Exercise 9 – *Bayesian inference and Data assimilation*

Due by: Tuesday, 20 June 2023, 23:59 (CEST)

Problem 1 Consider the Markov chain that takes values on $S = \{1, 2, 3, 4\}$. The transition probability from state j to state i is given by

$$[P_{ij}] = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

1. Let $X_0 = 1$. Compute the probability mass function of X_2.
2. Does the probability distribution of X_n converges as n goes to infinity? If so, find the limit. If not, explain why.

Problem 2 Consider the Markov chain that takes values on $S = \{1, 2, 3, 4\}$. The transition probability from state j to state i is given by

$$[P_{ij}] = \begin{pmatrix} 1/2 & 2/3 & 0 & 0 \\ 1/2 & 1/3 & 0 & 0 \\ 0 & 0 & 1/4 & 4/5 \\ 0 & 0 & 3/4 & 1/5 \end{pmatrix}$$

1. Suppose the Markov chain is initialized at $X_0 = 1$. What is the distribution of X_t may tend towards?
2. Suppose the Markov chain is initialized at $X_0 = 3$. What is the distribution of X_t may tend towards?
3. Characterize all the invariant measure.

Problem 3 We solve for a toy example of Bayesian inference problem. Consider the hidden state variable X with prior $X \sim N(1, 1)$. The observable

$$Y = X^2 + W$$

where $W \sim N(0, 1)$.

1. Given x and y, find the value w such that the event $[X = x, Y = y]$ is equivalent to the event $[X = x, W = w]$.
2. Use the Bayes formula, find the formula for $\pi_{X|Y=y}(x)$, the conditional density of X at x given $Y = y$. You may omit the explicit formula for the normalization constant.
3. Plot the conditional PDF given $y = 2$ and find the Maximum a-posteriori (MAP) estimator of X given $Y = y$.