

EL9343

Data Structure and Algorithm

Lecture 4: Introduction to sorting II: HeapSort, Quicksort

Instructor: Pei Liu

Last Lecture

- ▶ Divide-and-conquer algorithms
 - ▶ maximum subarray
- ▶ Insertion sort
 - ▶ Design approach: Incremental
 - ▶ Sorts in place: Yes
 - ▶ Best case: $\Theta(n)$
 - ▶ Worst case: $\Theta(n^2)$

Introduction to Sorting: Merge Sort

▶ Bubble Sort

- ▶ Design approach: Incremental
- ▶ Sorts in place: Yes
- ▶ Running time: $\Theta(n^2)$

▶ Merge Sort

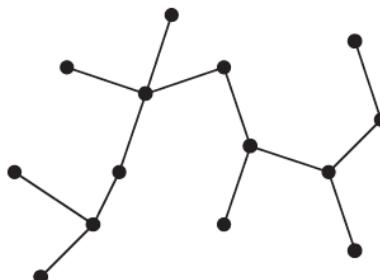
- ▶ Design approach: divide and conquer
- ▶ Sorts in place: No
- ▶ Running time: $\Theta(n \log n)$

Introduction to Sorting: HeapSort

- ▶ So far we've talked about several algorithms to sort an array of numbers
 - ▶ What is the advantage of merge sort?
 - ▶ Answer: $O(n \lg n)$ worst-case running time
 - ▶ What is the advantage of insertion sort?
 - ▶ Answer: sorts in place
 - ▶ Also: When array “nearly sorted”, runs fast in practice
- ▶ Next on the agenda: *Heapsort*
 - ▶ Combines advantages of both previous algorithms

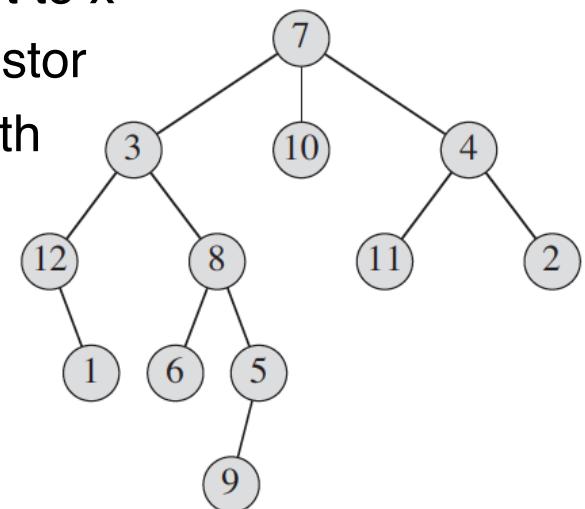
Data Structure: Tree

- ▶ **(Free) Tree**: connected, acyclic, undirected graph
- ▶ **Forest**: acyclic, undirected graph, possibly disconnected



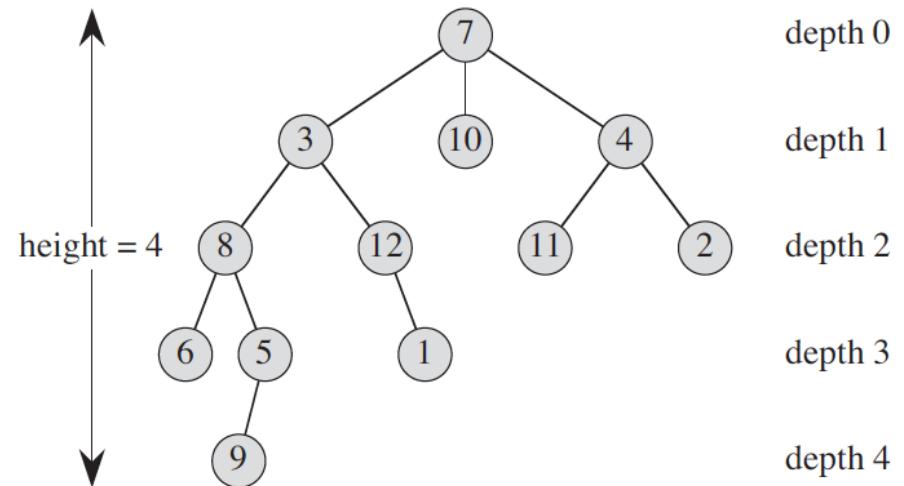
- ▶ **Rooted Tree**: a free tree with special **root** node

- **Ancestor** of node x: any node on the path from root to x
- **Descendant** of node x: any node with x as its ancestor
- **Parent** of node x: node immediately before x on path from root
- **Child** of node x: any node with x as its parent
- **Siblings** of node x: nodes sharing parent with x
- **Leaf/external node**: without child
- ▶ 5
- **Internal node**: with at least one child



Data Structure: Tree

- ▶ **degree** of x : number of children
- ▶ **depth** of x : length of the simple path from root to x
- ▶ **level** of a tree: all nodes at the same depth
- ▶ **height** of x : length of the longest simple path from x downward to some leaf node
- ▶ **height of a tree**: height of root
- ▶ **Ordered Tree**: rooted tree in which children of each node are ordered



Special Types of Trees

- ▶ **Binary Tree (recursive def.)**

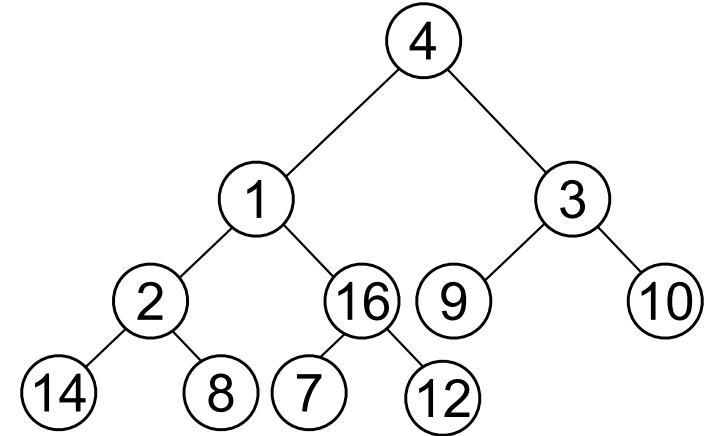
- ▶ Contains no node
- ▶ Or has a root node, left subtree (binary), right subtree (binary)

- ▶ **Full binary tree**

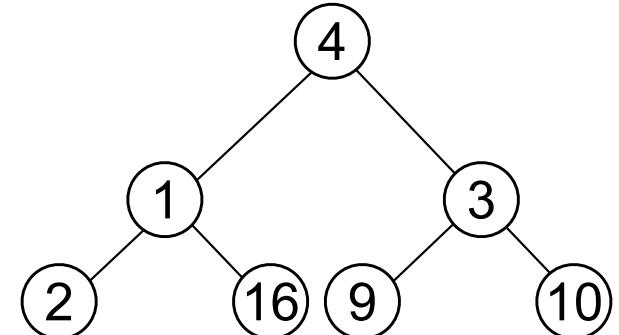
- ▶ A binary tree in which each node is either a leaf or has degree exactly 2.

- ▶ **Complete binary tree**

- ▶ A binary tree in which all leaves are on the same level and all internal nodes have degree 2.



