Exercise 6 – *Bayesian inference and Data assimilation*

Due by: Tuesday, 30 May 2023, 23:59 (CEST)

Problem 1 (70pts) In this exercise, we will implement different methods to find the area of the unit circle. Namely, we want to compute

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \tag{1}$$

We all know the true value is $\pi/4$. Let $f(x) = \sqrt{1-x^2}$.

- Implement uniform interval mid-point rule: For each M, the quadrature points are given by
$$b_i = \frac{1}{M}, \quad c_i = \frac{1}{2M} + \frac{i-1}{M}, \qquad i = 1, 2, \dots, M$$
Compute
$$\bar{f}_M := \sum_{i=1}^{M} b_i f(c_i)$$
for $M = 2, 3, \dots, 30$. Plot the error $|\bar{f}_M - \pi/4|$.

- The Gaussian quadrature points for $U[-1, 1]$ is given by
$$b_1 = \frac{4}{9}, \quad c_1 = 0, \quad b_2 = b_3 = \frac{5}{18}, \quad c_2, c_3 = \pm\sqrt{\frac{3}{5}}$$
Compute
$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$
using the given quadrature points. Compare the result with the mid-point rule above with the same number of evaluation.

- Consider the Monte Carlo method as follows: (1) Draw M pair of random numbers (x, y) on $[0, 1] \times [0, 1]$ uniformly. (2) If the point (x, y) lies in the circle, that is, $x^2 + y^2 \leq 1$, then we record this as a 'hit'. (3) The area is now approximated by
$$\bar{f}_{\mathrm{MC}} := \frac{\text{number of hits}}{M}$$
Perform a single simulation for each value of $M = 10, 20, 30, \dots, 500$. Plot the result versus M. Draw a horizontal line at the true value $f = \pi/4$.

- Draw 100,000 Monte-Carlo samples and compute the absolute value of the error. How many intervals M you would need to beat this error using mid-point rule?

Problem 2 (30pts) Let $X \sim \mathcal{N}(1, 3)$ and $f(x) = 1 + 2x + x^2$.

- Calculate $\mathbb{E}[f(X)]$ and $\mathrm{Var}[f(X)]$ by hand.
- Implement the Monte-Carlo method to approximate the expected value of f, i.e.

$$\mathbb{E}[f(X)] \approx f_M := \frac{1}{M} \sum_{i=1}^{M} f(x_i), \qquad x_i \sim \mathcal{N}(1, 3)$$

- Let $M = 1, 2, 4, \cdots, 256$. For each M, do $N = 10000$ simulations to approximate the expectation using f_M. For each M, calculate the mean and the variance of f_M over N rounds. Visualize your result.