# Nim Game.

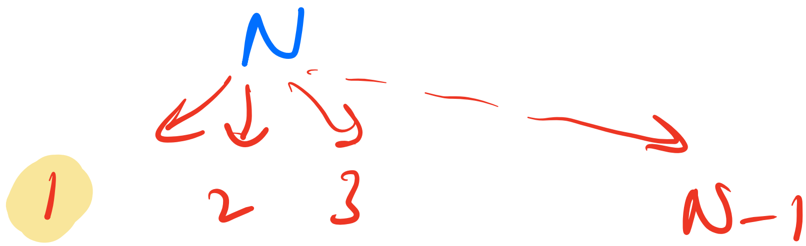
## Game Theory.

Game being played by 2 players. Player 1 & 2. Game is such that it will end, and there has to be winner ( no ties / draws )

Analysis of such games is game theory.

**Q1** Number N. 2 Players. Each player plays turn by turn.
In one turn, player can replace current no $x$ with any no $1 \le y < x$ $[1, x-1]$
Who will win the game.
Person who cannot make a move loses.

$$5 \xrightarrow{P_1} 3 \xrightarrow{P_2} 1$$



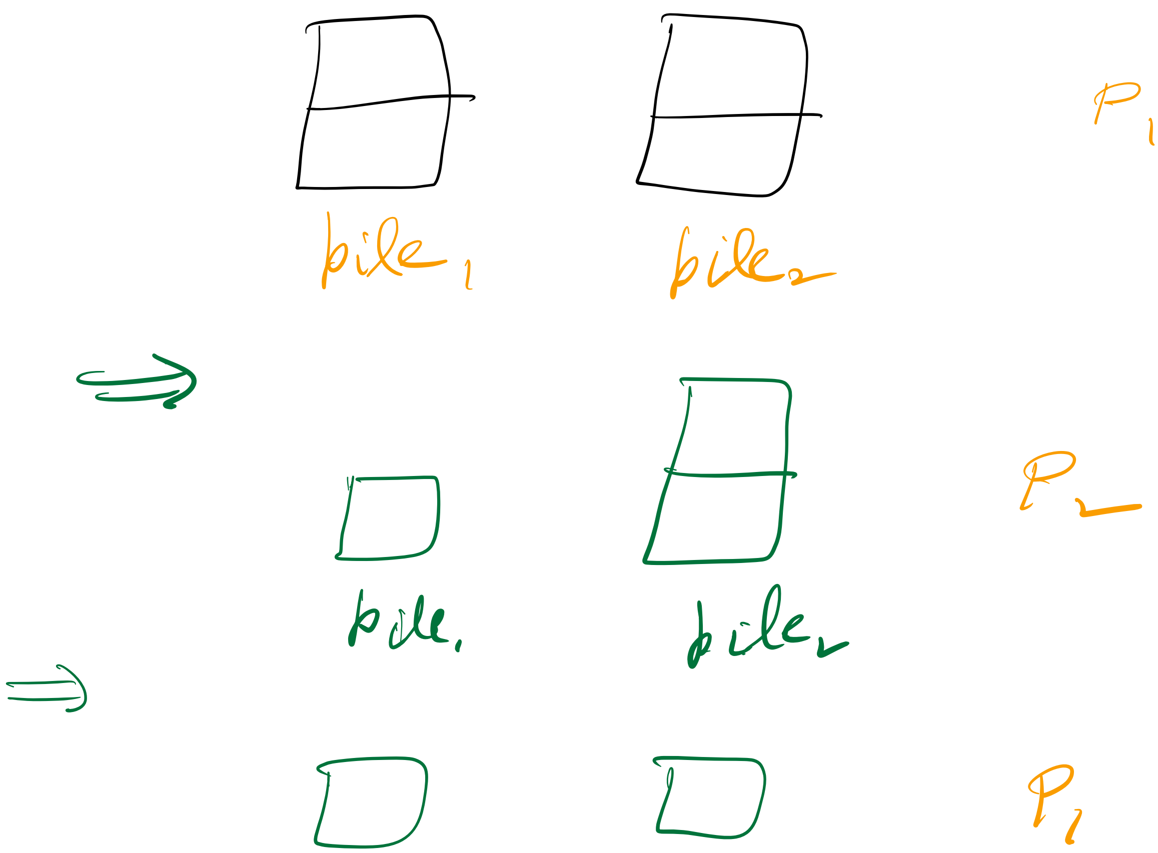

$N = 1$ win = $P_2$

$N != 1$ win = $P_1$

**Q2** There are $N$ piles, each with $K$ stones. In a turn, you can convert exactly 1 pile of size $x$ to $y$ where $1 \le y < x$ and $\gcd(x,y) = 1$

Who will win?

