459\ldots$

In principal, one can get the density for any x from the differential equation. In practice?

5.1 $f(x)=2\left(\frac{x}{R^2}\right)$ for $0<x<R$ because $F(x)=Pr\{X\le x|N=1\}=\left(\frac{x}{R}\right)^2$.

5.3 For $i=1,2,\ldots,n^2$ let $\xi_i=1$ if two or more points are in box i, and $\xi_i=0$ otherwise. Then the number of reactions is $\xi_1+\cdots+\xi_{n^2}$ and $p=Pr\{\xi=1\}=\frac{1}{2}\lambda^2d^2+o\left(d^2\right)$.

$$n^2p=\frac{1}{2}\lambda^2(td)^2+\cdots\to\frac{1}{2}\lambda^2\mu^2.$$

The number of reactions is asymptotically Poisson with parameter $\frac{1}{2}\lambda^2\mu^2$.

$$
\begin{aligned}
F_D(x)=Pr\{D\le x\} &= 1-Pr\{D>x\}\\
&= 1-Pr\{\text{No particles in circle of radius } x\}
\end{aligned}
$$

5.5 $$= 1-\exp\left\{-\nu\pi x^2\right\},\quad x>0.$$

5.7 The hint should be sufficient for (a). For (b)

$$\int_0^\infty \lambda(r)\,dr=2\pi\lambda\int_0^\infty r\int_r^\infty f(x)\,dx\,dr=\int_0^\infty 2\pi\lambda\left\{\int_0^x r\,dr\right\}f(x)\,dx=\pi\lambda\int_0^\infty x^2f(x)\,dx.$$

6.1 Nonhomogeneous Poisson, intensity $\lambda(t)=\lambda G(t)$. To see this, let $N(t)$ be the number of points $\le t$ in the relocated process $=\#$ point in $\triangle$.

6.3 From the shock model of Section 6.1, modified for the discrete nature of the damage process, we have

$$
\begin{aligned}
E[T] &= \frac{1}{\lambda}\sum_{n=0}^{\infty} G^{(n)}(a-1)\\
&= \frac{1}{\lambda}\left[1+\sum_{n=1}^{\infty}\sum_{k=0}^{a-1}\binom{n+k-1}{k}p^n(1-p)^k\right] \text{ (See I, (3.6))}\\
&= \frac{1}{\lambda}\left[1+\sum_{k=0}^{a-1}p(1-p)^k\left\{\sum_{n=1}^{\infty}\binom{n-1+k}{n-1}p^{n-1}\right\}\right]\\
&= \frac{1}{\lambda}\left[1+\sum_{k=0}^{a-1}p(1-p)^k(1=p)^{-k-1}\right] \text{ (See I, (6.21))}\\
&= \frac{1}{\lambda}\left[1+\frac{ap}{1-p}\right].
\end{aligned}
$$

6.5 $\frac{1}{2}T$ is exponentially distributed with rate parameter 2μ, whence

$$
\begin{aligned}
Pr\left\{X\left(\frac{1}{2}T\right)=k\right\} &= \int_0^{\infty}\frac{(\lambda t)^k e^{-\lambda t}}{k!}2\mu e^{-2\mu t}\,dt\\
&= \left(\frac{2\mu}{\lambda+2\mu}\right)\left(\frac{2\lambda}{\lambda+2\mu}\right)^k, \quad k=0,1,\ldots \quad \text{(See I, (6.4))}
\end{aligned}
$$

6.7 Refer to Exercise 6.3.

$$
Pr\{Z(t)>z|N(t)>0\}=\frac{e^{-\lambda z^{\alpha}t}-e^{-\lambda t}}{1-e^{-\lambda t}}; \quad 0<z<1.
$$

Let $Y(t)=t^{1/\alpha}Z(t)$. For large t, we have

$$
\begin{aligned}
Pr\{Y(t)>y|N(t)>0\} &= Pr\left\{Z(t)>\frac{y}{t^{1/\alpha}}|N(t)>0\right\}\\
&= \frac{e^{-\lambda y^{\alpha}}-e^{-\lambda t}}{1-e^{-\lambda t}} \xrightarrow[t\to\infty]{} e^{-\lambda y^{\alpha}} \quad \text{(Weibull).}
\end{aligned}
$$

6.9 To carry the methods of this section a little further, write $N(dt)=N(t+dt)-N(t)$ so $N(dt)=1$ if and only if $t<W_k\le t+dt$ for some W_k. Then

$$
\sum_{k=1}^{N(t)}(W_k)^2=\int_0^t x^2N(dx) \quad \text{so}
$$

$$
\begin{aligned}
E\left[\sum_{k=1}^{N(t)}(W_k)^2\right] &= E\left[\int_0^t x^2N(dx)\right]=\int_0^t x^2E[N(dx)]\\
&= \lambda\int_0^t x^2dx=\frac{\lambda t^3}{3}.
\end{aligned}
$$

Chapter 6

1.1 $$Pr\{X(U)=k\}=\int_0^1 e^{-\beta u}\left(1-e^{-\beta u}\right)^{k-1}du=\int_0^{1-e^{-\beta}} x^{k-1}\frac{dx}{\beta}=\frac{1}{\beta k}\left(1-e^{-\beta}\right)^k.$$

1.3 The probabilistic rate of increase in the infected population is jointly proportional to the number who can give the disease, and the number available to catch it.

$$\lambda_k=\alpha k(N-k),\quad k=0,1,\ldots,N.$$

1.5 The two possibilities $X(w_2)=O$ or $X(w_2)=1$ give us

$$\begin{aligned}Pr\{W_1>w_1,W_2>w_2\}&=Pr\{X(w_1)=0,X(w_2)=0\}+Pr\{X(w_1)=0,X(w_2)=1\}\\&=P_0(w_1)P_0(w_2-w_1)+P_0(w_1)P_1(w_2-w_1)\\&=e^{-\lambda_0 w_2}+\lambda_0 e^{-\lambda_0 w_1}\left[\frac{1}{\lambda_1-\lambda_0}e^{-\lambda_0(w_2-w_1)}+\frac{1}{\lambda_0-\lambda_1}e^{-\lambda_1(w_2-w_1)}\right]\\&=\frac{\lambda_1}{\lambda_1-\lambda_0}e^{-\lambda_0 w_2}+\frac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1 w_2}e^{-(\lambda_0-\lambda_1)w_1}\\&=\int_{w_1}^{\infty}\int_{w_2}^{\infty}f_{w_1,w_2}\left(w_1',w_2'\right)dw_1',dw_2',\end{aligned}$$

hence

$$f_{w_1,w_2}(w_1,w_2)=\frac{\partial}{\partial w_2}\frac{\partial}{\partial w_1}Pr\{W_1>w_1,W_2>w_2\}=\lambda_0\lambda_1 e^{-\lambda_0 w_1}e^{-\lambda_1(w_2-w_1)}$$

Setting $s_0=w_1,s_1=w_2-w_1(\text{Jacobean}=1)\ f_{s_0,s_1}(s_0,s_1)=(\lambda_0 e^{-\lambda_0 s_0})(\lambda_1 e^{-\lambda_1 s_1})$.

