ENPM 667 – Final Project

University of Maryland, College Park

Designing a LQR and a LQG controller for Crane with double pendulum

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I. Problem Statement

Consider a crane that moves along a one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m1 and m2, and the lengths of the cables are l1 and l2, respectively. The following figure depicts the crane and associated variables used throughout this project.

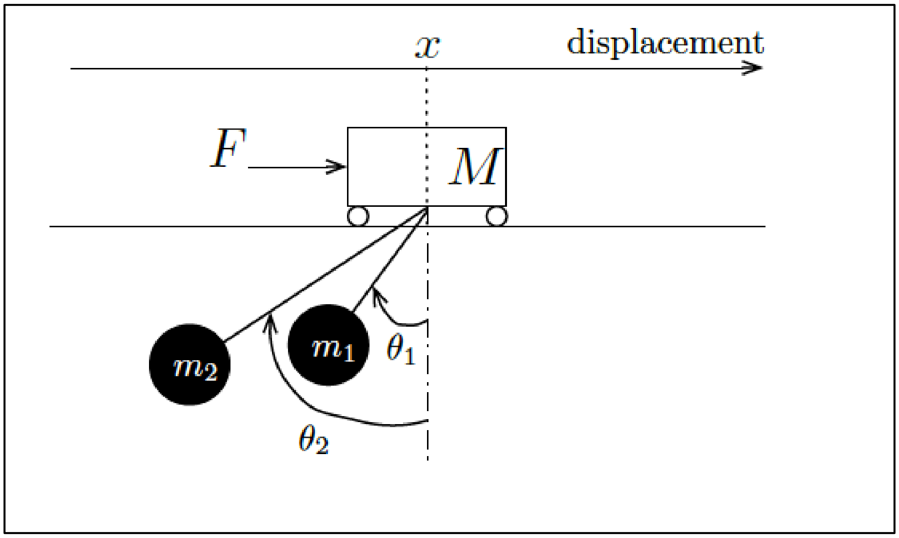


Figure 1: Crane along a one-dimensional track

II. Given Information

1. The mass of the cart M - $1000Kgs$
2. The masses of m_1, m_2 - $100Kgs$
3. The length of the cable l_1 - $20m$
4. The length of the cable $l_2 - 10m$
5. The cart moves along x direction with the applied force F
6. The cart is frictionless

III. First Component

a. Equations of motion and Non-linear state-space representation

The equations of motions follow the Euler-Lagrange method are given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F \;\; and \;\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Equation 1: Euler-Lagrange method for equations of motion

The Lagrange of the equation of the system is given by,

$$L = K - P$$

Equation 2: Lagrange of the equation

Here K is the Kinetic energy of the system and P is the Potential energy of the system.

The position of mass m1 and m2 as a function of θ_1 and θ_2 in the x-y plane can be given as,

$$\begin{gathered}\left(x_{m_1}, y_{m_1}\right) = \left(x - l_1 \sin(\theta_1), -l_1 \cos(\theta_1)\right) \\ \left(x_{m_2}, y_{m_2}\right) = \left(x - l_2 \sin(\theta_2), -l_2 \cos(\theta_2)\right)\end{gathered}$$

Equation 3: Position of masses m1 and m2

Differentiating the x and y components from Equation 3 would give the velocity components v_{m1} and v_{m2} of the two masses m_1 and m_2 as given in Equation 4. Since Kinetic energy of a system is given by $KE = \frac{1}{2}mv^2$, We know the velocity is given by $v^2 = \frac{dx}{dt}^2 + \frac{dy}{dt}^2$, therefore the velocities will be,

$$\begin{gathered}\dot{V_{m1}} = \frac{d}{dt}(x - l_1 \sin(\theta_1)) + \frac{d}{dt}(l_1 \cos(\theta_1)) = \dot{x} - l_1\dot{\theta}_1 \cos(\theta_1) + (-l_1\dot{\theta}_1 \sin(\theta_1)) \\ \dot{V_{m2}} = \frac{d}{dt}(x - l_2 \sin(\theta_2)) + \frac{d}{dt}(l_2 \cos(\theta_2)) = \dot{x} - l_2\dot{\theta}_2 \cos(\theta_2) + \left(-l_1\dot{\theta}_2 \sin(\theta_2)\right)\end{gathered}$$

Equation 4: Velocities of masses m1 and m2

The kinetic energy of the system can be calculated as shown in Equation 5 with M representing the mass of the cart and by using the velocity calculated in Equation 4.

$$\begin{gathered}K.E = \frac{1}{2}MV^2 + \frac{1}{2}m_1\left(V_{1x}^2 + V_{1y}^2\right) + \frac{1}{2}m_2\left(V_{2x}^2 + V_{2y}^2\right) \\ K.E = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_1 l_1 \dot{x}\dot{\theta}_1 \cos(\theta_1) - m_2 l_2 \dot{x}\dot{\theta}_2 \cos(\theta_2)\end{gathered}$$

Equation 5: Kinetic energy of the system

The potential energy of the masses m_1 and m_2 can be written as shown in Equation 5. The potential energy with respect to mass M will be zero as there is no change in height.

$$P.E = 0 - m_1 l_1 g\cos(\theta_1) - m_2 l_2 g\cos(\theta_2)$$

Equation 6: Potential energy of the system

Therefore, the Lagrange given in Equation 2 will become,

$$L = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_1 l_1\dot{x}\dot{\theta}_1\cos(\theta_1) - m_2 l_2\dot{x}\dot{\theta}_2\cos(\theta_2)$$
$$- \ (m_1 l_1 g\cos(\theta_1) + m_2 l_2 g\cos(\theta_2))$$

Equation 7: The Lagrange with kinetic and potential energies

The derivative of the Lagrangian with respect to the parameters $\dot{x}$, $\dot{\theta}_1$ & $\dot{\theta}_2$ are given in Equation 8.

$$\frac{\mathrm{d}}{dt}\left(\frac{\partial L}{\dot{x}}\right) - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{dt}\left(\frac{\partial L}{\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0$$
$$\frac{\mathrm{d}}{dt}\left(\frac{\partial L}{\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0$$

Equation 8: The Lagrangian derivatives

Differentiating Equation 7, with respect to x we get,

$$\frac{\partial L}{\partial x} = 0$$

Now differentiating Equation 7, with respect to $\dot{x}$ we get,

$$\frac{\partial L}{\partial\dot{x}} = \dot{x}(m_1 + m_2 + M) - m_1 l_1\dot{\theta}_1\cos(\theta_1) - m_2 l_2\dot{\theta}_2\cos(\theta_2)$$

