Exercise: 3

Problem: 1

(i)

Given $\mathbb{E}[X] = m$, and $\mathbb{E}[(X-m)^2] = Q$ for random variable X,

$$\mathbb{E}[W] = 0, \text{ and } Var(W) = R$$

and $\mathbb{E}[XW] = 0$, $Y = HX + W$

and affine estimator $\hat{X} = KY + b$

To find optimal b in terms of m, H and K, we will normalized the term $\mathbb{E}[(\hat{X} - X)^2]$.

$$\mathbb{E}[(\hat{X}-X)^2] = \mathbb{E}[((KY+b)-X)^2]$$ [Replace with the value of $\hat{X}$]

$$= \mathbb{E}[K^2Y^2 + 2KY(b-X) + (b-X)^2]$$

$$= \mathbb{E}[K^2Y^2 + 2KYb - 2KYX + b^2 - 2bX + X^2]$$

$$= \mathbb{E}[K^2(HX+W)^2 + 2Kb(HX+W) - 2K(HX+W)X + b^2 - 2bX + X^2]$$ [Replace with the value of Y]

$$= \mathbb{E}[K^2H^2X^2 + 2HK^2XW + K^2W^2 + 2KbHX + 2KbW - 2KHX^2 - 2KWX + b^2 - 2bX + X^2]$$

$$= K^2H^2\mathbb{E}[X^2] + 2HK^2\mathbb{E}[XW] + K^2\mathbb{E}[W^2] + 2KbH\mathbb{E}[X] + 2Kb\mathbb{E}[W] - 2KH\mathbb{E}[X^2] - 2K\mathbb{E}[XW] + b^2 - 2b\mathbb{E}[X] + \mathbb{E}[X^2]$$

$$\mathbb{E}[(\hat{x}-x)^2] = K^2H^2(Q+m^2) + K^2R + 2KbHm - 2KH(Q+m^2) + b^2 - 2bm + Q + m^2 \; ; \; \left[\because \mathbb{E}[X]=m, \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2 \;\; \therefore \mathbb{E}[X^2] = Q+m^2 \text{ and, } \mathbb{E}[W]=0, \mathbb{E}[W^2]=R \right]$$

Now,

we take the derivative of $\mathbb{E}[(\hat{x}-x)^2]$ with respect to b. and set the derivative to 0.

$$\frac{\partial}{\partial b}\left(\mathbb{E}[(\hat{x}-x)^2]\right) = 2KHm + 2b - 2m$$

$$\therefore \quad 2KHm + 2b - 2m = 0$$

$$b = m - KHm$$

This the optimal b in terms of m, H and K.

Problem : 1

(2) To verify $\hat{x}$ is unbiased regardles of K for the optimal b, we need to show that, $\cancel{\mathbb{E}[\hat{x}] - x}$ $\mathbb{E}[\hat{x}-x] = 0$.

$$\mathbb{E}[\hat{x}] = \mathbb{E}[KY+b] \quad [\text{Replace for } \hat{x}]$$

$$= K\mathbb{E}[Y] + b$$

$$= K\,\mathbb{E}[HX+W] + b$$

$$= KH\,\mathbb{E}[X] + K\,\mathbb{E}[W] + b$$

$$= KHm + b$$

$$= KHm + m - KHm \quad [\text{Replace the value of } b \text{ from problem 1}]$$

$$= m$$

Now ; $\mathbb{E}[\hat{x} - x]$

$= \mathbb{E}[\hat{x}] - \mathbb{E}[x]$

$= m - m$

$= 0$

which means that $\hat{x}$ is unbiased regardless of the K.

Problem : 1

(3)

Problem: 1

(3). To obtain optimal gain K, we need to minimize the mean squared error $E[(\hat{x}-x)^2]$ with respect to K.

by expanding $E[(\hat{x}-x)^2]$ we get,

$E[(KY+b-x)^2]$.

