MIE THEORY AND RADIATIVE TRANSFER

ASSIGNMENT - 1

Name: Jayendra Praveen Kumar Chorapalli

Master's Program : 2^{nd} semester Master's student in ESPACE Program, TU MUNICH

Matr. Number : 03736602

Task : 1 : Find $I(\tau, \mu < 0, \varphi)$ with boundary condition

$$I(\tau_A, \mu < 0, \varphi) = 0$$

Answer :

when the first boundary condition $I(\tau = 0, \mu > 0, \varphi) = 0$ is applied, to the downwelling radiance we obtained :

$$I(\tau, \mu > 0, \varphi) = \frac{F\omega}{4\pi} (P) \left(\frac{\mu_0}{\mu - \mu_0}\right) \left[e^{-\tau/\mu_0} - e^{-\tau/\mu}\right]$$

↳ singularity at $\mu = \mu_0$

Now, we need to find the upwelling radiance (reflection) with the boundary condition $I(\tau = \tau_A, \mu < 0, \varphi)$:

we knew ~~for~~ upwelling radiance: $\tau = \tau_A$ & $\mu < 0$:

$$I(\tau, \mu, \varphi) = e^{-\tau/\mu} F \cdot \frac{\omega}{4\pi\mu} \int_a^{\tau} e^{\tau'\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right)} P(\mu_0, \varphi_0)\, d\tau' \quad -①$$

with $\tau = 0$
$(a = 0)$

Hence, the above equation ① will change to the following after considering μ : $\mu < 0$

$\mu = -|\mu|$

$\tau = \tau_A$

$a = \tau_A$

$I(\tau_A, \mu<0, \varphi) =$

$$e^{-\tau/-|\mu|} \cdot F\frac{\omega}{4\pi \cdot -|\mu|} \int_{\tau_A}^{\tau} e^{\tau'\left(\frac{1}{-|\mu|} - \frac{1}{\mu_0}\right)} P(\mu_0, \varphi_0)\, d\tau'$$

$$= -e^{\tau/|\mu|} \frac{F\omega}{4\pi|\mu|} \int_{\tau_A}^{\tau} e^{-\tau'\left(\frac{1}{|\mu|} + \frac{1}{\mu_0}\right)} P(\mu_0, \varphi_0)\, d\tau'$$

$$= -e^{\tau/|\mu|} \frac{F\omega}{4\pi|\mu|} P \left[-\frac{(|\mu|\mu_0)}{|\mu| + \mu_0} e^{-\tau'\left(\frac{1}{|\mu|} + \frac{1}{\mu_0}\right)} \right]_{\tau_A}^{\tau}$$

$$= e^{\tau/|\mu|} \cdot \frac{F\omega}{4\pi|\mu|} \cdot P \left[\left(\frac{|\mu|\mu_0}{|\mu|+\mu_0}\right) e^{-\tau\left(\frac{1}{|\mu|} + \frac{1}{\mu_0}\right)} - e^{-\tau_A\left(\frac{1}{|\mu|} + \frac{1}{\mu_0}\right)} \right]$$

$$= e^{\tau/|\mu|} \cdot \frac{F\omega \cdot P}{4\pi} \cdot \frac{\mu_0}{|\mu|+\mu_0} \left[e^{-\tau/|\mu|} \cdot e^{-\tau/\mu_0} - e^{-\tau_A/|\mu|} \cdot e^{-\tau_A/\mu_0} \right]$$

$$= \frac{F\omega \cdot P}{4\pi} \frac{\mu_0}{|\mu|+\mu_0} \left[\underbrace{e^{\tau/|\mu|} \cdot e^{-\tau/|\mu|}}_{=1} \cdot e^{-\tau/\mu_0} - e^{\tau/|\mu|} \cdot e^{-\tau_A/|\mu|} \cdot e^{-\tau_A/\mu_0} \right]$$

$$= \frac{F\omega P}{4\pi} \frac{\mu_0}{(|\mu|+\mu_0)} \left[e^{-\tau/\mu_0} - e^{\tau/|\mu|} \cdot e^{-\tau_A\left[\frac{1}{|\mu|} + \frac{1}{\mu_0}\right]} \right]$$

Hence, the upwelling radiance (reflected radiation) for $I(\tau, \mu, \varphi)$ with bandary condition $I(\tau=\tau_A, \mu<0, \varphi)$ is