Full binary tree

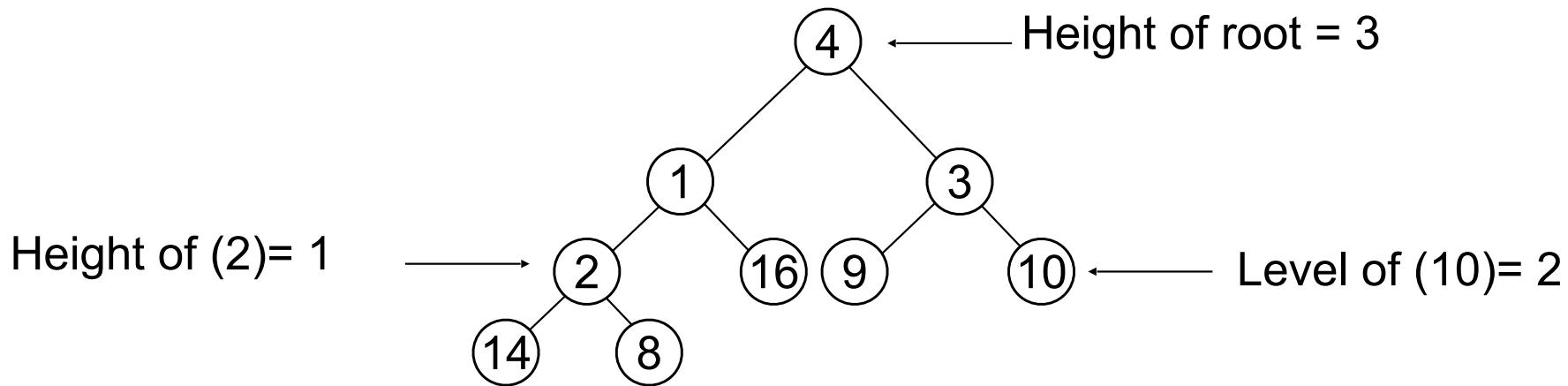


Complete binary tree

Useful Properties

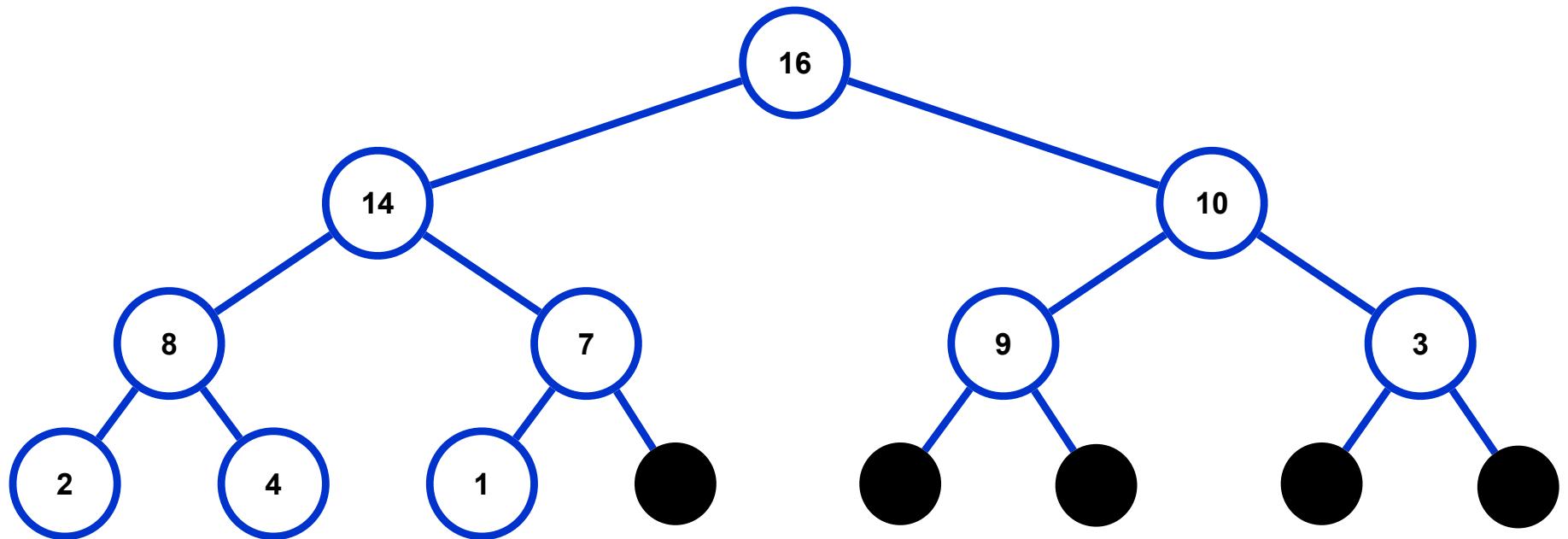
- ▶ There are at most 2^l nodes at level l of a binary tree
- ▶ A binary tree with depth d has at most $2^{d+1}-1$ nodes
- ▶ A binary tree with n nodes has depth at least $\lfloor \lg n \rfloor$

$$n \leq \sum_{l=0}^d 2^l = \frac{2^{d+1}-1}{2-1} = 2^{d+1} - 1$$



Data Structure: Heap

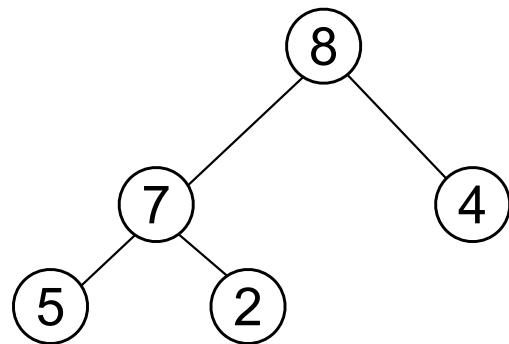
- ▶ A **heap** can be seen as a complete binary tree:



- ▶ The book calls them “nearly complete” binary trees; can think of unfilled slots as null pointers

The Heap Data Structure

- ▶ A **heap** can be seen as a complete binary tree with the following two properties:
 - ▶ **Structural property:** all levels are full, except possibly the last one, which is filled from **left to right**
 - ▶ **Order (heap) property:** for any node x : $\text{Parent}(x) \geq x$



Heap

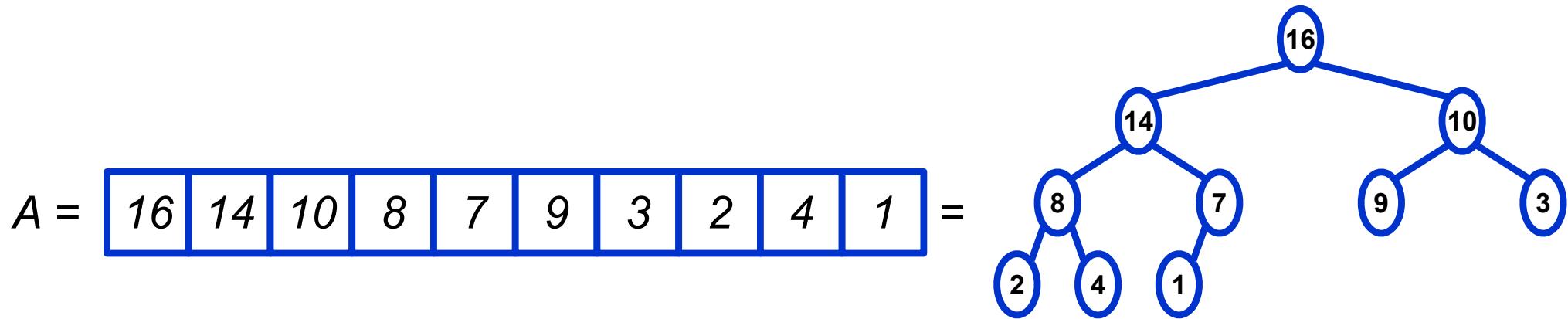
From the heap property, it follows that:

“The root is the maximum element of the heap!”

