Exercise 5 – *Bayesian inference and Data assimilation*

Due by: Tuesday, 23 May 2023, 23:59 (CEST)

Problem 1 (40pts) [Problem 2.7 of the lecture notes] Consider the two sets

$$\mathcal{X}_1 := \{a_1 = 1, a_2 = 2, a_3 = 3\} \quad \text{and} \quad \mathcal{X}_2 := \{b_1 = 1.5, b_2 = 2, b_3 = -1\}$$

with uniform probability mass: $\mathbb{P}(a_i) = \mathbb{P}(b_i) = 1/3$ for $i = 1, 2, 3$. A coupling is defined by a matrix $T \in \Re^{3\times 3}$ with $t_{ij} \geq 0$ and

$$\sum_{i=1}^{3} t_{ij} = \sum_{j=1}^{3} t_{ij} = 1/3$$

Find the coupling that minimizes

$$\mathsf{J}(T) = \sum_{i,j=1}^{3} t_{ij} |b_i - a_j|^2$$

What do you notice about the sparsity structure of the optimal coupling matrix T^*?

Problem 2a (30pts) Let $X \sim U[0, 1]$ be a uniform random variable.

1. Find a quadrature rule for $M = 1$ of order $p = 2$.
2. Find all quadrature rules for $M = 2$ that have order $p = 3$.

Problem 2b (30pts) [Problem 3.3 of the lecture notes] Determine the ANOVA decomposition for

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

and compute the associated variances σ_1^2, σ_2 and σ_{12}^2. The underlying measure is uniform measure on $[0, 1]^2$. (See also pp.71–72 for explanation about ANOVA decomposition.)