by taking the derivative with respect to K, we get

$$\frac{\partial}{\partial K}(E[(\hat{x}-x)^2] = 2E[(KY+b-x)Y]$$

Setting this equal to zero and solving for K, we get,

$$K = \frac{E[XY]-bE[Y]}{E[Y^2]}$$

$$K = \frac{E[X(HX+W)]-bE[HX+W]}{E[(HX+W)^2]}$$

$$K = \frac{HE[X^2]+E[XW]-b(HE[X]+E[W])}{(H^2E[X^2]+2HE[XW]\}+E[W^2])}$$

$$K = \frac{H(Q+m^2)-bmH}{H^2(Q+m^2)+R}$$

$$K = \frac{HQ+Hm^2-bmH}{H^2Q+H^2m^2+R}$$

$$K = \frac{(Q+m^2)-bm}{H(Q+m^2)+R}$$

Problem : 2

Here, given that $z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\bar{z} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$

and $\Sigma = \begin{bmatrix} \delta_{11}^2 & \delta_{12}^2 \\ \delta_{21}^2 & \delta_{22}^2 \end{bmatrix}$

$$|\Sigma| = \delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \delta_{21}^2$$

$$\Sigma^{-1} = \frac{1}{\delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \delta_{21}^2} \begin{bmatrix} \delta_{22}^2 & -\delta_{12}^2 \\ -\delta_{21}^2 & \delta_{11}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \delta_c^{-2} & \delta_c^{-2} \frac{\delta_{12}^2}{\delta_{22}^2} \\ \delta_c^{-2} \frac{\delta_{12}^2}{\delta_{22}^2} & \delta_c^{-2} \frac{\delta_{11}^2}{\delta_{22}^2} \end{bmatrix}$$

where, $\delta_c^2 = \delta_{11}^2 - \delta_{12}^2 \delta_{22}^{-2} \delta_{21}^2$

We need to show that,

$$n(z : \bar{z}, \Sigma) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(z - \bar{z})^T \Sigma^{-1} (z - \bar{z})\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\delta_c} \exp\left(-\frac{1}{2\delta_c^2}(x_1 - \bar{x}_c)^2\right) \frac{1}{\sqrt{2\pi}\,\delta_{22}} \exp\left(-\frac{1}{2\delta_{22}^2}(x_2 - \bar{x}_2)^2\right)$$

To solve the non-exponential part we can write,

$$\frac{1}{2\pi |\Sigma|^{1/2}} = \frac{1}{\sqrt{2\pi}\,\delta_c} \cdot \frac{1}{\sqrt{2\pi}\,\delta_{22}}$$

Also, $\frac{1}{2\pi |\Sigma|^{1/2}} = \frac{1}{2\pi (\delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \cdot \delta_{21}^2)^{1/2}}$

where $|\Sigma| = \delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \delta_{21}^2$

We know that,

$$\delta_c^2 = \delta_{11}^2 - \frac{\delta_{12}^2 \delta_{21}^2}{\delta_{22}^2}$$

$$\delta_c^2 = \frac{\delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \delta_{21}^2}{\delta_{22}^2}$$

$$\Rightarrow \delta_c^2 \delta_{22}^2 = \delta_{11}^2 \delta_{22}^2 - \delta_{12}^2 \delta_{21}^2$$

Then, $\dfrac{1}{2\pi |\Sigma|^{1/2}} = \dfrac{1}{2\pi (\delta_c^2 \delta_{22}^2)^{1/2}}$

$$= \frac{1}{\sqrt{2\pi}\,\delta_c} \cdot \frac{1}{\sqrt{2\pi}\,\delta_{22}}$$

For the solution of exponential part we need to show that,

$$\exp\left(-\frac{1}{2}(z-\bar{z})^T \Sigma^{-1} (z-\bar{z})\right) = \exp\left(-\frac{1}{2\delta_c^2}(x_1-\bar{x}_c)^2 - \frac{1}{2\delta_{22}^2}(x_2-\bar{x}_2)^2\right)$$

From L.H.S,

$$(z-\bar{z})^T \Sigma^{-1} (z-\bar{z}) = [x_1-\bar{x}_1, \; x_2-\bar{x}_2] \begin{bmatrix} \delta_c^{-2} & \delta_c^{-2}\dfrac{\delta_{12}^2}{\delta_{22}^2} \\ \delta_c^{-2}\dfrac{\delta_{12}^2}{\delta_{22}^2} & \delta_c^{-2}\dfrac{\delta_{11}^2}{\delta_{22}^2} \end{bmatrix} \begin{bmatrix} x_1-\bar{x} \\ x_2-\bar{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-\bar{x}_1)\delta_c^{-2} + (x_2-\bar{x}_2)\delta_c^{-2}\dfrac{\delta_{12}^2}{\delta_{22}^2} \\ (x_1-\bar{x}_1)\delta_c^{-2}\dfrac{\delta_{12}^2}{\delta_{22}^2} + (x_2-\bar{x})\delta_c^{-2}\dfrac{\delta_{11}^2}{\delta_{22}^2} \end{bmatrix} \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{bmatrix}$$