A heap is a binary tree that is filled in order

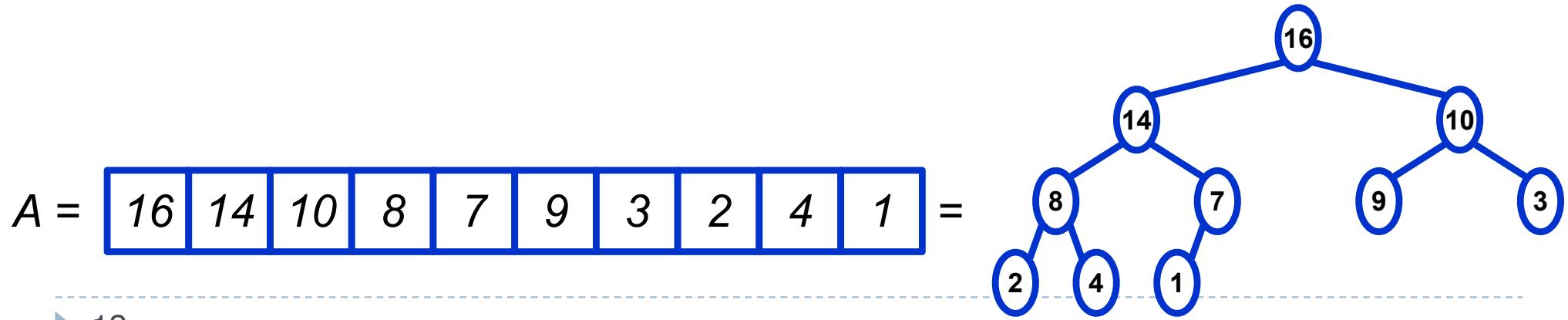
The Heap Data Structure

- In practice, heaps are usually implemented as arrays:



Array Representation of Heaps

- ▶ A heap can be stored as an array A .
 - ▶ Root of tree is $A[1]$
 - ▶ Node i is $A[i]$
 - ▶ Left child of node $i = A[2i]$
 - ▶ Right child of node $i = A[2i + 1]$
 - ▶ Parent of node $i = A[\lfloor i/2 \rfloor]$
 - ▶ Heapsize[A] \leq length[A]
- ▶ The elements in the subarray $A[\lfloor n/2 \rfloor + 1 .. n]$ are leaves

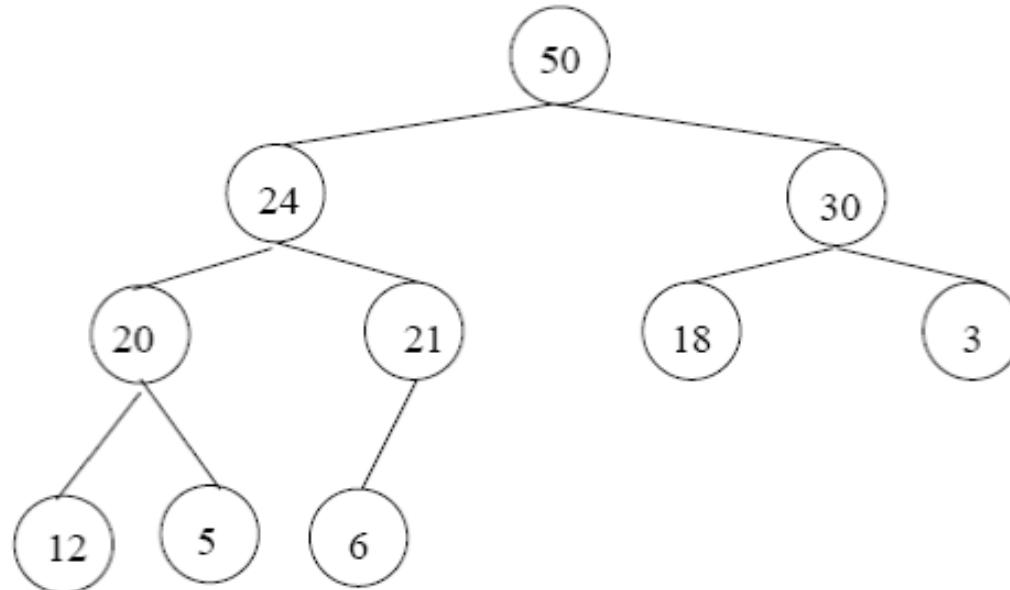


Heap Types

- ▶ **Max-heaps** (largest element at root), have the *max-heap property*:
 - ▶ For all nodes i , excluding the root:
$$A[\text{PARENT}(i)] \geq A[i]$$
- ▶ **Min-heaps** (smallest element at root), have the *min-heap property*:
 - ▶ For all nodes i , excluding the root:
$$A[\text{PARENT}(i)] \leq A[i]$$

Adding/Deleting Nodes

- ▶ New nodes are always inserted at the bottom level (**left to right**)
- ▶ Nodes are removed from the bottom level (**right to left**)



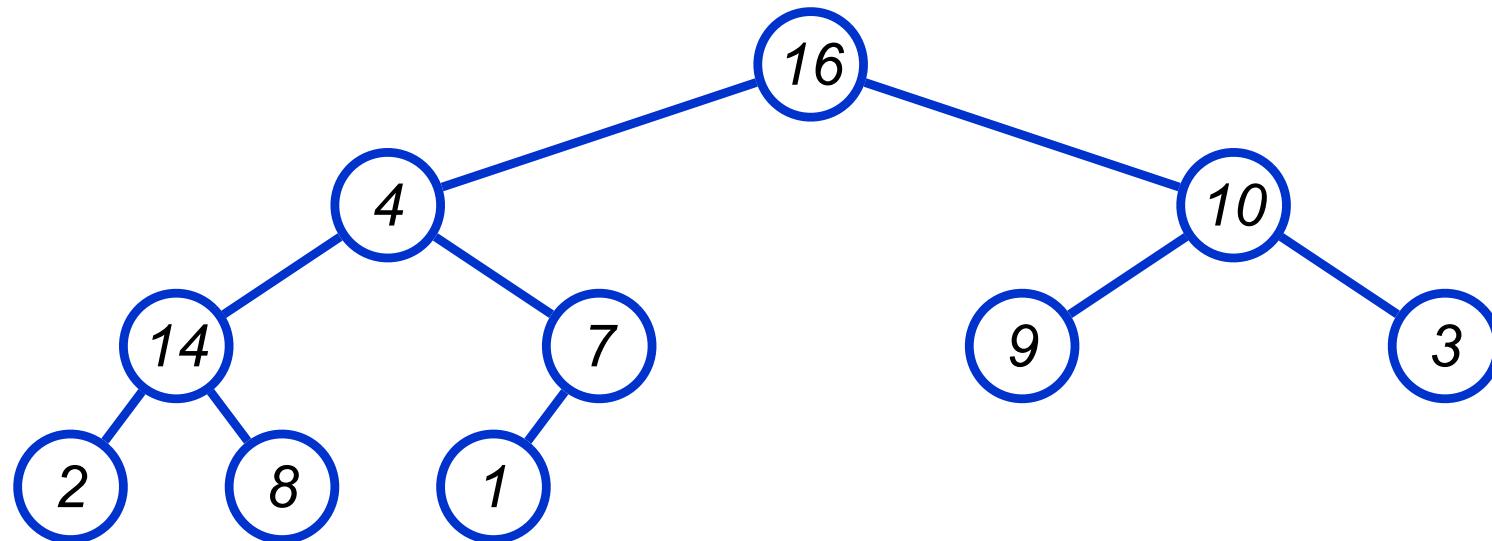
Operations on Heaps

- ▶ Maintain/Restore the max-heap property
 - ▶ MAX-HEAPIFY
- ▶ Create a max-heap from an unordered array
 - ▶ BUILD-MAX-HEAP
- ▶ Sort an array in place
 - ▶ HEAPSORT
- ▶ Priority queues

