EL9343

Data Structure and Algorithm

Lecture 8: Graph Basics

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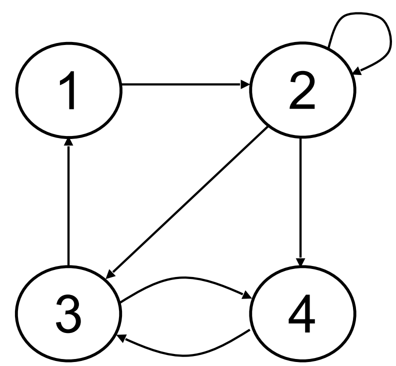
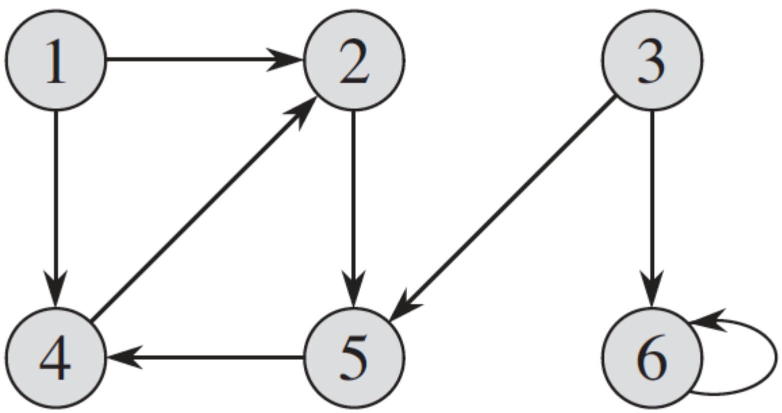
Graphs

- **Definition** = a set of vertices (nodes) with edges (links) between them.
  - $G = (V, E)$ - graph
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

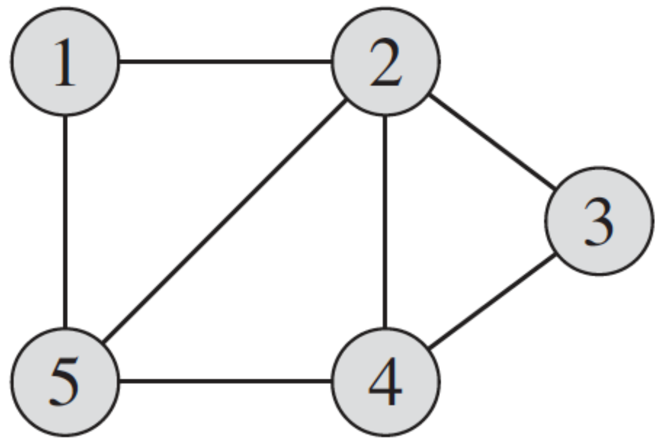


Directed Graphs (Digraph) VS. Undirected Graph

Directed Graphs (digraphs)
(ordered pairs of vertices)



Undirected Graphs
(unordered pairs of vertices)



in-degree of v: # of edges entering v
out-degree of v: # of edges leaving v

v is **adjacent** to u if there is an edge (u,v)

degree of v: # of edges incident on v

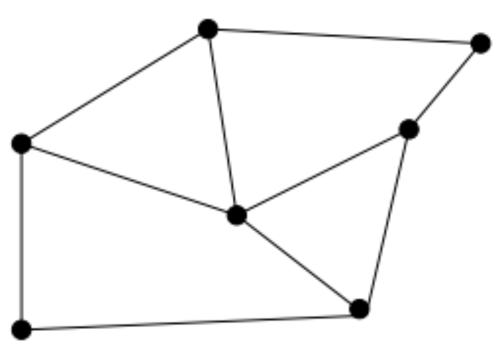
v is **adjacent** to u and u is adjacent to v is if there is an edge between v and u

More Graph variations

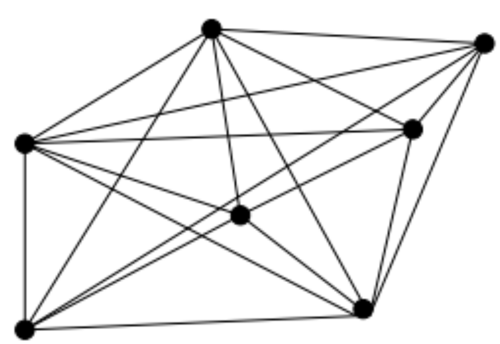
- A *weighted graph* associates weights with either the edges or the vertices
  - e.g., a road map: edges might be weighted w/ distance
- A *multigraph* allows multiple edges between the same pair of vertices
  - e.g., the call graph in a program (a function can get called from multiple points in another function)

Sparse VS. Dense Graphs

- We will typically express running times in terms of $|E|$ and $|V|$
  - If $|E| \approx |V|^2$ the graph is *dense*
    - have a quadratic number of edges
  - If $|E| \approx |V|$ the graph is *sparse*
    - linear in size, only a small fraction of the possible number of vertex pairs actually have edges defined between them



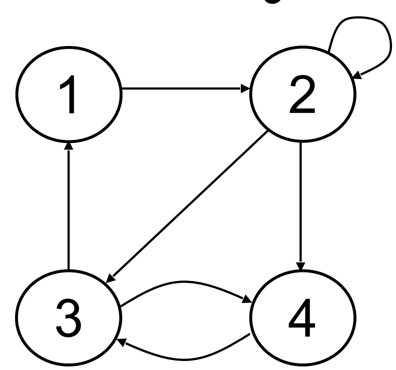
sparse



dense

Terminology

- Complete graph
  - A graph with an edge between each pair of vertices
- Subgraph
  - A graph (V', E') such that V'⊆V and E'⊆E
- Path from v to w
  - A sequence of vertices $<v_0, v_1, \ldots, v_k>$ such that $v_0=v$ and $v_k=w$
- Length of a path
  - Number of edges along the path

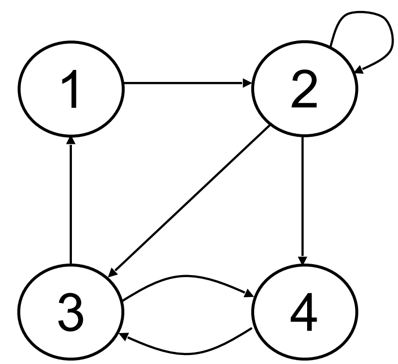


path from v_1 to v_4

$<v_1, v_2, v_4>$

Terminology (cont'd)

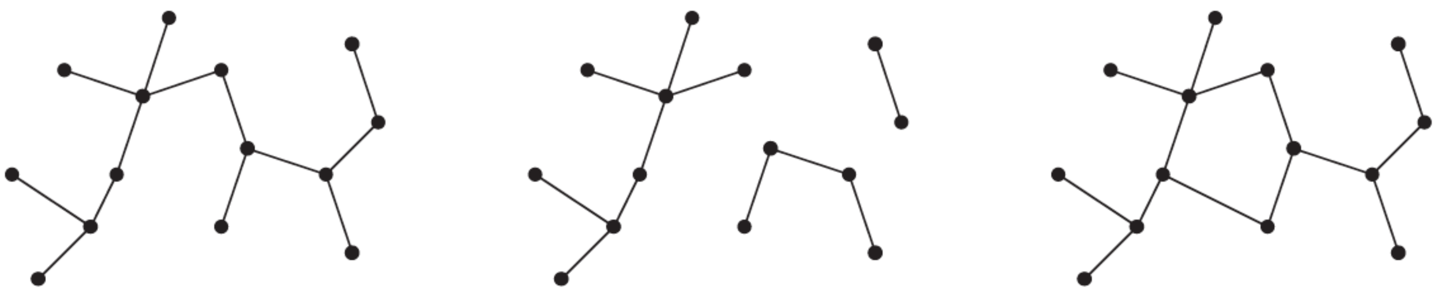
- w is **reachable** from v
  - If there is a path from v to w
- **Simple** path
  - All the vertices in the path are distinct
- Cycles
  - A path $<v_0, v_1, \ldots, v_k>$ forms a cycle if $v_0=v_k$ and $k\geq 2$
- Acyclic graph
  - A graph without any cycles



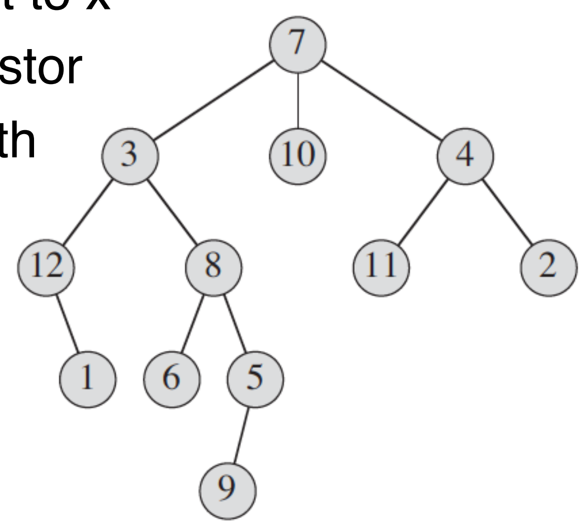
cycle from v_1 to v_1
$<v_1, v_2, v_3, v_1>$

Special case: Tree

- **(Free) Tree**: connected, acyclic, undirected graph
- **Forest:** acyclic, undirected graph, possibly disconnected



- **Rooted Tree**: a free tree with special **root** node
  - **Ancestor** of node x: any node on the path from root to x
  - **Descendant** of node x: any node with x as its ancestor
  - **Parent** of node x: node immediately before x on path from root
  - **Child** of node x: any node with x as its parent
  - **Siblings** of node x: nodes sharing parent with x
  - **Leaf/external node**: without child
  - **Internal node**: with at least one child

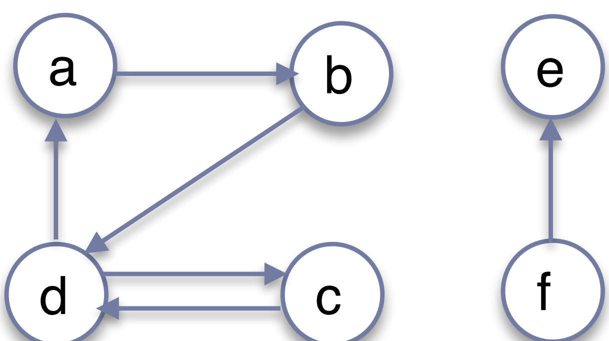


Strongly connected VS. Connected

Directed Graphs

Strongly connected: every two vertices are reachable from each other

Strongly connected components: all possible strongly connected subgraphs

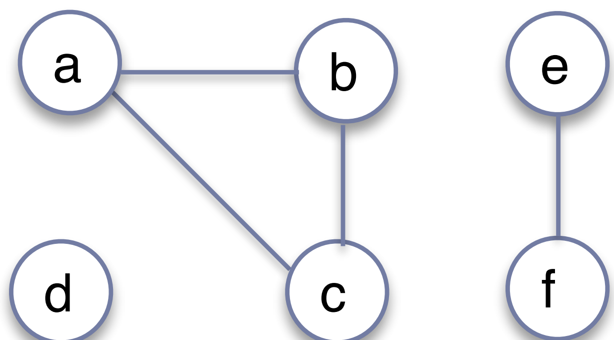


strongly connected components:
{a,b,c,d},{e},{f}

Undirected Graphs

connected: every pair of vertices are connected by a path

connected components: all possible connected subgraphs



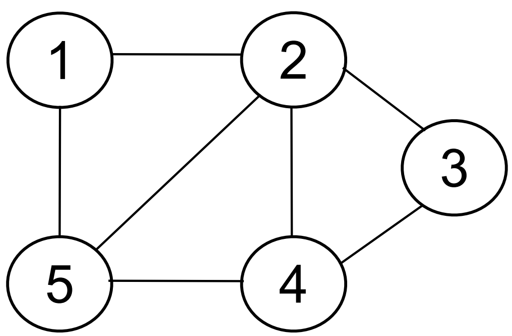
connected components:
{a,b,c},{d},{e,f}

Representing Graphs

- **Adjacency matrix representation** of G = (V, E)
  - Assume vertices are numbered 1, 2, … $|V|$
  - The representation consists of a matrix $A_{|V| \times |V|}$:
  - $a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$

Graphs: Adjacency Matrix

- Example



Undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

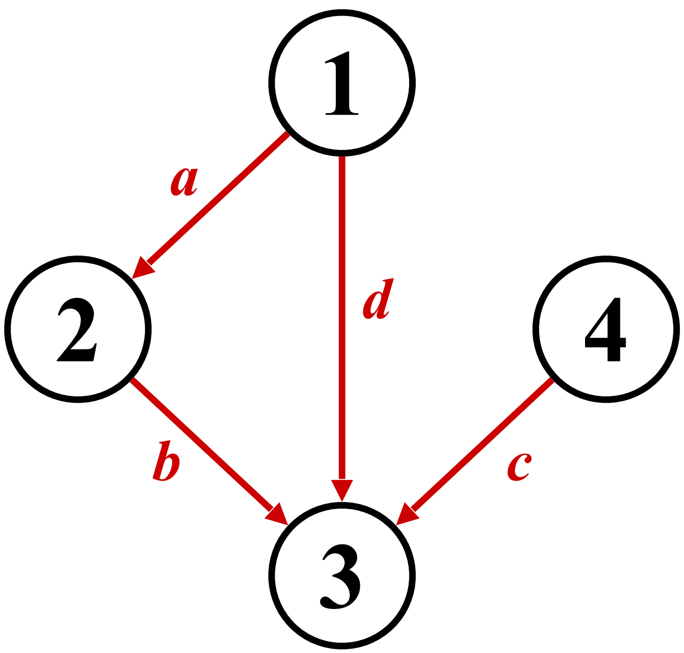
For undirected graphs, matrix A is symmetric:

$a_{ij} = a_{ji}$

$A = A^T$

Graphs: Adjacency Matrix

- Another Example



directed graph

	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Properties of Adjacency Matrix Representation

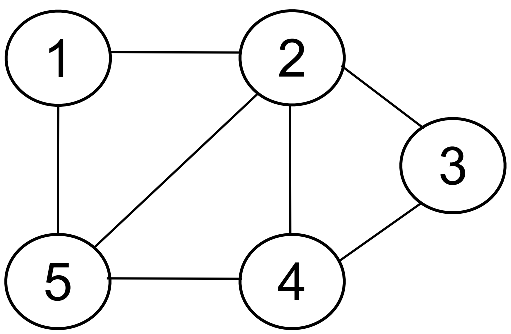
- Memory required
  - $\Theta(V^2)$, independent on the number of edges in G
- Preferred when
  - The graph is **dense:** $|E|$ is close to $|V|^2$
  - We need to quickly determine if there is an edge between two vertices
- Time to determine if $(u, v) \in E$:
  - $\Theta(1)$
- Disadvantage
  - No quick way to determine the vertices adjacent to a vertex
- Time to list all vertices adjacent to u:
  - $\Theta(V)$

Graph Adjacency List

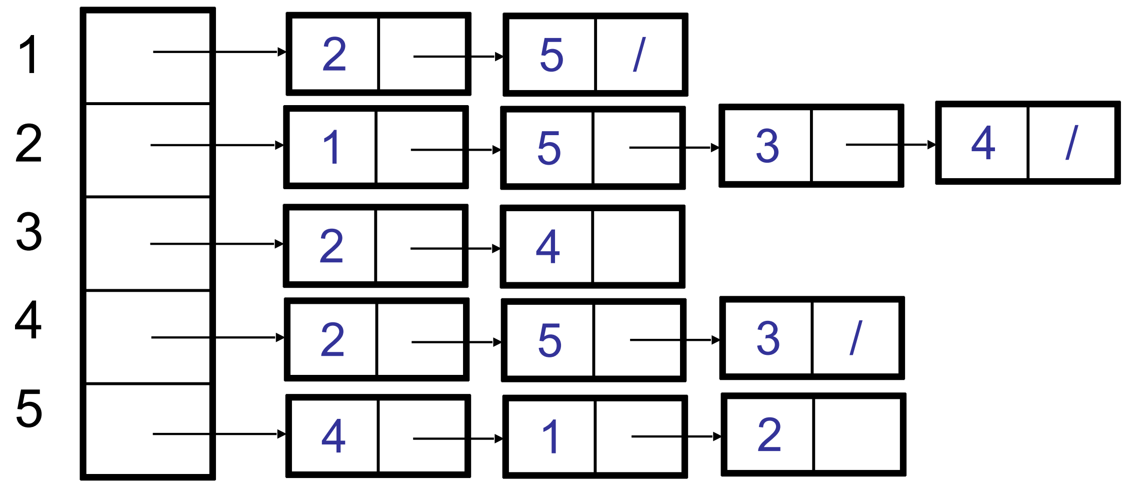
- **Adjacency list representation** of G = (V, E)
  - An array of $|V|$ lists, one for each vertex in V
  - Each list Adj[u] contains all the vertices v that are adjacent to u (i.e., there is an edge from u to v)
  - Can be used for both directed and undirected graphs

Graph Adjacency List

- Example

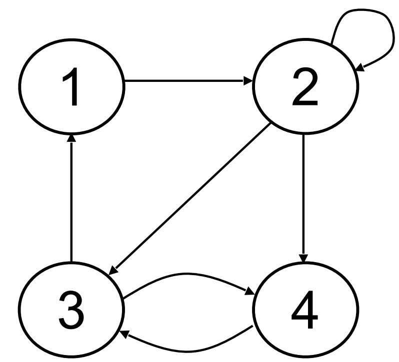


Undirected graph

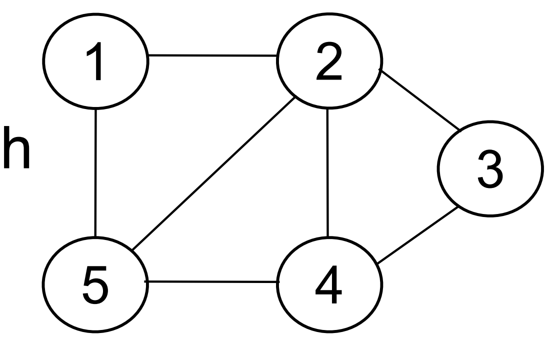


Properties of Adjacency-List Representation

- Sum of "lengths" of all adjacency lists
  - Directed graph: $|E|$
    - edge (u, v) appears only once (i.e., in the list of **u**)
  - Undirected graph: $2|E|$
    - edge (u, v) appears twice (i.e., in the lists of both **u** and **v**)