Differentiate the above equation with respect to dt.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \ddot{x}(m_1 + m_2 + M) - m_1 l_1\left(\ddot{\theta}_1\cos(\theta_1) - \dot{\theta}_1^2\sin(\theta_1)\right) - m_2 l_2\left(\ddot{\theta}_2\cos(\theta_2) - \theta_2^2\sin(\theta_2)\right)$$

Equation 9: The Lagrangian derivatives for component x

Differentiating Equation 7, with respect to θ_1

$$\frac{\partial L}{\partial\theta_1} = m_1 l_1\dot{x}\dot{\theta}_1\sin(\theta_1) - m_1 g l_1\sin(\theta_1)$$

Now differentiating Equation 7, with respect to $\dot{\theta}_1$ we get,

$$\frac{\partial L}{\partial\dot{\theta}_1} = m_1 l_1^2\dot{\theta}_1 - m_1 l_1\dot{x}\cos(\theta_1)$$

Differentiate the above equation with respect to dt

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1\big(\ddot{x}\cos(\theta_1) - \dot{x}\dot{\theta}_1 \sin(\theta_1)\big)$$

Equation 10: The Lagrangian derivatives for component θ_1

Differentiating Equation 7, with respect to θ_2

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_2 \dot{x}\cos(\theta_2)$$

Now differentiating Equation 7, with respect to $\dot{\theta_2}$ we get,

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2 m_2 \dot{x}\dot{\theta}_2 \sin(\theta_2) - m_2 g l_2 \sin(\theta_2)$$

Differentiate the above equation with respect to dt

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2\big(\ddot{x}\cos(\theta_2) - \dot{x}\dot{\theta}_2 \sin(\theta_2)\big)$$

Equation 11: The Lagrangian derivatives for component θ_2

Substituting the components obtained in Equation 9, Equation 10, Equation 11 back in Equation 8 we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1\big(\ddot{x}\cos(\theta_1) - \dot{x}\dot{\theta}_1 \sin(\theta_1)\big) - m_1 l_1 m_1 \dot{x}\dot{\theta}_1 \sin(\theta_1) + m_1 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \ddot{x}\cos(\theta_1) + m_1 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2\big(\ddot{x}\cos(\theta_2) - \dot{x}\dot{\theta}_2 \sin(\theta_2)\big) - m_2 l_2 m_2 \dot{x}\dot{\theta}_2 \sin(\theta_2) + m_2 g l_2 \sin(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x}\cos(\theta_2) + m_2 g l_2 \sin(\theta_2)$$

Equation 12: The derived Lagrangian derivatives

From Equation 9 and Equation 12 we get,

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \left(\ddot{\theta_1}\cos(\theta_1) - \dot{\theta_1}^2 \sin(\theta_1)\right) - m_2 l_2 \left(\ddot{\theta_2}\cos(\theta_2) - \dot{\theta_2}^2 \sin(\theta_2)\right)$$
$$m_1 {l_1}^2 \ddot{\theta}_1 - m_1 l_1 \cos(\theta_1)\,\ddot{x} + m_1 g l_1 \sin(\theta_1) = 0$$
$$m_2 {l_2}^2 \theta_2 - m_2 l_2 \cos(\theta_2)\,\ddot{x} + m_2 g l_2 \sin(\theta_2) = 0$$

Equation 13: Solution of Lagrangian derivatives

Rewriting the above equation in-terms of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ we get,

$$\ddot{\theta}_1 = \frac{\ddot{x}\cos(\theta_1) - g\sin(\theta_1)}{l_1}$$
$$\ddot{\theta}_2 = \frac{\ddot{x}\cos(\theta_2) - g\sin(\theta_2)}{l_2}$$

Substituting $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in Equation 13 of F we can rewrite $\ddot{x}$ as in Equation 14

$$\ddot{x} = \frac{F - m_1\big(g\sin(\theta_1)\cos(\theta_1) + l_1\sin(\theta_1)\,\dot{\theta}_1^2\big) - m_2\big(g\sin(\theta_2)\cos(\theta_2) + l_2\sin(\theta_2)\,\dot{\theta}_2^2\big)}{\big(M + m_1(\sin^2(\theta_1)) + m_2(\sin^2(\theta_2))\big)}$$

Equation 14: Linear acceleration term

To get $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in terms of θ_1 and θ_2, substitute the value of $\ddot{x}$ back in $\ddot{\theta}_1$ and $\ddot{\theta}_2$.

$$\ddot{\theta}_1 = \frac{\cos(\theta_1)}{l_1}\left[\frac{F - m_1\big(g\sin(\theta_1)\cos(\theta_1) + l_1\dot{\theta}_1^2\sin(\theta_1)\big) - m_2\big(g\sin(\theta_2)\cos(\theta_2) + l_2\dot{\theta}_2^2\sin(\theta_2)\big)}{\big(M + m_1(\sin^2(\theta_1)) + m_2(\sin^2(\theta_2))\big)}\right] - g\frac{\sin(\theta_1)}{l_1}$$

$$\ddot{\theta}_2 = \frac{\cos(\theta_2)}{l_2}\left[\frac{F - m_1\big(g\sin(\theta_1)\cos(\theta_1) + l_1\dot{\theta}_1^2\sin(\theta_1)\big) - m_2\big(g\sin(\theta_2)\cos(\theta_2) + l_2\dot{\theta}_2^2\sin(\theta_2)\big)}{\big(M + m_1(\sin^2(\theta_1)) + m_2(\sin^2(\theta_2))\big)}\right] - g\frac{\sin(\theta_2)}{l_2}$$

Equation 15: Angular acceleration terms

The nonlinear state-space of any system is given by the equation,

$$\dot{X} = AX + BU$$

For the given crane system, the state-space representation will be,

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{\theta} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x \\ \dfrac{F - m_1 g\sin\,\theta_1\cos\,\theta_2 - m_2 g\sin\,\theta_2\cos\,\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\,\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\,\theta_2}{M + m_1\sin^2\theta_1 + m_2\sin^2\,\theta_2} \\ \theta_1 \\ \dfrac{F - m_1 g\sin\,\theta_1\cos\,\theta_2 - m_2 g\sin\,\theta_2\cos\,\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\,\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\,\theta_2}{(M + m_1\sin^2\theta_1 + m_2\sin^2\,\theta_2)l_1} - \dfrac{g\sin\,\theta_1}{l_1} \\ \theta_2 \\ \dfrac{F - m_1 g\sin\,\theta_1\cos\,\theta_2 - m_2 g\sin\,\theta_2\cos\,\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\,\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\,\theta_2}{(M + m_1\sin^2\theta_1 + m_2\sin^2\,\theta_2)l_2} - \dfrac{g\sin\,\theta_2}{l_2} \end{bmatrix}$$

