sat sun mon tue wed thu fri Date: / /

Exercise 4

problem 1a:

① X and Z are uncorrelated if their covariance is equal to zero.
The covariance between X and Z is given by,

$$Cov(X, Z) = E[XZ] - E[X]E[Z].$$

Since,

$$Z = aX^2 + bX + c,$$

we have $E[Z] = aE[X^2] + bE[X] + c$.
Substituting this to the covariance,

$$Cov(X, Z) = E[aX^3 + bX^2 + cX] - E[X](aE[X^2] + bE[X] + c)$$

Since, f is an even function,

we have, $E[X] = 0$
and, $E[X^3] = 0$.

Therefore, $Cov(X, Z) = bE[X^2]$.

So, X and Z are uncorrelated if and only if $b = 0$.

a and c can take any values from real number.

HEART'S

problem Ia.

(2) Independance between two random variables means that the joint probability distributions of the two variables is same as the product of their marginal probability distributions.

So, $f_{XZ}(x,z) = f_X(x) f_Z(z)$ for all x and z.

Here,

$$Z = aX^2 + bX + c,$$

$$f_{XZ}(x,z) = f_X(x) f_Z(z)$$
$$= f_X(x) f_Z(ax^2 + bx + c)$$

For X and Z to be independant, we need the right side of the equation to factorize into a product of seperate functions of x and z. This factorization is only possible if $a = 0$.

So, X and Z to be independant, $a = 0$,
b and c can take any real values.

Problem 1b.

① This statement is False. Two random variables can be uncorrelated but still dependent:

Let X be a random variable uniformly distributed on the interval $[-1, 1]$ and $Z = X^2$.

Here, $\mathbb{E}[X] = 0$.

$$\mathbb{E}[XZ] = \mathbb{E}[X^3] = 0.$$

So $Cov(X, Z) = 0$ and X, Z are uncorrelated. But, Z depends on the value of X, which means they are not independent.

② This statement is True.

Here $Cov(X, Z) = \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z]$

if X and Z are independent,

$$\mathbb{E}[XZ] = \mathbb{E}[X]\mathbb{E}[Z].$$

So, $Cov(X, Z) = 0$.

sat sun mon tue wed thu fri Date : / /

(3) This statement is True.

Here, $Var(X+Z) = \mathbb{E}[(X+Z)^2] - (\mathbb{E}[X+Z])^2$

$= \mathbb{E}[X^2+2XZ+Z^2] - (\mathbb{E}[X]+\mathbb{E}[Z])^2$.

$= \mathbb{E}[X^2]+2\mathbb{E}[XZ]+\mathbb{E}[Z^2] -$

$(\mathbb{E}[X])^2 - 2\mathbb{E}[X]\mathbb{E}[Z] - (\mathbb{E}[Z])^2$

$= (\mathbb{E}[X^2]-(\mathbb{E}[X])^2) + (\mathbb{E}[Z^2]-(\mathbb{E}[Z])^2)$

$+ 2(\mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z])$

$= var(X) + var(Z) + 2\,cov(X,Z)$

if, X and Z are independent, $cov(X,Z) = 0$.

So, $var(X+Z) = var(X) + var(Z)$.

Therefore, if the variance of X and Z are finite, and $var(X+Z) = var(X) + var(Z)$ then,

X and Z are uncorrelated.

sat sun mon tue wed thu fri Date: / /

Problem 2

(1) To characterize all possible couplings between X and Y, we need to define a joint probability distribution, which is as follows:

$p(X=0, Y=0) = (1/2 \times 1/3) = 1/6$

$p(X=0, Y=1) = (1/2 \times 2/3) = 1/3$

$P(X=1, Y=0) = (1/2 \times 1/3) = 1/6$

$P(X=1, Y=1) = (1/2 \times 2/3) = 1/3$.

(2) The correlation between X and Y are defined as:

$$\text{Corr}(X, Y) = \text{cov}(X, Y) / (\sigma(X) \times \sigma(Y))$$

Where, $\text{Cov}(X, Y)$ is the covariance of X and Y, $\sigma(X)$ and $\sigma(Y)$ are the standard deviations for X and Y respectively.