$I(\tau=\tau_A, \mu<0, \varphi) =$

$$\frac{F\omega P}{4\pi} \frac{\mu_0}{|\mu|+\mu_0} \left[e^{-\tau/\mu_0} - e^{\tau/|\mu|} \cdot e^{-\tau_A\left[1/|\mu| + 1/\mu_0\right]} \right]$$

Task: 2: Say what happens when $\mu \to \mu_0$

Answer:

$$I(\tau, \mu>0, \phi) = \frac{F\omega}{4\pi} P \cdot \frac{\mu_0}{\mu - \mu_0} \left[e^{-\tau/\mu_0} - e^{-\tau/\mu} \right] \quad -(1)$$

↳ singularity at $\mu = \mu_0$.

The singularity when $\mu = \mu_0$ can be solved by considering taylor series to expand the exponents.

Taylor series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$

Taylor series expansion for $e^{-\tau/\mu_0}$:

$$1 - \frac{\tau}{\mu_0} + \frac{\tau^2}{2!\mu_0^2} - \frac{\tau^3}{3!\mu_0^3} + \dots \quad -(2)$$

Taylor series expansion for $e^{-\tau/\mu}$:

$$1 - \tau/\mu + \frac{\tau^2}{2!\mu^2} - \frac{\tau^3}{3!\mu^3} + \dots \quad -(3)$$

Hence, from (2) & (3)

$$e^{-\tau/\mu_0} - e^{-\tau/\mu} = 1 - \frac{\tau}{\mu_0} + \frac{\tau^2}{2!\mu_0^2} - \frac{\tau^3}{3!\mu_0^3} + \dots$$
$$- \left(1 - \frac{\tau}{\mu} + \frac{\tau^2}{2!\mu^2} - \frac{\tau^3}{3!\mu^3} + \dots \right)$$

$$= \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \left(\frac{1}{\mu_0^{k-1}} + \frac{1}{\mu_0^{k-2} \cdot \mu} + \frac{1}{\mu_0^{k-3} \cdot \mu^2} \cdots \frac{1}{\mu_0 \mu^{k-2}} + \frac{1}{\mu_0^{k-1}} \right)$$

where k goes from 1 to ∞.

$-(4)$

$$= \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \sum_{k=1}^{\infty} \left(\frac{\tau^k}{k!} \sum_{i=0}^{k-1} \frac{1}{\mu^{k-1-i} \mu_0^i} \right) \quad -(5)$$

substituting value ⑤ in ①

$$= \frac{F\omega}{4\pi} \frac{P\mu_0}{\mu_1 - \mu_0} \left[\left(\frac{1}{\mu_0} - \frac{1}{\mu_1} \right) \sum_{K=1}^{\infty} \left(\frac{\tau^K}{K!} \sum_{i=0}^{K-1} \frac{1}{\mu_1^{K-1-i} \mu_0^i} \right) \right.$$

$$= \frac{F\omega}{4\pi} \cdot \frac{P\cancel{\mu_0}}{(\cancel{\mu_1 - \mu_0})} \left(\frac{\cancel{\mu_1 - \mu_0}}{\mu_1 \cancel{\mu_0}} \right) \sum_{K=1}^{\infty} \left(\frac{\tau^K}{K!} \sum_{i=0}^{K-1} \frac{1}{\mu_1^{K-1-i} \mu_0^i} \right)$$

$$= \frac{F\omega P}{4\pi \mu_1} \sum_{K=1}^{\infty} \left(\frac{\tau^K}{K!} \sum_{i=0}^{K-1} \frac{1}{\mu_1^{K-1-i} \mu_0^i} \right)$$

now, when $\mu_1 \to \mu_0$, the above equation will be:

$$= \frac{F\omega P}{4\pi \mu_0} \sum_{K=1}^{\infty} \left(\frac{\tau^K}{K!} \sum_{i=0}^{K-1} \frac{1}{\mu_0^{K-1} \cdot \underbrace{\mu_0^{-i} \cdot \mu_0^i}_{=1}} \right)$$

=> As the value after second summation is now independent of i, we can remove that summation.

$$= \frac{F\omega P}{4\pi \mu_0} \sum_{K=1}^{\infty} \left(\frac{\tau^K}{K!} \cdot \frac{1}{\mu_0^{K-1}} \right)$$

final answer.