Heap Operations: MAX-HEAPIFY

- ▶ Maintain the max-heap property: **MAX-HEAPIFY**
- ▶ Suppose a node is smaller than a child
 - ▶ Left and Right subtrees of i are max-heaps
- ▶ To eliminate the violation:
 - ▶ Exchange with larger child
 - ▶ Move down the tree
 - ▶ Continue until node is not smaller than children

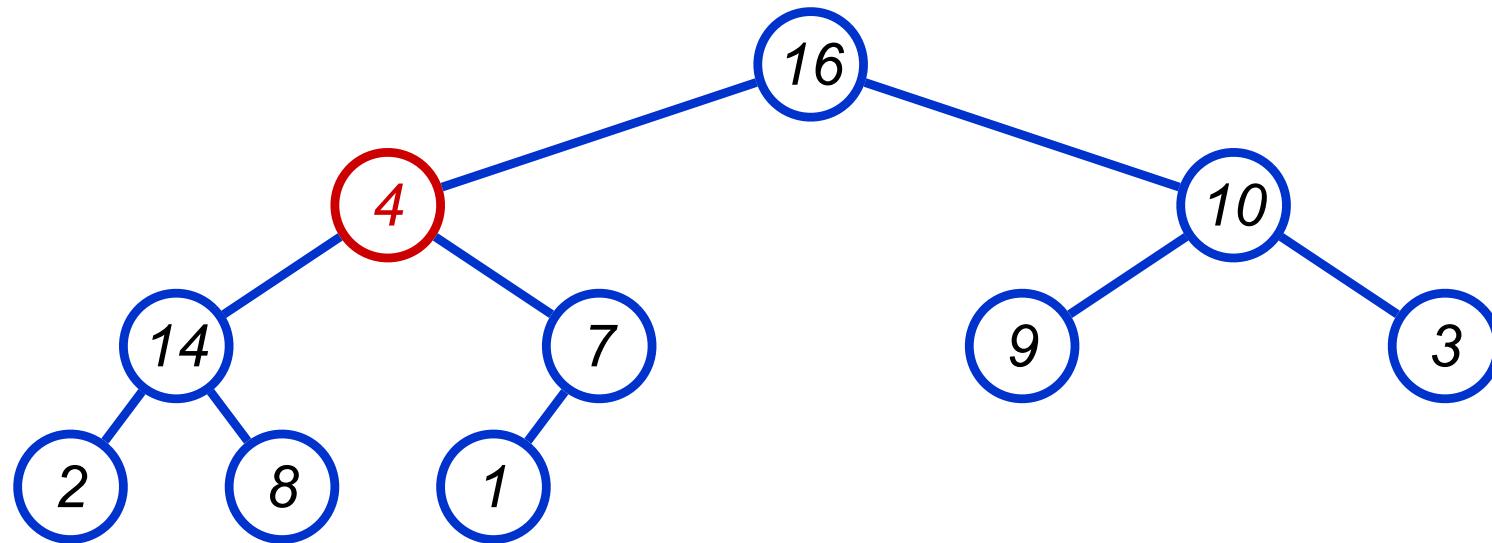
MAX-HEAPIFY Example



$A =$

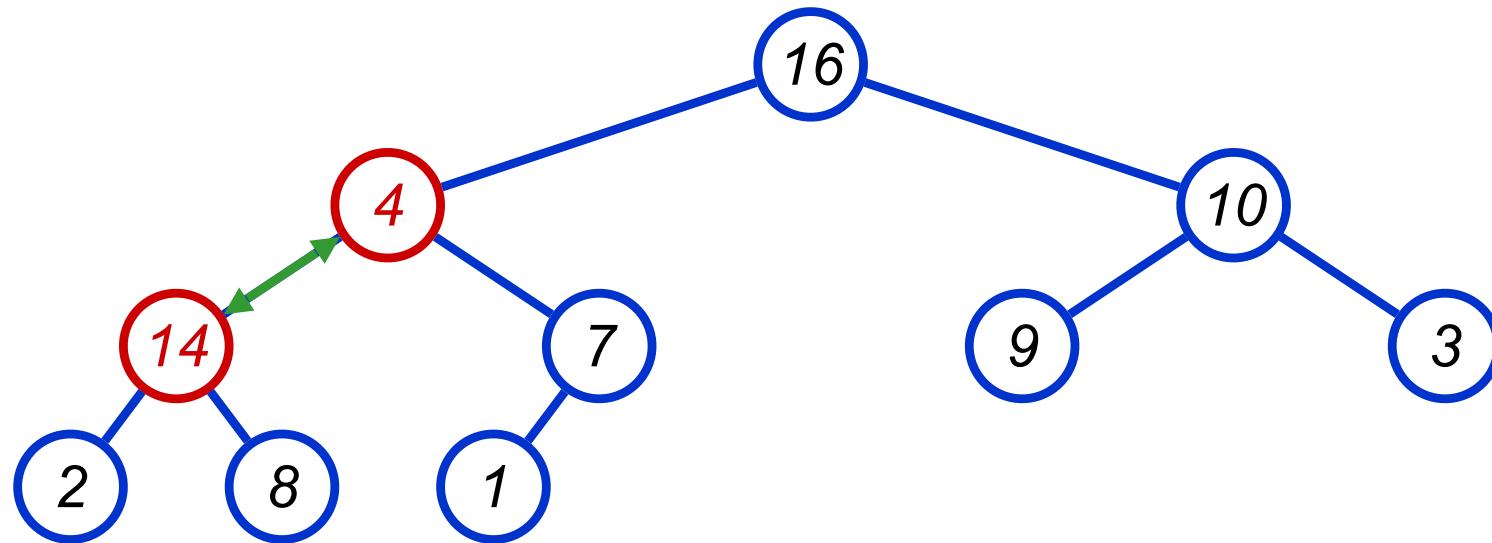
16	4	10	14	7	9	3	2	8	1
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MAX-HEAPIFY Example