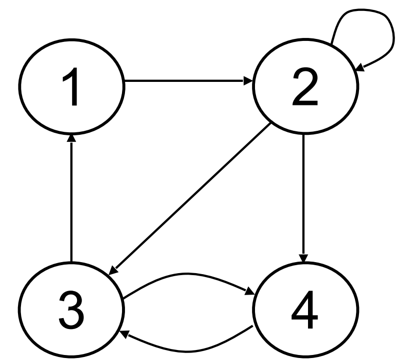
Directed graph



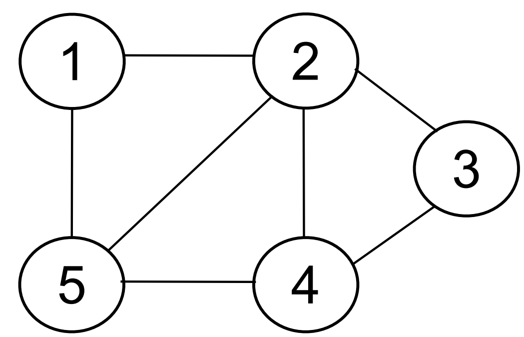
Undirected graph

Properties of Adjacency-List Representation

- Memory required
  - $\Theta(V + E)$
- Preferred when
  - The graph is **sparse**: $|E| << |V|^2$
  - We need to quickly determine the nodes adjacent to a given node.
- Disadvantage
  - No quick way to determine whether there is an edge between node u and v
- Time to determine if $(u, v) \in E$:
  - $O(\text{degree}(u))$
- Time to list all vertices adjacent to u:
  - $\Theta(\text{degree}(u))$



Directed graph



Undirected graph

Graph Search

- Given: a graph G = (V, E), directed or undirected
  - In general, given a vertex s, we want to locate some vertex t.
    - Find a path in G
  - We want to visit all vertices in a "local" organized manner

Breadth-First Search (BFS)

- "Explore" a graph, turning it into a tree
  - start with a *source vertex,* explore all other vertices *reachable* from the source, one vertex at a time
  - expand frontier of explored vertices across the *breadth* of the frontier
  - compute the distance (smallest number of edges) from source to each reachable vertex
- Builds a breadth-first tree over the graph
  - source is the root, cover all reachable vertices
  - find ("discover") its children, then their children, etc.
    - discover vertices at distance k from source before discovering vertices at distance k+1
  - the path from source to a vertex in breadth-first tree is the shortest path in the original graph

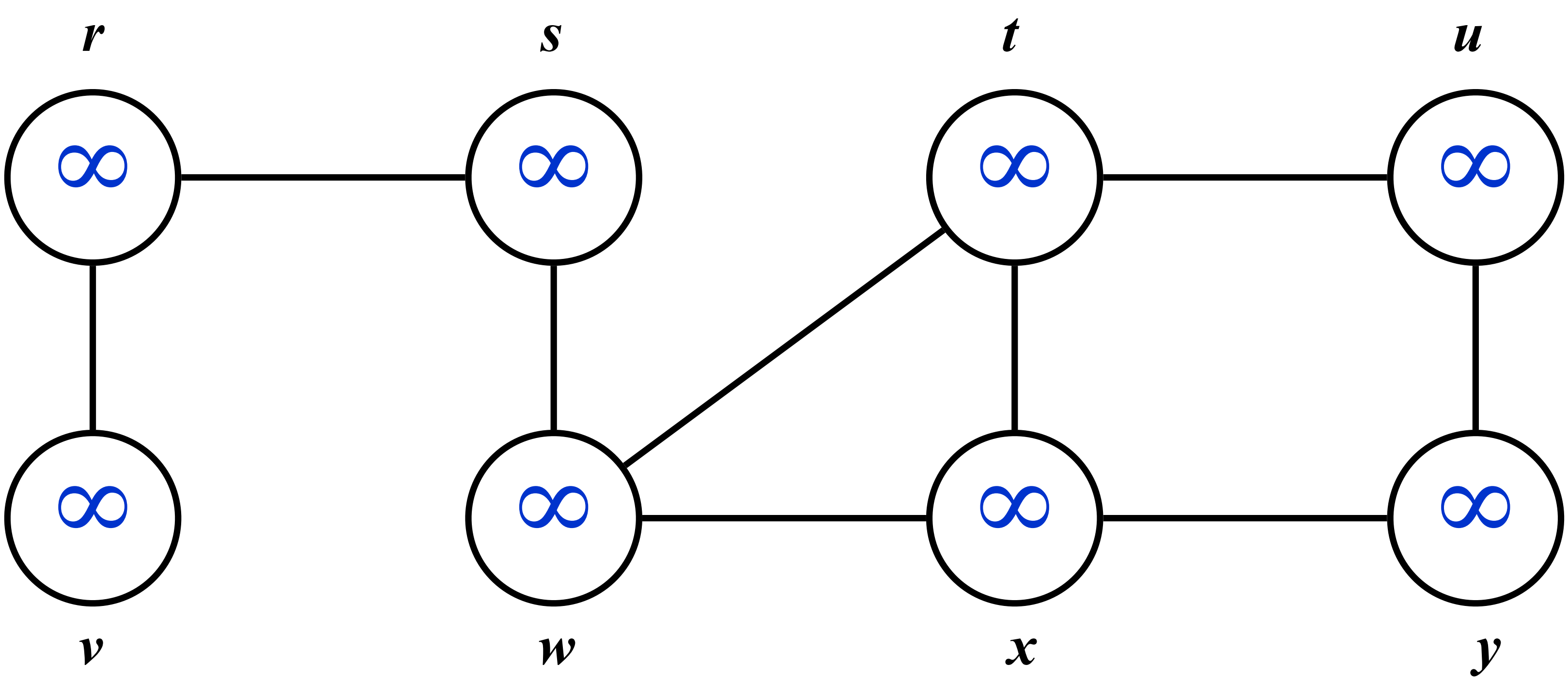
Breadth-First Search (BFS)

- Associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Gray vertices are discovered but not fully explored
    - They may have some adjacent white vertices
  - Black vertices are discovered and fully explored
    - adjacent vertices of a black vertices are either black or gray
- Explore vertices by scanning adjacency list of gray vertices

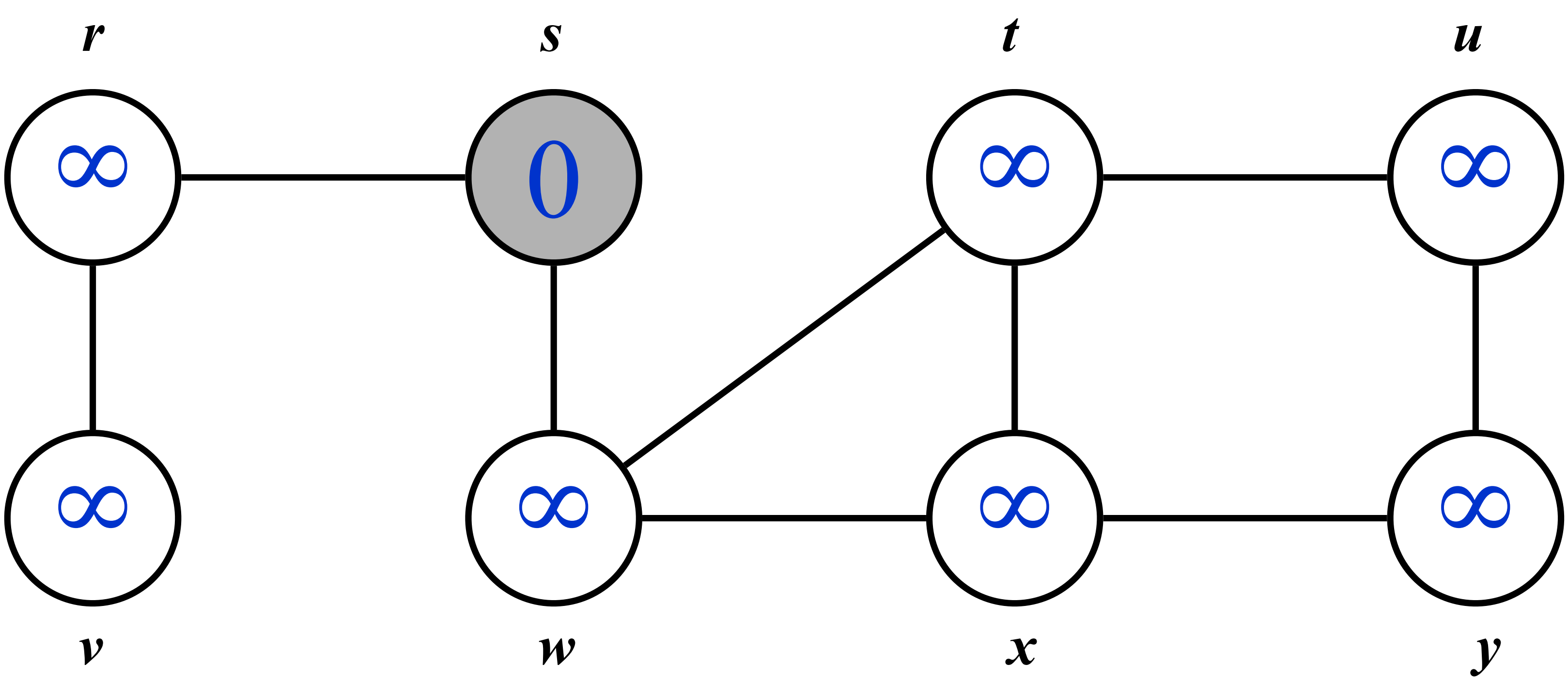
Breadth-First Search (BFS)

```
BFS(G,s) {
    for each u in V {                        // initialization
        color[u] = white
        d[u]     = infinity
        pred[u]  = null
    }
    color[s] = gray                          // initialize source s
    d[s] = 0
    Q = {s}                                  // put s in the queue
    while (Q is nonempty) {
        u = Q.Dequeue()                      // u is the next to visit
        for each v in Adj[u] {
            if (color[v] == white) {         // if neighbor v undiscovered
                color[v] = gray              // ...mark it discovered
                d[v]     = d[u]+1            // ...set its distance
                pred[v]  = u                 // ...and its predecessor
                Q.Enqueue(v)                 // ...put it in the queue
            }
        }
        color[u] = black                     // we are done with u
    }
}
```