1.7

(a) $Pr\{S_0\le t\}=1-e^{-\lambda_0 t};Pr\{S_0>t\}=e^{-\lambda_0 t}.$

$$Pr\{S_0+S_1 \le t\} = \int_0^t \left[1-e^{-\lambda_1(t-x)}\right]\lambda_0 e^{-\lambda_0 x}\,dx = 1-e^{-\lambda_0 t} - \left(\frac{\lambda_0}{\lambda_0-\lambda_1}\right)e^{-\lambda_1 t}\left[1-e^{-(\lambda_0-\lambda_1)t}\right]$$

$$= 1-\frac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1-\lambda_0}e^{-\lambda_0 t} = 1 - Pr\{S_0+S_1 > t\}.$$

$$Pr\{S_0+S_1+S_2 \le t\} = \int_0^t \left\{1-\frac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1(t-x)} - \frac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_0(t-x)}\right\}\lambda_2 e^{-\lambda_2 x}\,dx$$

$$= 1-\frac{\lambda_0\lambda_2}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 t} - \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} = \frac{\lambda_0\lambda_1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}.$$

$$\begin{aligned}
P_2(t) &= \frac{\lambda_0\lambda_2}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)} - e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{\lambda_0\lambda_1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t} \\
&\quad - \frac{\lambda_0}{(\lambda_0-\lambda_1)}e^{-\lambda_1 t} - \frac{\lambda_1}{(\lambda_1-\lambda_0)}e^{-\lambda_0 t} \\
\text{(b)} \qquad &= \lambda_0\left[\frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{1}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}\right].
\end{aligned}$$

1.9 Equations (1.2) become

$$\begin{aligned}
P_0'(t) &= -\beta P_0(t) \\
P_n'(t) &= -\beta P_n(t) + \alpha P_{n-1}(t), \quad n = 2,4,6,\ldots \\
P_n'(t) &= -\alpha P_n(t) + \beta P_{n-1}(t), \quad n = 1,3,5,\ldots
\end{aligned}$$

Multiply the n th equation by n, sum, collect terms and simplify to get

and (See Problem 1.8 above)

$$M(t) = \frac{2\alpha\beta}{\alpha+\beta}t + \left(\frac{\beta-\alpha}{\beta+\alpha}\right)\left(\frac{\beta}{\alpha+\beta}\right)\left[1-e^{-(\alpha+\beta)t}\right].$$

1.11

$$P_1(t)=\lambda_0 e^{-\lambda_1 t}\int_0^t e^{\lambda_1 x}e^{-\lambda_0 x}dx=\frac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1 t}+\frac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_0 t}$$

$$P_2(t)=\lambda_1 e^{-\lambda_2 t}\int_0^t\left[\frac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1 x}+\frac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_0 x}\right]e^{\lambda_2 x}dx$$

$$=\lambda_0\lambda_1\left[\frac{1}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 t}+\frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t}+\left\{\frac{1}{(\lambda_0-\lambda_1)(\lambda_1-\lambda_2)}+\frac{1}{(\lambda_1-\lambda_0)(\lambda_0-\lambda_2)}\right\}e^{-\lambda_2 t}\right.$$

and

$$\frac{1}{(\lambda_0-\lambda_1)(\lambda_1-\lambda_2)}+\frac{1}{(\lambda_1-\lambda_0)(\lambda_0-\lambda_2)}=\frac{1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}$$

$$P_3(t)=\lambda_0\lambda_1\lambda_2 e^{-\lambda_3 t}\int_0^t\left[\frac{1}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 x}+\frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 x}+\frac{1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 x}\right]e^{\lambda_3 x}dx$$

$$=\lambda_0\lambda_1\lambda_2\left[\frac{1}{(\lambda_3-\lambda_0)(\lambda_2-\lambda_0)(\lambda_1-\lambda_0)}e^{-\lambda_0 t}+\frac{1}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)(\lambda_0-\lambda_1)}e^{-\lambda_1 t}\right.$$
$$\left.+\frac{1}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)(\lambda_0-\lambda_2)}e^{-\lambda_2 t}+\frac{1}{(\lambda_0-\lambda_3)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 t}\right]$$

1.13 Let $Q_n(t)=e^{\lambda t}P_n(t)$. Then $Q_0(t)\equiv 1$ and (1.5) becomes $Q_n'(t)=\lambda Q_{n-1}(t)$ which solves to give $Q_1(t)=\lambda t;Q_2(t)=\frac{1}{2}(\lambda t)^2\ldots Q_n(t)=\frac{(\lambda t)^n}{n!}$ and $P_n(t)=\frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

2.1 Using the memoryless property (Section I, 5.2)

$$Pr\{X(T)=0\}=Pr\{T>W_n\}=\prod_{i=1}^N Pr\{T>W_i|T>W_{i-1}\}=\prod_{i=1}^N\left(\frac{\mu_i}{\mu_i+\theta}\right).$$

2.3 Refer to Figure 2.1 to see that $\text{Area}=W_N+\cdots+W_1$

Hence $$E[Area]=E[W_N]+\cdots+E[W_1]=NE[S_N]+(N-1)E[S_{N-1}]+\cdots+E[S_1]=\sum_{n=1}^{N}\frac{n}{\mu_n}.$$

2.5 Breakdown rule is "exponential breakdown".

$$\mu_k=kK\left[\frac{NL}{k}\right]=k\sinh\left[\frac{NL}{k}\right]$$

$$E[W_N]=\sum_{k=1}^N\frac{1}{k\sinh\left[\frac{NL}{k}\right]}=\sum_{k=1}^N\frac{1}{\left(\frac{k}{N}\right)\sinh\left(\frac{L}{R/N}\right)}\frac{1}{N}\cong\int_0^1\frac{dx}{x\sinh\left(\frac{L}{X}\right)}\text{ (Riemann approx.)}$$

3.1
$$\begin{aligned}Pr\{X(t+h)&=1|X(t)=0\}=\lambda h+o(h)\text{ so }\lambda_0=\lambda.\\ Pr\{X(t+h)&=0|X(t)=1\}=\lambda(1-\alpha)h+o(h)\text{ so }\mu_1=\lambda(1-\alpha).\end{aligned}$$

The Markov property requires the independent increments of the Poisson process.

$$\begin{aligned} Pr\{V(t)=1\} &= \pi \text{ for all } t \text{ (See Exercise 3.3)} \\ E[V(s)V(t)] &= Pr\{V(s)=1, V(t)=1\} = Pr\{V(s)=1\} \times Pr\{V(t)=1 | V(s)=1\} \\ &= \pi P_{11}(t-s) = \pi[1-P_{10}(t-s)] \end{aligned}$$

3.3 $$Cov[V(s)V(t)] = E[V(s)V(t)] - E[V(s)]E[V(t)] = \pi P_{11}(t-s) - \pi^2 = \pi(1-\pi)e^{-(\alpha+\beta)(t-s)}.$$

4.1 *Single repairman* $R=1$

$$\lambda_k = 2 \quad \text{for} \quad k=0,1,2,3,4, \quad \mu_k = k \quad \text{for} \quad k=0,1,2,3,4,5.$$

$$\theta_0 = 1, \theta_1 = 2, \theta_2 = 2, \theta_3 = \frac{4}{3}, \theta_4 = \frac{2}{3}, \theta_5 = \frac{4}{15}.$$

$$\sum_{k=0}^{5} \theta_k = \frac{218}{30} = \frac{109}{15}$$

Two repairman $R=2$

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 4; \lambda_4 = 2 \quad \mu_k = k$$

$$\theta_0 = 1, \theta_1 = 4, \theta_2 = 8, \theta_3 = \frac{32}{3}, \theta_4 = \frac{32}{3}, \theta_5 = \frac{64}{15},$$

$$\sum_{k=0}^{5} \theta_k = \frac{579}{15} = \frac{193}{15}.$$

	π_0	π_1	π_2	π_3	π_4	π_5
R=1	$\frac{15}{109}$	$\frac{30}{109}$	$\frac{30}{109}$	$\frac{20}{109}$	$\frac{10}{109}$	$\frac{4}{109}$
R=2	$\frac{15}{579}$	$\frac{60}{579}$	$\frac{120}{579}$	$\frac{160}{579}$	$\frac{160}{579}$	$\frac{64}{579}$
	$(a)\sum k\pi_k$	$(b)\frac{1}{N}\sum k\pi_k$	**(c)**			
R=1	1.93	.39	$\pi_5 = .0367$			
R=2	3.01	.06	$2\pi_5 + \pi_4 = .4974$			

4.3 Repairman Model $M=N=5, R=1, \mu=.2, \lambda=.5$

k	=	0	1	2	3	4	5
λ_k	=	.5	.5	.5	.5	.5	0
μ_k	=	0	.2	.4	.6	.8	.10
θ_k	=	1	$\frac{5}{2}$	$\frac{25}{8}$	$\frac{125}{48}$	$\frac{625}{384}$	$\frac{628}{768}$
π_k	=	$\frac{768}{8963}$	$\frac{1920}{8963}$	$\frac{2400}{8963}$	$\frac{2000}{8963}$	$\frac{1250}{8963}$	$\frac{625}{8963}$
	=	.086	.214	.268	.223	.139	.070

Fraction of time repairman is $\text{idle} = \pi_5 = .07$

4.5 If $X(t) = k$, there are $N - k$ unbonded A molecules and an equal number of unbonded B molecules. $\lambda_k = \alpha(N-k)^2,\ \mu_k = \beta k$.