Eg $N=2$ $K=2$




$\Rightarrow$ $P_2$ wins

$N = 3 \qquad K = 1$

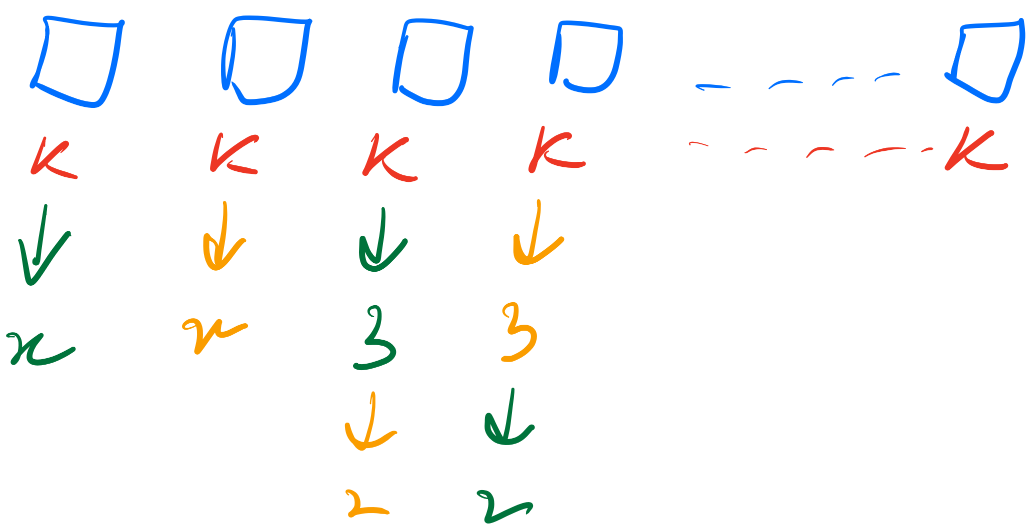
$P_2$ wins.

$N = 6 \qquad K = 1$

$P_2$

if (K == 1)
$P_2$ win

N is even



$N = 4$ $K = 3$

| | | | | |
|---|---|---|---|---|
| | 3 | 3 | 3 | 3 |
| $\Rightarrow$ | 3 | 2 | 3 | 3 |
| | 2 | 2 | 3 | 3 |
| | 2 | 2 | 2 | 3 |
| | 2 | 2 | 2 | 2 |
| | 1 | 2 | 2 | 2 |
| | 1 | 1 | 2 | 2 |
| | 1 | 1 | 1 | 2 |
| | 1 | 1 | 1 | 1 |

$P_2$

if ( N is even )

$P_2$ wins.

$N=5$ $K=3$

$$\begin{array}{ccccc} 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 3 \end{array} \quad P_2$$

reduces to $N=$ even case

hence $P_1$ wins.

if $(K==1)$
$P_2$

if ( $N$ is even)
$p_2$
else
$p_1$

# Q3 Make Palindrome.

string of lowercase alphabets.
In 1 move, a player can remove exactly 1 letter.
If the player, before his turn, can reorder s into a palindrome, that player wins.

Eg- aab P1
Eg ab ⇒ P2

How to get a palindrome.

even a a b b cc aa
all chars even no of times

odd a a b b k cc aa
only 1 char odd no of times

if no of odd freq characters $\leq 1$

you can directly convert to palin

what if odd freq chars $= 2$

P2 wins

abca

$a \Rightarrow \cancel{2}\ \cancel{1}\ 0$

$b \Rightarrow \cancel{1}\ 0$

$c \Rightarrow 1$

$\Rightarrow$ aab $P_2$

$\Rightarrow$ abc $P_2 \Rightarrow$ bc $\Rightarrow P_1$

$\Rightarrow$ c $P_2$

odd freq $= 3$ $P_1$

$x \rightarrow$ even

$y$

$z$

odd $= 2$ P2 chance

odd freq = 4 $P_2$

abcd →

→

By obs, :

if odd freq = 0

$P_1$

if the odd freq is even

$P_2$

else

$P_1$

Eg - aabb abba

aabbc

# Nim Game

N piles of stones, each with diff amount

$a_0 \quad a_1 \quad a_2 \quad \text{----} \quad a_{N-1}$

In a move, a player can take any positive no of stones from any pile & throw them away

Person who cannot make a move, loses

Eg - 3 $\rightarrow$ 0 $\quad$ $P_1$ win

3, 3 $\rightarrow$ $\quad$ $P_2$ win

$\rightarrow$

3, 5 $\rightarrow$ 3, 3 $\quad$ $P_2$ chance

$P_1$ win

Solution to nim game.

Current player wins if
$a_0 \wedge a_1 \wedge a_2 \text{----} \wedge a_{n-1}$ is non-zero

Proof

Obs: If xor is 0, then it is
a losing state

$a_0 \quad a_1 \quad a_2 \quad \text{------} \quad a_{n-1}$ $\quad P_1$

$\Downarrow$

xor $= 0$ $\quad P_2$

Let current xor $s = a_0 \wedge a_1 \wedge a_2 \wedge \text{.....} \wedge a_{n-1}$
$s$ will have some largest
set bit

Take number $a_i$ where this
largest bit is set

Convert $a_i \Rightarrow a_i \wedge s$

$a_0 \quad a_1 \quad a_2 \; --- \; a_i \; --- \; a_{n-1}$

$a_0 \quad a_1 \quad a_2 \; --- \; S \wedge a_i \; --- \; a_{n-1}$

$$\underbrace{a_0 \wedge a_1 \wedge a_2 \; --- \wedge a_{n-1}}_{S} \wedge S = 0$$

- Variations:
  In a move, apart from removing stones, you can also add stones.

Ans: No change, same solution applies.

$a_0 \quad a_1 \quad a_2 \; ---- \; a_{n-1}$

Other player can reverse the move.

**Q4** Given N piles. On one move you can remove only 1, 2 or 3 stones. The player who cannot move loses.

⇒ Try analysis for only pile

| N: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Win | $P_1$ | $P_1$ | $P_1$ | $P_2$ | $P_1$ | $P_1$ | $P_1$ | $P_2$ | $P_1$ | $P_1$ | $P_1$ | $P_2$ |

**Obs:** $P_2$ wins when $N \% 4 == 0$

Thus we can reduce (N) any number to $N \% 4$

Piles ⇒ 10, 12, 7, 6, 16

2 0 3 2 0

Sprague Grundy Theorem

Once replaced with equivalent number this can be considered as the eq Nim Game.

If you want to explore more

Sprague Grundy Theorem

<done>