Breadth-First Search (BFS)-Example

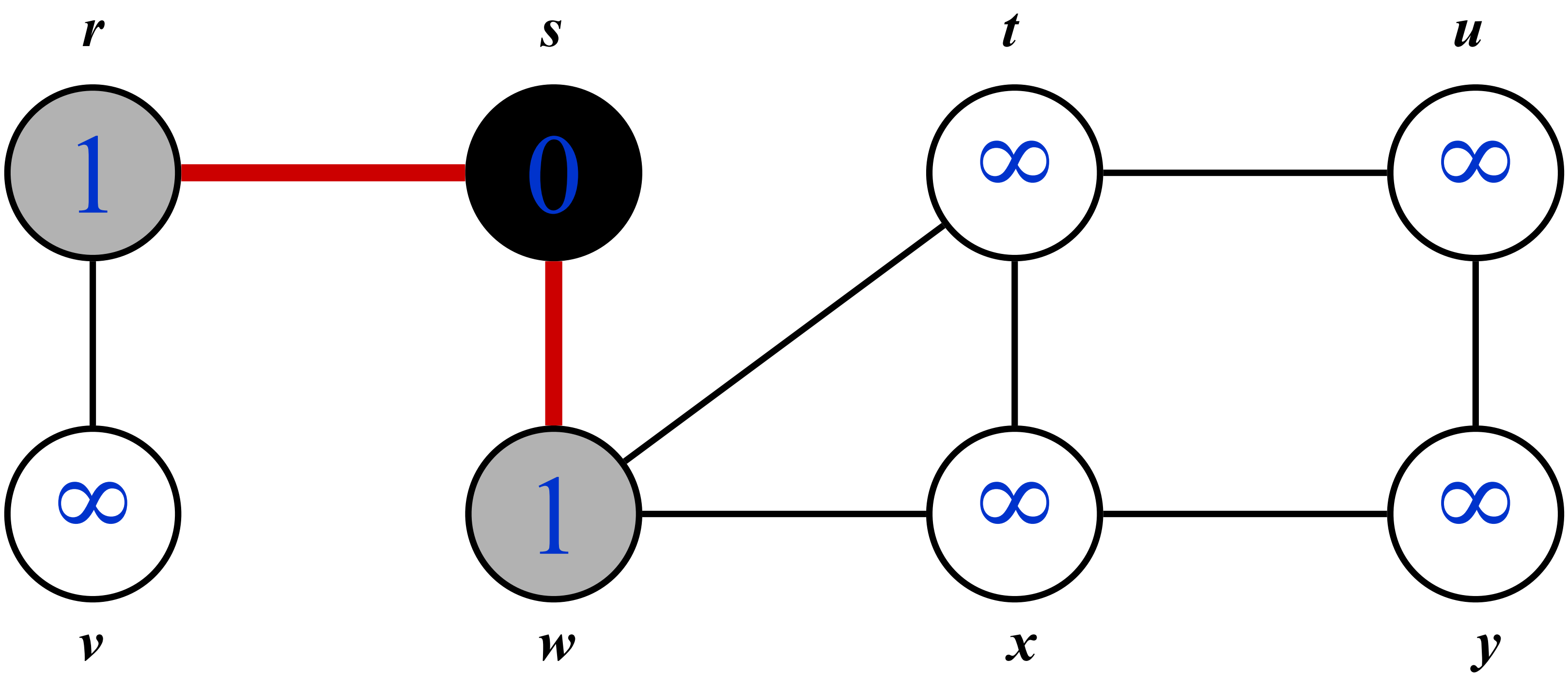


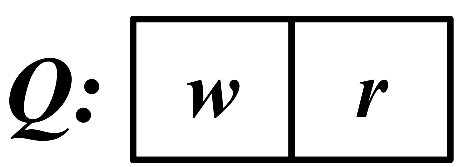
Breadth-First Search (BFS)-Example



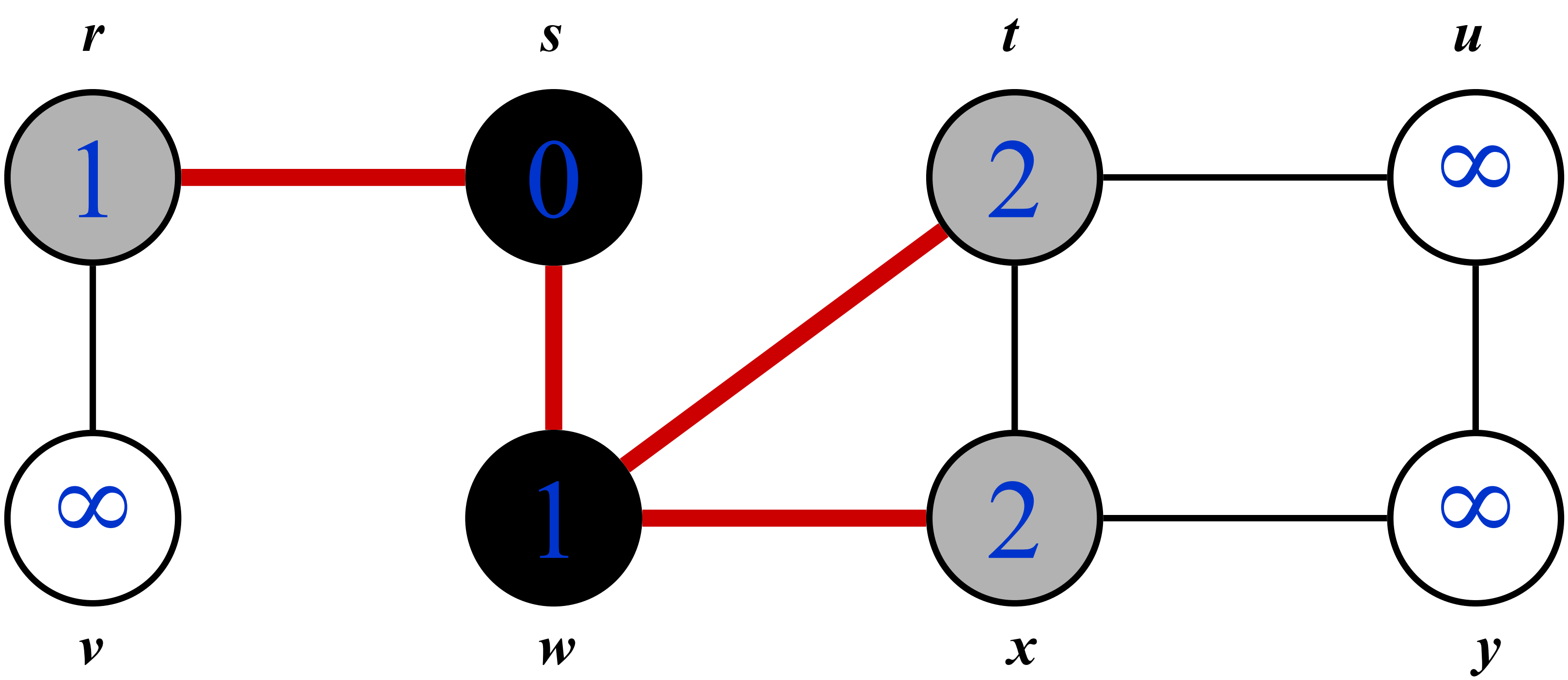
Q: $\boxed{s}$

Breadth-First Search (BFS)-Example



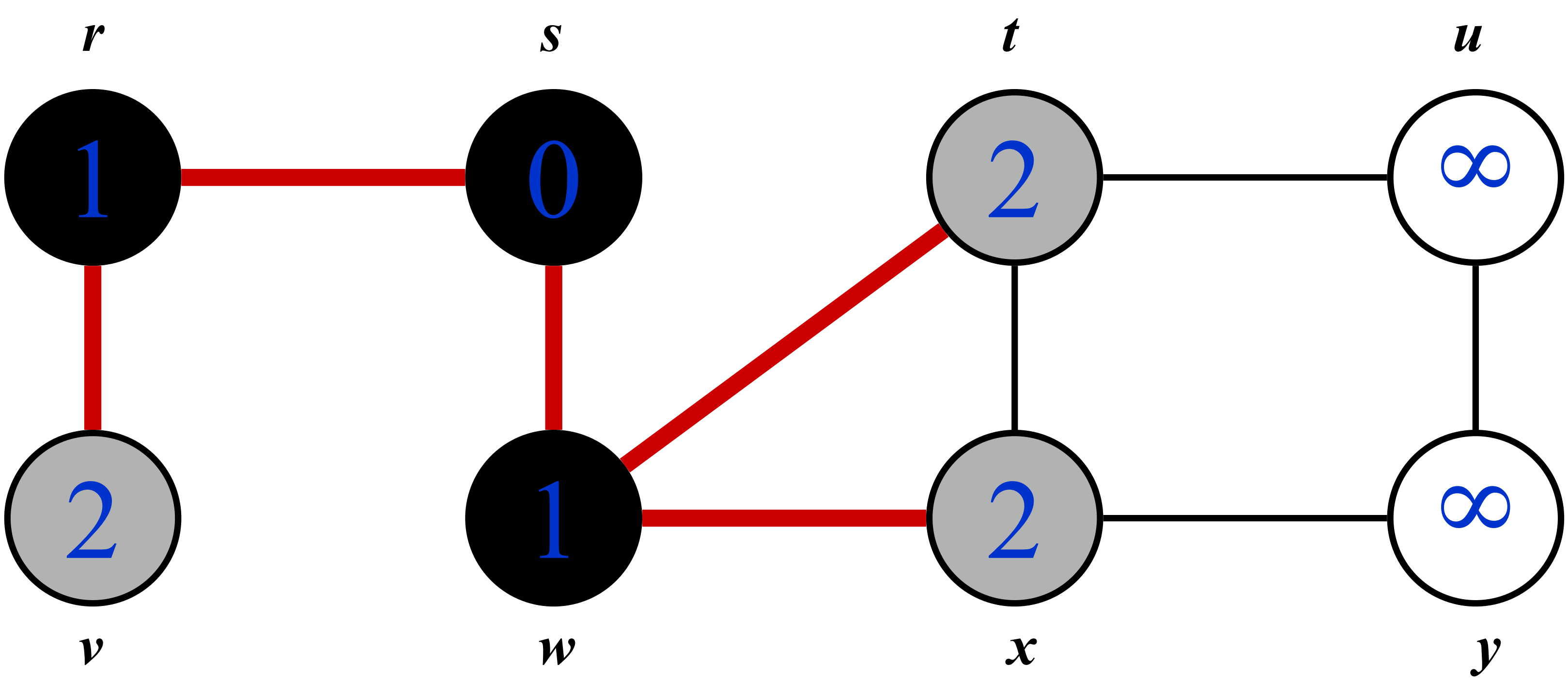


Breadth-First Search (BFS)-Example



Q:	r	t	x

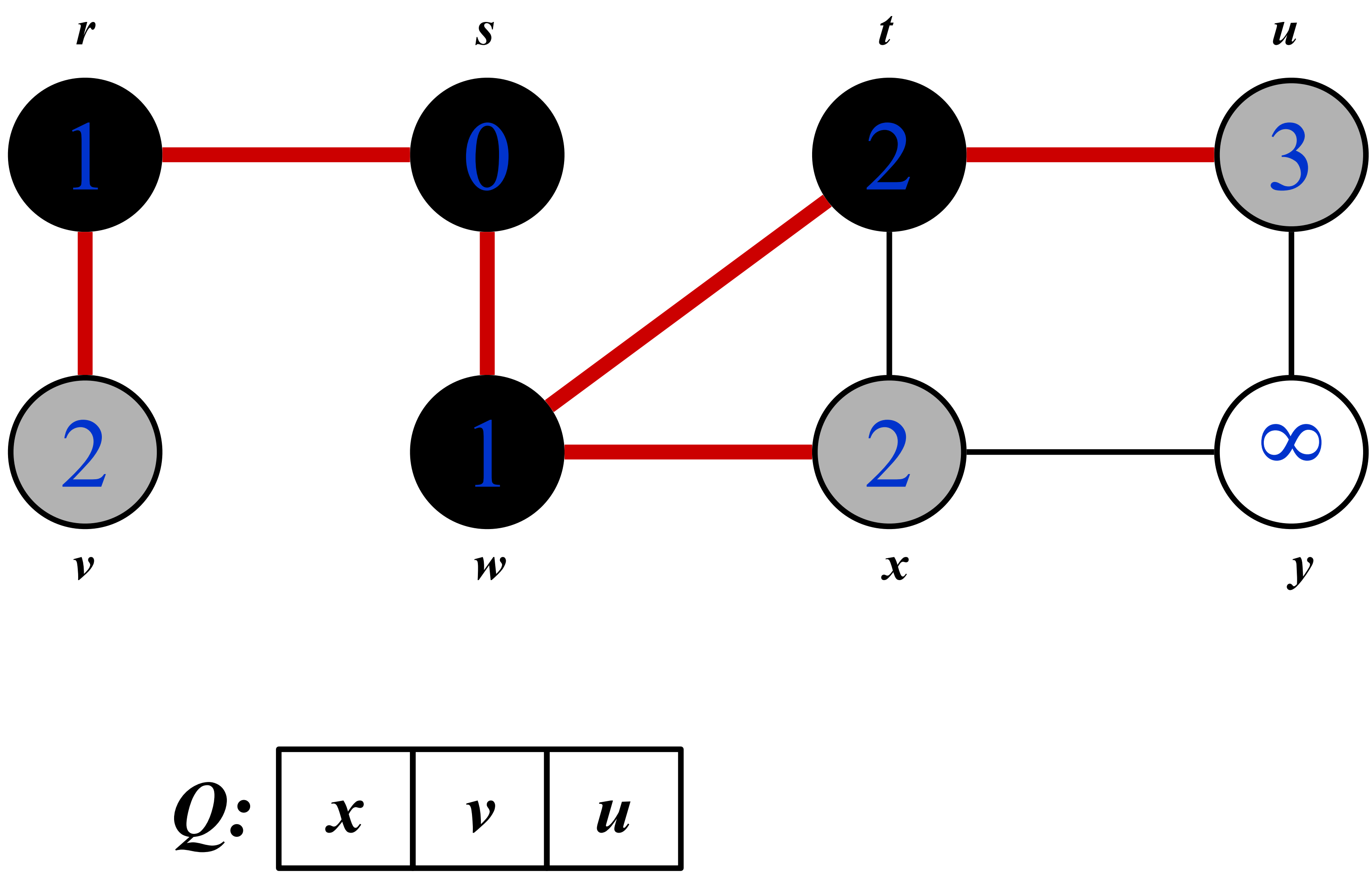
Breadth-First Search (BFS)-Example



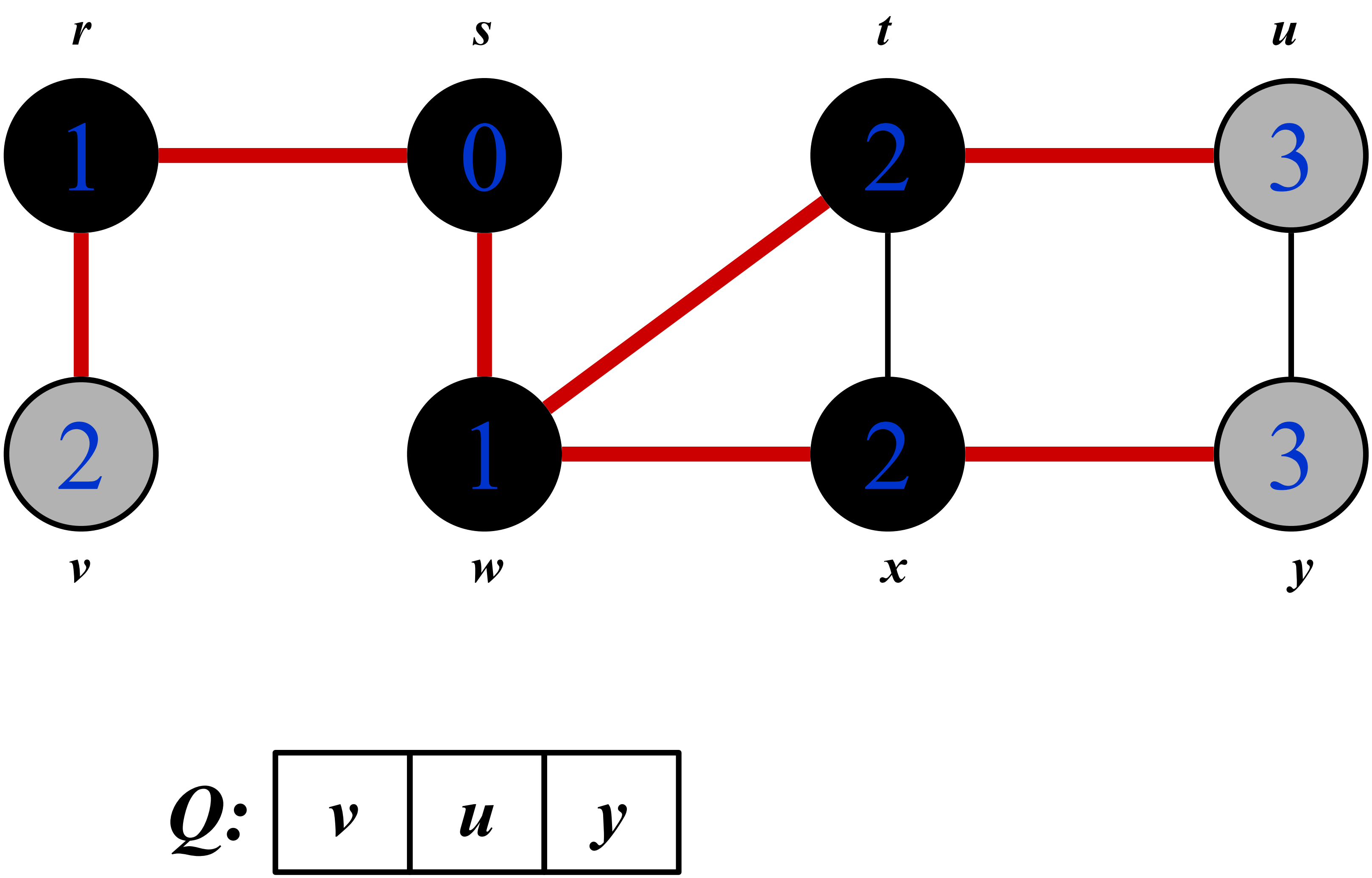
Q:

t	*x*	*v*

Breadth-First Search (BFS)-Example



Breadth-First Search (BFS)-Example



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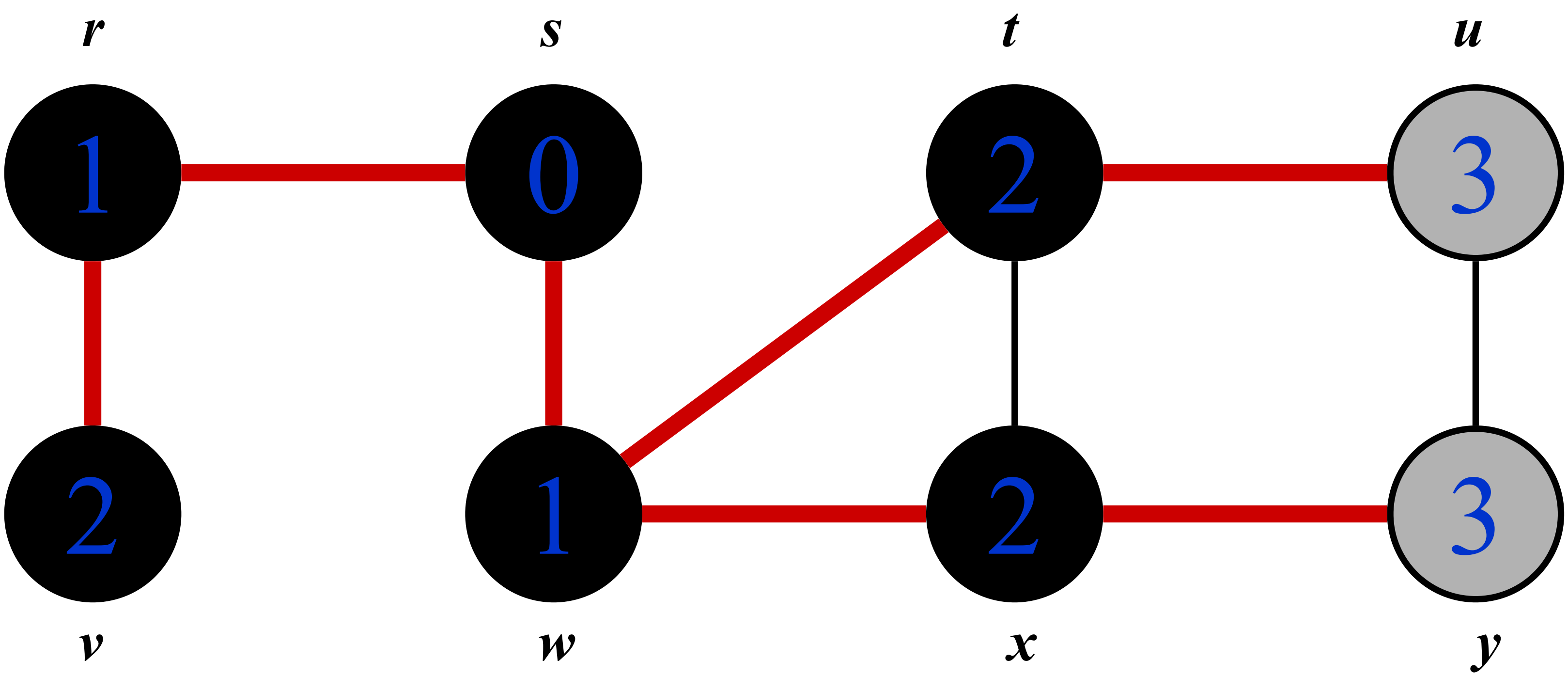
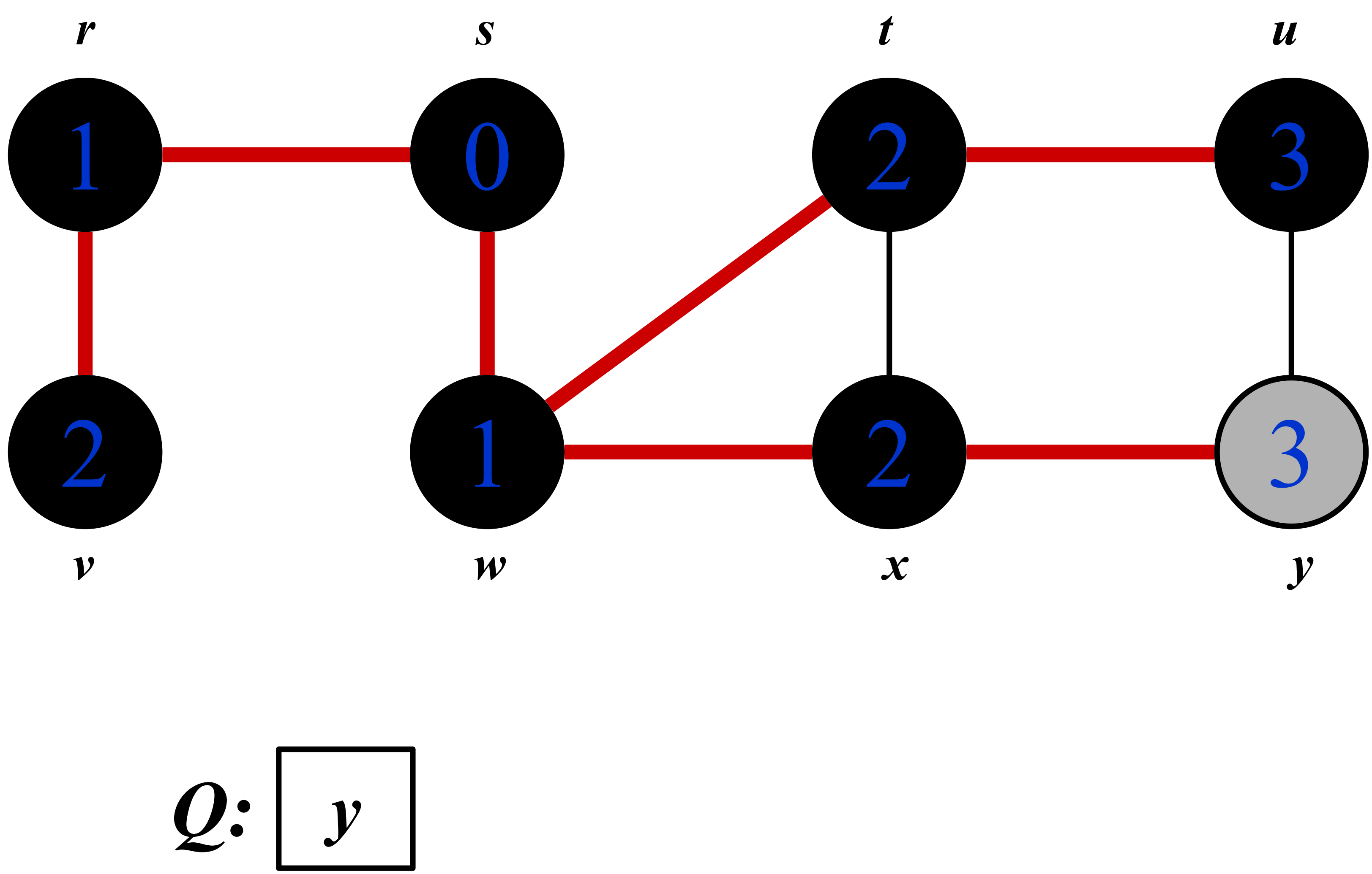