4.7 The repairman model with $N = 3, R = 2$

$M = 2$	$\mu = \frac{1}{5} = .2,$		$\lambda = \frac{1}{4} = .25$	
k	0	1	2	3
λ_k	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	0
μ_k	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
θ_k	1	$\frac{5}{2}$	$\frac{25}{8}$	$\frac{125}{64}$
π_k	$\frac{64}{549}$	$\frac{160}{549}$	$\frac{200}{549}$	$\frac{125}{549}$
$\sum \theta_k = 1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{64} + \frac{549}{64}.$				

Long run avg. # Machines $\text{operating} = 0\pi_0 + 1\pi_1 + 2\pi_2 + \pi_3 = \frac{810}{549} = 1.48$; Avg. $\text{Output} = 148$ items per hour.

5.1 Change K into an absorbing state by setting $\lambda_K = \mu_K = 0$. Then $Pr\{\text{Absorption in } 0\} = Pr\{\text{Reach } 0 \text{ before } K\}$. Because $\rho_i = 0$ for $i \geqq K$, (5.7) becomes $\mu_m = \frac{\sum_{i=m}^{K-1} \tilde{\mathbf{n}}_i}{1 + \sum_{i=1}^{K-1} \tilde{\mathbf{n}}_i}$, as desired.

6.1 For $i \neq j, Pr\{X(t+h) = j | X(t) = i\} = \lambda P_{ij} h + o(h)$ for $h \approx 0$, The independent increments of the Poisson process are needed to establish the Markov property. 21

6.3
$$\begin{aligned} Pr\{Z(t+h) &= k+1 | Z(t) = k\} = (N-k)\lambda h + o(h), \\ Pr\{Z(t+h) &= k-1 | Z(t) = k\} = k\mu h + o(h). \end{aligned}$$

7.1

(a) $(1-\pi)P_{01}(t) + \pi P_{11}(t) = (1-\pi)\pi(1-e^{-\beta t}) + \pi[\pi + (1-\pi)e^{-\beta t}] = \pi - \pi^2 + \pi^2 = \pi$.

(b) Let $N(dt) = 1$ if there is an even in $(t, t+dt]$, and zero otherwise. Then $N((0,t]) = \int_0^t N(ds)$ and $E[N((0,t])] = E\left[\int_0^t N(ds)\right] = \int_0^t E[N(ds)] = \int_0^t \pi\lambda ds = \pi\lambda t$.

7.3 $T > t$ if and only if $N((0,t]) = 0$. $Pr\{T > t\} = Pr\{N((0,t]) = 0\} = f(t,\lambda)$ hence $\phi(t) = -\frac{d}{dt} f(t;\lambda) = c_+ \mu_+ e^{-\mu_+ t} + c_- \mu_- e^{-\mu_- t}$. When $\alpha = \beta 1$ and $\lambda = 2$, then

$$\mu_\pm = 2 \pm \sqrt{2}, c_\pm = \frac{1}{4}\left(2 \mp \sqrt{2}\right) \text{ and } \phi(t) = e^{-2t}\frac{e^{-\sqrt{2}t} + e^{\sqrt{2}t}}{2} = e^{-2t}\cosh\sqrt{2}t.$$

$Pr\{N((t,t+s]) = 0 | N((0,t]) = 0\}$

$$= \frac{f(t+s;\lambda)}{f(t;\lambda)} = \frac{c_+ e^{-\mu_+(t+s)} + c_- e^{-\mu_-(t+s)}}{c_+ e^{-\mu_+ t} + c_- e^{-\mu_- t}}$$

7.5 $$= \frac{c_+ e^{-(\mu_+-\mu_-)t}}{c_+ e^{-(\mu_+-\mu_-)t} + c_-} e^{-\mu_+ s} + \frac{c_-}{c_+ e^{-(\mu_+-\mu_-)t} + c_-} e^{-\mu_- s} \to e^{-\mu_- s} \text{ as } t \to \infty.$$

7.7 When $\alpha = \beta = 1$ and $\lambda = 2(1-\theta)$ then

$$\mu_\pm = (z-\theta) \pm R, \quad c_\pm = \frac{1}{2} \mp \frac{1}{2R}$$

and

$$g(t;\theta) = e^{-(2-\theta)t} \left\{ \left(\frac{1}{2} - \frac{1}{2R} \right) e^{-Rt} + \left(\frac{1}{2} + \frac{1}{2R} \right) e^{Rt} \right\} = e^{-(2-\theta)t} \left\{ \cosh Rt + \frac{1}{R} \sinh Rt \right\}.$$

$$\frac{dg}{d\theta} = e^{-(2-\theta)t} t \left\{ \cosh Rt + \frac{1}{R} \sinh Rt \right\} + e^{-(2-\theta)t} \left\{ t \sinh Rt + \frac{t}{R} \cosh Rt - \frac{1}{R^2} \sinh Rt \right\} \frac{dr}{d\theta}$$

$$R \Big|_{\theta=0} = \sqrt{2}, \quad \frac{dR}{d\theta} \Big|_{\theta=0} = -\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

$$\frac{dg}{d\theta}\Big|_{\theta=0} = e^{-2t} t \left\{ \cosh \sqrt{2}t + \frac{1}{2}\sqrt{2} \sin \sqrt{2}t \right\} - e^{-2t} t \left[\left\{ \sin \sqrt{2}t + \frac{1}{2}\sqrt{2} \cosh \sqrt{2}t \right\} - \frac{1}{2} \sinh \sqrt{2}t \right] \frac{\sqrt{2}}{2}$$

$$= \frac{1}{2} e^{-2t} \left\{ t \cosh \sqrt{2}t + \frac{\sqrt{2}}{2} \sinh \sqrt{2}t \right\} = Pr\{N((0,t]) = 1\}.$$

Students with symbol manipulating computer skills may test them on $Pr\{N((0,t]) = 2\} = \frac{1}{2} \frac{d^2 g}{d\theta^2}\Big|_{\theta=0}$.

7.9 The initial conditions are easy to verify: $f_0(0) = a_+ + a_- = 1 - \pi, f_1(0) = b_+ + b_- = \pi$. We wish to check that

$$f'_0(t) = -\alpha f_0(t) + \beta f_1(t)$$
$$f'_1(t) = \alpha f_0(t) - (\beta + \lambda) f_1(t)$$

Now

$$f'_0(t) = -a_+ \mu_+ e^{-\mu_+ t} - a_- \mu_- e^{-\mu_- t}$$

While

$$-\alpha f_0(t) = -\alpha a_+ e^{-\mu_+ t} - \alpha a_- e^{-\mu_- t}$$
$$\beta f_1(t) = \beta b_+ e^{-\mu_+ t} + \beta b_- e^{-\mu_- t}$$

Upon equating coefficients, we want to check that $-a_+ \mu_+ \stackrel{?}{=} -\alpha a_+ + \beta b_+$ and $-a_- \mu_- \stackrel{?}{=} -\alpha a_- + \beta b_-$. Starting with the first, is $0 \stackrel{?}{=} a_+(\mu_+ - \alpha) + \beta b_+$

$$
\begin{aligned}
a_+(\mu_+-\alpha) &= \frac{1}{4}(1-\pi)\left[(\lambda-\alpha+\beta)+R-\frac{(\lambda-\alpha+\beta)(\lambda+\alpha+\beta)}{R}-(\alpha+\beta+\lambda)\right] \\
&= \frac{1}{4}(1-\pi)\left[-2\alpha+R-\frac{(\lambda-\alpha+\beta)(\lambda+\alpha+\beta)}{R}\right] \\
\beta b_+ &= \frac{1}{2}\pi\beta\left[1+\frac{\lambda-\alpha-\beta}{R}\right], \quad \pi=\frac{\alpha}{\alpha+\beta}
\end{aligned}
$$

$$
\begin{aligned}
&2R(\alpha_-+\beta)[a_+(\mu_+-\alpha)+\beta b_+] \\
&= \beta R^2-\beta(\lambda-\alpha+\beta)(\lambda+\alpha+\beta)+2\alpha\beta(\lambda-\alpha-\beta) \\
&= \beta\left[(\alpha+\beta+\lambda)^2-4a\lambda-(\lambda-\alpha+\beta)(\lambda+\alpha+\beta)+2\alpha(\lambda-\alpha-\beta)\right] \\
&= \beta[(\alpha+\beta+\lambda)\{\alpha+\beta+\lambda-\lambda+\alpha-\beta\}-4\alpha\lambda+2\alpha(\lambda-\alpha-\beta)] \\
&= \beta[2\alpha(\alpha+\beta+\lambda)-4\alpha\lambda+2\alpha(\lambda-\alpha-\beta)] \\
&= \beta\alpha[2(\alpha+\beta)+2(-\alpha-\beta)]=0
\end{aligned}
$$