$$= (x_1-\bar{x}_1)^2 \delta_c^{-2} + (x_2-\bar{x}_2)\delta_c^{-2}\frac{\delta_{12}^2}{\delta_{22}^2}(x_1-\bar{x}_1) + (x_1-\bar{x}_1)\delta_c^{-2}\frac{\delta_{12}^2}{\delta_{22}^2}(x_2-\bar{x}_2) + (x_2-\bar{x}_2)^2 \delta_c^{-2}\frac{\delta_{11}^2}{\delta_{22}^2}$$

$$= \delta_c^{-2}\Big((x_1-\bar{x}_1)^2 + 2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\frac{\delta_{12}^2}{\delta_{22}^2} + (x_2-\bar{x})^2 \frac{\delta_{11}^2}{\delta_{22}^2}$$

Now from R.H.S we get,

$$\frac{(x_1-\bar{x}_c)^2}{\delta_c^2}+\frac{(x_2-\bar{x}_2)^2}{\delta_{22}^2}=\frac{(x_1-\bar{x}_c)^2\delta_{22}^2+(x_2-\bar{x}_2)^2\delta_c^2}{\delta_c^2\delta_{22}^2}$$

$$=\frac{1}{\delta_c^2}\left((x_1-\bar{x}_c)^2+(x_2-\bar{x}_2)^2\frac{\delta_c^2}{\delta_{22}^2}\right)$$

$$=\delta_c^{-2}\left((x_1-\bar{x}_c)^2+(x_2-\bar{x}_2)^2\left(\frac{\delta_{11}^2}{\delta_{22}^2}-\frac{\delta_{12}^2\delta_{21}^2}{\delta_{22}^2\delta_{22}}\right)\right)$$

where, $\delta_c^2=\delta_{11}^2-\frac{\delta_{12}^2\delta_{21}^2}{\delta_{22}^2}$

$$=\delta_c^{-2}\left((x_1-\bar{x}_1-\delta_{12}^2\delta_{22}^{-2}(x_2-\bar{x}_2))^2+(x_2-\bar{x}_2)^2\frac{\delta_{11}^2}{\delta_{22}^2}-(x_2-\bar{x}_2)^2\frac{\delta_{12}^4}{\delta_{22}^4}\right.$$

[since $\delta_{12}=\delta_{21}$ and $\bar{x}_c=\bar{x}_1+\delta_{12}^2\delta_{22}^{-2}(x_2-\bar{x}_2)$]

$$=\delta_c^{-2}\left((x_1-\bar{x}_1)^2+2(x_1-\bar{x}_1)\frac{\delta_{12}^2}{\delta_{22}^2}(x_2-\bar{x}_2)+(x_2-\bar{x}_2)^2\frac{\delta_{12}^4}{\delta_{22}^4}+\right.$$

$$\left.(x_2-\bar{x}_2)^2\frac{\delta_{11}^2}{\delta_{22}^2}-(x_2-\bar{x}_2)^2\frac{\delta_{12}^4}{\delta_{22}^4}\right.$$

$$=\delta_c^{-2}\left((x_1-\bar{x}_1)^2+2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\frac{\delta_{12}^2}{\delta_{22}^2}+(x_2-\bar{x}_2)^2\frac{\delta_{11}^2}{\delta_{22}^2}\right)$$

which is equal to the L.H.S.

putting both exponential and non-exponential we get,

$$\frac{1}{2\pi|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(z-\bar{z})^T\Sigma^{-1}(z-\bar{z})\right)$$

$$=\frac{1}{\sqrt{2\pi}\,\delta_c}\exp\left(-\frac{1}{2\delta_c^2}(x_1-\bar{x}_c)^2\right)\frac{1}{\sqrt{2\pi}\,\delta_{22}}\exp\left(-\frac{1}{2\delta_{22}^2}(x_2-\bar{x}_2)^2\right)$$

