Problem 1 (70pts) In this exercise, we will implement different methods to find the area of the unit circle. Namely, we want to compute

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

We all know the true value is $\pi/4$. Let $f(x)=\sqrt{1-x^2}$.

- Implement uniform interval mid-point rule: For each M, the quadrature points are given by

$$b_i=\frac{1}{M},\quad c_i=\frac{1}{2M}+\frac{i-1}{M},\quad i=1,2,\ldots,M$$

Compute

$$\bar{f}_M := \sum_{i=1}^{M} b_i f\left(c_i\right)$$

for $M=2,3,\ldots,30$. Plot the error $\left|\bar{f}_M-\pi/4\right|$.

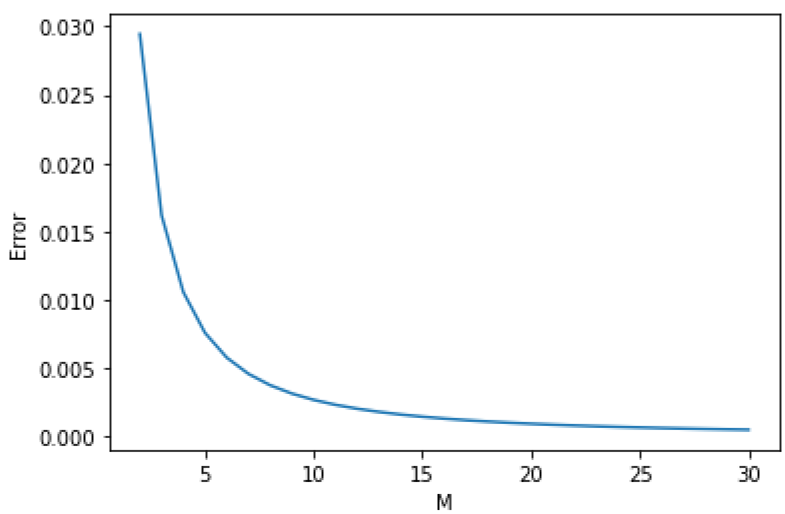
Solution:

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
# Define the midpoint rule
def midpoint_rule(M):
    b = 1/M
    c = np.array([(0.5/M) + (i-1)/M for i in range(1,M+1)])
    f_M = np.sum(b*f(c))
    return f_M
def f(x):
    return np.sqrt(1-x**2)

M_values = range(2, 31)
errors = []

for M in M_values:
    error = abs(midpoint_rule(M) - (np.pi/4))
    errors.append(error)

plt.plot(M_values, errors)
plt.xlabel('M')
plt.ylabel('Error')
plt.show()
```



- The Gaussian quadrature points for $U[-1, 1]$ is given by

$$b_1 = \frac{4}{9}, \quad c_1 = 0, \quad b_2 = b_3 = \frac{5}{18}, \quad c_2, c_3 = \pm\sqrt{\frac{3}{5}}$$

Compute

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^1 \sqrt{1-x^2}\,\mathrm{d}x$$

using the given quadrature points. Compare the result with the mid-point rule above with the same number of evaluation.

Solution:

In [2]:
```
# Gaussian quadrature
b = np.array([4/9, 5/18, 5/18])
c = np.array([0, np.sqrt(3/5), -np.sqrt(3/5)])
result_gq = 0.5 * sum([b[i]*f(c[i]) for i in range(len(b))])
print(f"Gaussian quadrature result: {result_gq}")

# Mid-point rule
M = 3 #3 quadrature points given by c1, c2, and c3.
result_mp = midpoint_rule(M)
print(f"Mid-point rule result: {result_mp}")
```