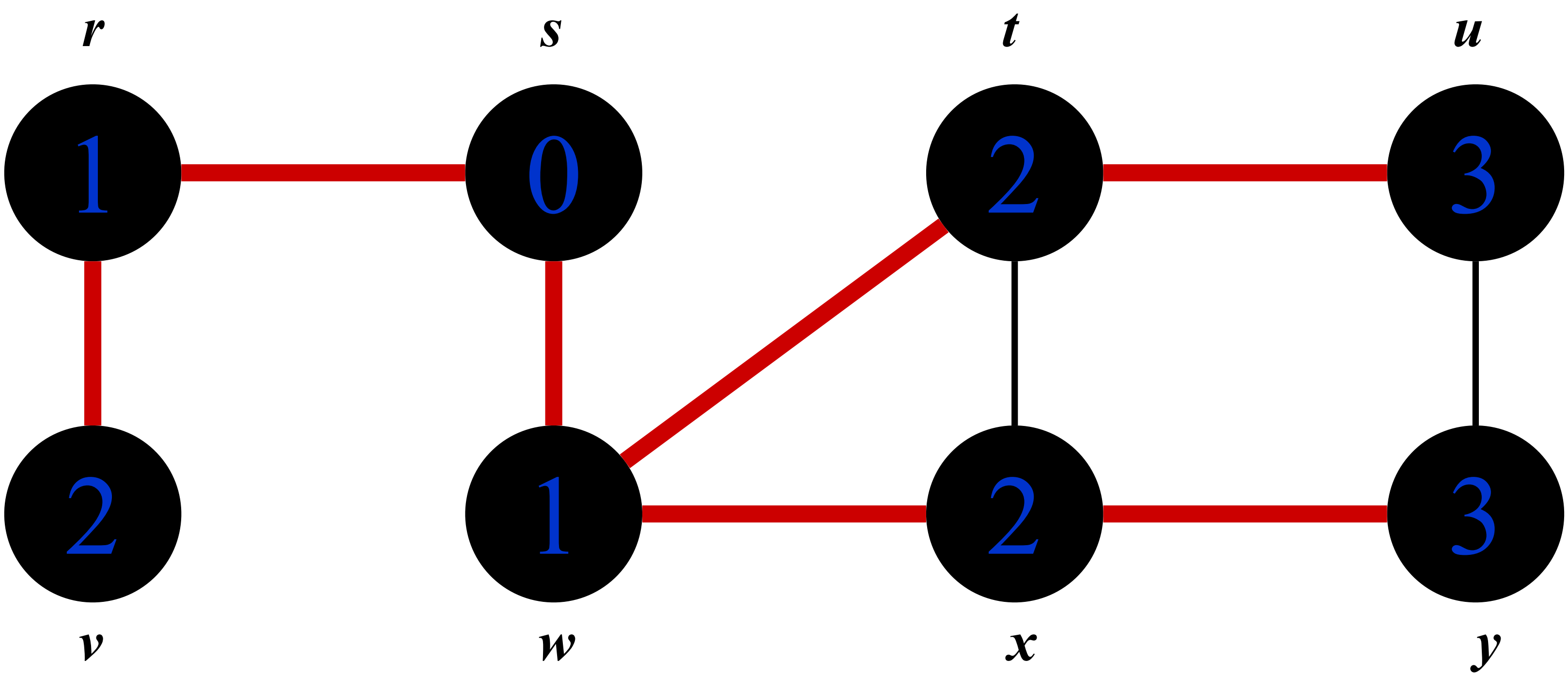
Q:

u	*y*

Breadth-First Search (BFS)-Example



Breadth-First Search (BFS)-Example



Q: Ø

BFS Properties

- BFS calculates a *shortest-path distance* from the source node to all other nodes
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
  - $d(v) = \delta(s,v)$, see proof in the book
- BFS builds a *breadth-first tree*
  - s is the root, pred(v) is the predecessor/parent of v in breadth-first tree (relative to s)
  - path from s to v in tree is a shortest path from s to v in G
  - Thus can use BFS to calculate shortest path from one vertex to another in $O(V+E)$ time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered

Depth-First Search

Again will associate vertex “colors” to guide the algorithm

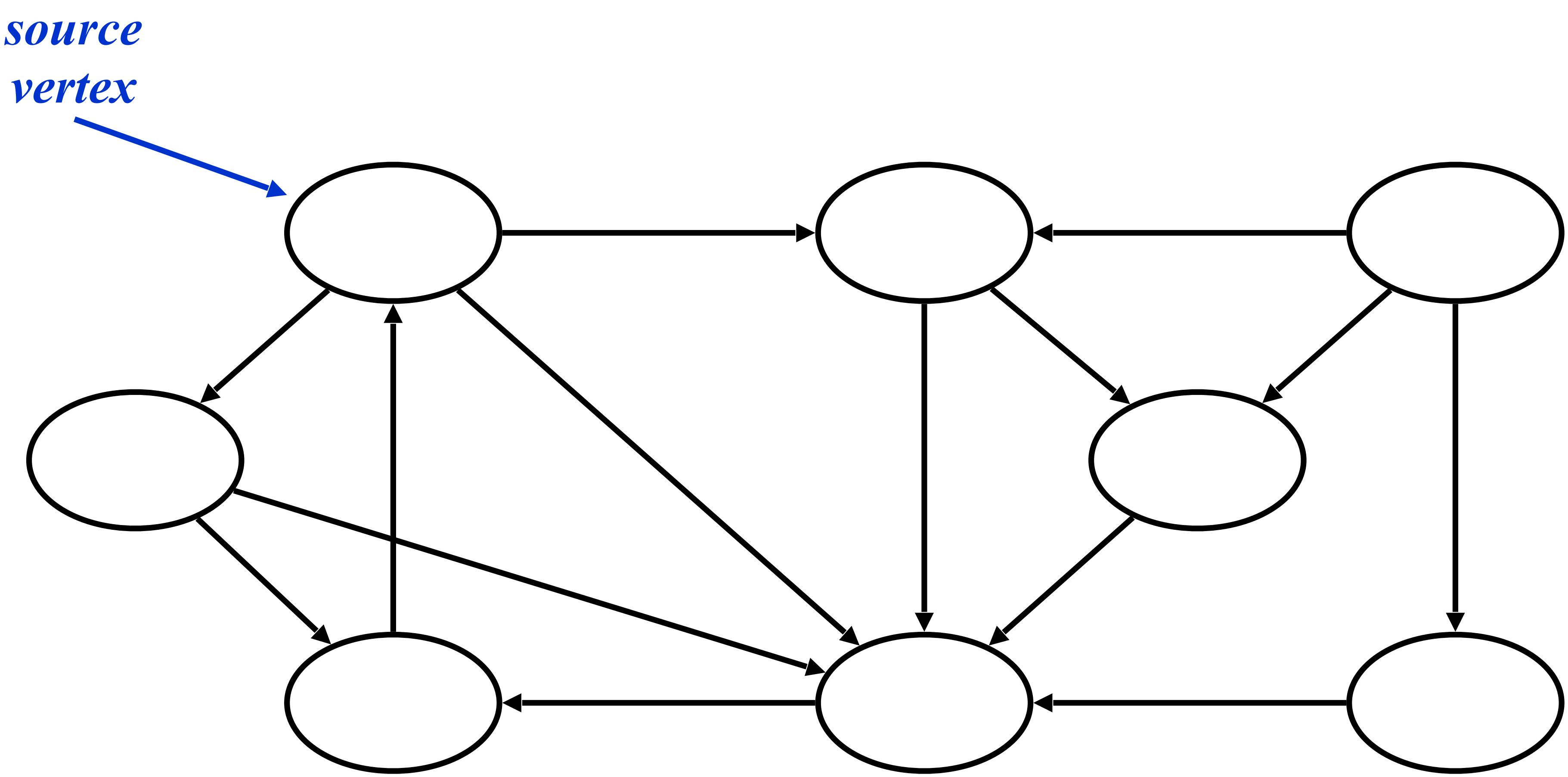
- Vertices initially colored white
- Then colored gray when discovered, not finished
- Then black when finished

Depth-First Search (DFS)

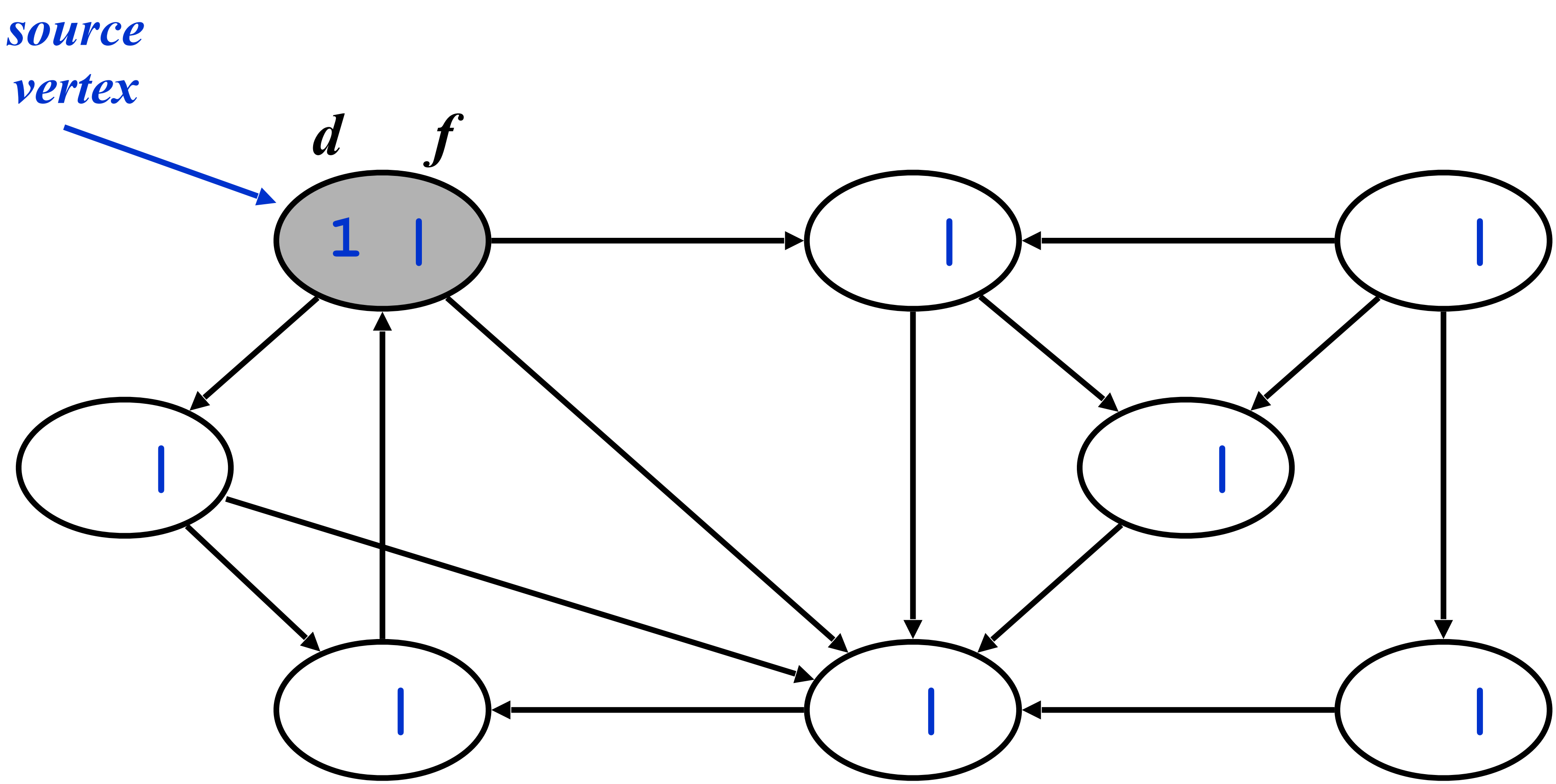
```
DFS(G) {                                        // main program
    for each u in V {                           // initialization
        color[u] = white;
        pred[u]  = null;
    }
    time = 0;
    for each u in V
        if (color[u] == white)                  // found an undiscovered vertex
            DFSVisit(u);                        // start a new search here
}

DFSVisit(u) {                                   // start a search at u
    color[u] = gray;                            // mark u visited
    d[u] = ++time;
    for each v in Adj(u) do
        if (color[v] == white) {                // if neighbor v undiscovered
            pred[v] = u;                        // ...set predecessor pointer
            DFSVisit(v);                        // ...visit v
        }
    color[u] = black;                           // we're done with u
    f[u] = ++time;
}
```