To check: $0 \stackrel{?}{=} a_-(\mu_--\alpha)+\beta b_-$

$$
\begin{aligned}
a_-(\mu_--\alpha) &= \frac{1}{4}(1-\pi)\left[1+\frac{\alpha+\beta+\lambda}{R}\right][(\lambda-\alpha+\beta)-R] \\
&= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[\lambda-\alpha+\beta-R+\frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R}-\alpha-\beta-\lambda\right] \\
&= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[-2\alpha-R+\frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R}\right] \\
\beta b_- &= \frac{1}{2}\frac{\alpha\beta}{\alpha+\beta}\left[1-\frac{\lambda-\alpha-\beta}{R}\right] \\
&= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[2\alpha-\frac{2\alpha(\lambda-\alpha-\beta)}{R}\right] \\
a_-(\mu_--\alpha)+\beta b_- &= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[-R+\frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R}-\frac{2\alpha(\lambda-\alpha-\beta)}{R}\right] \\
&= \frac{1}{4R}\left(\frac{\beta}{\alpha+\beta}\right)\left[-(\alpha+\beta+\lambda)^2+4\alpha\lambda+(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)-2\alpha(\lambda-\alpha-\beta)\right] \\
&= \frac{1}{4R}\left(\frac{\beta}{\alpha+\beta}\right)[4\alpha\lambda-2\alpha(\alpha+\beta+\lambda)-2\alpha(\lambda-\alpha-\beta)]=0
\end{aligned}
$$

Verifying the second differential equation reduces to checking if $0 \stackrel{?}{=} aa_+ + b_+(\mu_+-\beta-\lambda)$ and $0 \stackrel{?}{=} aa_- + b_-(\mu_--\beta-\lambda)$. The algebra is similar to that used for the first equation.

7.11 $V(u)$ plays the role of $\lambda(u)$, and $S(t)$, that of $\Lambda(t)$.

Chapter 7

$$
\begin{aligned}
Pr\{N(t-x)=N(t+y)\} &= \sum_{k=0}^{\infty} Pr\{N(t-x)=k=N(t+y)\} \\
&= \sum_{k=0}^{\infty} Pr\left\{W_k \le t-x, W_{k+1} > t+y\right\} \\
&= [1-F(t+y)] + \sum_{k=1}^{\infty} Pr\left\{W_k \le t-x, W_k + X_{k+1} > t+y\right\} \\
&= [1-F(t+y)] + \sum_{k=1}^{\infty} \int_0^{t-x} [1-F(t+y-z)]\,dF_k(z).
\end{aligned}
$$

1.1

In the exponential case:

$$
\begin{aligned}
&= e^{-\lambda(t+y)} + \sum_{k=1}^{\infty} \int_0^{t-x} e^{-\lambda(t+y-z)} \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z} dz \\
&= e^{-\lambda(t+y)} + \int_0^{t-x} e^{-\lambda(t+y-z)} \lambda e^{\lambda z} e^{-\lambda z} dz \\
&= e^{-\lambda(t+y)} + e^{-\lambda(t+y)} \left[e^{\lambda(t-x)} - 1\right] = e^{-\lambda(x+y)}.
\end{aligned}
$$

1.3 $E[\gamma_t] = E\left[W_{N(t)+1} - t\right] = E[X_1][M(t)+1] - t$.

2.1

Block Period	**Cost** $= \dfrac{\mathbf{4+5}M(K-I)}{K}$
K	**Θ (*K*)**
1	4.00
2	2.25
3	2.183
4	2.1183
5	2.1231
*Replace on failure: $\Theta = 1.923$, is best	

$$
\begin{aligned}
M(1) &= p_1 = \beta \\
M(2) &= p_1 + p_2 + p_1 M(1) = \beta + \beta(1-\beta) + \beta^2 = 2\beta \\
M(3) &= p_1 + p_2 + p_3 + p_1 M(2) + p_2 M(1) \\
&= \beta + \beta(1-\beta) + \beta(1-\beta)^2 + \beta(2\beta) + \beta(1-\beta)\beta \\
&= \beta + \beta - \beta^2 + \beta - 2\beta^2 + \beta^3 + 2\beta^2 + \beta^2 - \beta^3 = 3\beta
\end{aligned}
$$

2.3

In general, $M(n) = n\beta$.

3.1 $E\left[\frac{1}{N(t)+1}\right] = \sum_{\infty}^{k=0} \frac{1}{k+1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{1}{\lambda t} e^{-\lambda t} \sum_{\infty}^{j=1} \frac{(\lambda t)^j}{j!} = \frac{1}{\lambda t} e^{-\lambda t}\left(e^{\lambda t} - 1\right)$.

Using the independence established in Exercise 3.3,

$$E\left[\frac{W_{N(t)+1}}{N(t)+1}\right] = E\left[W_{N(t)+1}\right] E\left[\frac{1}{N(t)+1}\right] = \left(t+\frac{1}{\lambda}\right)\frac{1}{\lambda t}\left(1-e^{-\lambda t}\right) = \frac{1}{\lambda}\left(1-\frac{1}{\lambda t}\right)\left(1-e^{-\lambda t}\right).$$

3.3 For $t > \tau, p(t) = Pr\{\text{No arrivals in } (t-\tau, t]\} = e^{-\lambda\tau}$. For $t \le \tau, p(t) = Pr\{\text{No arrivals in } (0,t]\} = e^{-\lambda t}$. Thus $p(t) = e^{-\lambda \min(\tau,t)}$.

$$\begin{aligned}
Pr[D(t) > x] &= Pr\{\text{No birds in } (t-x, t+x]\} = \begin{cases} e^{-2\lambda x} & \text{for } 0 < x < t \\ e^{-\lambda(x+t)} & \text{for } 0 < t \le x. \end{cases} \\
E[D(t)] &= \int_0^\infty Pr\{D(t) > x\}dx = \int_0^t e^{-2\lambda x}dx + \int_t^\infty e^{-\lambda(x+t)}dx = \frac{1}{2\lambda}\left(1+e^{-2\lambda t}\right). \\
f_T(t) &= -\frac{d}{dx}Pr\{D(t) > x\} = \begin{cases} 2\lambda e^{-2\lambda x} & \text{for } 0 < x < t \\ \lambda e^{-\lambda(x+t)} & \text{for } 0 < t \le x. \end{cases}
\end{aligned}$$

3.5

4.1 $\frac{1}{t}M(t) \to \frac{1}{\mu}$ implies $\mu = 1$

$$M(t) - \frac{t}{\mu} \to \frac{\sigma^2 - \mu^2}{2\mu^2} = 1 \quad \text{implies} \quad \sigma^2 = 3$$

4.3 $1 - F(y) = \exp\left\{-\int_0^y \theta x \; dy\right\} = e^{-\frac{1}{2}\theta y^2}, y \geqq 0$

$$\mu = \int_0^\infty [1-F(y)]dy = \int_0^\infty e^{-\frac{1}{2}\theta y^2}dy = \sqrt{\frac{\pi}{2\theta}}.$$

$$\sigma^2 + \mu^2 = \int_0^\infty y^2 dF(y) = \frac{2}{\theta}.$$

Limiting mean age $\frac{\sigma^2+\mu^2}{2\mu} = \sqrt{\frac{2}{\pi\theta}}$.

$$\begin{aligned}
m_0 &= 1 + .3m_0 \quad \Rightarrow \quad m_0 = \frac{1}{.7} = \frac{10}{7} \\
m_2 &= 1 + .5m_2 \quad \Rightarrow \quad m_2 = 2
\end{aligned}$$

4.5

Successive visits to state 1 form renewal instants for which $\mu = 1 + .6m_0 + .4m_2 = \frac{93}{35}$. And $\pi_1 = \frac{35}{93}\left(\pi_0 = \frac{30}{93}, \pi_2 = \frac{28}{93}\right)$.