Equation 16: The nonlinear state-space representation

b. Linearizing state space equation

The nonlinear state-space representation has been derived in Equation 16. The goal is to linearize the system around the equilibrium point x=0, θ_1=0, θ_2=0.
Assuming $\sin\theta_1 \approx \theta_1$,$\sin\theta_2 \approx \theta_2$, $cos\theta_1 \approx \theta_1$, $\cos\theta_2 \approx \theta_2$, $\dot{\theta}_1^2 = \dot{\theta}_2^2 \approx 0$ the equations can be simplified as in Equation 17

$$\ddot{x} = \frac{1}{M}(-m_1 g\theta_1 - m_2 g\theta_2 + F)$$
$$\ddot{\theta}_1 = \frac{1}{Ml_1}(-m_1 g\theta_1 - m_2 g\theta_2 - Mg\theta_1 + F)$$
$$\ddot{\theta}_2 = \frac{1}{Ml_2}(-m_1 g\theta_1 - m_2 g\theta_2 - Mg\theta_2 + F)$$

Equation 17: Linearized state equations

Thus, the linear state space representation can be written as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g(-M-m_1)}{Ml_1} & 0 & -\frac{gm_2}{M_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & \frac{g(-M-m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} u$$

Equation 18: The Linear state-space representation

The output vector of the state-space representation is given by,

$$Y = CX + DU$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g(-M-m_1)}{Ml_1} & 0 & -\frac{gm_2}{M_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & \frac{g(-M-m_2)}{Ml_2} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D = 0$$

c. To obtain conditions on M_1, m_1, m_2, l_1, l_2 for which the linearized system is controllable

The controllability of the system can be obtained by deriving the rank of the controllability matrix as in equation.

$$C = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

Equation 19: Equation for checking controllability

Code to check for controllability is as in the appendix. A matrix is full rank if its determinant is not equal to zero. Taking the determinant of Equation 19 results in Equation 20.

$$\mathrm{Det}\,(C) = -\frac{g^6 l_1^2 - 2g^6 l_1 l_2 + g^6 l_2^2}{M^6 l_1^6 l_2^6} = -\frac{g^6 (l_1 - l_2)^2}{M^6 l_1^6 l_2^6}$$

Equation 20: Determinant of controllability matrix

The determinant of the matrix is zero when either M, l_1, l_2 are equal to zero or when $l_1 = l_2$

d. LQR Controller design and simulations

Substituting M=1000 Kg, m1=m2=100 Kg, $l_1 = 20m$ and $l_2 = 10m$ in Equation 18the A and B matrix is obtained in Equation 21. The code to design the LQR are in appendix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.98 & 0 & -0.98 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.539 & 0 & -0.049 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.098 & 0 & -1.078 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ 0.00005 \\ 0 \\ 0.0001 \end{bmatrix}$$

Equation 21: A and B state matrices

Choosing the Q matrix and R values as in Equation 22 gave the best results for both the linear and the non-linear systems.

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30000 \end{bmatrix}$$

$$R = 1$$

Equation 22: Q and R weight matrices

Using Lyapunov's indirect stability criterion the real part of the eigen values should lie on the left half plane. The eigen values obtained are in the negative plane, as shown below.

$$\begin{bmatrix} -0.0061 + 0.7285i \\ -0.0061 - 0.7285i \\ -0.0129 + 1.0430i \\ -0.0129 + 1.0430i \\ -0.0646 + 0.0645i \\ -0.0646 + 0.0645i \end{bmatrix}$$

The responses of the linear and non-linear system are shown in Figure 2 and Figure 3. Code is present in Appendix V.a.

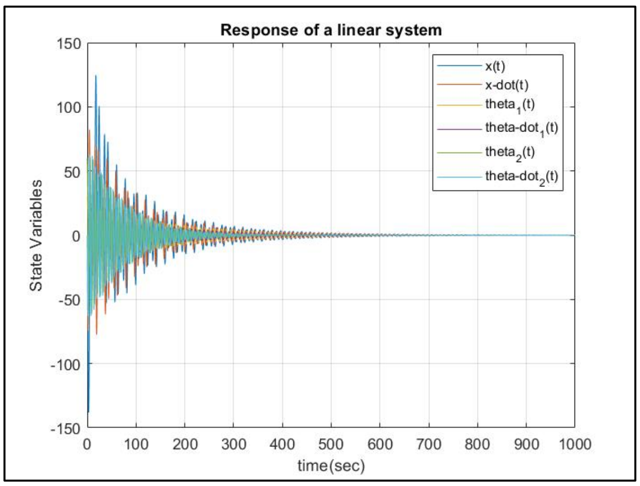


Figure 2: Response of linear system

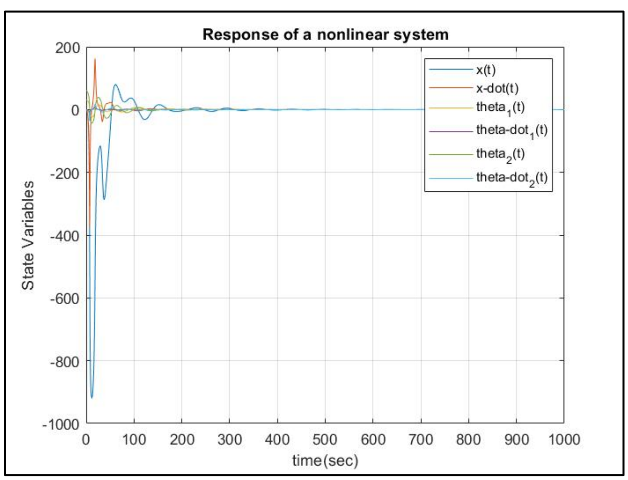


Figure 3: Response of nonlinear system

IV. Second Component

a. Observability

Given 4 sets of output vectors $x(t), (\theta_1(t), \theta_2(t)), (x(t), \theta_2(t))\ and\ (x(t), \theta_1(t), \theta_2(t))$. The script to compute the observability can be found in appendix V.b.

The output vectors are observable if the below matrix yields a rank of 6.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}$$

Equation 23: Observability matrix

Observability for $x(t)$, can be obtained by taking C as:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix is 6. Therefore, the system with the $x(t)$ is observable.

Observability for $\theta_1(t), \theta_2(t)$ can be obtained by considering C as below

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of the matrix is less than 6. Therefore, the system with inputs $\theta_1(t), \theta_2(t)$ is not observable.