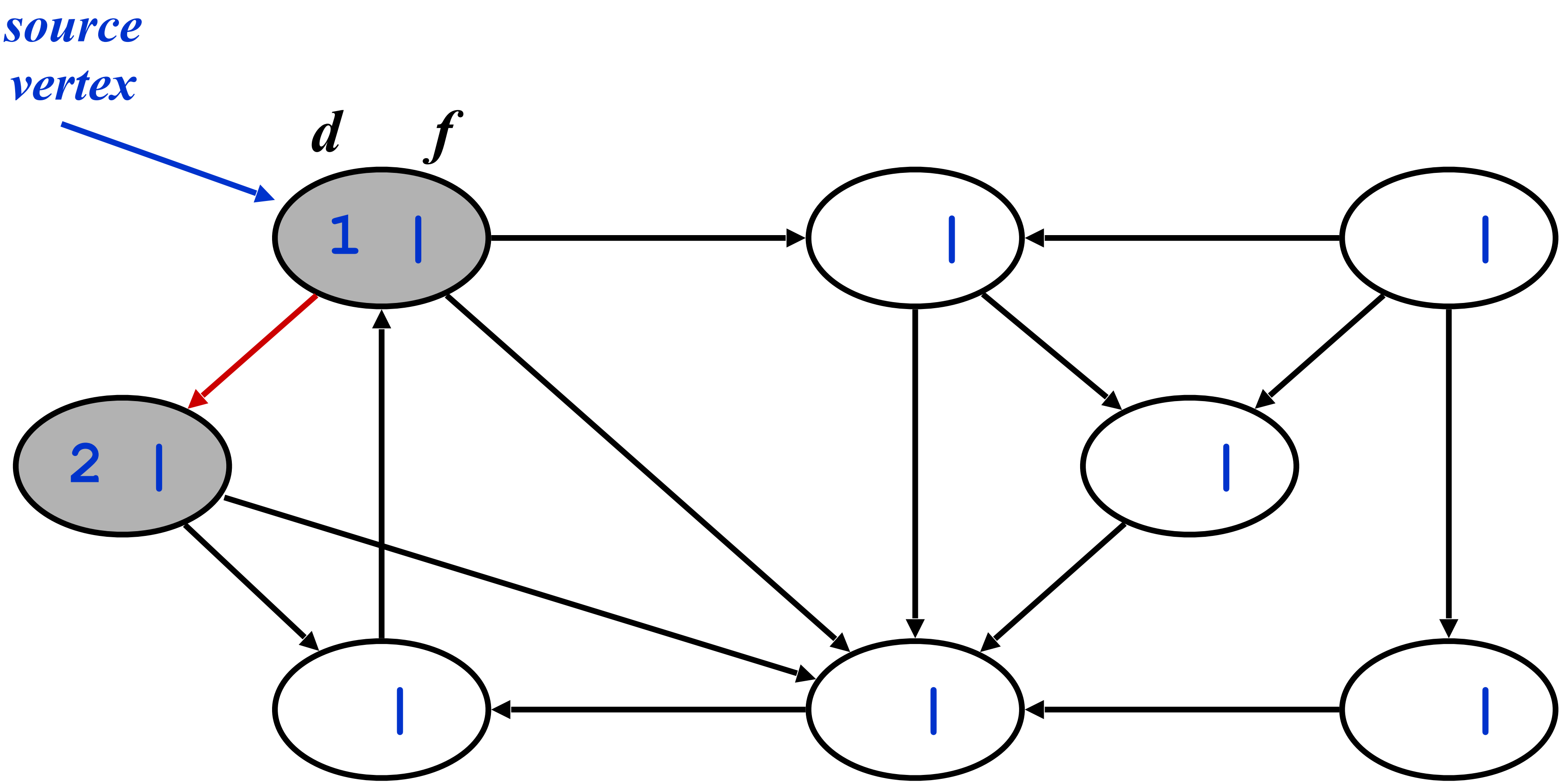
DFS Example



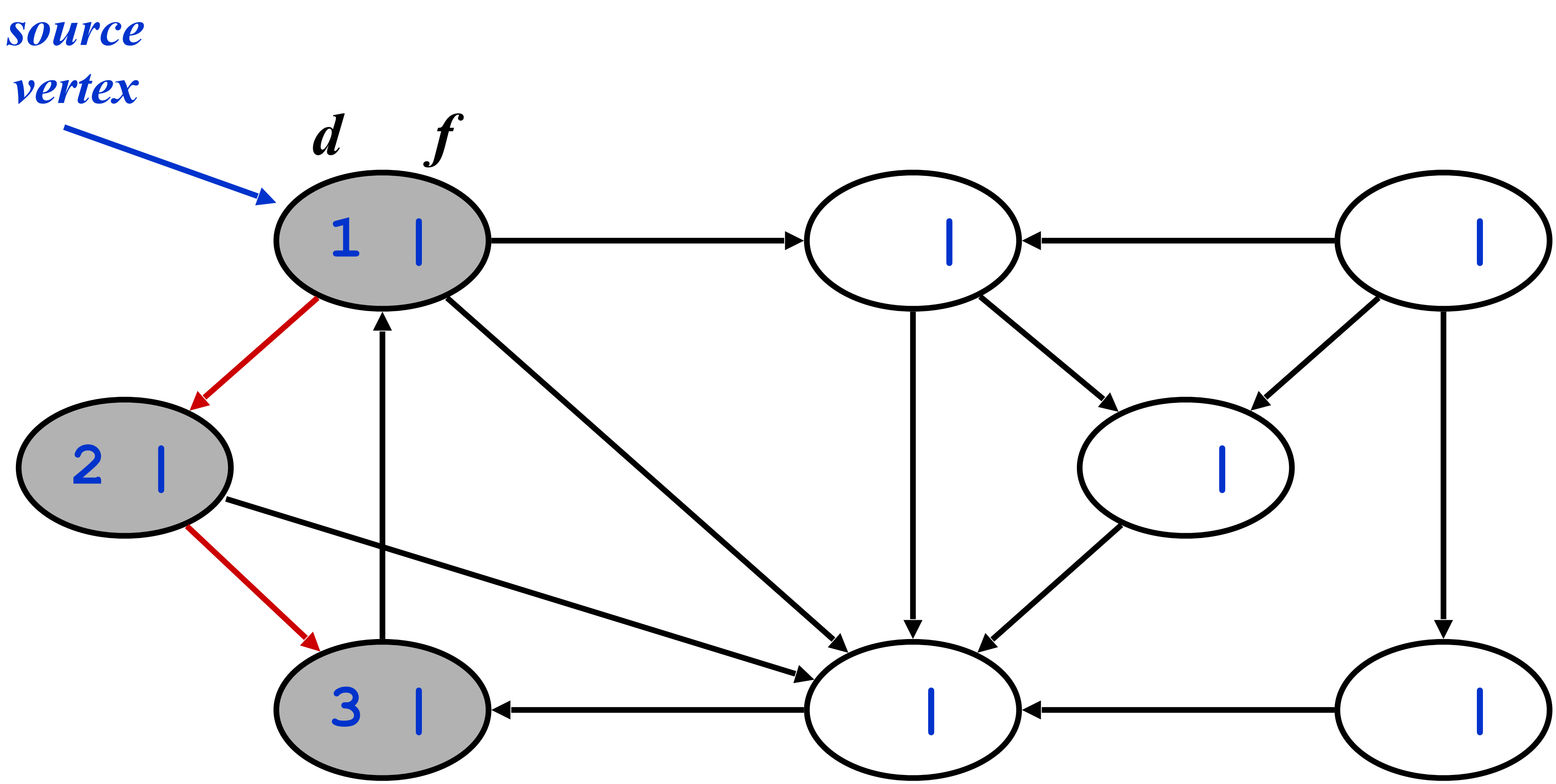
DFS Example



DFS Example



DFS Example



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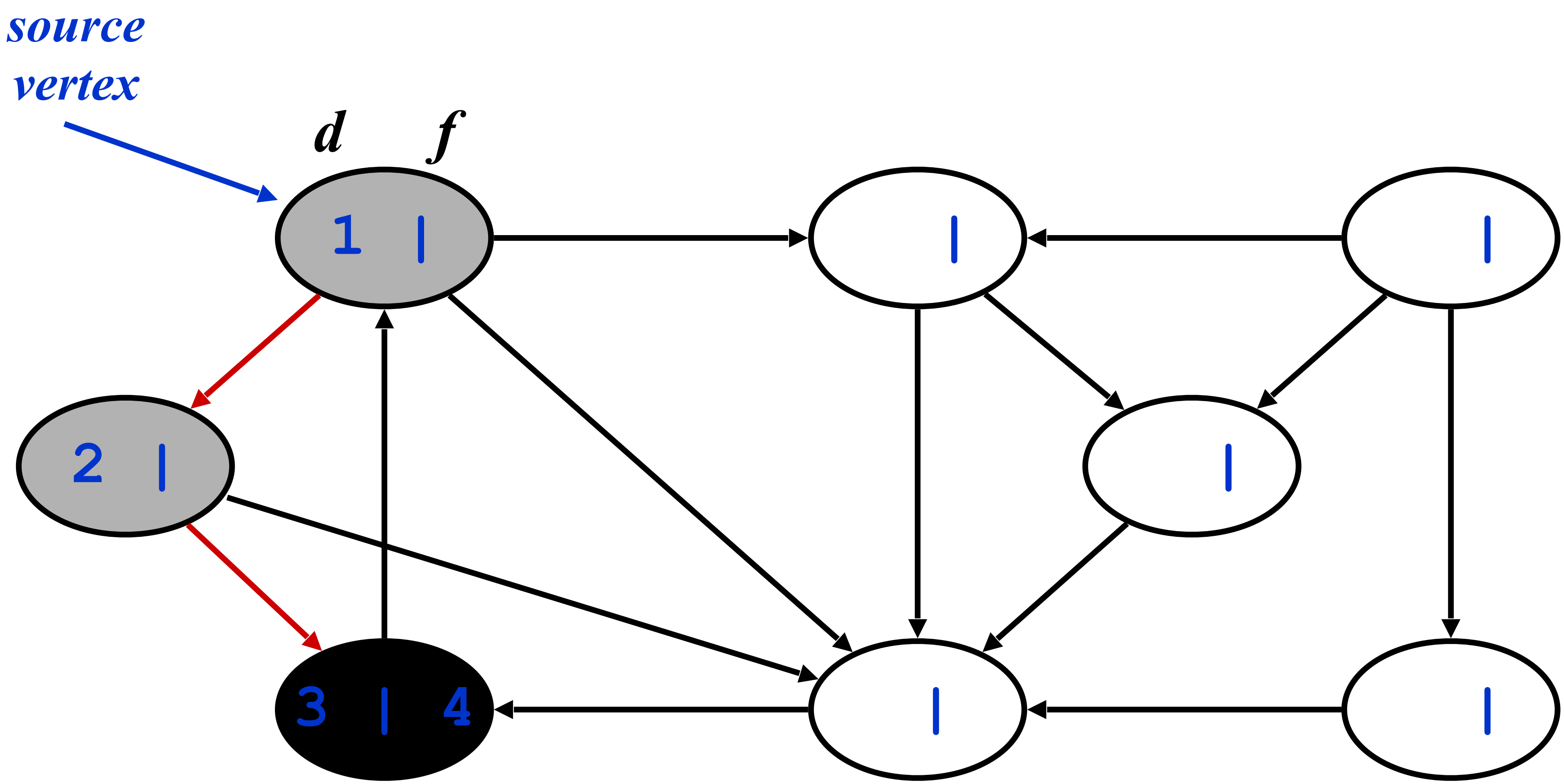
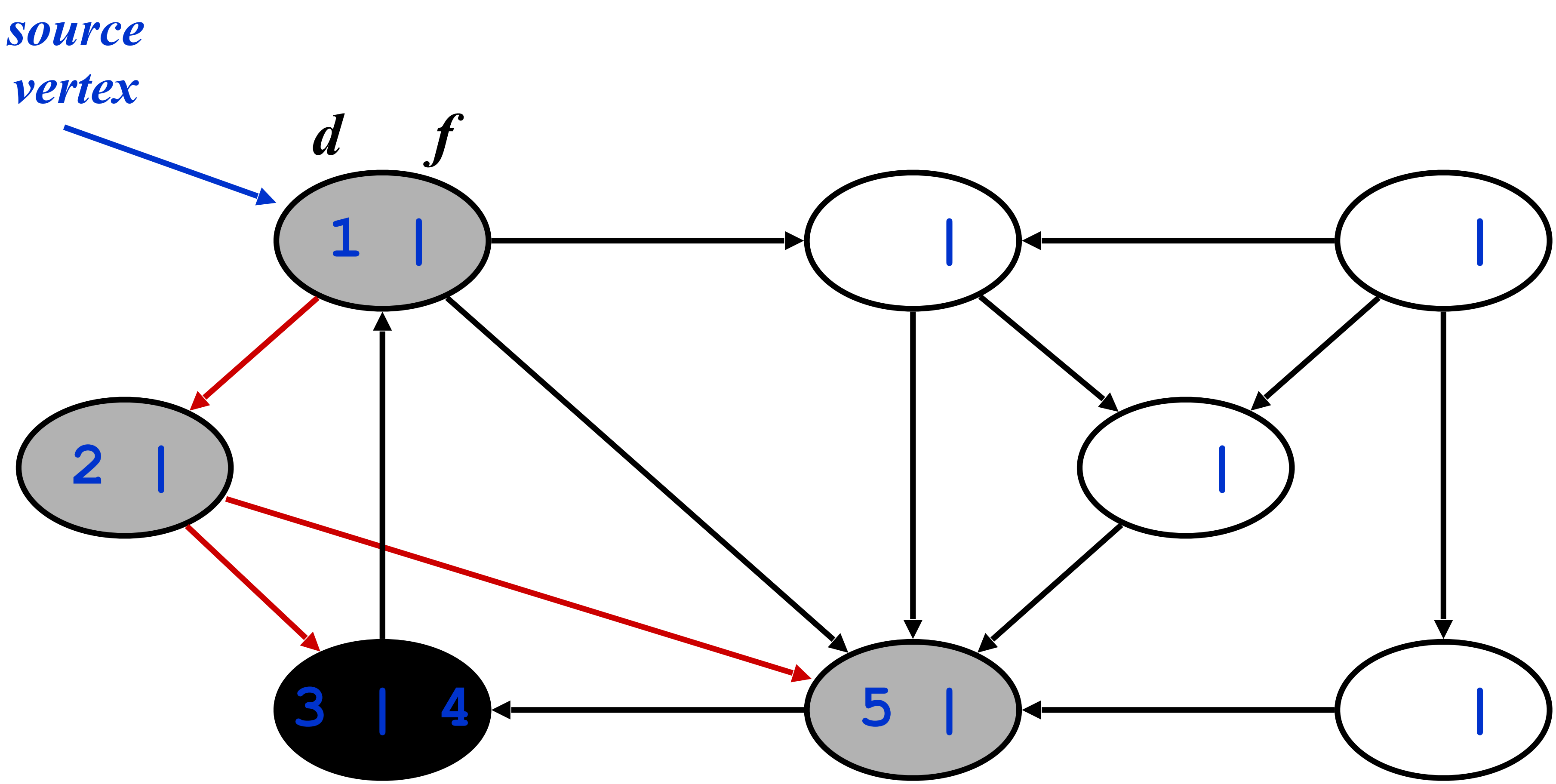


DFS Example



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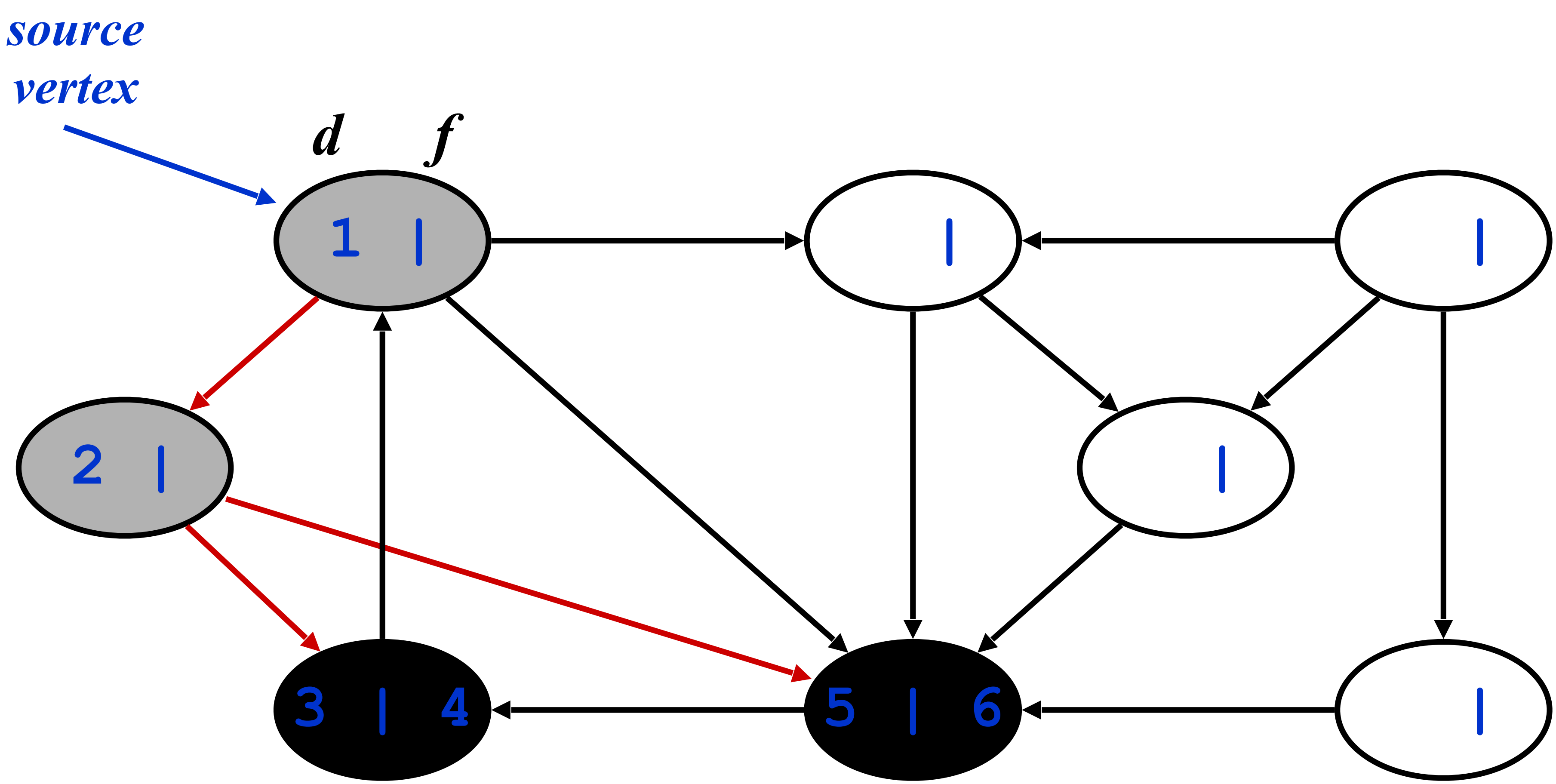
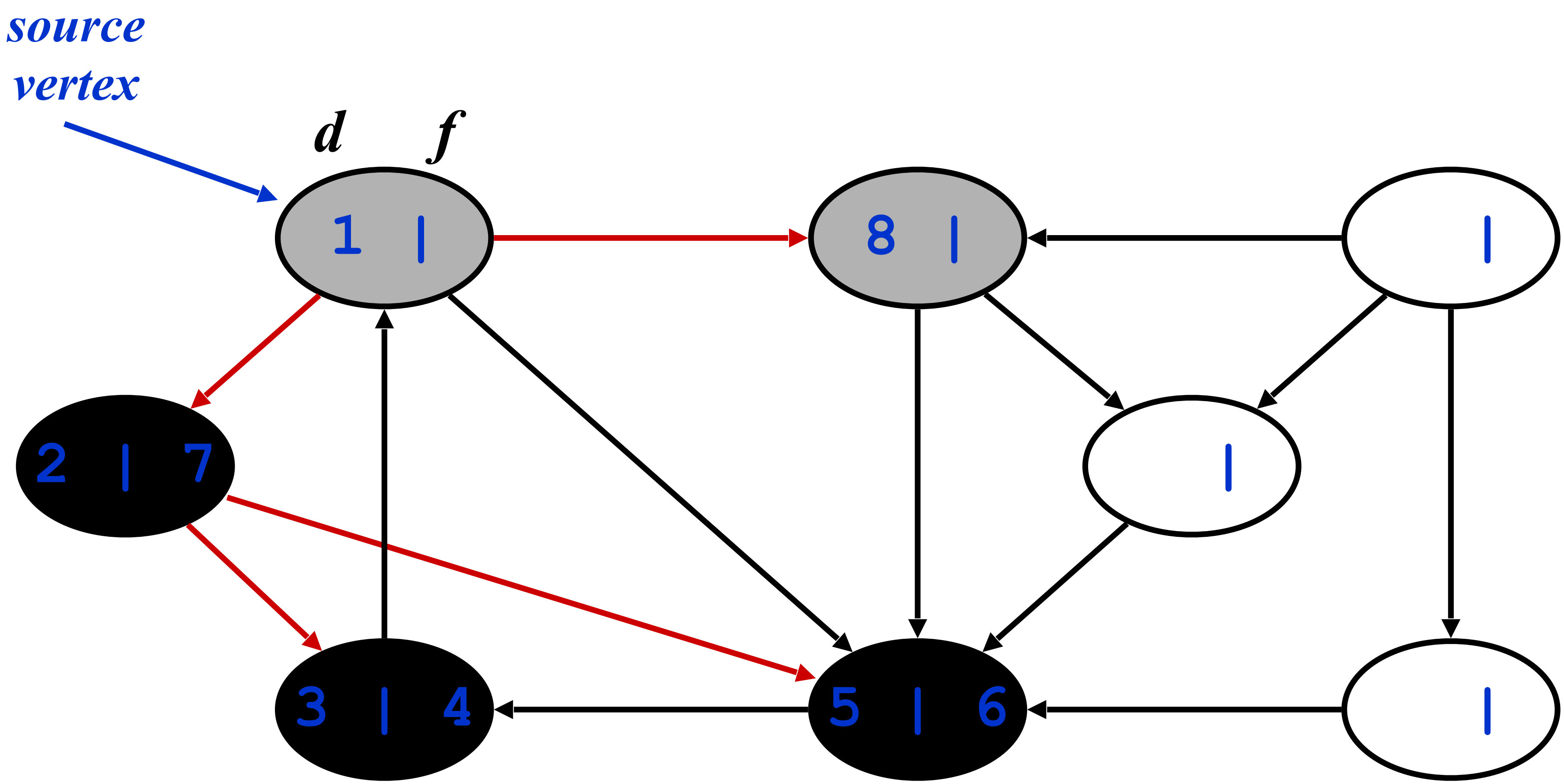
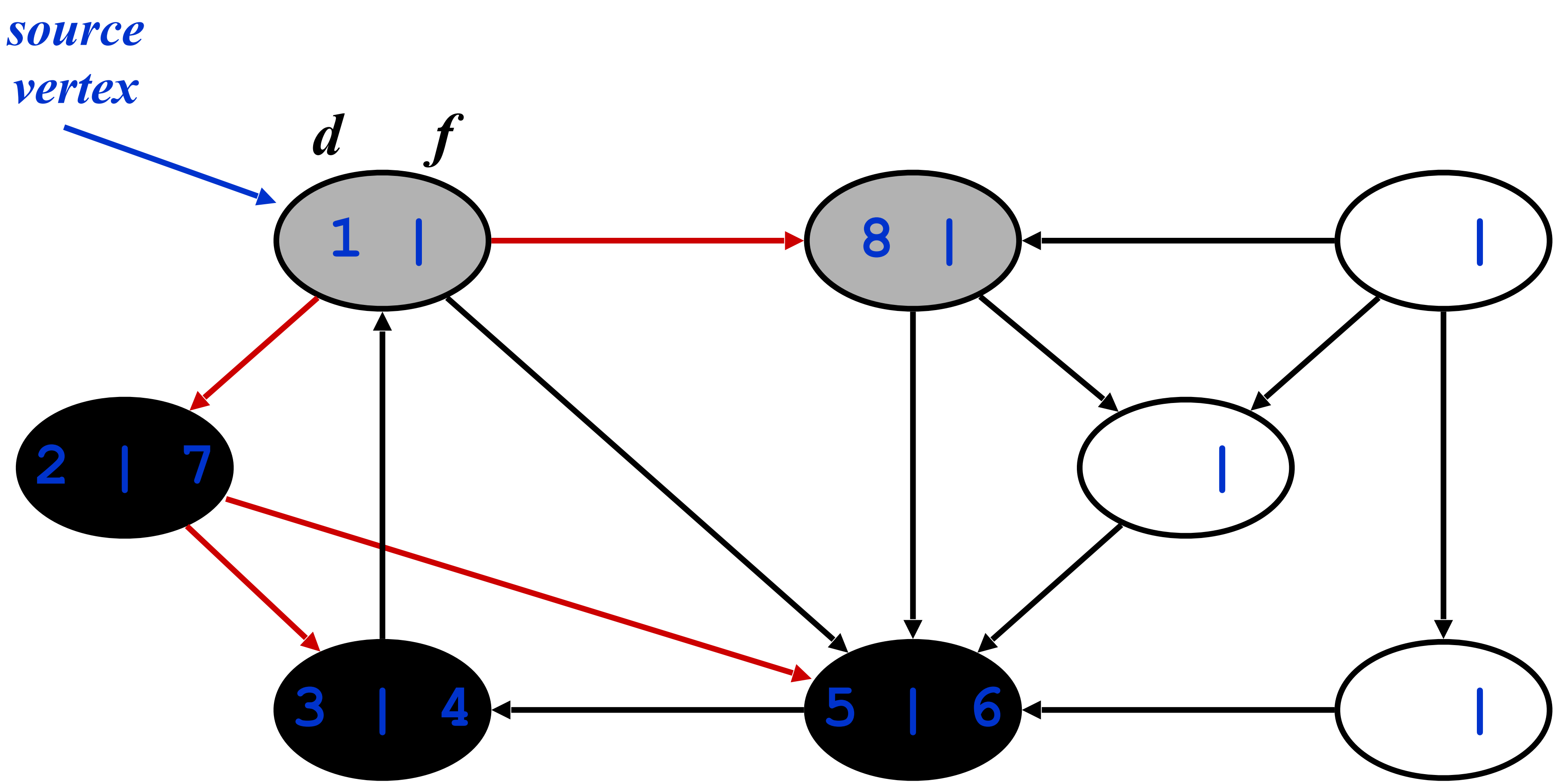