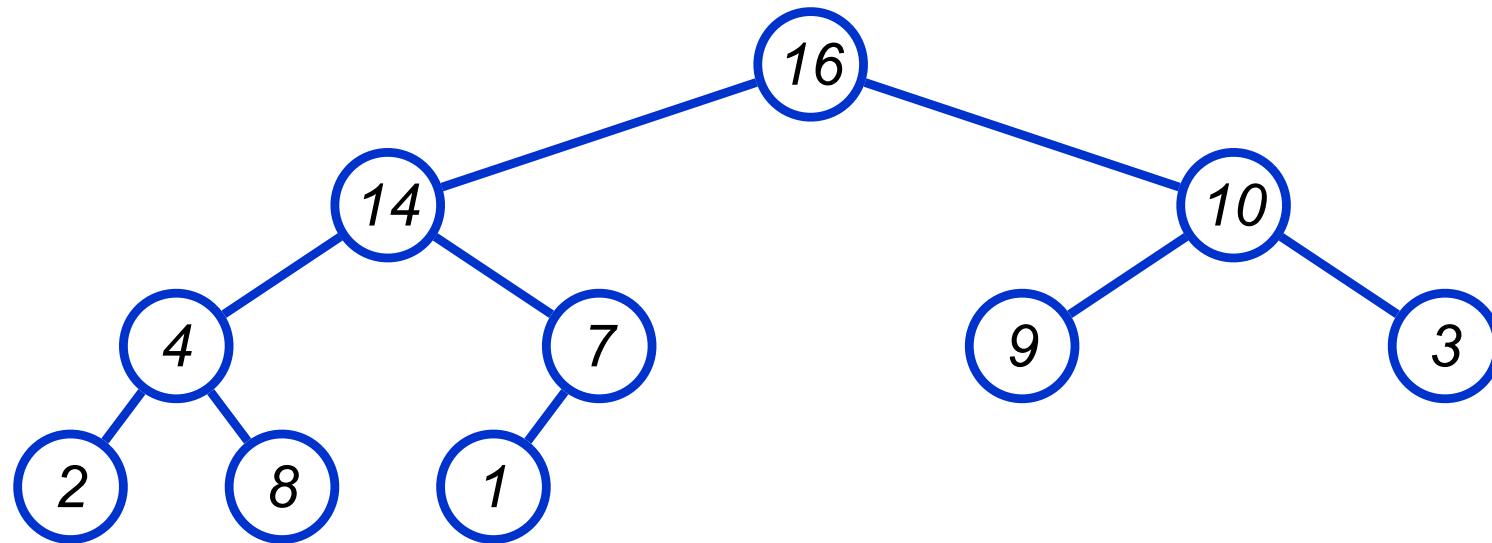
$A = \boxed{16 \quad 4 \quad 10 \quad 14 \quad 7 \quad 9 \quad 3 \quad 2 \quad 8 \quad 1}$

MAX-HEAPIFY Example



$A = [16 \ 4 \ 10 \ 14 \ 7 \ 9 \ 3 \ 2 \ 8 \ 1]$

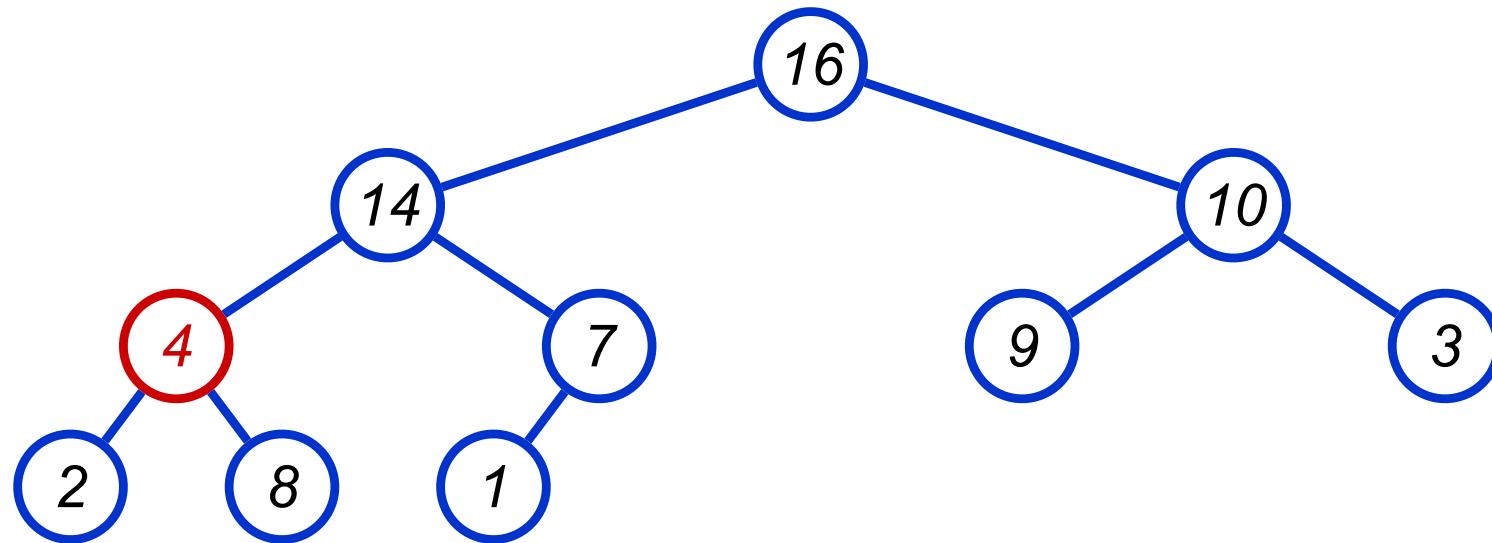
MAX-HEAPIFY Example



$A =$

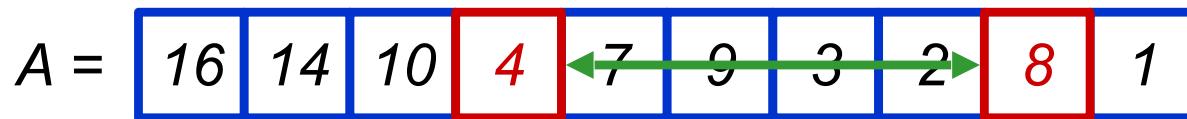
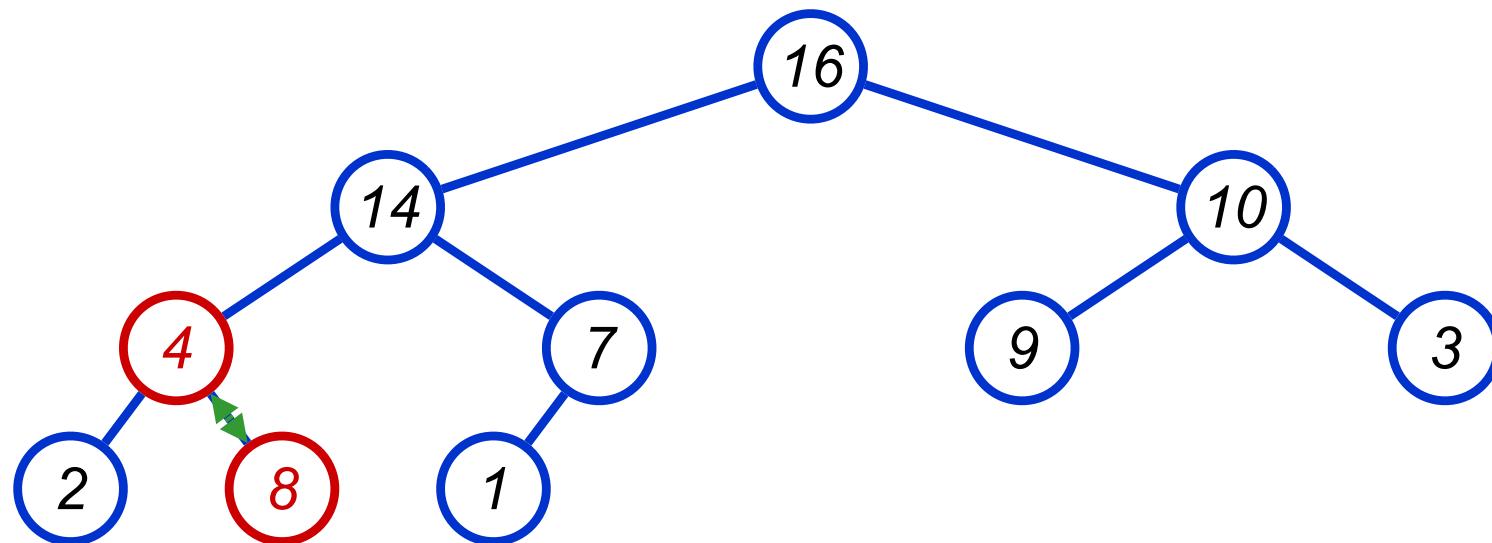
16	14	10	4	7	9	3	2	8	1
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MAX-HEAPIFY Example

