**Exercise 4** – *Bayesian inference and Data assimilation*

**Due by:** Tuesday, 16 May 2023, 23:59 (CEST)

**Problem 1a** (20pts) Let $f$ be the pdf of a random variable $X$. The variance of $X$ is denoted by $\sigma$ and $f$ is an even function (that is, $f(-x) = f(x)$). Define another random variable $Z = aX^2 + bX + c$.

1. For which value of $a$, $b$ and $c$, $X$ and $Z$ are uncorrelated?
2. For which value of $a$, $b$ and $c$, $X$ and $Z$ are independent?

**Problem 1b** (25pts) Which of the following statements are true? Prove the true ones and give counterexamples for the false ones. Let $X$ and $Z$ be random variables.

1. If $X$ and $Z$ are uncorrelated, then $X$ and $Z$ are independent.
2. If $X$ and $Z$ are independent, then $X$ and $Z$ are uncorrelated.
3. Suppose $\mathrm{var}(X)$ and $\mathrm{var}(Z)$ are finite. Then $\mathrm{var}(X + Z) = \mathrm{var}(X) + \mathrm{var}(Z)$ if and only if $X$ and $Z$ are independent.

**Problem 2** (30pts) Let $X$, $Y$ and $Z$ be Bernoulli random variables defined by

$$X = \begin{cases} 0, & \text{w.p. } 1/2 \\ 1, & \text{w.p. } 1/2 \end{cases} \quad Y = \begin{cases} 0, & \text{w.p. } 1/3 \\ 1, & \text{w.p. } 2/3 \end{cases} \quad Z = \begin{cases} 0, & \text{w.p. } 1 \\ 1, & \text{w.p. } 0 \end{cases}$$

1. Design appropriate parameters to characterize all possible coupling between $X$ and $Y$.
2. Which coupling maximizes the correlation? Which coupling minimizes the correlation? Do you have an intuitive explanation why these couplings are the ones that maximize (minimize) the correlation?
3. Which coupling makes the two random variables uncorrelated?
4. Repeat the task for $X$ and $Z$. Are $X$ and $Z$ independent?

**Problem 3** (25pts) Let $X_1 \sim N(\bar{x}_1, \sigma_1)$ and $X_2 \sim (\bar{x}_2, \sigma_2)$ be two Gaussian random variables, and $\pi_{X_1}(x_1)$ and $\pi_{X_2}(x_2)$ are pdf of $X_1$ and $X_2$, respectively.

- Derive an explicit formula for Wasserstein distance between $\pi_{X_1}$ and $\pi_{X_2}$. (See Eq. 2.16 and Example 2.31 in the lecture notes)
- Derive an explicit formula for KL divergence of $\pi_{X_1}$ from $\pi_{X_2}$ given by

$$D_{KL}(\pi_{X_1} \| \pi_{X_2}) := \int_{\mathbb{R}} \log \frac{\pi_{X_1}(x)}{\pi_{X_2}(x)} \pi_{X_1}(x) \, \mathrm{d}x$$