The conditional probability distribution $\pi_{x_1|x_2}(x_1|x_2)$ represents the distribution of variable x_1 given the value of x_2. Using the formula obtained in the previous step, we have

$$\pi_{x_1|x_2}(x_1|x_2) = 1/(\sqrt{(2\pi\delta_c)}) \cdot \exp(-(1/(2\delta_c^2))(x_1-\bar{x}_c)^2)$$

substituting the values of $\bar{x}_c$ and δ_c, we get

$$\pi_{x_1|x_2}(x_1|x_2) = 1/(\sqrt{(2\pi(\delta_{11}/\sqrt{2}))}) \cdot \exp(-(1/(2(\delta_{11}/\sqrt{2}))(x_1-\bar{x}_1)^2)$$

furthermore,

$$\pi_{x_1|x_2}(x_1|x_2) = 1/(\sqrt{(\pi\delta_{11})}) \cdot \exp(-(1/2\delta_{11}))(x_1-\bar{x}_1)^2)$$

Therefore the conditional probability distribution $\pi_{x_1|x_2}(x_1|x_2)$ is a one dimentional Gaussian PDF. ~~with mean~~

Problem: 3

① Let, $Z = (X, Y)$, the value for x and Y we get from problem one.

Now,

The mean of Z is given by the vector

$$\mathbb{E}[Z] = (\mathbb{E}[X], \mathbb{E}[Y])$$

$$= (m, \mathbb{E}[HX + W]) \quad [\text{replace by the value of } Y.]$$

$$= (m, H\,\mathbb{E}[X] + \mathbb{E}[W])$$

$$= (m, Hm) \quad [\mathbb{E}[W] = 0, \text{ from problem one}]$$

And, The covariance matrix of Z is given by

$$Cov(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^T]$$

Expanding this expression and simplyfying, we get,

$$Cov(Z) = \mathbb{E}[[(X-m), (Y-Hm)][(X-m), (Y-Hm)]^T]$$

$$= \begin{bmatrix} \mathbb{E}[(X-m)^2] & \mathbb{E}[(X-m)(Y-Hm)] \\ \mathbb{E}[(Y-Hm)(X-m)] & \mathbb{E}[(Y-Hm)^2] \end{bmatrix}$$

$$= \begin{bmatrix} Q & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

Substitution the expressions for X and Y and simplifying further, we get

$$Cov(Z) = \begin{bmatrix} Q & H\,Cov(X,X) \\ Cov(X,X)H^T & H\,Cov(X,X)H^T + R \end{bmatrix}$$

$$= \begin{bmatrix} Q & HQ \\ QH^T & HQH^T + R \end{bmatrix} \quad Z$$

$$E[X_1^2 X_2 | X_2 = a] = aE[X_1^2 | X_2 = a]$$ [We can move X_2 outside of the expectation as $X_2 = a$]

$$= a \int_{-\alpha}^{\alpha} x_1^2 \pi_{X_1}(x_1 | X_2 = a)\, dx_1.$$

$$= a \int_{-\alpha}^{\alpha} \frac{x_1^2 \pi_{X_1 X_2}(x_1, X_2 = a)}{\pi_{X_2}(X_2 = a)}\, dx_1 = a \frac{\int_{-\alpha}^{\alpha} x_1^2 \pi_{X_1 X_2}(x_1, X_2 = a)\, dx_1}{\int_{-\alpha}^{\alpha} \pi_{X_1 X_2}(x_1, X_2 = a)\, dx_1}$$

$$= a \frac{\frac{1}{Z} \int_{-\alpha}^{\alpha} x_1^2 e^{-x_1^2 - a^2 - a^2 x_1^2}\, dx_1}{\frac{1}{Z} \int_{-\alpha}^{\alpha} e^{-x_1^2 - a^2 - a^2 x_1^2}\, dx_1} = a \frac{e^{-a^2} \int_{-\alpha}^{\alpha} x_1^2 e^{-x_1^2(1+a^2)}\, dx_1}{e^{-a^2} \int_{-\alpha}^{\alpha} e^{-x_1^2(1+a^2)}\, dx_1}$$

Taking $n = 1 + a^2$ and $I = \int_{-\alpha}^{\alpha} e^{-n x_1^2}\, dx_1$ — (I)