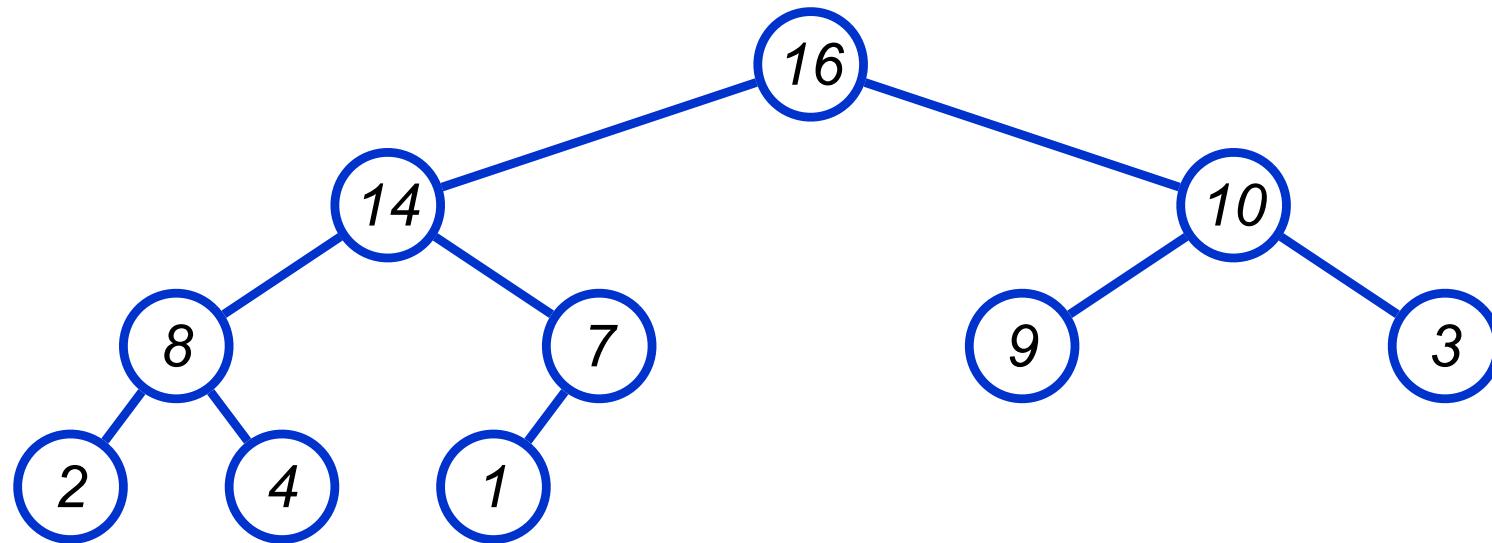


$A = \boxed{16 \ 14 \ 10 \ \boxed{4} \ 7 \ 9 \ 3 \ 2 \ 8 \ 1}$

MAX-HEAPIFY Example



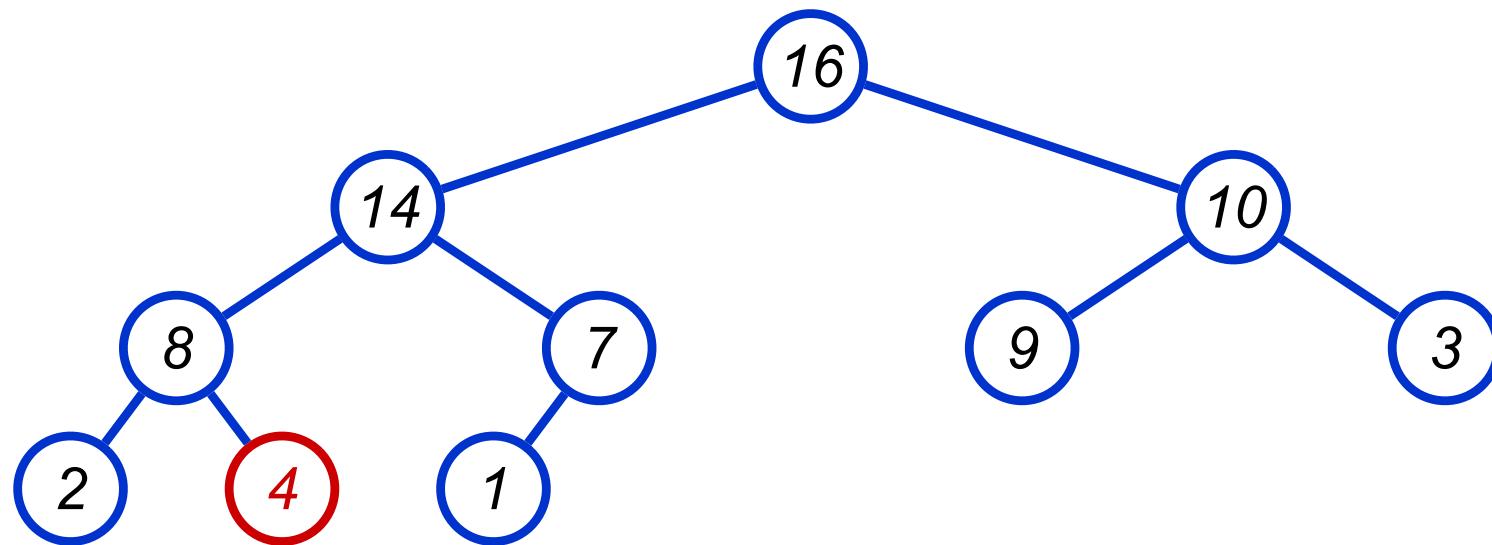
MAX-HEAPIFY Example



$A =$

16	14	10	8	7	9	3	2	4	1
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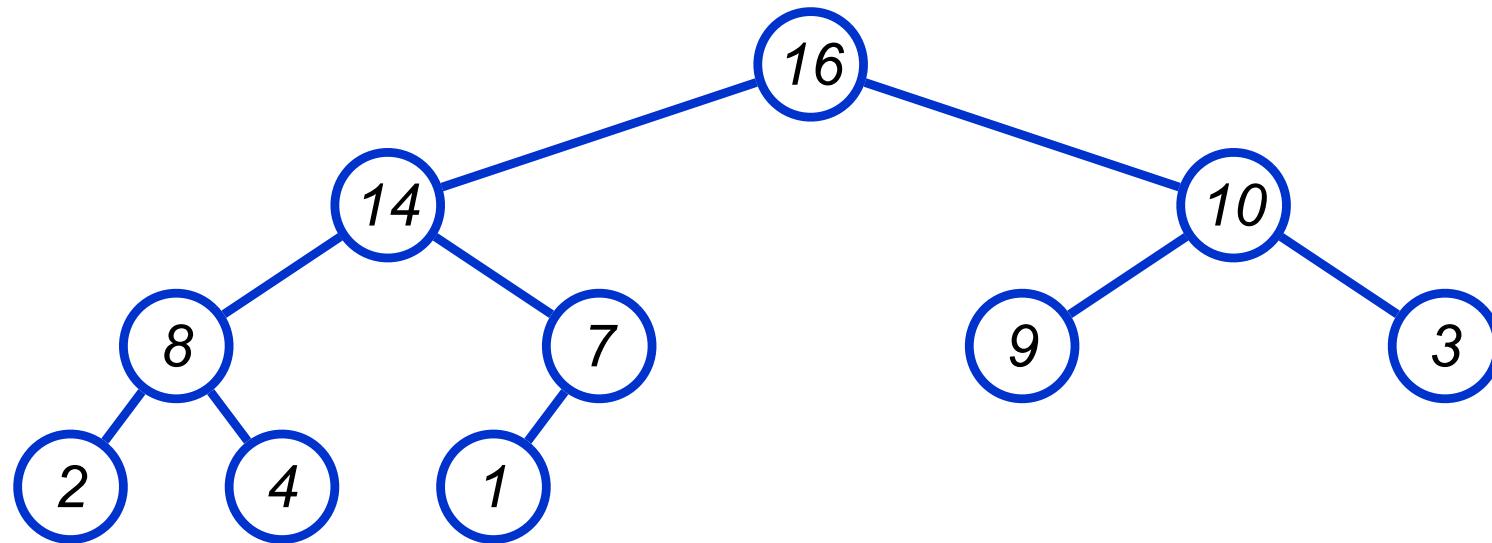
MAX-HEAPIFY Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

MAX-HEAPIFY Example



$A =$

16	14	10	8	7	9	3	2	4	1
----	----	----	---	---	---	---	---	---	---

Heap Operations: MAX-HEAPIFY

```
Max-Heapify(A, i)
{
    l = Left(i); r = Right(i);
    if (l <= heap_size(A) && A[l] > A[i])
        largest = l;
    else
        largest = i;
    if (r <= heap_size(A) && A[r] > A[largest])
        largest = r;
    if (largest != i)
        Swap(A, i, largest);
        Heapify(A, largest);
}
```

Assumptions:

- ▶ Left and Right subtrees of i are max-heaps
- ▶ $A[i]$ may be smaller than its children

Analyzing MAX-HEAPIFY - Informal

- ▶ Intuitively
 - ▶ It traces a path from the root to a leaf (longest path length: h)
 - ▶ At each level, it makes exactly 2 comparisons
 - ▶ Total number of comparison is $2h$
 - ▶ Running time is $O(h)$ or $O(\lg n)$
- ▶ Running time of MAX-HEAPIFY is $O(\lg n)$