Observability for $x(t)\ and\ \theta_2(t)$ can be obtained by considering C as:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Observability for $x(t), (\theta_1(t), \theta_2(t))$, can be obtained by considering C as below

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of the matrices with outputs $x(t)\ and\ \theta_2(t)$ and $x(t), (\theta_1(t), \theta_2(t))$ is 6. Therefore, the system with these state outputs is observable. Figure 4 is output of the code snippet.

```
System is observable, when only x(t) is output
System is not observable, when only theta1(t) and theta2(t) is output
System is observable, when only x(t) and theta2(t) is output
System is observable, when x(t), theta1(t) and theta2(t) is output
```

Figure 4: Observability output from code

b. Leunberger observer for each of the observable output vectors

We know that the standard state-space representation is given by,

$$\dot{X}(t) = AX(t) + Bu(t)$$
$$y(t) = CX(t) + Du(t)$$

Equation 24: Standard state-space representation

Adding the Leunberger observer to the state-space representation will change the state-space representation as shown in equation below,

$$\dot{\hat{X}}(t) = A\hat{x}(t) + BU(t) + L(Y(t) - C\hat{x}(t))$$

Equation 25: State-space representation with Leunberger observer

Here L is the observer gain matrix and $Y(t) - C\hat{x}(t)$ is a correction term and $\hat{x}(0) = 0$

The estimation error $X_e(t) = X(t) - \hat{X}(t)$ has the following state space representation

$$\dot{X}_e(t) = AX_e(t) - L\big(Y(t) - C\hat{x}(t)\big) + B_D U_D(t)$$

Assume, $D = 0, Y = Cx(t)$

$$\dot{X}_e(t) = [A - LC]X_e(t) + B_D U_D(t)$$

Equation 26: State space representation of the estimation error

Simulating the best Leunberger observer for each of the output vectors that were observable will result in the graphs shown in Figure 5, Figure 6, and Figure 7. As shown in the graphs, the tracking of the observer is approximately same as the estimated response. The code is present in the appendix V.c.

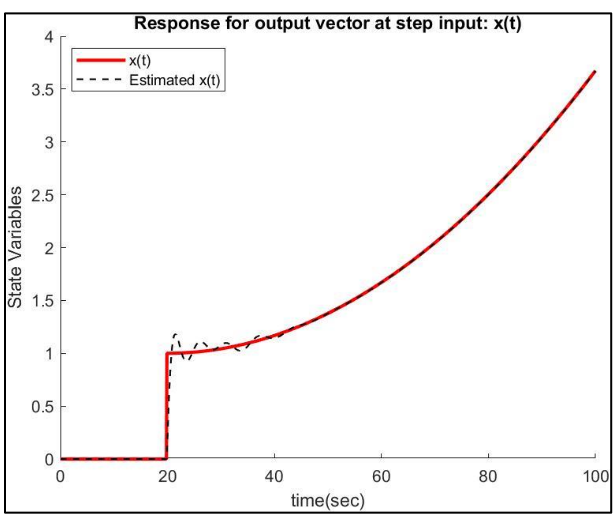


Figure 5: Output Response with $x(t)$

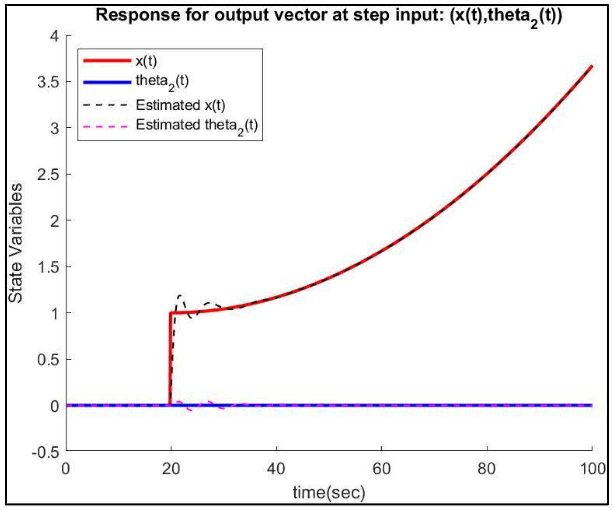


Figure 6: Output Response with $x(t)$ and $\theta_2(t)$

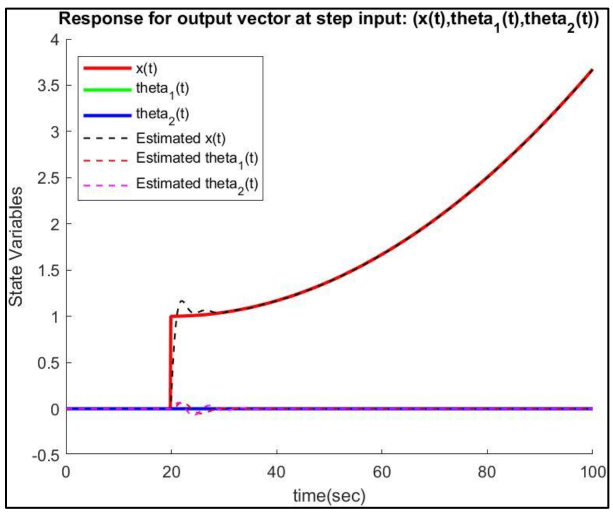


Figure 7: Output Response with $x(t)$, $\theta_1(t)$ and $\theta_2(t)$

Now simulating the state-space representation given in Equation 26 for both nonlinear (Equation 16) and linear systems (Equation 18) for the observable output vectors will produce the following results.