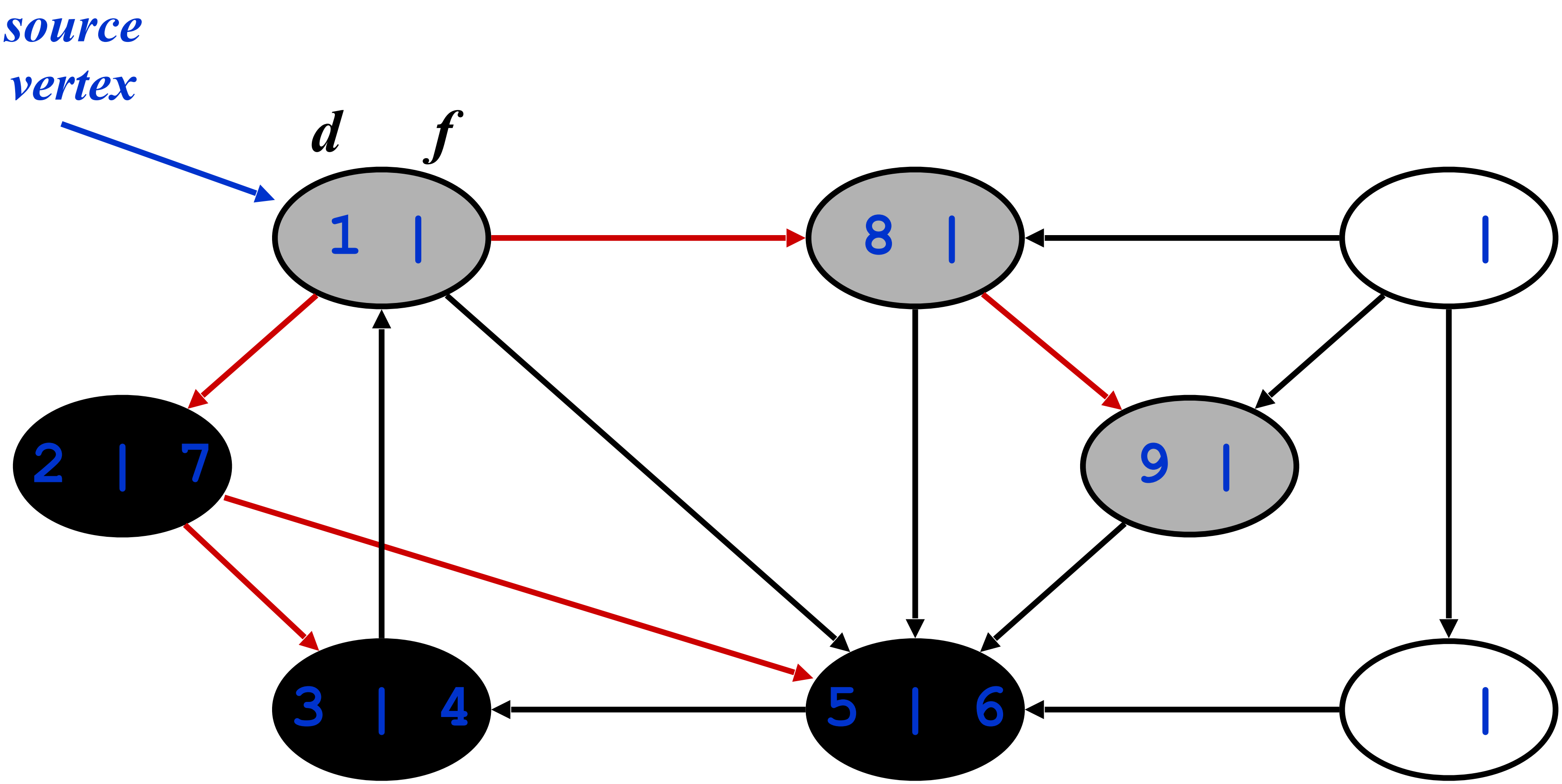
DFS Example



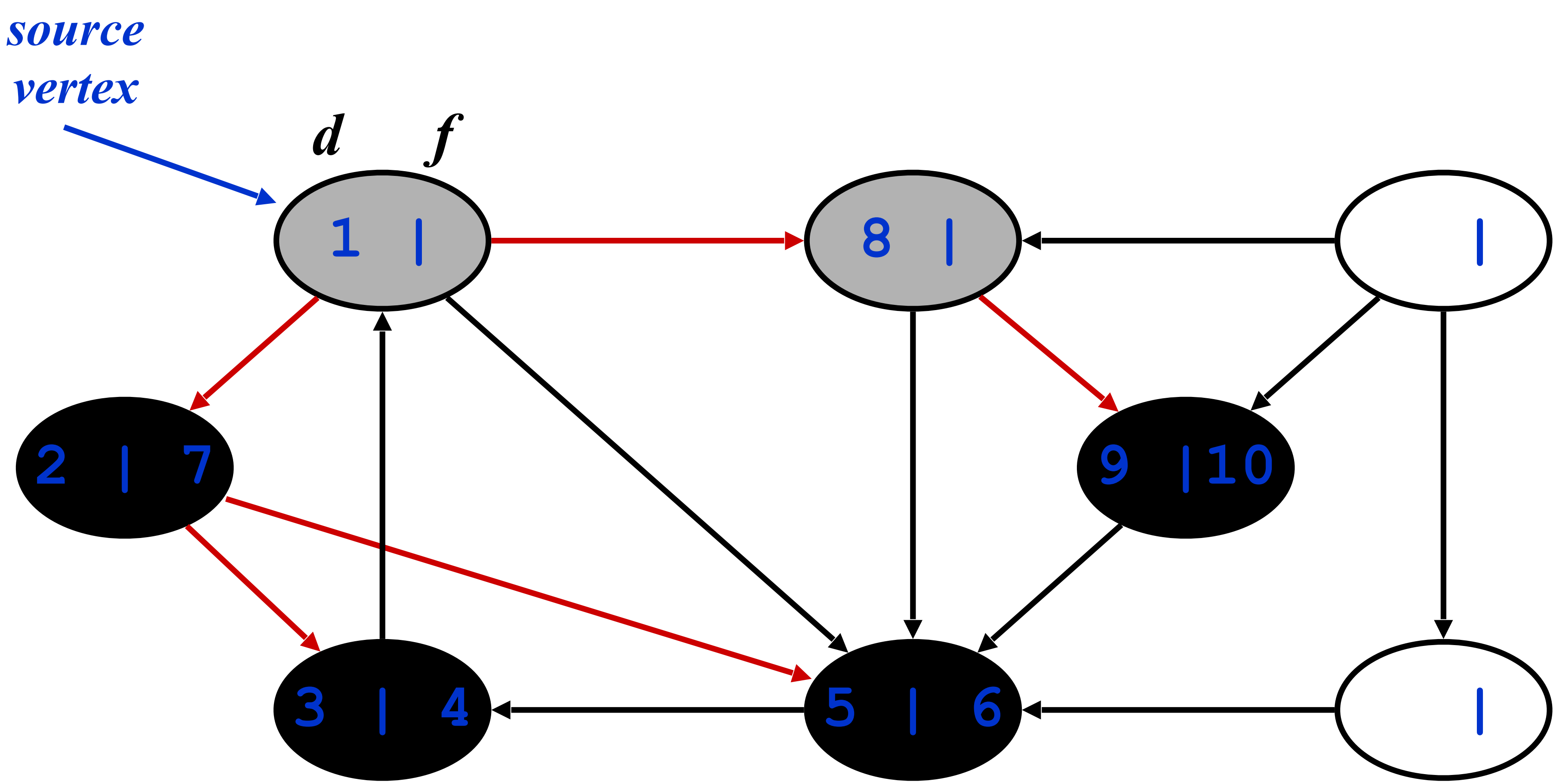
DFS Example



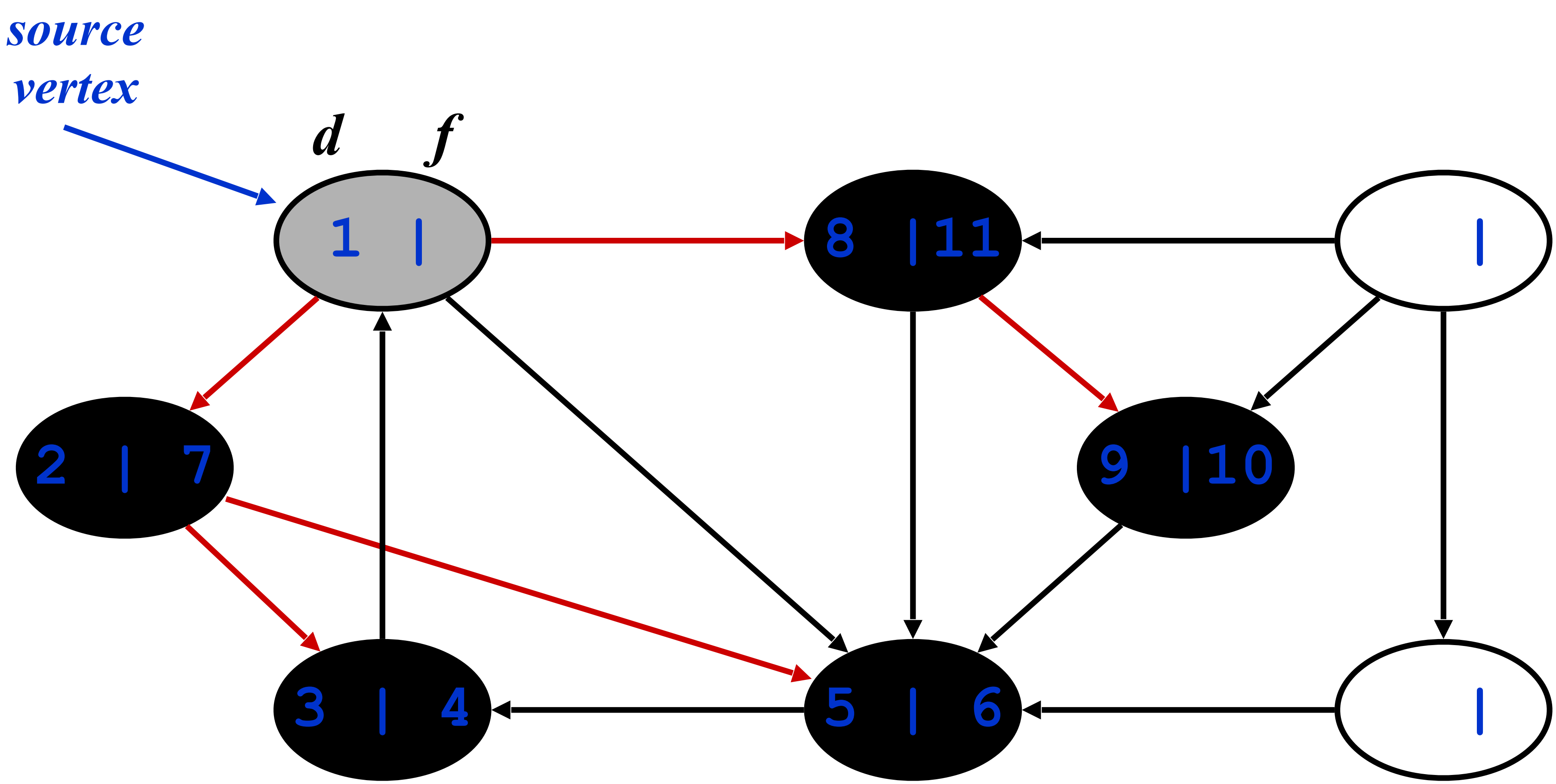
DFS Example



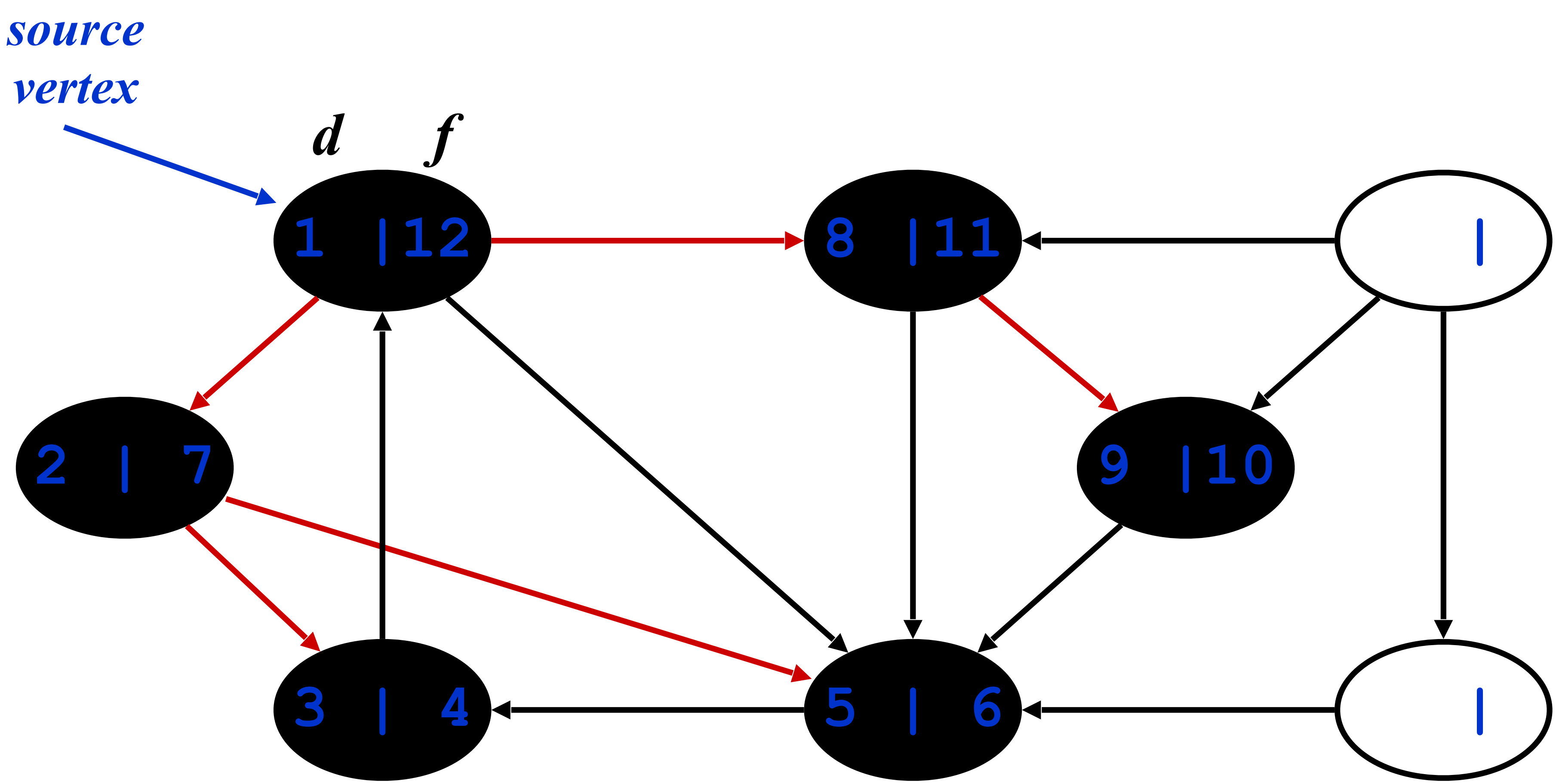
DFS Example



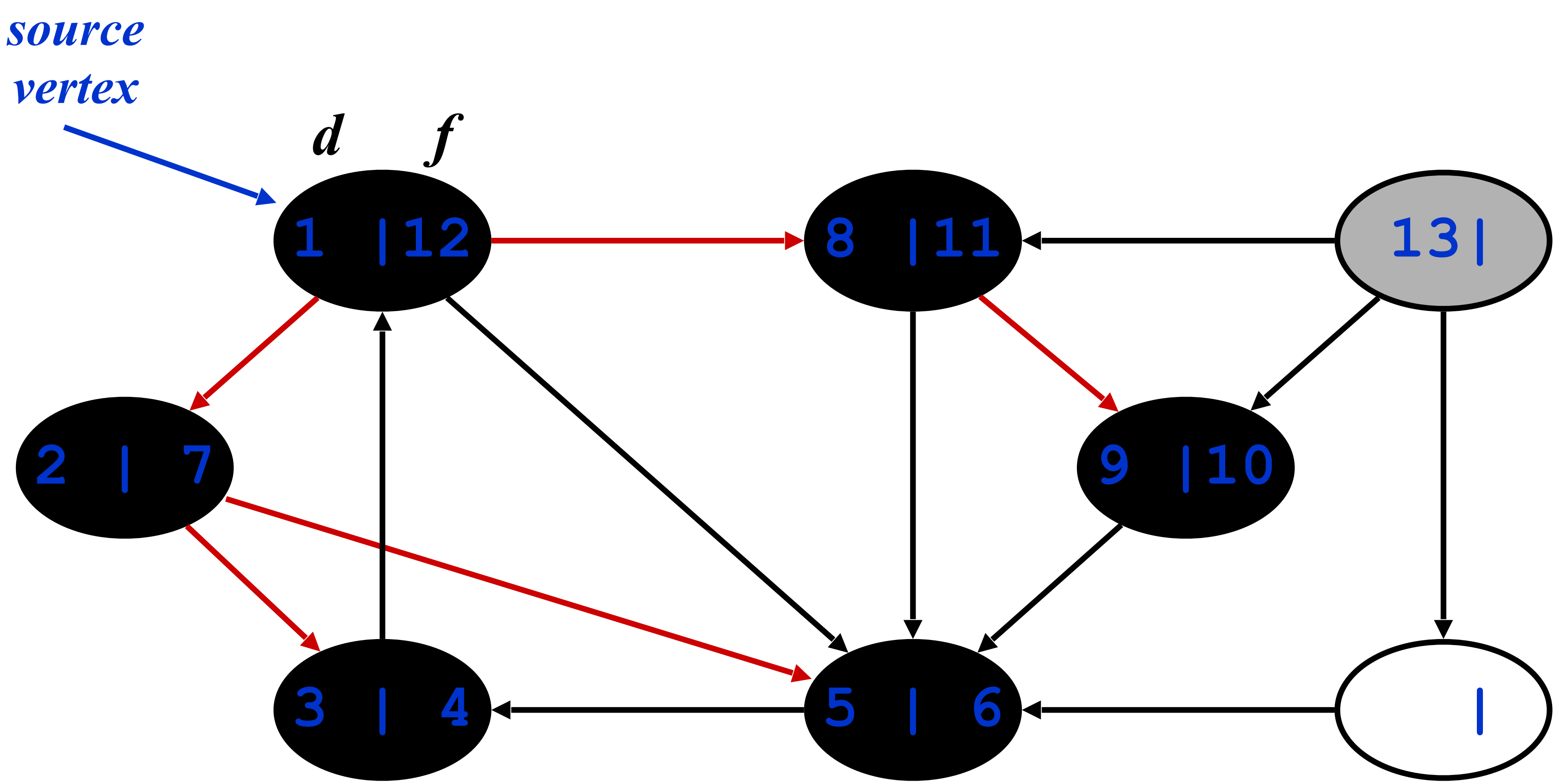
DFS Example



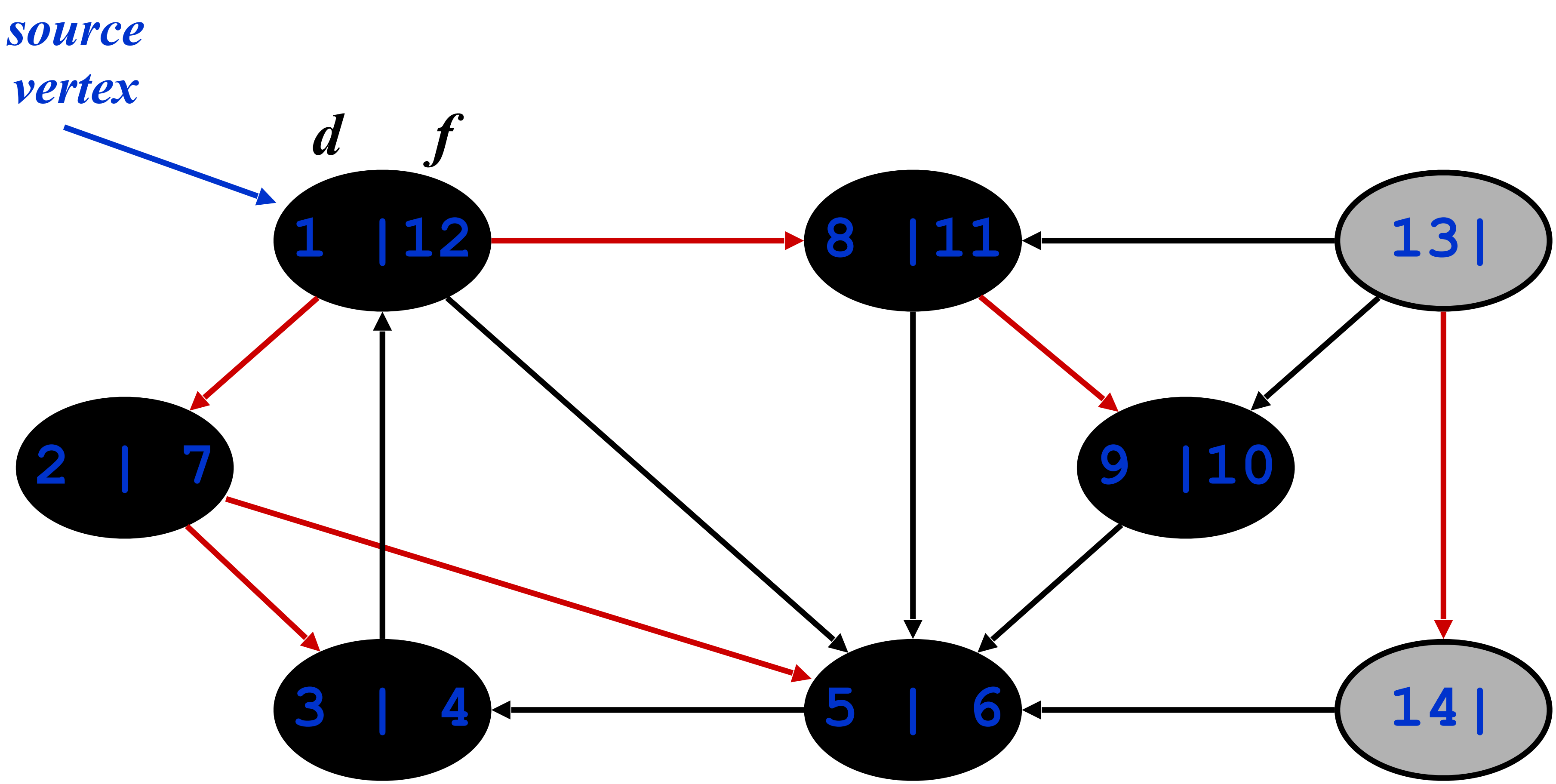
DFS Example



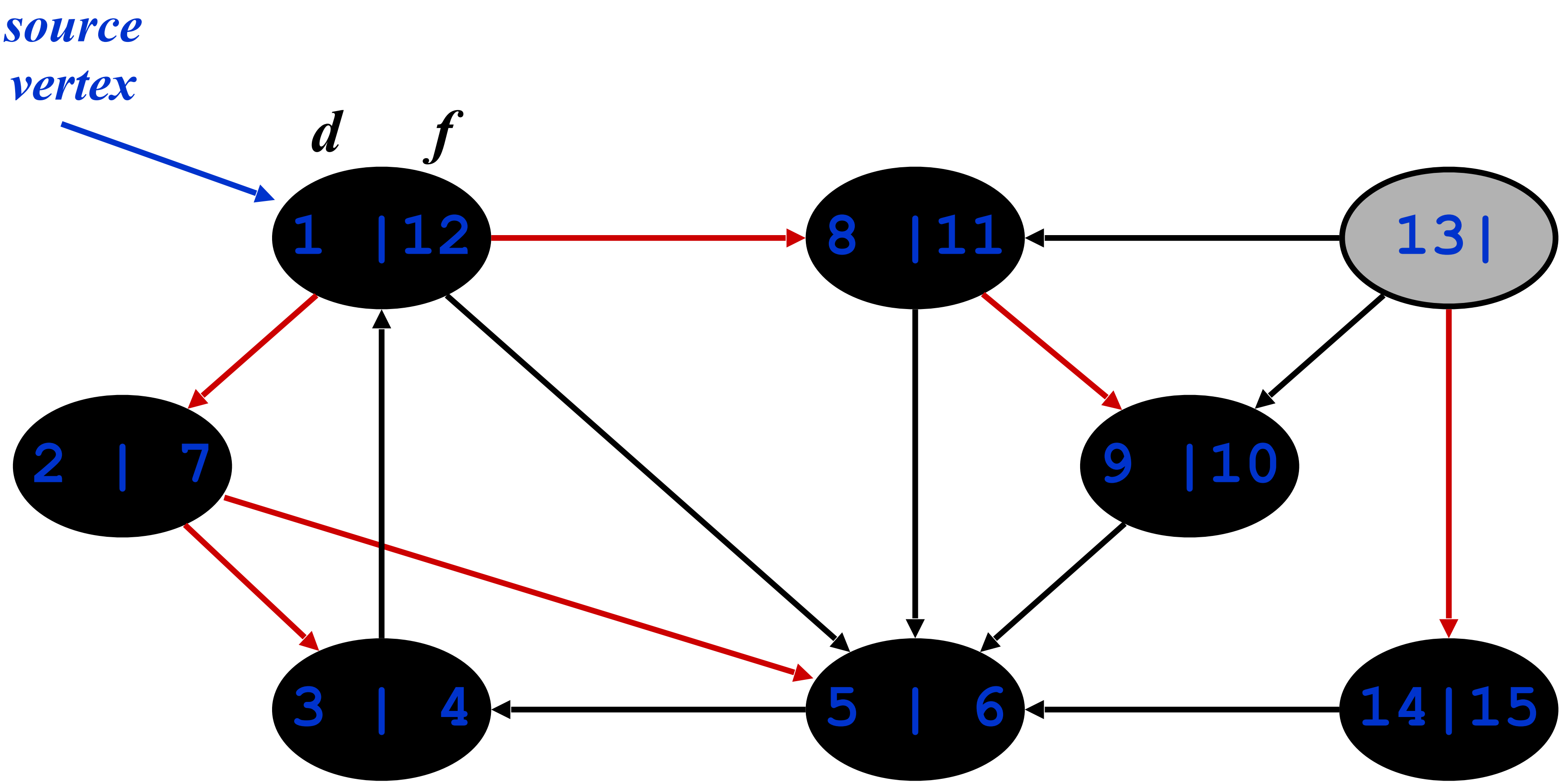
DFS Example



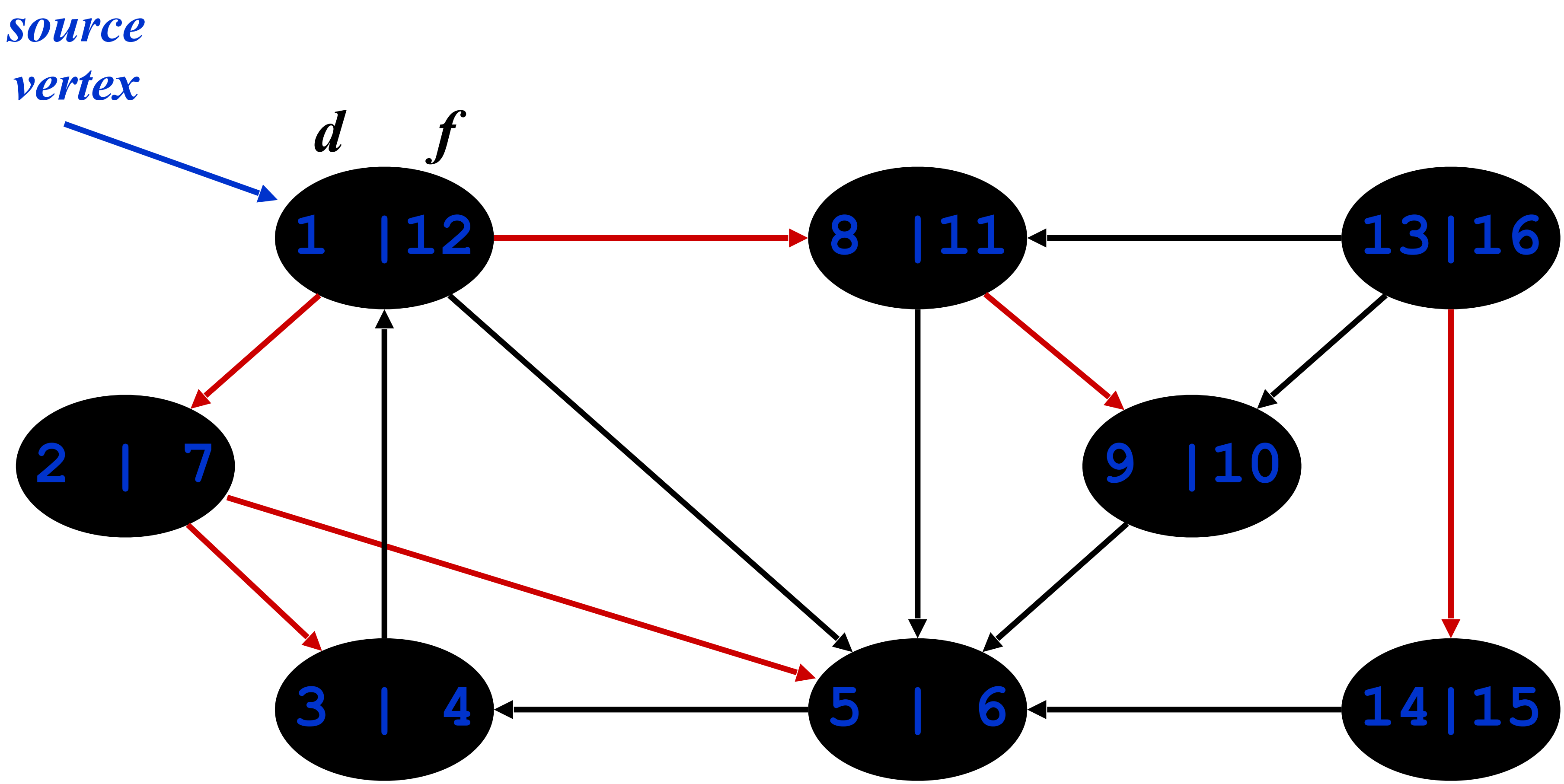
DFS Example



DFS Example