We can also write by changing X_1 to Y_1,

$$I = \int_{-\alpha}^{\alpha} e^{-n y_1^2}\, dY_1 \quad \text{— (II)}$$

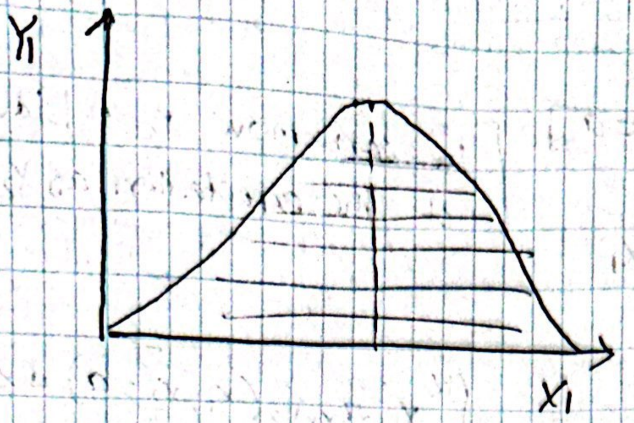
From Multiplication we get,

$$I^2 = \int_{-\alpha}^{\alpha} e^{-n x_1^2}\, dx_1 \int_{-\alpha}^{\alpha} e^{-n Y_1^2}\, dY_1 = \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} e^{-n(x_1^2 + Y_1^2)}\, dx_1\, dY_1 \quad \text{— (III)}$$

We can transform this to polar co-ordinate

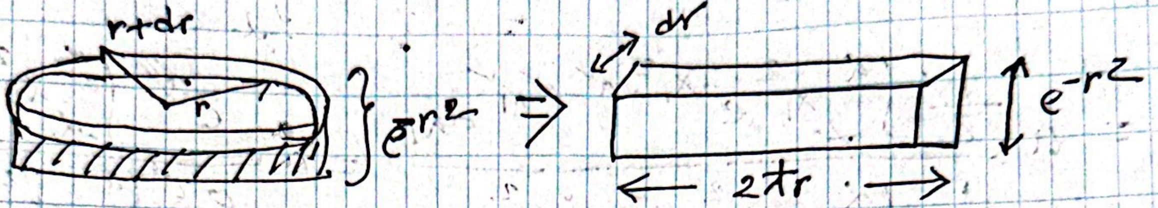
Hence, $r^2 = x_1^2 + Y_1^2$

This is the equation of circle with radious r.



In order to solve the equation (III) we need to find the area under this curve.

We can consider the area under this is made up with hollow cylinders:



· The rectangular cube has width $2\pi r$ and height $= e^{-r^2}$ and depth $= dr$.

So the volume is: $e^{-r^2} \cdot 2\pi r \cdot dr$

From (III) we can get,

$$I^2 = \int_0^{\infty} e^{-nr^2} \cdot 2\pi r \cdot dr = 2\pi \int_0^{\infty} e^{-nr^2} r \cdot dr$$

Taking $u = nr^2$ $\quad \frac{du}{dr} = 2nr \Rightarrow \frac{du}{2n} = r\,dr$

$$\therefore I^2 = 2\pi \int_0^{\infty} e^{-u} \frac{du}{2n} = \frac{\pi}{n} \int_0^{\infty} e^{-u} du = \left[-e^{-u}\right]_0^{\infty} \frac{\pi}{n}$$

$$= \frac{\pi}{n}$$

$$\therefore I = \sqrt{\frac{\pi}{n}}$$

Hence we get $\int_{-\infty}^{\infty} e^{-x_1^2(1+a^2)} dx_1 = \frac{\sqrt{\pi}}{\sqrt{1+a^2}}$

Again, By differentiating eq(1) with respect n we get

$$\int_{-\alpha}^{\alpha} \frac{\delta}{\delta n} e^{-n x_1^2} dx_1 = \frac{\delta}{\delta n}\sqrt{\frac{\pi}{n}}$$

$$\Rightarrow \int_{-\alpha}^{\alpha} -x_1^2 e^{-n x_1^2} dx_1 = -\frac{\sqrt{\pi}}{2n^{3/2}}$$

$$\Rightarrow \int_{-\alpha}^{\alpha} x_1^2 e^{-n x_1^2} dx_1 = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$$

$$\therefore \int_{-\alpha}^{\alpha} x_1^2 e^{-x_1^2(1+a^2)} dx_1 = \frac{1}{2(1+a^2)}\sqrt{\frac{\pi}{1+a^2}}$$

Finally we can write,

$$E[x_1^2 x_2 / x_2 = a] = a \frac{\frac{1}{2(1+a^2)}\cancel{\sqrt{\frac{\pi}{1+a^2}}}}{\cancel{\sqrt{\frac{\pi}{1+a^2}}}} = \frac{a}{2(1+a^2)}$$