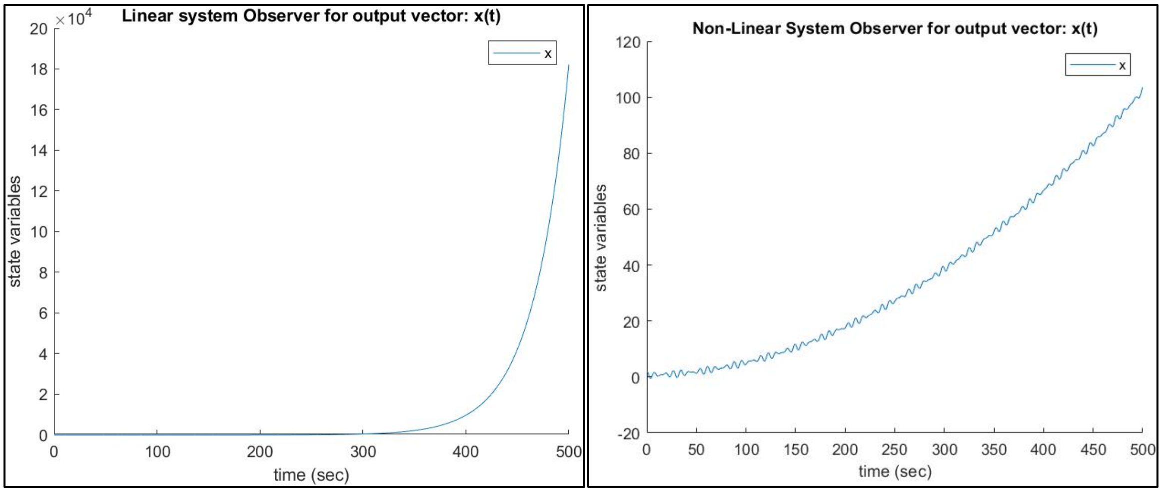


Figure 8: Linear and Nonlinear observer for the output vector $x(t)$

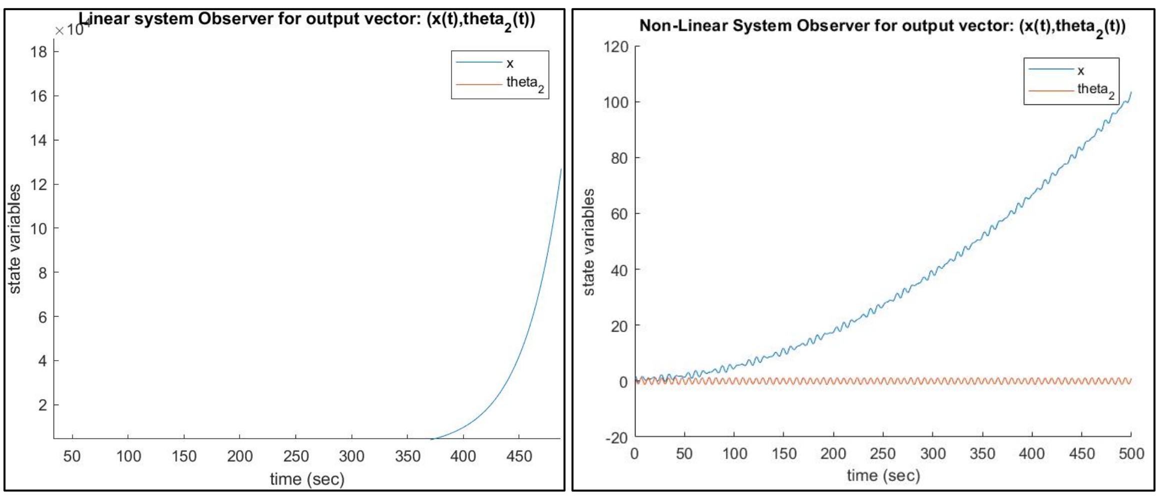


Figure 9: Linear and Nonlinear observer for the output vector $x(t)$ and $\theta_2(t)$

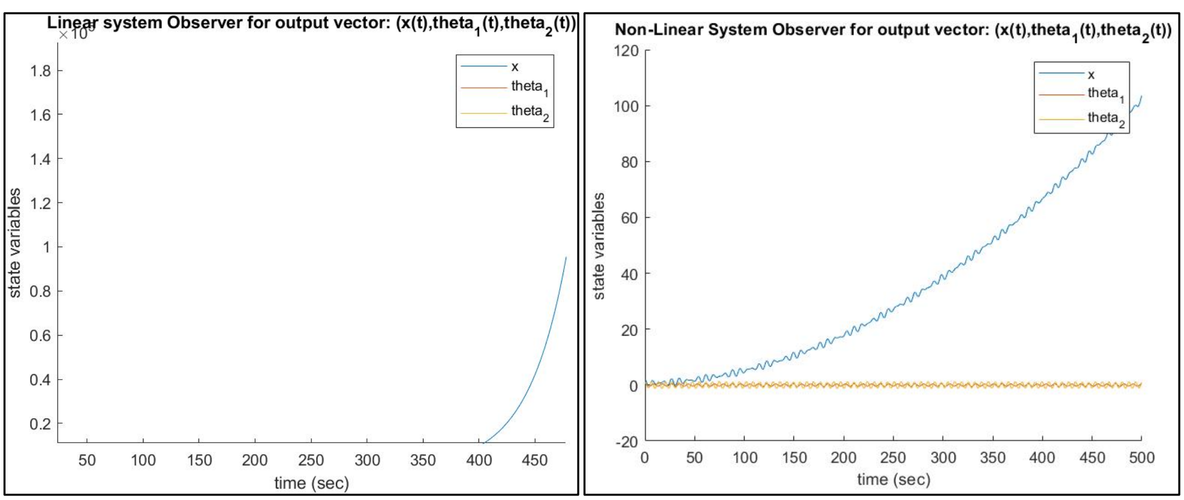


Figure 10: Linear and Nonlinear observer for the output vector $x(t)$, $\theta_1(t)$ and $\theta_2(t)$

c. LQG Control

From the given set of output vectors, $x(t)$ is the smallest output vector. So, the LQG design will comprise of the LQR controller along with a Kalman filter for a nonlinear system. Since we have only the $x(t)$ term, we can substitute $Q = 100$ and $R = 1$. For the Kalman filter, we can consider the process noise to be $V_d = 0.1$ and measurement noise to be $V_n = 0.01$. The code for LQG control is in the appendix V.d. The output response is shown in Figure 11

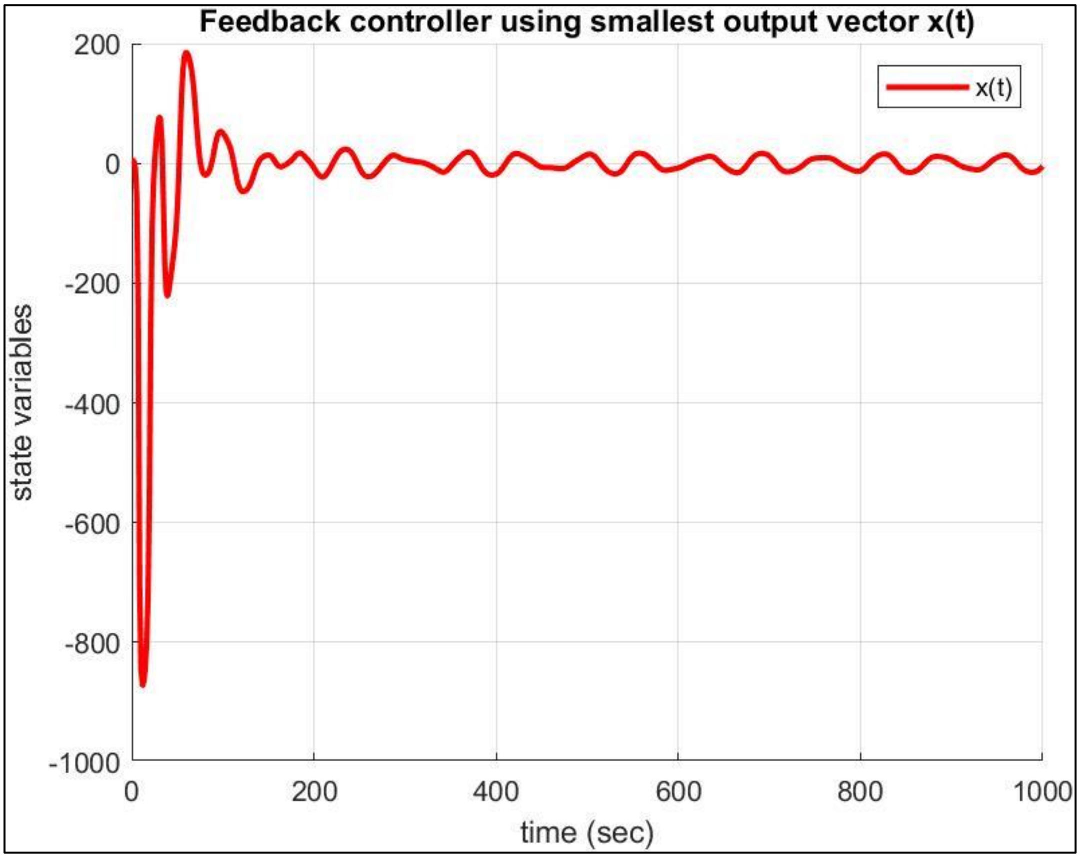
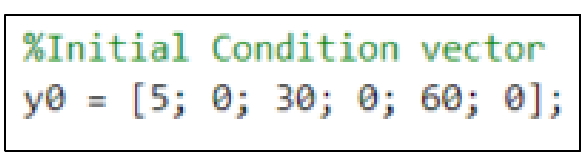