DFS Example

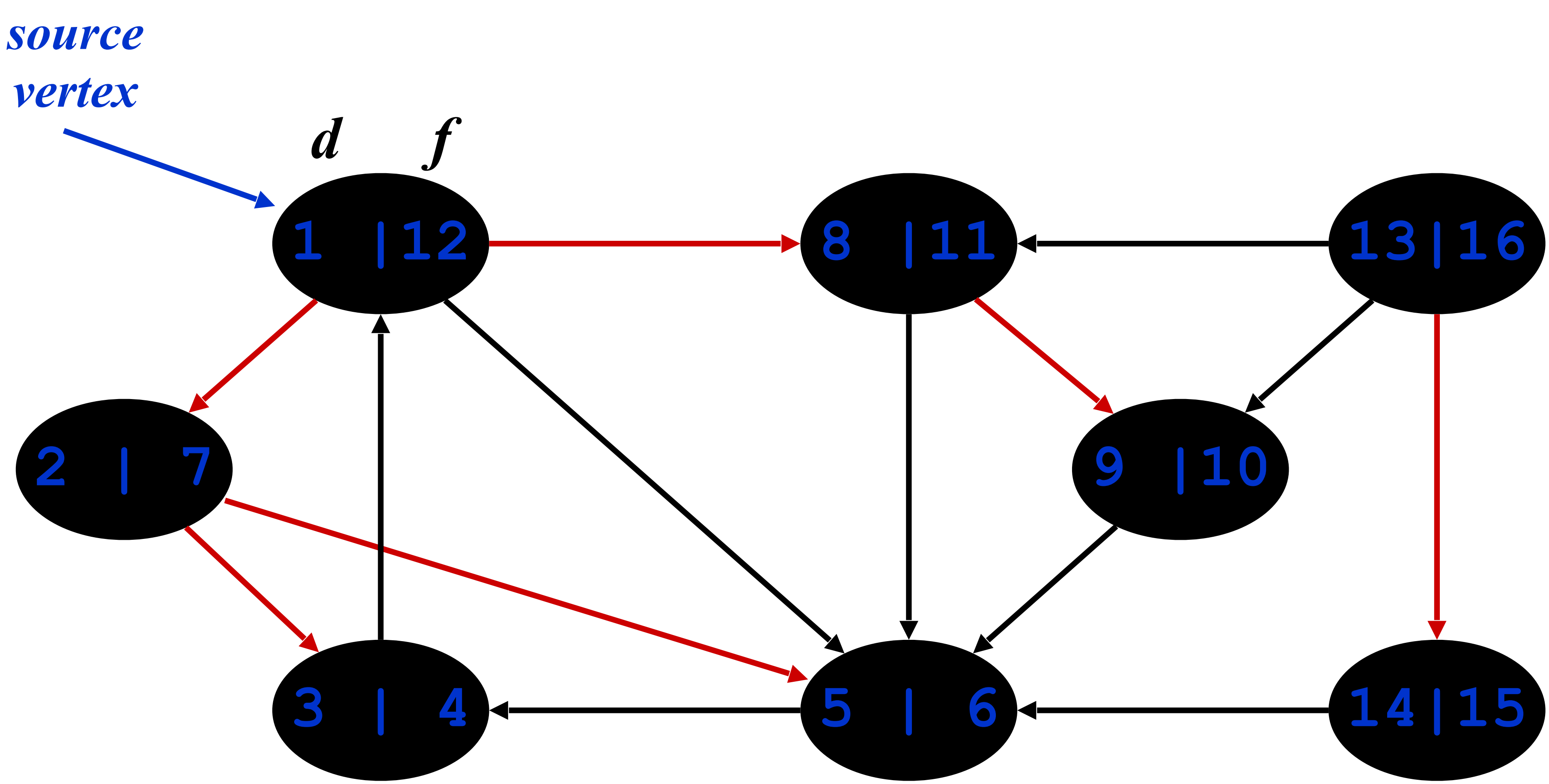


DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest, called depth-first forest consisting of depth-first trees
      - pred(v) is the parent of v in its depth-first tree

```
DFSVisit(u) {
    color[u] = gray;
    d[u] = ++time;
    for each v in Adj(u) do
        if (color[v] == white) {
            pred[v] = u;
            DFSVisit(v);
        }
    color[u] = black;
    f[u] = ++time;
}
```