- ▶ Can be written in terms of the height of the heap, as being $O(h)$
 - ▶ Since the height of the heap is $\lfloor \lg n \rfloor$

Analyzing MAX-HEAPIFY - Formal

- ▶ Fixing up relationships between i , l , and r takes $\Theta(1)$ time
- ▶ *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
 - ▶ Answer: $2n/3$ (worst case: bottom row 1/2 full)
- ▶ So time taken by MAX-HEAPIFY is given by
 - ▶ $T(n) \leq T(2n/3) + \Theta(1)$

Analyzing MAX-HEAPIFY - Formal

- ▶ So we have
 - ▶ $T(n) \leq T(2n/3) + \Theta(1)$
- ▶ By case 2 of the Master Theorem,
 - ▶ $T(n) = O(\lg n)$
- ▶ Thus, MAX-HEAPIFY takes logarithmic time

Building a Heap

- ▶ We can build a heap in a bottom-up manner by running MAX-HEAPIFY on successive subarrays
 - ▶ Convert an array $A[1 \dots n]$ into a max-heap ($n = \text{length}[A]$)
 - ▶ The elements in the subarray $A[(\lfloor n/2 \rfloor + 1) \dots n]$ are leaves
 - ▶ Apply MAX-HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$

Alg: BUILD-MAX-HEAP(A)

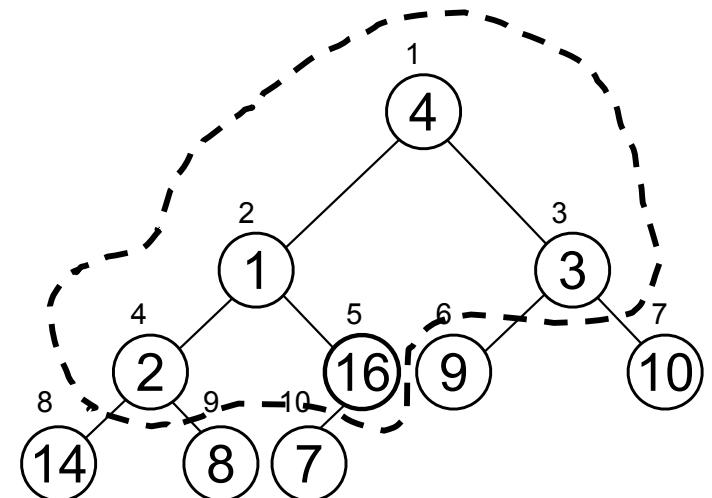
$n = \text{length}[A]$

for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1

do MAX-HEAPIFY(A, i, n)

$A:$

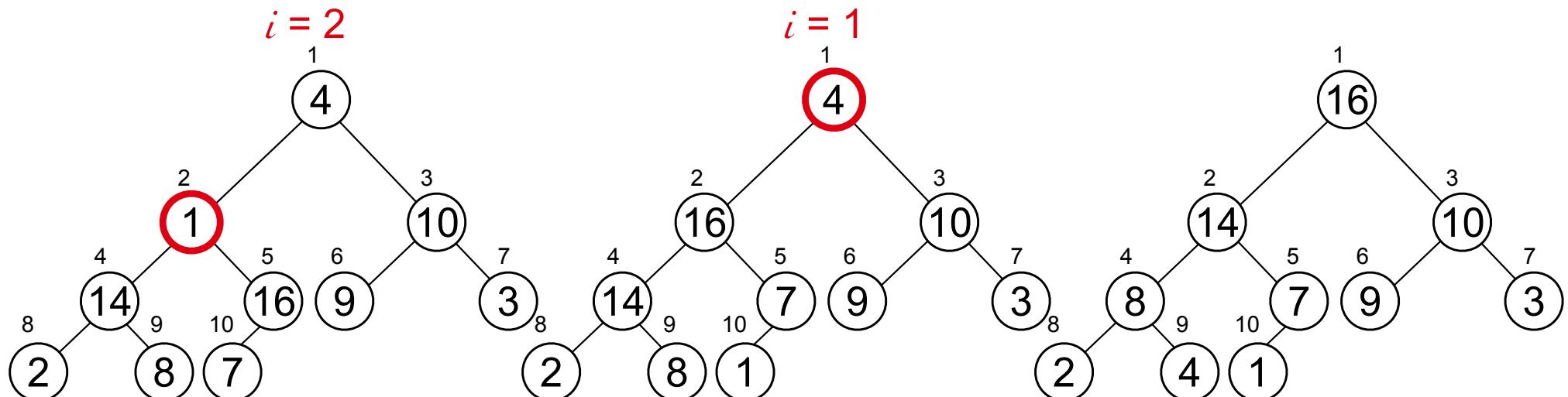
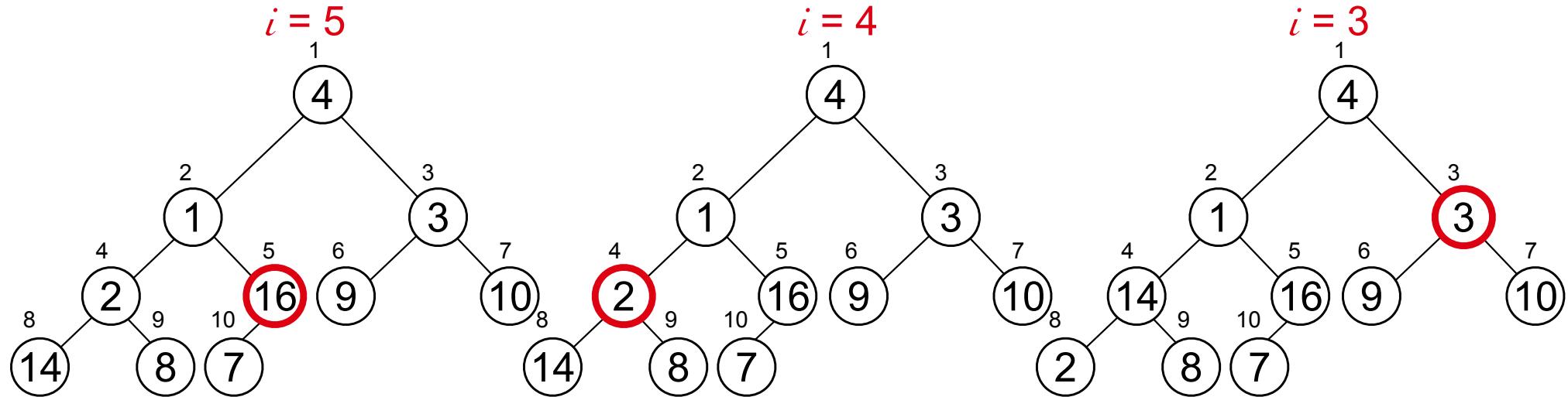
4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---