Figure 11: The output response of LQG controller using the nonlinear output vector $x(t)$

We can reconfigure the LQR controller to track a constant reference on x by changing the value of x in the initial conditions vector, which is the 1st array element of the initial condition vector as shown in Figure 12.



```
%Initial Condition vector
y0 = [5; 0; 30; 0; 60; 0];
```

Figure 12: Initial condition vector

Yes, our design can reject the disturbances applied on the cart by varying the process noise to be V_d in the Kalman filter.

V. Appendix

a. Code for LQR controller

```
%clearing all the previous outputs
clc
clear all
close all

% Given information
global M m1 m2 l1 l2 g
M=1000;%Mass of the cart
m1=100;%mass of Pendulum 1
m2=100;%mass of Pendulum 2
l1=20;%length of the string of Pendulum 1
l2=10;%length of the string of Pendulum 2
g=9.81; %declaring the value of the accelertaion due to gravity in m/

global A
A=[0 1 0 0 0 0;
0 0 -(m1*g)/M 0 -(m2*g)/M 0;
0 0 0 1 0 0;
0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
0 0 0 0 0 1;
0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

global B
B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

% Checking for the controllability of the given system
if (rank(ctrb(A,B))==6)
disp("Rank of ctrb matches order of A, system is controllable")
else
disp("Rank of ctrb doesnt matche order of A, system is uncontrollable")
end

global Q
Q=[100 0 0 0 0 0;
0 100 0 0 0 0;
0 0 30000 0 0 0;
0 0 0 30000 0 0;
0 0 0 0 30000 0;
0 0 0 0 0 30000];

global R
R=1;

global C
C = eye(6);% To form a 6 X 6 identity matrix

global D
D = 0; % Initialising the D matrix to be Zero

global K_val
```

```
disp("Now, seeing the results using an LQR controller")
[K_val, P_mat, Poles] = lqr(A,B,Q,R);%In-built MATLAB code
Poles

y0 = [0; 0; 30; 0; 60; 0];
t_int = 0:0.001:1000;%defining the timespan
[t1,y1] = ode45(@pendlinear,t_int,y0); %Linearization with initial conditions
[t2,y2] = ode45(@pendnonlinear,t_int,y0); %Non-linear systems
figure
plot(t1,y1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','x-dot(t)', 'theta_1(t)', 'theta-dot_1(t)', 'theta_2(t)',
'theta-dot_2(t)')
title('Response of a linear system')
grid on

figure
plot(t2,y2)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','x-dot(t)', 'theta_1(t)', 'theta-dot_1(t)', 'theta_2(t)',
'theta-dot_2(t)')
title('Response of a nonlinear system')
grid on

function dydt = pendlinear(t,y)
global A B K_val
u = -K_val * y;
dydt = A*y + B*u;
end

function dydt = pendnonlinear(t,y)
global K_val g m1 m2 l1 l2 M
F =-K_val*y;
dydt=zeros(6,1);
dydt(1) = y(2);
dydt(2)=(F-(g/2)*(m1*sind(y(3))+m2*sind(2*y(5)))-(m1*l1*(y(4)^2)*sind(y(3)))-
(m2*l2*(y(6)^2)*sind(y(5))))/(M+m1*((sind(y(3)))^2)+m2*((sind(y(5)))^2));%xDD
dydt(3)= y(4);%theta 1D;
dydt(4)= (dydt(2)*cosd(y(3))-g*(sind(y(3))))/l1';%theta 1 Ddot;
dydt(5)= y(6);%theta 2D
dydt(6)= (dydt(2)*cosd(y(5))-g*(sind(y(5))))/l2;%theta 2Ddot;
end
```

b. Code for Observability

```
%clearing all previous outputs and variables
clc
clear all

syms M m1 m2 l1 l2 g;

A=[0 1 0 0 0 0;
0 0 -(m1*g)/M 0 -(m2*g)/M 0;
0 0 0 1 0 0;
0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
0 0 0 0 0 1;
0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];%Initializing the B matrix

C1 = [1 0 0 0 0 0]; %Corresponding to x component
C2 = [0 0 1 0 0 0; 0 0 0 0 1 0]; %corresponding to theta1 and theta2
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0]; %cooresponding to x and theta2
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0]; %cooresponding to x,theta and
theta2

%Matrix to check teh Observability Condition
Ob1 = [C1' A'*C1' A'*A'*C1' A'*A'*A'*C1' A'*A'*A'*A'*C1' A'*A'*A'*A'*A'*C1'];
if rank(Ob1)==6
disp('System is observable, when only x(t) is output')
else
disp('System is not observable, when only x(t) is output')
end

%Matrix to check teh Observability Condition
Ob2 = [C2' A'*C2' A'*A'*C2' A'*A'*A'*C2' A'*A'*A'*A'*C2' A'*A'*A'*A'*A'*C2'];
if rank(Ob2)==6 %condition for system observability i.e when rank = 6
disp('System is observable, when only theta1(t) and theta2(t) is output')
else
disp('System is not observable, when only theta1(t) and theta2(t) is output')
end
%Matrix to check teh Observability Condition
Ob3 = [C3' A'*C3' A'*A'*C3' A'*A'*A'*C3' A'*A'*A'*A'*C3' A'*A'*A'*A'*A'*C3'];
if rank(Ob3)==6%condition for system observability i.e when rank = 6
disp('System is observable, when only x(t) and theta2(t) is output')
else
disp('System is not observable, when only x(t) and theta2(t) is output')
end
%Matrix to check teh Observability Condition
Ob4 = [C4' A'*C4' A'*A'*C4' A'*A'*A'*C4' A'*A'*A'*A'*C4' A'*A'*A'*A'*A'*C4'];
if rank(Ob4)==6%condition for system observability i.e when rank = 6
    disp('System is observable, when x(t), theta1(t) and theta2(t) is
output')
else
    disp('System is not observable, when x(t), theta1(t) and theta2(t) is
output')
end
```