5.1

(a) $Pr\{\text{A down}\} = \frac{\beta_A}{\alpha_A + \beta_A}; Pr\{\text{B down}\} = \frac{\beta_B}{\alpha_B + \beta_B}.$

$$Pr\{\text{System down}\} = \left(\frac{\beta_A}{\alpha_A + \beta_A}\right)\left(\frac{\beta_B}{\alpha_B + \beta_B}\right).$$

(b) System leaves the failed state upon first component repair. E [Sojourn System down] $= \frac{1}{\alpha_A + \alpha_B}$. 23

(c) $Pr[\text{System down}] = \frac{E[\text{Sojourn System Down}]}{E[\text{Cycle}]}$ so

$$E[\text{Cycle}] = \frac{1/(\alpha_A + \alpha_B)}{\left(\frac{\beta_A}{\alpha_A} + \beta_A\right)\left(\frac{\beta_B}{\alpha_B} + \beta_B\right)}.$$

(d)
$$\begin{aligned} E[\text{System Sojourn Up}] &= E[\text{Cycle}] - E[\text{Sojourn down}] \\ &= \left(\frac{1}{\alpha_A + \alpha_B}\right)\left\{\frac{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}{\beta_A \beta_B} - 1\right\} \end{aligned}$$

5.3 If successive fees are $Y_1, Y_2, \ldots$ then $W(t) = \Sigma_{k=1}^{N(t)+1} Y_k$ when $N(t)$ is the number of customers arriving in $(0, t]$.

$$\lim_{t \to \infty} \frac{E[W(t)]}{t} = \frac{E[Y]}{E[X]} = \frac{\int_0^\infty [1 - G(y)]dy}{\int_0^\infty [1 - F(x)]dx}$$

6.1

(a)
$$\begin{aligned} u_0 &= 1 \\ u_1 &= p_1 u_0 = \alpha \\ u_2 &= p_2 u_0 + p_1 u_1 = \alpha(1-\alpha) + \alpha^2 = \alpha \\ u_3 &= p_3 u_0 + p_2 u_1 + p_1 u_2 = \alpha(1-\alpha)^2 + \alpha[\alpha + \alpha(1-\alpha)] = \alpha \end{aligned}$$

We guess $u_n = \alpha$ for all n

$$\begin{aligned} \text{so} \quad \alpha &\stackrel{?}{=} \alpha(1-\alpha)^{n-1} + \alpha(p_1 + p_2 + \cdots + (p_{n-1}) \\ &= \alpha(1-\alpha)^{n-1} + \alpha^2\left(1 + (1-\alpha) + \cdots + (1-\alpha)^{n-2}\right) \\ &= \alpha(1-\alpha)^{n-1} + \alpha\left(1 - (1-\alpha)^{n-1}\right) = \alpha \end{aligned}$$

(b) For excess life $b_n = p_{n+m} = \alpha(1-\alpha)^{n+m-1} = p_m(1-\alpha)^n$

$$V_n = \sum_{k=0}^{n} b_{n-k} u_k = p_m(1-\alpha)^n \alpha \sum_{k=1}^{n} (1-\alpha)^{n-k} p_m = p_m\left[(1-\alpha)^n + 1 - (1-\alpha)^n\right] = p_m.$$

6.3 That delaying the age of first birth, even at constant family size, would lower population growth rates, was an important conclusion in early work with this model. $\Sigma_{v=0}^{\infty} m_v s^v = 2s^2 + 2s^3 = 1$ solves to gives $s = .5652$ whence $\lambda = \frac{1}{s} 1.77$ compared to $\lambda = 2.732$ in the example.

Chapter 8

1.1 Let

$$
\begin{aligned}
T_n &= \min\{t \geqq 0; \quad B_n(t) \leqq -a \text{ or } B_n(t) \geqq b\} \\
&= \min\{t \geqq 0; \quad S_{[nt]} \leqq -a\sqrt{n} \text{ or } S_{[nt]} \geqq b\sqrt{n}\} \\
&= \frac{1}{n}\min\{k \geqq 0; \quad S_k \leqq -a\sqrt{n} \text{ or } S_k \geqq b\sqrt{n}\} \\
E[T_n] &= \frac{1}{n}\left[a\sqrt{n}\prod b\sqrt{n}\right] \qquad \text{(III, Section 5.3)} \\
&\to ab \text{ as } n \to \infty.
\end{aligned}
$$

1.3 $Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} = Pr\{|B(t)| > \varepsilon t\} = 2\{1-\Phi_t(\varepsilon t)\} = 2\left\{1-\Phi\left(\varepsilon\sqrt{t}\right)\right\}$ (1.8)

$$
\begin{aligned}
&Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} \to 0 \quad \text{as } t \to \infty \text{ because } \Phi\left(\varepsilon\sqrt{t}\right) \to 1 \\
&Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} \to 1 \quad \text{as } t \to 0 \text{ because } \Phi\left(\varepsilon\sqrt{t}\right) \to \frac{1}{2}.
\end{aligned}
$$

1.5

(a) $Pr\{M_\tau = 0\} = Pr\{S_n \text{ drops to } -a < 0 \text{ before rising 1 unit}\} = \frac{1}{1+\alpha}$ (Using III, 5.3).

(b) In order that $M_\tau \geqq 2$, we must first have $M_\tau \geqq 1$ followed by moving ahead to 2, starting afresh from $S' = 1$, before dropping a units below 1. The spatial homogeneity gives this second move the same probability as $Pr\{M(\tau) \geqq 1\}$. Thus $Pr\{M_\tau \geqq 2\} = [Pr\{M_\tau \geqq 1\}]^2 = \left(\frac{a}{1+a}\right)^2$. The argument repeats to give $Pr\{M_\tau \geqq k\} = \left(\frac{a}{1+a}\right)^k$.

(c) Divide the state space into increments of length $\frac{1}{n}$ and observe the Brownian motion as it crosses lines $\frac{k}{n}$. That is, $\tau_1 = \min\{t \geqq 0; B(t) = \frac{1}{n} \text{ or } B(t) = -\frac{1}{n}\}$, and if, for example, $B(\tau_1) = \frac{1}{n}$, then let $\tau_2 = \min\{t \geqq \tau_1; B(t) = \frac{2}{n} \text{ or } B(t) = \frac{0}{n}\}$, etc. $B(\tau_1), B(\tau_2), \ldots$ has the same distribution as $\frac{1}{n}S_1$, $\frac{1}{n}S_2 \ldots$ We apply part (b) to this approximating Brownian motion to see

$$
Pr\{M_n(\tau) > x\} = \left(\frac{na}{1+na}\right)^{nx} \to e^{-x/a}
$$

As $n \to \infty$, the partially observed Brownian gets closer to the Brownian motion. We conclude that $M(\tau)$ is exponentially distributed with mean a 25 (Rate parameter $\frac{1}{a}$).

1.7

(a) $E[B(n+1)\mid B(0),\ldots,B(n)] = E[B(n+1)-B(n)\mid B(0),\ldots,B(n)] + B(n) = B(n).$

$$\begin{aligned} E[B(n+1)^2-(n+1)|B(n)^2-n] &= E[\{B(n+1)^2-B(n)^2-1\}+B(n)^2-n|B(n)^2-n] \\ &= B(n)^2-n+E[B(n+1)^2-B(n)^2]-1 = B(n)^2-n. \end{aligned}$$

(b)

2.1

$$\begin{aligned} Pr\{B(u)\neq 0, t<u<t+b\,|\,B(u)\neq 0, t<u<t+a\} &= \frac{Pr\{B(u)\neq 0, t<u<t+b\}}{Pr\{B(u)\neq 0, t<u<t+a\}} = \frac{1-\xi(t,t+b)}{1-\xi(t,t+a)} \\ &= \frac{\text{arc sin}\sqrt{t/(t+b)}}{\text{arc sin}\sqrt{t/(t+a)}} 0<a<b. \end{aligned}$$

2.3 $Pr\{M(t)>a\} = 2\{1-\Phi_t(a)\} = Pr\{B(t)>a\}$. But the joint distributions clearly differ (For $0<s<t$, it must be $M(t)\geqq M(s)$).

