Exercise 3 – *Bayesian inference and Data assimilation*

Due by: Tuesday, 9 May 2023, 23:59 (CEST)
Please refer to the assignment submission guideline on Moodle.

Problem 1 (30pts) This example can be done in multi-dimensional setting, but let us assume everything is scalar for simplicity. Let

- X be a random variable with mean $\mathbb{E}[X] = m$ and covariance $\mathbb{E}\big[(X-m)^2\big] = Q$;
- W be a random variable with mean zero and covariance R;
- $\mathbb{E}[XW] = 0$.

Let

$$Y = HX + W$$

where $H \neq 0$ is a given constant. The goal is to construct an affine estimator of the form

$$\hat{X} = KY + b$$

where the gain K and bias b are to be determined by minimizing the mean squared error:

$$\min_{K,b} \mathbb{E}\big[(\hat{X} - X)^2\big]$$

1. Fix K and solve for optimal b in terms of m, H and K.
2. Verify that for the optimal b, $\hat{X}$ is unbiased regardless of K. That is, $\mathbb{E}[\hat{X} - X] = 0$.
3. Obtain the optimal gain K and corresponding mean squared error value.

Problem 2 (25pts) Consider the two-dimensional Gaussian PDF $n(z; \bar{z}, \Sigma)$ where $z = (x_1, x_2)$ with mean $\bar{z} = (\bar{x}_1, \bar{x}_2)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{pmatrix}$$

The Gaussian PDF is given by

$$n(z; \bar{z}, \Sigma) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\Big(-\frac{1}{2}(z-\bar{z})^{\mathrm{T}}\Sigma^{-1}(z-\bar{z})\Big)$$

where $|\Sigma|$ denotes the determinant.

1. Obtain constants $\bar{x}_c$ and σ_c such that
$$n(z; \bar{z}, \Sigma) = \frac{1}{\sqrt{2\pi}\sigma_c} \exp\Big(-\frac{1}{2\sigma_c^2}(x_1 - \bar{x}_c)^2\Big) \frac{1}{\sqrt{2\pi}\sigma_{22}} \exp\Big(-\frac{1}{2\sigma_{22}^2}(x_2 - \bar{x}_2)^2\Big)$$
2. Discuss about the conditional probability distribution $\pi_{X_1|X_2}(x_1 \mid x_2)$ using the formula obtained in the previous step.

Problem 3 (20pts) Assume X and W from the problem 1 are independent Gaussian random variables with the given means and variances. Then it is known that Y is also a Gaussian random variable.

1. Let $Z = (X, Y)$. Compute the mean and covariance matrix of Z.
2. Use the formula you obtained from problem 2 to obtain the conditional probability distribution of X given Y.
3. Compare the result with the optimal affine estimator you obtained in problem 1.

Problem 4 (25pts) Let X_1 and X_2 be two random variables with joint PDF given by

$$\pi_{X_1,X_2}(x_1, x_2) = \frac{1}{Z} \exp\big(- x_1^2 - x_2^2 - x_1^2 x_2^2 \big)$$

where Z is a normalization constant. Evaluate $\mathbb{E}[X_1^2 X_2 \mid X_2 = a]$ for some constant a.

(Hint: compute the marginal probability density $\pi_{X_2}(x_2)$ to obtain the conditional probability density $\pi_{X_1|X_2}(x_1 \mid a)$.)