c. Code for Leunberger observer

```
%clearing all the previous outputs
clc
clear all
close all

% Given information
global M m1 m2 L1 L2
M=1000;%Mass of the cart
m1=100;%mass of Pendulum 1
m2=100;%mass of Pendulum 2
L1=20;%length of the string of Pendulum 1
L2=10;%length of the string of Pendulum 2

global g
g=9.81; %declaring the value of the accelertaion due to gravity in m/

global A
A=[0 1 0 0 0 0;
0 0 -(m1*g)/M 0 -(m2*g)/M 0;
0 0 0 1 0 0;
0 0 -((M+m1)*g)/(M*L1) 0 -(m2*g)/(M*L1) 0;
0 0 0 0 0 1;
0 0 -(m1*g)/(M*L2) 0 -(g*(M+m2))/(M*L2) 0];

global B
B=[0; 1/M; 0; 1/(M*L1); 0; 1/(M*L2)];

% Checking for the controllability of the given system
if (rank(ctrb(A,B))==6)
disp("Rank of ctrb matches order of A, system is controllable")
else
disp("Rank of ctrb doesnt matche order of A, system is uncontrollable")
end

global C
C = eye(6);% To form a 6 X 6 identity matrix
global D
D = 0; % Initialising the D matrix to be Zero
y0 = [5; 0; 30; 0; 60; 0];
t_int = 0:0.001:1000;%defining the timespan

Bd = 0.1*eye(6);
Vn = 0.01;

global c1
c1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
global c3
c3 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 1 0];
global c4
c4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
```

```
tspan = 0:0.1:500;
q0 = [2 0 deg2rad(30) 0 deg2rad(60) 0];

Bd = 0.1*eye(6); %Process Noise
Vn = 0.01; %Measurement Noise
[Lue1,~,~] = lqe(A,Bd,c1,Bd,Vn*eye(3));
[Lue3,~,~] = lqe(A,Bd,c3,Bd,Vn*eye(3));
[Lue4,~,~] = lqe(A,Bd,c4,Bd,Vn*eye(3));
Ac1 = A-(Lue1*c1);
Ac3 = A-(Lue3*c3);
Ac4 = A-(Lue4*c4);
e_sys1 = ss(Ac1,[B Lue1],c1,0);
e_sys3 = ss(Ac3,[B Lue3],c3,0);
e_sys4 = ss(Ac4,[B Lue4],c4,0);

unitStep = 0*tspan;
unitStep(200:length(tspan)) = 1;
d = [1;0;0];
sys1 = ss(A,B,c1,d);
sys3 = ss(A,B,c3,d);
sys4 = ss(A,B,c4,d);
[y1,t] = lsim(sys1,unitStep,tspan);
[x1,t] = lsim(e_sys1,[unitStep;y1'],tspan);
[y3,t] = lsim(sys3,unitStep,tspan);
[x3,t] = lsim(e_sys3,[unitStep;y3'],tspan);
[y4,t] = lsim(sys4,unitStep,tspan);
[x4,t] = lsim(e_sys4,[unitStep;y4'],tspan);


figure();
hold on
plot(t,y1(:,1),'r','Linewidth',2)
plot(t,x1(:,1),'k--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: x(t)')
hold off

figure();
hold on
plot(t,y3(:,1),'r','Linewidth',2)
plot(t,y3(:,3),'b','Linewidth',2)
plot(t,x3(:,1),'k--','Linewidth',1)
plot(t,x3(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_2(t)','Estimated x(t)','Estimated theta_2(t)')
title('Response for output vector at step input: (x(t),theta_2(t))')
hold off
```

```
figure();
hold on
plot(t,y4(:,1),'r','Linewidth',2)
plot(t,y4(:,2),'g','Linewidth',2)
plot(t,y4(:,3),'b','Linewidth',2)
plot(t,x4(:,1),'k--','Linewidth',1)
plot(t,x4(:,2),'r--','Linewidth',1)
plot(t,x4(:,3),'m--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','theta_1(t)','theta_2(t)','Estimated x(t)','Estimated
theta_1(t)','Estimated theta_2(t)')
title('Response for output vector at step input:
(x(t),theta_1(t),theta_2(t))')
hold off

[t,q1] = ode45(@(t,q)linearObs1(t,q,Lue1),tspan,q0);
figure();
hold on
plot(t,q1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: x(t)')
legend('x')
hold off

[t,q3] = ode45(@(t,q)linearObs3(t,q,Lue3),tspan,q0);
figure();
hold on
plot(t,q3(:,1))
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off
```

```
[t,q3] = ode45(@(t,q)nonLinearObs3(t,q,1,Lue3),tspan,q0);
figure();
hold on
plot(t,q3(:,1))
plot(t,q3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: (x(t),theta_2(t))')
legend('x','theta_2')
hold off


[t,q4] = ode45(@(t,q)nonLinearObs4(t,q,1,Lue4),tspan,q0);
figure();
hold on
plot(t,q4(:,1))
plot(t,q4(:,3))
plot(t,q4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector:
(x(t),theta_1(t),theta_2(t))')
legend('x','theta_1','theta_2')
hold off

function dQe = linearObs4(t,Qe,Lue4)
global A B c4
y4 = [Qe(1); Qe(3); Qe(5)];
K = 1; % feedback = 1;
dQe = (A+B*K)*Qe + Lue4*(y4 - c4*Qe);
end

function dQe = linearObs1(t,Qe,Lue1)
global A B c1
y1 = [Qe(1); 0; 0];
K = 1; % feedback = 1;
dQe = (A+B*K)*Qe + Lue1*(y1 - c1*Qe);
end

function dQe = linearObs3(t,Qe,Lue3)
global A B c3
y3 = [Qe(1); 0; Qe(5)];
K = 1; % feedback = 1;
dQe = (A+B*K)*Qe + Lue3*(y3 - c3*Qe);
end
```

```
function dQ = nonLinear(t,y,F)
global M m1 m2 L1 L2 g
x = y(1);
dx = y(2);
t1 = y(3);
dt1 = y(4);
t2 = y(5);
dt2 = y(6);
dQ=zeros(6,1);
dQ(1) = dx;
dQ(2) = (F-((m1*sin(t1)*cos(t1))+(m2*sin(t2)*cos(t2)))*g -
(L1*m1*(dQ(3)^2)*sin(t1)) - (L2*m2*(dQ(5)^2)*sin(t2)))/(m1+m2+M-
(m1*(cos(t1)^2))-(m2*(cos(t2)^2)));
dQ(3) = dt1;
dQ(4) = (cos(t1)*dQ(2)-g*sin(t1))/L1;
dQ(5) = dt2;
dQ(6) = (cos(t2)*dQ(2)-g*sin(t2))/L2;
end

function dQ = nonLinearObs1(t,y,F,Lue1)
global M m1 m2 L1 L2 g
x = y(1);
dx = y(2);
t1 = y(3);
dt1 = y(4);
t2 = y(5);
dt2 = y(6);
dQ=zeros(6,1);
y1 = [x; 0; 0];
c1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
sum = Lue1*(y1-c1*y);
dQ(1) = dx + sum(1);
dQ(2) = (F-((m1*sin(t1)*cos(t1))+(m2*sin(t2)*cos(t2)))*g -
(L1*m1*(dQ(3)^2)*sin(t1)) - (L2*m2*(dQ(5)^2)*sin(t2)))/(m1+m2+M-
(m1*(cos(t1)^2))-(m2*(cos(t2)^2)))+sum(2);
dQ(3) = dt1+sum(3);
dQ(4) = ((cos(t1)*dQ(2)-g*sin(t1))/L1) + sum(4);
dQ(5) = dt2 + sum(5);
dQ(6) = (cos(t2)*dQ(2)-g*sin(t2))/L2 + sum(6);
end
```

```
function dQ = nonLinearObs3(t,y,F,Lue3)
global M m1 m2 L1 L2 g c3
x = y(1);
dx = y(2);
t1 = y(3);
dt1 = y(4);
t2 = y(5);
dt2 = y(6);
dQ=zeros(6,1);
y3 = [x; 0; t2];
sum = Lue3*(y3-c3*y);
dQ(1) = dx + sum(1);
dQ(2) = (F-((m1*sin(t1)*cos(t1))+(m2*sin(t2)*cos(t2)))*g -
(L1*m1*(dQ(3)^2)*sin(t1)) - (L2*m2*(dQ(5)^2)*sin(t2)))/(m1+m2+M-
(m1*(cos(t1)^2))-(m2*(cos(t2)^2)))+sum(2);
dQ(3) = dt1+sum(3);
dQ(4) = ((cos(t1)*dQ(2)-g*sin(t1))/L1) + sum(4);
dQ(5) = dt2 + sum(5);
dQ(6) = (cos(t2)*dQ(2)-g*sin(t2))/L2 + sum(6);
end

function dQ = nonLinearObs4(t,y,F,Lue4)
global M m1 m2 L1 L2 g c4
x = y(1);
dx = y(2);
t1 = y(3);
dt1 = y(4);
t2 = y(5);
dt2 = y(6);
dQ=zeros(6,1);
y4 = [x; t1; t2];
sum = Lue4*(y4-c4*y);
dQ(1) = dx + sum(1);
dQ(2) = (F-((m1*sin(t1)*cos(t1))+(m2*sin(t2)*cos(t2)))*g -
(L1*m1*(dQ(3)^2)*sin(t1)) - (L2*m2*(dQ(5)^2)*sin(t2)))/(m1+m2+M-
(m1*(cos(t1)^2))-(m2*(cos(t2)^2)))+sum(2);
dQ(3) = dt1+sum(3);
dQ(4) = ((cos(t1)*dQ(2)-g*sin(t1))/L1) + sum(4);
dQ(5) = dt2 + sum(5);
dQ(6) = (cos(t2)*dQ(2)-g*sin(t2))/L2 + sum(6);
end
```