2.5 The Jacobian is one, whence

$$f_{M(t),Y(t)}(z,y) = f_{M(t),B(t)}(z,z-y) = \frac{2}{t}\left(\frac{z+y}{\sqrt{t}}\right)\varphi\left(\frac{z+y}{\sqrt{t}}\right)$$

3.1

$$\begin{aligned} Pr\left\{\frac{R(t)}{\sqrt{t}}>z\right\} &= \int\int_{\sqrt{x^2+y^2}>z} \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}dxdy = \int_z^{\infty}\int_0^{2\pi}\frac{1}{2\pi}re^{-\frac{1}{2}r^2}d\theta dr = -\int_z^{\infty} d\left\{e^{-\frac{1}{2}r^2}\right\} = e^{-\frac{1}{2}z^2}. \\ E\left[\frac{R(t)}{\sqrt{t}}\right] &= \int_0^{\infty} Pr\left\{\frac{R(t)}{\sqrt{t}}>z\right\}dz = \int_0^{\infty} e^{-\frac{1}{2}z^2} = \sqrt{\frac{\pi}{2}} \text{ and } E[R(t)] = \sqrt{\frac{\pi t}{2}}. \end{aligned}$$

3.3 We need only show that B (1) and $B(u)-uB(1)$ are uncorrelated (Why?)

$$E[B(1)\{B(u)-uB(1)\}] = E[B(1)B(u)] - uE\left[B(1)^2\right] = u-u = 0.$$

(a) $B(t) = \{B(t)-tB(1)\}+tB(1)$

The conditional distribution of $B(t)-tB(1)$, given $B(1)$, is the same as the unconditional distribution, by independence, where as $tB(1)$, given $B(1)=0$, is zero.

$$\begin{aligned} E[B^\circ(s)B^\circ(t)] &= E[\{B(s)-sB(1)\}\{B(t)-tB(1)\}] \\ &= E[B(s)B(t)] - sE[B(1)B(t)] - tE[B(s)B(1)] + stE\left[B(1)^2\right] \\ &= s-st-st+st = s(1-t) \quad \text{for } 0<s<t<1. \end{aligned}$$

(b)

3.5 $\displaystyle\int_0^{\infty} y[\phi_t(y-x)-\varphi_t(y+x)]dy = \int_{-\infty}^{+\infty} y\varphi_t(y-x)dy = x.$

3.7 The same calculation as in Problem 3.5 showes that $A(t)$ is a martingale. For the (continuous time) martingale $A(t)$, the maximal inequality is an equality.

3.9

(a) $E[B^{\circ}(F(s))B^{\circ}(F(t))] = F(s)[1-F(t)]$ for $s<t$.

(b) The approximation is

$$F_N(t) \approx F(t) + \frac{1}{\sqrt{N}} B^{\circ}(F(t)).$$

4.1 $Pr\{\max[B(t)-bt] > a\} = e^{-2ab}$.

4.3

$$\begin{aligned} Pr\{\max_{0\le u\le 1} B^{\circ}(u) > a\} &= Pr\left\{\max_{t>0}(1+t)B^{\circ}\left(\frac{t}{1+t}\right) - a(1+t) > 0\right\} \\ &= Pr\left\{\max_{t>0} B(t) - at > a\right\} = e^{-2a^2}. \end{aligned}$$

4.5 $Pr\{\max X^A(t) > B\} = \dfrac{e^{-2\mu x/\sigma^2}-1}{e^{-2\mu B/\sigma^2}-1}$.

4.7

$$\begin{aligned} Pr\{Z(\tau) > a|Z(0) = z\} &= Pr\{\log Z(\tau) > \log a|\log Z(0) = \log z\} \\ &= 1 - \Phi\left(\frac{\log\frac{a}{z}\left(a-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

4.9 The differential equation is $\frac{d^2}{dx^2}w(x) = 2\lambda w(x)$ for $x>0$. The solution is $w(x) = c_1 e^{\sqrt{2\lambda x}} + c_2 e^{-\sqrt{2\lambda x}}$. The constants are evaluated via the boundary conditions $w(0) = 1, \lim_{x\to\infty} w(x) = 0$ whence $w(x) = e^{-1\sqrt{2\lambda x}}$. There are many other ways to evaluate $w(x)$, including martingale methods.

5.1

(a) The formula follows easily from iteration. It is a discrete analog of (5.22).

(b) Sum $\Delta V_n = V_n - V_{n-1} = \beta V_{n-1} + \xi_n$ to get the given formula. It is a discrete analog to (5.24).

5.3 This is merely the result of Exercise 4.6 applied to the position process.

Chapter 9

1.1 Let $X(t)$ be the number of trucks at the loader at time t. Then $X(t)$ is birth and death process for which $\lambda_0=\lambda_1=\lambda$ and $\mu_1=\mu_2=\mu\cdot(\lambda_2=\mu_0=0)$. Then $\theta_0=1, \theta_1=\tilde{n}=\dfrac{\lambda}{\mu}$, and $\theta_2=\tilde{n}^2$ $\pi_0=$ Long run fraction of time of no trucks are at the loader $=\dfrac{1}{1+\tilde{n}+\tilde{n}^2}$. Fraction of time trucks are loading $=1-\pi_0=\dfrac{\tilde{n}+\tilde{n}^2}{1+\tilde{n}+\tilde{n}^2}$.

Since trucks load at a rate of μ per unit time, long run loads per unit time $=\mu(1-\pi_0)=\mu\left(\dfrac{\tilde{n}+\tilde{n}^2}{1+\tilde{n}+\tilde{n}^2}\right)=\lambda\left(\dfrac{1+\tilde{n}}{1+\tilde{n}+\tilde{n}^2}\right)$.

2.1 For the M/M/2 system, $\theta_k=2\tilde{n}^k$ for $k\geqq 1$.

$$\begin{aligned}
\sum\theta_k &= 2\sum n^k-1=\frac{2}{1-\tilde{n}}-1=\frac{1+\tilde{n}}{1-\tilde{n}}.\\
\pi_0 &= \frac{1-\tilde{n}}{1+\tilde{n}};\quad \pi_k=\frac{2(1-\tilde{n})}{1+\tilde{n}}\tilde{n}^k,\quad k\geqq 1\\
L_0 &= \frac{2\tilde{n}^3}{1-\tilde{n}^2},\quad L=\frac{2\tilde{n}}{1-\tilde{n}^2}.\\
L_0 &, L\to\infty \quad as \quad \tilde{n}\to 1.
\end{aligned}$$

2.3 For the M/M/2 system $\theta_0=1,\ \theta_k=2\,n^k$ for $k\geqq 1, \pi_0=\dfrac{1-\tilde{n}}{1+\tilde{n}}, \pi_k=2\left(\dfrac{1-\tilde{n}}{1+\tilde{n}}\right)\tilde{n}^k, k\geqq 1$.

$$\begin{aligned}
L &= \sum k\pi_k=2\left(\frac{1-\tilde{n}}{1+\tilde{n}}\right)\sum_{\varpi}^{k=1}k\tilde{n}^k=\frac{2\tilde{n}}{1-\tilde{n}^2}.\\
W &= \frac{1}{\mu}\pi_0+\frac{1}{\mu}\pi_1+\left(\frac{1}{\mu}+\frac{1}{2\mu}\right)\pi_2+\left(\frac{1}{\mu}+\frac{2}{2\mu}\right)\pi_2+\cdots\\
&= \frac{1}{\mu}(\pi_0+\pi_1+\pi_2+\cdots)+\frac{1}{2\mu}\sum_{\varpi}^{k=2}(k-1)\pi_k\\
W &= \frac{1}{\mu}+\frac{1}{\mu}\left(\frac{1-\tilde{n}}{1+\tilde{n}}\right)\sum_{\varpi}^{k=2}(k-1)\tilde{n}^k=\frac{1}{\mu}\left(\frac{1}{1-\tilde{n}^2}\right)
\end{aligned}$$

and $L=\lambda W$.