To maximize the correlation, we need to find a coupling which maximizes the numerator of the correlation formula.

(P.T.O.)

If we use the joint probabilities defined in part 1, we can see that, when $X = Y$, i.e.

$$P(X=0, Y=0)$$

$$P(X=1, Y=1),$$

are the couplings that maximizes the numerator.

for $P(X=0, Y=0)$

$$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$$

$$= (0-\tfrac{1}{2})(0-\tfrac{2}{3}) \times \tfrac{1}{6}$$

$$= \tfrac{1}{36}$$

for, $P(X=1, Y=1)$

$$Cov(X,Y) = (1-\tfrac{1}{2})(1-\tfrac{2}{3}) \times \tfrac{1}{3}$$

$$= \tfrac{1}{18}$$

These results in a positive correlation.

Conversely, for $X \neq Y$, the cov results in negative which implies a negative correlation (minimize), which are

$$P(X=0, Y=1)$$

$$P(X=1, Y=0).$$

Explanation:

The maximization of correlation occurs when the two variables have a deterministic relationship, where the value of one variable completely determines the value of the other.

In this case, when $X=0, Y=0$ and
$X=1, Y=1,$
resulting in a positive correlation.

Similarly, minimum correlation occurs when there is a deterministic relationship of the opposite sign: when $X=0, Y=1,$
$X=1, Y=0,$
resulting in a negative correlation.

These couplings maximize and minimize the correlation because they have the deterministic relationship between the variables.

(3) To make the two variables uncorrelated, we need the covariance resulting in zero.

As we have seen, for, $(X=0, Y=0)$ and $(X=1, Y=1)$, $\text{cov}(X,Y) \neq 0$.

for, $(X=0, Y=1)$, $\text{cov}(X,Y) = (0-\frac{1}{2})(1-\frac{2}{3}) \cdot \frac{1}{3}$
$= -\frac{1}{18}$

(P.T.O)

for, $(X=1, Y=0)$, $\text{cov}(X,Y) = (1-\frac{1}{2})(0-\frac{1}{3}) \cdot \frac{1}{6}$

$= -\frac{1}{36}$.

So, we can say, there is no couplings that makes the random variables X and Y uncorrelated.

(4)

(i) All the possible couplings:

$P(X=0, Z=0) = \frac{1}{2} \times 1 = \frac{1}{2}$

$P(X=0, Z=1) = \frac{1}{2} \times 0 = 0$

$P(X=1, Z=0) = \frac{1}{2} \times 1 = \frac{1}{2}$

$P(X=1, Z=1) = \frac{1}{2} \times 0 = 0$.

(ii) $\text{cov}(X,Z) = E[(X-E[X])(Z-E[Z])]$

for, $P(X=0, Z=0)$,

$\text{Cov}(X,Z) = (0-\frac{1}{2}) \times (0-1) \times \frac{1}{2} = \frac{1}{4}$

for, $P(X=0, Z=1)$

$\text{Cov}(X,Z) = 0$

for, $P(X=1, Z=0)$

$\text{Cov}(X,Z) = (1-\frac{1}{2}) \cdot (0-1) \times \frac{1}{2} = -\frac{1}{4}$

for, $P(X=1, Z=1)$, $\text{Cov}(X,Z) = 0$.

Here, X and Z are independent.

As we can see,

$P(X=0, Z=0) = \frac{1}{2} = P(X=0) \times P(Z=0) = \frac{1}{2} \times 1 = \frac{1}{2}$

$P(X=0, Z=1) = 0 = P(X=0) \times P(Z=1) = \frac{1}{2} \times 0 = 0$

$P(X=1, Z=0) = \frac{1}{2} = P(X=1) \times P(Z=0) = \frac{1}{2} \times 1 = \frac{1}{2}.$

$P(X=1, Z=1) = 0 = P(X=1) \times P(Z=1) = \frac{1}{2} \times 0 = 0.$

if X and Z are independent then they are uncorrelated (proved in problem 16).