d. Code for the design of LQG controller

```
%clearing all the previous outputs
clc
clear all
close all

% Given information
global M m1 m2 l1 l2 g
M=1000;%Mass of the cart
m1=100;%mass of Pendulum 1
m2=100;%mass of Pendulum 2
l1=20;%length of the string of Pendulum 1
l2=10;%length of the string of Pendulum 2
g=9.81; %declaring the value of the accelertaion due to gravity in m/

global A
A=[0 1 0 0 0 0;
0 0 -(m1*g)/M 0 -(m2*g)/M 0;
0 0 0 1 0 0;
0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
0 0 0 0 0 1;
0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

global B
B=[0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

global C
C = [1 0 0 0 0 0];

global D
D = 0; % Initialising the D matrix to be Zero

%Initial Condition vector
y0 = [5; 0; 30; 0; 60; 0];

t_int = 0:0.001:1000;%defining the timespan

Bd = 0.1*eye(6);
Vn = 0.01;

global Q
Q=[100 0 0 0 0 0;
0 0 0 0 0 0;
0 0 0 0 0 0;
0 0 0 0 0 0;
0 0 0 0 0 0;
0 0 0 0 0 0];
```

```
global R
R=1;

vd = 0.1*eye(6);
vn = 0.01;

global K_val
[K_val, P_mat, Poles] = lqr(A, B, Q, R);
Ac1 = A-(K_val'*C);
e_sys1 = ss(Ac1,[B K_val'],C, 0);

global kalman_gain
kalman_gain = lqe(A, vd, C, vd, vn);
%kalman_gain = kalman(e_sys1, 0.1, 0.1, 0.1)

[t2,y2] = ode45(@pendnonlinear,t_int,y0);

figure();
hold on
plot(t_int,y2(:,1),'r','Linewidth',2)
ylabel('state variables')
xlabel('time (sec)')
title('Feedback controller using smallest output vector x(t)')
legend('x(t)')
grid on
hold off

function dydt = pendnonlinear(t,y)

global K_val g m1 m2 l1 l2 M kalman_gain C
F =-K_val*y;
sum = kalman_gain * (y(1) - C * y(1));
dydt=zeros(6,1);
dydt(1) = y(2) + sum(1);
dydt(2)=(F-(g/2)*(m1*sind(y(3))+m2*sind(2*y(5)))-(m1*l1*(y(4)^2)*sind(y(3)))-
(m2*l2*(y(6)^2)*sind(y(5))))/(M+m1*((sind(y(3)))^2)+m2*((sind(y(5)))^2));%xDD
dydt(3)= y(4) + sum(2);
dydt(4)= (dydt(2)*cosd(y(3))-g*(sind(y(3))))/l1';
dydt(5)= y(6) + sum(3);
dydt(6)= (dydt(2)*cosd(y(5))-g*(sind(y(5))))/l2;
end
```