2.5 Observe that $\mu\pi_k=\lambda\pi_{k-1}$. Following the hint, we get for $j\geqq 1$

$$\pi_0 P_{0j}'(t)=\lambda\pi_0P_{1j}(t)-\lambda\pi_0P_{0j}(t)=\mu\pi_1P_{1j}(t)-\lambda\pi_0P_{0j}(t)$$

and

$$\pi_k P'_{kj}(t) = \mu\pi_k P_{k-1,j-1}(t) + \lambda\pi_k P_{k+1,j}(t) - (\lambda+\mu)\pi_k P_{k,j}(t)$$
$$= \lambda\pi_{k-1} P_{k-1,j-1}(t) + \mu\pi_{k+1} P_{k+1,j}(t) - (\lambda+\mu)\pi_k P_{k,j}(t).$$

Upon summing the above, the μ terms drop out and we get

$$P'_j(t) = \lambda P_{j-1}(t) - \lambda P_j(t), \quad j \geqq 1$$

Similar analysis yields

$$P'_0(t) = -\lambda P_0(t).$$

Set $Q_n(t) = e^{\lambda t} P_n(t)$. Then $Q'_n(t) = Q_{n-1}(t)$

The solution (using $Q_0(0) = P_0(0) = 1$) is

$$Q_n(t) = \frac{(\lambda t)^n}{n!}, n = 0, 1, 2, \ldots \text{hence}$$
$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Compare with Theorem 5.1.

2.7 Since $X(0) = 0$ we set $P_j(t) = P_{0_j}(t)$. The forward equations in this case are 28

$$P'_j(t) = \lambda P_{j-1}(t) + \mu(j+1)P_{j+1}(t) - (\lambda+\mu j)P_j(t). \text{ Multiply by } j$$
$$jP'_j(t) = \lambda(j-1+1)P_{j-1}(t) + \mu\left[(j+1)^2 - (j+1)\right]P_{j+1}(t) - \lambda jPj(t) - \mu j^2 P_j(t), \quad j \geqq 0.$$

Sum:

$$M'(t) = \lambda M(t) + \lambda - \mu M(t) + \mu P_1(t) - \mu P_1(t) - \lambda M(t) = \lambda - \mu M(t), \quad t \geqq 0$$
$$M(0) = 0. \text{ Let } Q(t) = e^{\mu t} M(t).$$
$$Q'(t) = e^{\mu t} M'(t) + e^{\mu t}\mu M(t) = e^{\mu t}\left[M'(t) + \mu M(t)\right] = \lambda e^{\mu t}, \quad t \geqq 0$$
$$Q(t) = \frac{\lambda}{\mu}\int_0^t \mu e^{\mu t} dt = \frac{\lambda}{\mu}\left(e^{\mu t} - 1\right)$$
$$M(t) = e^{-\mu t} Q(t) = \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right).$$

3.1 $X(t)$ and $Y(t)$ are independent random variables, each Poisson distributed where

$$E[X(t)] = \lambda \int_0^t [1 - G(y)]dy$$

$$E[Y(t)] = \lambda \int_0^1 G(y)dy.$$

Observe that $Y(t)$ is the number of points (W_k, V_k) in the triangle $B_t = \{(w,v) : 0 \le w \le t, 0 \le v \le t - w\}$ and that A_t and B_t are disjoint. Apply Theorem V, 6.1.

4.1 When the faster server is first

$$D = 1400 \text{ and } \pi_{(1,1)} = .3977$$

When the slower is first

$$D = 1600 \text{ and } \pi_{(1,1)} = .4082$$

Slightly more customers are lost when the slower server is first.

4.3 A birth and death process with $\lambda_n = \lambda$ and $\mu_n = \mu + r_n$

$$\sum \theta_k = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{\mu_1 \mu_2 \cdots \mu_k}, \quad \pi_0 = \frac{1}{\sum \theta_k} \pi_k = \pi_0 \left(\frac{\lambda^k}{\mu_1 \mu_2 \cdots \mu_k} \right).$$

Rate at which customers depart prior to service is $\Sigma \pi_k r_k$.

4.5 $\lambda_0 = \lambda_1 = \lambda_2 = \lambda, \mu_1 = \mu, \mu_2 = \mu_3 = 2\mu.$

$$\theta_0 = 1, \quad \theta_1 = \frac{\lambda}{\mu} \quad \theta_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \quad \theta_3 = \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3.$$

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3} \quad \pi_k = \theta_k \pi_0$$

5.1

$$Pr\{X_3 = k\} = \left(1 - \frac{10}{15}\right)\left(\frac{10}{15}\right)^k, \quad k \geqq 0$$

$$Pr\{X_3 > k\} = \left(\frac{10}{15}\right)^k$$

$$\text{Want } c \text{ such that } \left(\frac{10}{15}\right)^c \leq .01$$

$$\text{such that } c\log\frac{2}{3} \leq \log .01$$

$$\text{such that } c \geqq \frac{\log .01}{\log \frac{2}{3}}$$

$$c \geqq 11.36$$

$$c^* = 12.$$

6.1 Let x be the rate of feedback. Then $x+\lambda$ go in to Server #1, and $.6(x+\lambda)$ go in and out and Server #2. But $x = .2$ of the output of Server #2. Therefore $x = (.2)(.6)(x+\lambda) = .12x + .12\lambda$

$$.88x = .12\lambda \quad x = \frac{12}{88}\lambda = \frac{3}{11}.$$

Server #2: Arrival rate

$$= .6(x+\lambda) = \frac{6}{10}\left(\frac{3}{11} + 2\right) = \frac{15}{11}.$$

$$\pi_{20} = 1 - \frac{15/11}{3} = \frac{6}{11}.$$

Server #3: Arrival rate $= .4(x+\lambda) = \frac{10}{11}$

$$\pi_{30} = 1 - \frac{10/11}{2} = \frac{6}{11}$$

Long run $Pr\{X_2 = 0, X_3 > 0\} = \frac{6}{11} \times \frac{5}{11} = \frac{30}{121}.$

10.1

(a) Since the rows of the matrix $P(t)$ add to 1, we may represent $P(t)$ in the form

$$P(t)=\begin{pmatrix} \phi(t) & 1-\phi(t) \\ 1-\psi(t) & \psi(t) \end{pmatrix} \quad \text{and set} \quad Q=\begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

where $\phi(0)=1, \phi'(0)=-a, \psi(0)=1, \psi'(0)=-b$. The (1,1) term of the matrix equation $P'=PQ$ is the first-order linear equation

$$\phi'(t)=-(a+b)\phi(t)+b.$$

The general solution is the sum of a particular solution and a general solution of the homogeneous equation, obtained by a family of exponentials; in detail

$$\phi(t)=\frac{b}{a+b}+ce^{-(a+b)t} \tag{1}$$

The constant c is obtained from the first derivative at $t=0$ in (1); thus

$$-a=\phi'(0)=-c(a+b) \Longrightarrow c=\frac{a}{a+b}$$

which gives the result

$$\phi(t)=\frac{b}{a+b}+\frac{a}{a+b}e^{-(a+b)t}, \quad 1-\phi(t)=\frac{a}{a+b}-\frac{a}{a+b}e^{-(a+b)t}.$$

Similarly one obtains $\psi(t)$ by interchanging a and b.

(b) We can use the result of part (a) to obtain

$$\begin{aligned} E[V(t)|V(0)=v_1] &= v_1P_{11}(t)+v_2P_{12}(t) \\ &= v_1\left(\frac{a+be^{-\mu t}}{a+b}\right)+v_2\left(\frac{b-be^{-\mu t}}{a+b}\right) \\ &= \frac{av_1+bv_2}{a+b}+e^{-\mu t}\frac{bv_1-bv_2}{a+b} \end{aligned}$$

(c) Interchange a with b and v_1 with v_2 in (1b), to obtain the indcated result.

(d) Replace v_1 with v_1^2 and v_2 with v_2^2 in the result of part (b).

(e) Interchange a with b and replace v_1 with v_1^2, v_2 with v_2^2. in the result of part (d).

10.3 The ordinary differential equation for g is linear and homogeneous with constant coefficients. The form of the solution depends on whether $|\mu|<1$, $|\mu|=1$ or $|\mu|>1$. In every case we have the quadratic equation $r^2+2r+\mu^2=0$ whose solution is $r=-1\pm\sqrt{1-\mu^2}$ which leads to the following three-fold system:

$$
\begin{aligned}
|\mu|<1 \ \textit{implies}\ \ & g(t)=e^{-t}\left(A\cosh t\sqrt{1-\mu^2}+B\sinh t\sqrt{1-\mu^2}\right)\\
|\mu|=1 \ \textit{implies}\ \ & g(t)=e^{-t}(A+Bt)\\
|\mu|>1 \ \textit{implies}\ \ & g(t)=e^{-t}\left(A\cos(t\sqrt{\mu^2-1})+B\sin(t\sqrt{\mu^2-1})\right)
\end{aligned}
$$

for suitable constants A,B.