```
Gaussian quadrature result: 0.3979043144537988
Mid-point rule result: 0.8016031664534248
```

The true value of the definite integral is π/4≈0.7853981633974483. It seems that in this case, the Mid-point rule result result is closer to the true value than the Gaussian quadrature result.

- Consider the Monte Carlo method as follows: (1) Draw M pair of random numbers (x, y) on $[0, 1] \times [0, 1]$ uniformly. (2) If the point (x, y) lies in the circle, that is, $x^2 + y^2 \leq 1$, then we record this as a 'hit'. (3) The area is now approximated by

$$\bar{f}_{\text{MC}} := \frac{\text{number of hits}}{M}$$

Perform a single simulation for each value of $M = 10, 20, 30, \ldots, 500$. Plot the result versus M. Draw a horizontal line at the true value $f = \pi/4$.

Solution:

In [3]:

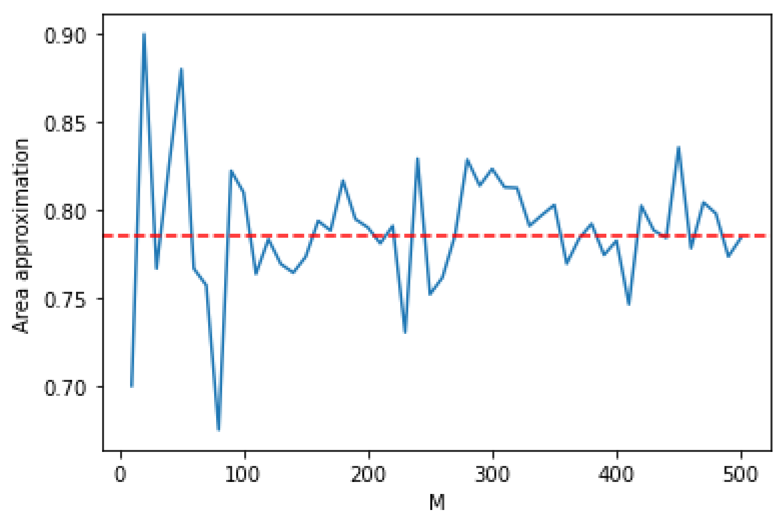
```
# Set the number of simulations
M_values = np.arange(10, 510, 10)
results = []

for M in M_values:
    # Draw M pairs of random numbers
    x = np.random.uniform(0, 1, M)
    y = np.random.uniform(0, 1, M)

    # Count the number of hits
    hits = np.sum(x**2 + y**2 <= 1)

    # Compute the approximation of the area
    f_MC = hits / M
    results.append(f_MC)

# Plot the results
plt.plot(M_values, results)
plt.axhline(np.pi/4, color='r', linestyle='--')
plt.xlabel('M')
plt.ylabel('Area approximation')
plt.show()
```



- Draw 100,000 Monte-Carlo samples and compute the absolute value of the error. How many intervals M you would need to beat this error using mid-point rule?

Solution:

In [13]:
```
# Define the Monte Carlo method
def monte_carlo(M):
    x = np.random.uniform(0,1,M)
    y = np.random.uniform(0,1,M)
    hits = np.sum(x**2 + y**2 <= 1)
    f_MC = hits/M
    return f_MC

# Draw 100000 Monte Carlo samples
M = 100000
f_MC = monte_carlo(M)
error_MC = abs(f_MC - np.pi/4)

# Find the number of intervals M needed to beat this error using the midpoint rule
M_values = range(2,31)
errors = [abs(midpoint_rule(M) - np.pi/4) for M in M_values]
M_beat_MC = next(M for M, error in zip(M_values, errors) if error < error_MC)

print(f"With {M} Monte Carlo samples, the absolute value of the error is {error_MC:.6f}.
print(f"To beat this error using the midpoint rule, we need {M_beat_MC} intervals.")
```