DFS Example

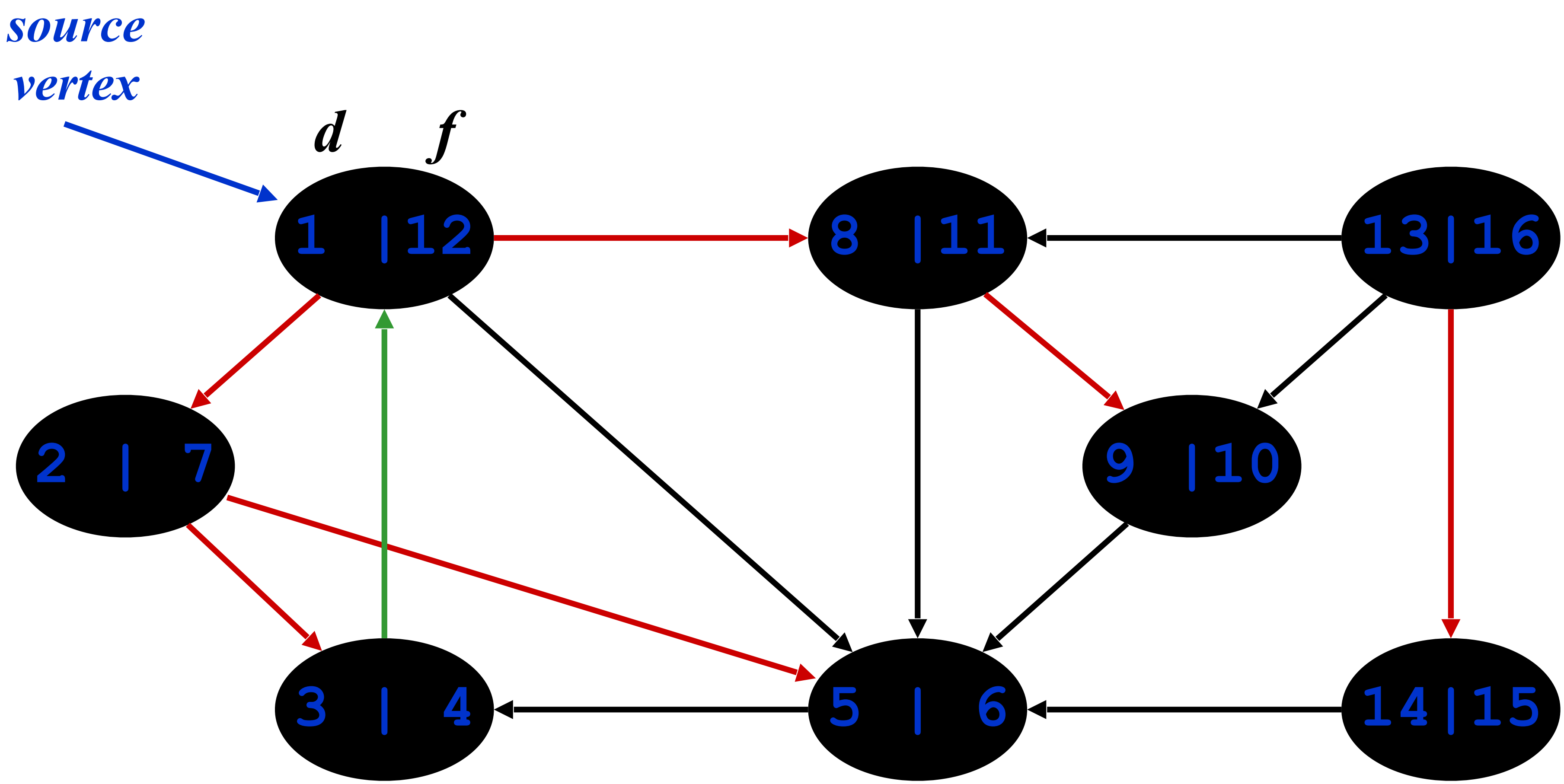


Tree edges

DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor (w.r. depth-first tree)
    - Encounter a grey vertex (grey to grey)

DFS Example

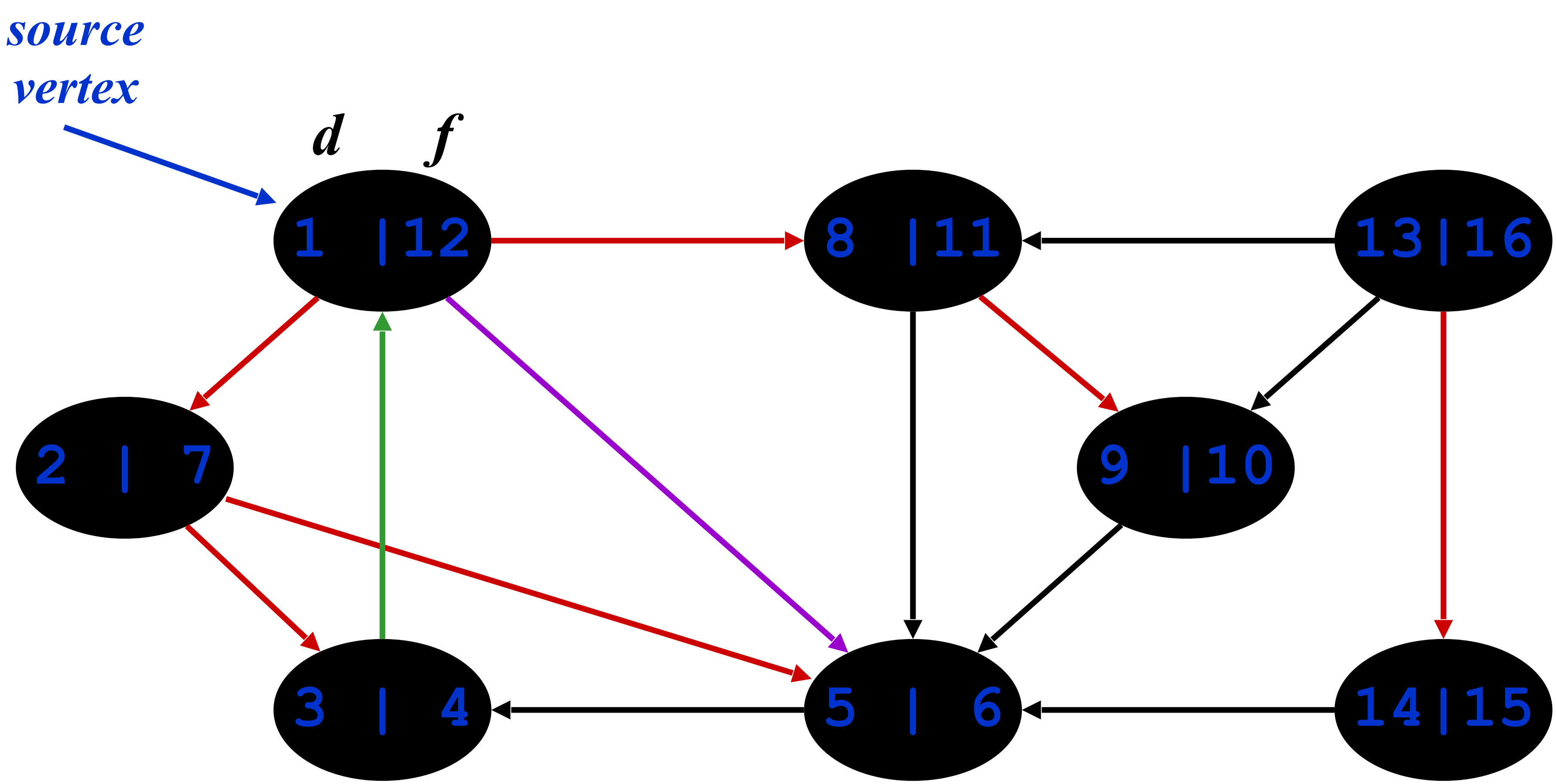


Tree edges *Back edges*

DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor (w.r. depth-first tree)
  - *Forward edge*: from ancestor to descendent (w.r. depth-first tree)
    - not a tree edge, though
    - from grey node to black node

DFS Example

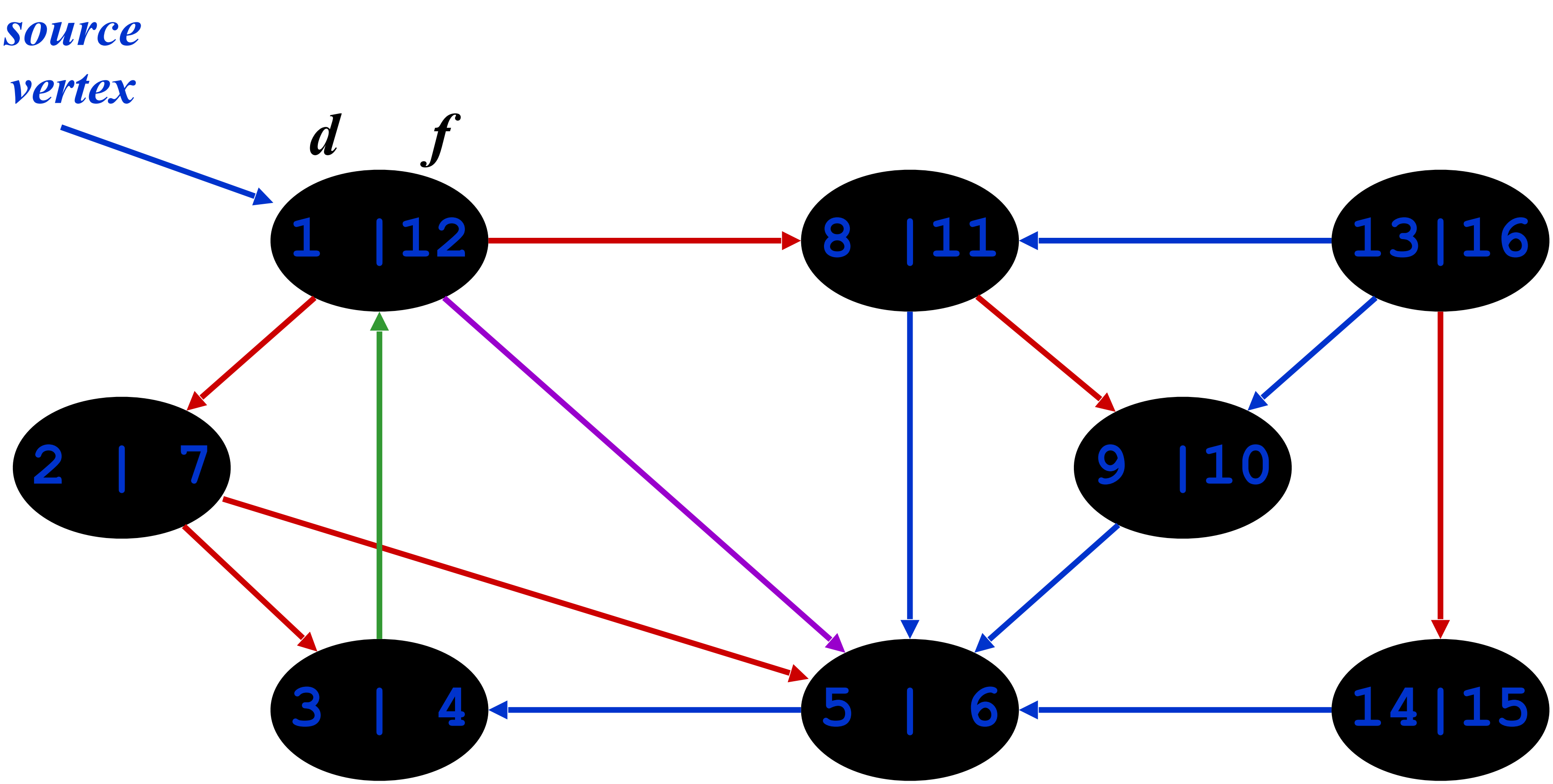


Tree edges *Back edges* *Forward edges*

DFS: Kinds of Edges

- DFS introduces an important distinction among edge (u,v) in the original graph:
  - *Tree edge*: encounter new (white) vertex, edge from parent to child in depth-first tree
    - v is white color when (u,v) is first explored
  - *Back edge*: from descendant to ancestor in depth-first tree
    - v is gray color when (u,v) is first explored
  - *Forward edge*: from ancestor to descendant in depth-first tree
    - v is black color when (u,v) is first explored
  - *Cross edge*: between two nodes w/o ancestor-descendant relation in a depth-first tree or two nodes in two different depth-first trees
    - v is black color when (u,v) is first explored
- Note: tree & back edges are important; most algorithms don't distinguish forward & cross

DFS Example



Tree edges *Back edges* *Forward edges* *Cross edges*

Parenthesis Theorem

Theorem: In any DFS of directed or undirected graph, for any two vertices u and v, one of the following three conditions holds:

1. intervals $[d(u), f(u)]$ and $[d(v), f(v)]$ are disjoint;
2. $[d(u), f(u)]$ entirely inside $[d(v), f(v)]$, i.e. $d(u) < d(v) < f(v) < f(u)$;
3. $[d(v), f(v)]$ entirely inside $[d(u), f(u)]$, i.e. $d(v) < d(u) < f(u) < f(v)$.

Proof: Without loss of generality (w.l.o.g), assume u is discovered first, $d(u)<d(v)$. There are two subclasses:

a. $d(v)<f(u)$: v was discovered while u was still gray => v is a descendant of u in DFS tree, search will return to u after all outgoing edge of v are explored, so that $f(v)<f(u)$, this is case 2 in the theorem.
b. $d(v)>f(u)$: v was discovered after u was fully explored, case 1 in theorem.

Corollary: Nesting of Descendant's Intervals
v is a proper descendant of u in DFS forest if and only if $d(u) < d(v) < f(v) < f(u)$

White-path Theorem

Theorem: Vertex v is a descendant of u in a DFS tree **if and only if** at time d(u) that u was discovered, vertex v can be reached from u along a path consisting entirely of white vertices.

Proof: **=> (only if),** let w be any vertex on the path between u and v in DFS tree, w is a descendant of u, then according to Parenthesis theorem, d[w]>d[u], i.e., w was white at time d[u]. The path between u and v in DFS tree corresponds to a white path from u to v in the original graph.

<= (if) let p be the white path at time d(u), w.l.o.g., let w1 be the node which is the closest to u on p but not a descendant of u in DFS, let w2 be the predecessor of w1 on p, then w2 is a descendant of u in DFS, according to Nesting Corollary, d(u)<d(w2)<f(w2)<f(u), since (w2,w1) is an edge in G, w1 will be discovered before w2 is finished, so d(u)<d(w1)<f(w2)<f(u), according to Parenthesis theorem, d(u)<d(w1)<f(w1)<f(u), w1 is a descendant of u in DFS according to Nesting Corollary.

DFS in Undirected Graph

- Theorem: In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.
- Proof: for any (u,v), w.l.o.g, assume d(u)<d(v), then d(v)<f(v)<f(u). (white path theorem)
  1) if the first time (u,v) is processed, it is from u's adjacency list, then v must not have been discovered (v is white), then (u,v) is a tree edge.
  2) if the first time (u,v) is processed, it is from v's adjacency list, then u is still gray, then (u,v) is a back edge.

DFS & Graph Cycle: undirected

- Theorem: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle)
  - If no back edges, acyclic
    - No back edges implies only tree edges
    - Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic

DFS & Graph Cycle: undirected

- *What will be the running time?*
- A: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

DFS & Graph Cycle: directed

- Theorem: A directed graph is *acyclic* iff a DFS yields no back edges
- Proof: (sketch, details in book)
  => DFS produces a back edge (u,v), v is an ancestor of u in depth-first tree, then in G there is a path from v to u, then back edge (u,v) completes a cycle
  <= suppose G has a cycle c, let v be the first vertex to be discovered by DFS, u is the predecessor of v in cycle c, at time d(v), all vertices of c are white, form a white path from v to u, then u becomes a descendant of v in depth-first tree, (white path theorem), (u,v) is a back edge.