Exercises (10.4) and (10.5) can be paraphrased as follows: the idea is to charcterize those second-order differential operators which arise from[30] random evolutions in one dimension. Consider the following set-up: let $\mathscr{L}$ be the set of second order differential operators of the form (10.32) for some $v_1<v_2, q_1>0, q_2>0$. Let $\mathscr{M}$ be the set of differential operators of the form (10.33)

$$a_{20}=1,\ a_{12}^2<4a_{02}a_{20},\ a_{10}>0,\ v_1<a_{01}/a_{10}<v_2,\ a_{00}=0 \tag{2}$$

The revised exercises are written as follows:

Exercise (10.4'). Let $L\in\mathscr{L}$ for some $v_1<v_2, q_1>0, q_2>0$. Then $L\in\mathscr{M}$ with

$$a_{20}=1, a_{11}=v_1+v_2, a_{02}=v_1v_2, a_{10}=q_1+q_2, a_{01}=q_1v_2+q_2v_1, a_{00}=0.$$

Exercise (10.5'). Let $L\in\mathscr{M}$ for some (a_{ij}) satisfying the above conditions (2). Then $L\in\mathscr{L}$.

Thus we have a necessary and sufficient condition for a second order operator to be associated with a random evolution process.

Solution of Exercises (10.4'), (10.5'): If $L\in\mathscr{L}$, equation (2) holds and we can make the suitable identifications:

$$a_{20}=1,\ a_{11}=v_1+v_2,\ a_{02}=v_1v_2,\ a_{10}=q_1+q_2,\ a_{01}=v_1q_2+v_2q_1$$

It is immediate that these satisfy the conditions (2) hence $L\in\mathscr{M}$.

Conversely, suppose that $L\in\mathscr{M}$. Let v_1, v_2 be the roots of the equation $\lambda^2+a_{11}\lambda+a_{02}=0$. From the hypotheses, both roots are real and can be labeled so that $v_1<v_2$.

Next, we define q_1, q_2 as the solution of the 2×2 system

$$a_{10}=q_1+q_2,\quad a_{01}=q_1v_2+q_2v_1$$

The unique solution is

$$q_1=(a_{01}-v_1a_{10})/(v_2-v_1),\quad q_2=(v_2a_{10}-a_{01})/(v_2-v_1).$$

In both formulas the denominator is positive; in addition we have the hypothesis $v_1 < a_{01}/a_{10} < v_2$, which shows that the numerators are also positive. Hence the ratios are positive, so that, in particular, $q_1+q_2>0$. Finally we need to verify the condition on a_{01}/a_{10}. But this follows by dividing the second equation by q_1+q_2 to obtain

$$v_1 < \frac{a_{01}}{a_{10}} < v_2$$

But a_{01}/a_{10} is a convex combination of v_1, v_2, hence it must lie on the segment from v_1 to v_2. The proof is complete.•

For completeness, we include the self-contained proof of (10.5):

10.5 The polynomial equation is $\lambda^2 - \lambda(v_1+v_2)+v_1v_2=0$. By inspection the roots are $\lambda = v_1, v_2$, where we may assume that $v_1 < v_2$.

The coefficient $a_{10} = q_1+q_2>0$. Finally $a_{01}/a_{10} = (v_1q_2+v_2q_1)/(q_1+q_2)$, a convex combination of v_1, v_2. Hence this fraction belongs to the interval (v_1, v_2), as required.

10.7 Since the roots are distinct, the associated cubic polynomial separates into three linear factors: $\lambda^3 + a_{21}\lambda^2 + a_{12}\lambda + a_{03} = (\lambda - v_1)(\lambda - v_2)(\lambda - v_3)$, where we label the v's so that $v_1 < v_2 < v_3$, which proves i').

To prove ii'), note that a_{11}/a_{20} is a convex combination of the three quantities $v_1+v_2, v_2+v_3, v_1+v_3$. which are also in increasing order. This set can equally well be described as the open interval (v_1+v_2, v_2+v_3), which was to be proved.

To prove the second part of ii') note that $a_{02}/a_{20} = (q_1v_2v_3+q_2v_1v_3+q_3v_1v_2)/(q_1+q_2+q_3)$ is a convex combination of the three products v_1v_2, v_1v_3, v_2v_3. This convex combination belongs to the interval (m, M) where $m = \min_{i,j} v_iv_j, M = \max_{i,j} v_iv_j$ which proves the second half of ii').

Chapter 11

11.1 The moment of order k of the normal distribution is obtained through the formula $\sqrt{2\pi} m_k = \int_R x^k e^{-x^2/2}\,dx$. If k is odd the integrand is an odd function, hence m_k is zero if k is odd. If k is even, we can write $k = 2n$ for some $n = 1, 2, \vdots$. Next, we integrate by parts:

$$
\begin{aligned}
\sqrt{2\pi} m_{2n} &= \int_{-\infty}^{\infty} x^{2n-1} x e^{-x^2}\,dx \\
&= -\int_{-\infty}^{\infty} x^{2n-1} d\left(e^{-x^2/2}\right) \\
&= +\int_{-\infty}^{\infty} (2n-1) x^{2n-2} e^{-x^2/2}\,dx \\
&= \sqrt{2\pi}(2n-1) m_{2n-2}
\end{aligned}
$$

For example $m_0 = 1, m_2 = m_0, m_4 = 3m_2 = 3m_0$ and in general

$$
m_{2n} = (2n-1)(2n-3)\cdots m_0 = \frac{(2n)!}{2^n n!}
$$

11.3 Write $\phi_X(t) = f(t) = u(t) + iv(t)$ where u, v are continuous functions with $u(t)^2 + v(t)^2 \le 1$.

Then

$$
\begin{aligned}
f(t) &= u(t) + iv(t) \\
\int f(t)\,dt &= \int u(t) + i \int v(t)\,dt \\
&= Re^{i\theta}, \text{ where } R = \left|\int f(t) dt\right| \\
e^{-i\theta} \int f(t)\,dt &= \cos\theta \int u(t) dt + \sin\theta \int v(t) dt
\end{aligned}
$$

Now use Cauchy-Schwarz as follows:

$$
\begin{aligned}
\left|\cos\theta \int u(t) dt + \sin\theta \int v(t)\,dt\right| &= \left|\int [\cos\theta\, u(t) + \sin\theta\, v(t)]\,dt\right| \\
&\le \int \sqrt{u(t)^2 + v(t)^2}\,dt \\
&= \int |f(t)|\,dt \le 1
\end{aligned}
$$

which was to be proved.

11.5 The complex exponential is defined by the power series

$$e^{iz} = \sum_{n=0}^{\infty} (iz)^n/n!.$$

Moving the first three terms to the left side, we have

$$e^{iz} - 1 - iz - (iz)^2/2 = \sum_{n=3}^{\infty} (iz)^n/n!$$

$$\frac{e^{iz} - 1 - iz - (iz)^2/2}{z^2} = \sum_{n=3}^{\infty} (iz)^{n-2}/n!$$

The term on the right is bounded by $|z|e^{|z|}$. Hence

$$\left|\frac{e^{iz} - 1 - iz}{z^2} + \frac{1}{2}\right| \le |z|e^{|z|} \to 0, \quad z \to 0$$

11.7 For $0 \le \theta \le \pi/2$, $\cos\theta \le 1$, so that

$$\sin\theta = \int_0^{\theta} \cos t\, dt \le \int_0^{\theta} dt = \theta.$$

Meanwile $\theta \to \sin\theta$ is a convex function so that it always lies above its chord with respect to any two points, esp. on the interval $[0, \pi/2]$ where the chord is the line with equation $y = (2/\pi)\theta$. Combining these two estimates, we have the two-sided estimate

$$\frac{2}{\pi}\theta \le \sin\theta \le \theta, \qquad 0 \le \theta \le \frac{\pi}{2}$$