```
With 100000 Monte Carlo samples, the absolute value of the error is 0.000962.
To beat this error using the midpoint rule, we need 20 intervals.
```

In case of "StopIteration:" please run the cell above again.

Problem 2.

We have $f(x) = 1 + 2x + x^2$

Now, $E[f(x)] = E[1] + E[2x] + E[x^2]$ [Linearity of expectation]

$= 1 + 2E[x] + E[x^2]$

$= 1 + 2(1) + Var[x] + (E[x])^2$ $[Var[x] = E[x^2] - (E[x])^2]$

$= 1 + 2(1) + 3 + 1^2$ $[x \sim N(1,3)]$

$= 7$.

Now, $Var[f(x)] = E[(f(x))^2] - (E[f(x)])^2$

$= E[(1+2x+x^2)^2] - 7^2$ $[E[f(x)] = 7]$

$= E[1 + 4x + 6x^2 + 4x^3 + x^4] - 7^2$

$= E[1] + E[4x] + E[6x^2] + E[4x^3] + E[x^4] - 7^2$

$= 1 + 4E[x] + 6E[x^2] + 4E[x^3] + E[x^4] - 7^2$ [Linearity of expectation]

$= 1 + 4(1) + 6 \times (3 + 1^2) + 4E[x^3] + E[x^4] - 7^2$

$= 29 + 4E[x^3] + E[x^4] - 7^2$ ——— (1)

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at, $t=0$, nth moment of a normal distribution with mean μ and variance σ^2 is given by

$$\mathbb{E}[x^n] = \mu^n + n(n-1)\sigma^2 \frac{\mu^{n-2}}{2}$$

So, $\mathbb{E}[x^3] = 1^3 + 3(3-1)(3)\left(\frac{1^{3-2}}{2}\right)$

$= 1+9$

$= 10$

$\mathbb{E}[x^4] = 1^4 + 4(4-1)(3)\left(\frac{1^{4-2}}{2}\right)$

$= 1+18$

$= 19$

Plugging the values to (1), we get,

$\text{Var}[f(x)] = 29 + 4(10) + 19 - 7^2$

$= 39$.

So, $\mathbb{E}[f(x)] = 7$,

$\text{var}[f(x)] = 39$.

Problem 2 (30pts) Let $X \sim \mathcal{N}(1,3)$ and $f(x) = 1 + 2x + x^2$.

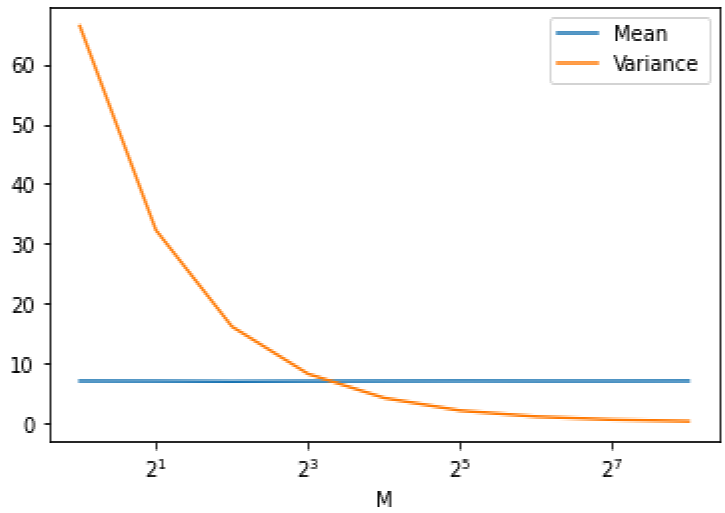
- Implement the Monte-Carlo method to approximate the expected value of f, i.e.

$$\mathbb{E}[f(X)] \approx f_M := \frac{1}{M}\sum_{i=1}^{M} f(x_i), \quad x_i \sim \mathcal{N}(1,3)$$

In [19]:
```python
import numpy as np

def monte_carlo_approximation(M):
    # Define the function f(x)
    def f(x):
        return 1 + 2*x + x**2

    # Generate M samples from the normal distribution with mean 1 and variance 3
    samples = np.random.normal(loc=1, scale=np.sqrt(3), size=M)

    # Evaluate f(x) for each sample
    evaluations = f(samples)

    # Take the average of all evaluations to approximate the expected value of f(X)
    expected_value_approximation = np.mean(evaluations)

    return expected_value_approximation
```

- Let $M = 1, 2, 4, \cdots, 256$. For each M, do $N = 10000$ simulations to approximate the expectation using f_M. For each M, calculate the mean and the variance of f_M over N rounds. Visualize your result.

In [20]:
```
def monte_carlo_simulation(N):
    # Define the function f(x)
    def f(x):
        return 1 + 2*x + x**2

    # Initialize lists to store the results
    M_values = []
    means = []
    variances = []

    # Iterate over values of M
    for M in [2**i for i in range(9)]:
        # Initialize a list to store the results of N simulations
        fM_values = []

        # Perform N simulations
        for _ in range(N):
            # Generate M samples from the normal distribution with mean 1 and variance 3
            samples = np.random.normal(loc=1, scale=np.sqrt(3), size=M)

            # Evaluate f(x) for each sample
            evaluations = f(samples)

            # Take the average of all evaluations to approximate the expected value of f
            expected_value_approximation = np.mean(evaluations)

            # Store the result of this simulation
            fM_values.append(expected_value_approximation)

        # Calculate the mean and variance of fM over N rounds
        mean = np.mean(fM_values)
        variance = np.var(fM_values)

        # Store the results
        M_values.append(M)
        means.append(mean)
        variances.append(variance)

    # Visualize the results
    plt.plot(M_values, means, label="Mean")
    plt.plot(M_values, variances, label="Variance")
    plt.xscale("log", base=2)
    plt.xlabel("M")
    plt.legend()
    plt.show()

# Run the simulation with N=10000
monte_carlo_simulation(10000)
```



Mean
Variance
60
50
40
30
20
10
0
2^1
2^3
2^5
2^7
M