Example:

A

4	1	3	2	16	9	10	14	8	7
---	---	---	---	----	---	----	----	---	---



Analyzing BUILD MAX HEAP

Alg: BUILD-MAX-HEAP(A)

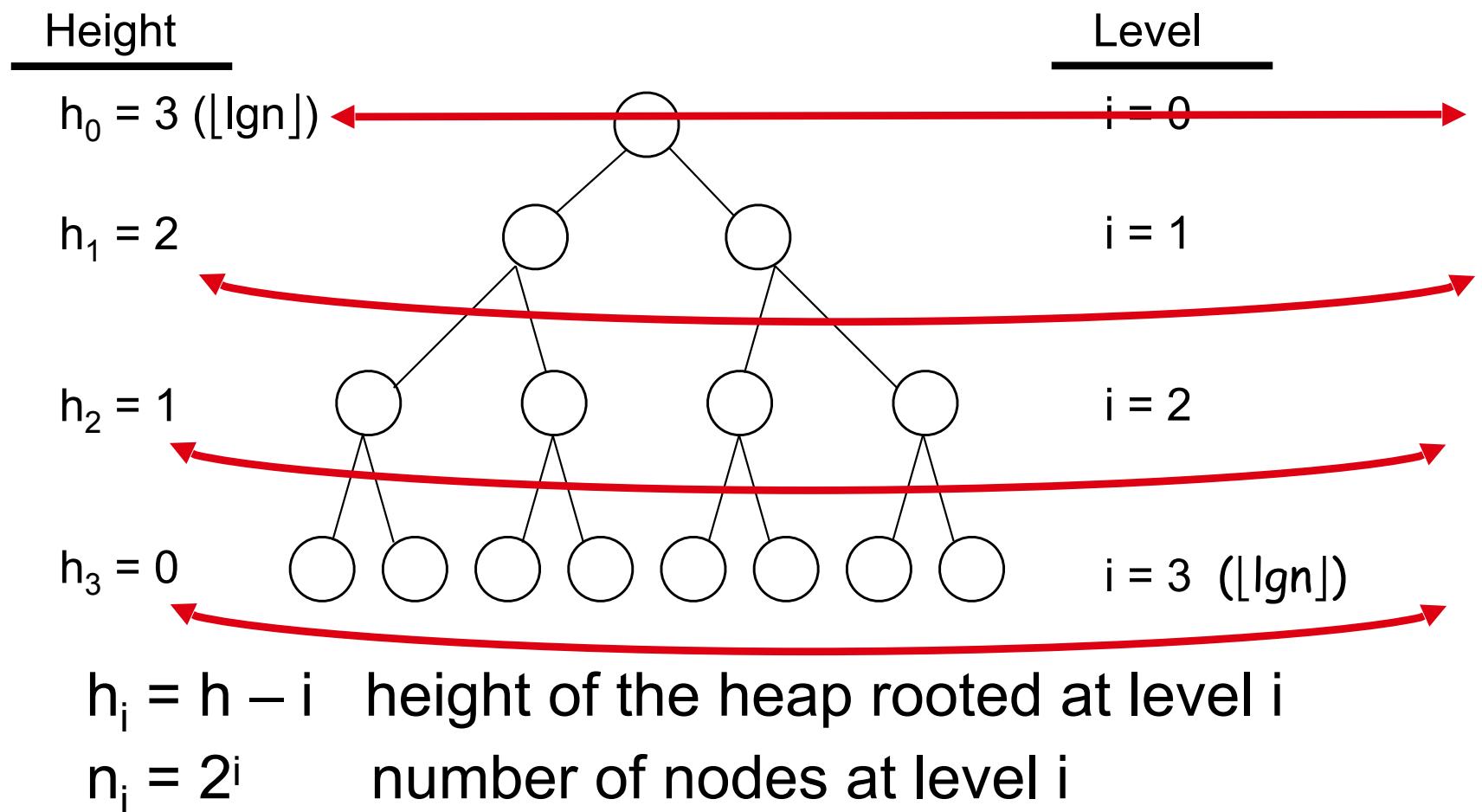
1. $n = \text{length}[A]$
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1
3. **do** MAX-HEAPIFY(A, i, n)

$O(\lg n)$ } $O(n)$

- ▶ Each call to MAX-HEAPIFY takes $O(\lg n)$ time
- ▶ There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)
- ▶ Thus the running time is $O(n \lg n)$
 - ▶ *Is this a correct asymptotic upper bound? YES*
 - ▶ *Is this an asymptotically tight bound? NO*
- ▶ A tighter bound is $O(n)$

Running Time of BUILD MAX HEAP

- ▶ HEAPIFY takes $O(h)$ ⇒ the cost of HEAPIFY on a node i is proportional to the height of the node i in the tree



Running Time of BUILD MAX HEAP

$$T(n) = \sum_{i=0}^h n_i h_i \quad \text{Cost of HEAPIFY at level } i * \text{number of nodes at that level}$$

$$= \sum_{i=0}^h 2^i (h - i) \quad \text{Replace the values of } n_i \text{ and } h_i \text{ computed before}$$

$$= \sum_{i=0}^h \frac{h - i}{2^{h-i}} 2^h \quad \text{Multiply by } 2^h \text{ both at the numerator and denominator and write } 2^i \text{ as } \frac{1}{2^{-i}}$$

$$= 2^h \sum_{k=0}^h \frac{k}{2^k} \quad \text{Change variables: } k = h - i$$

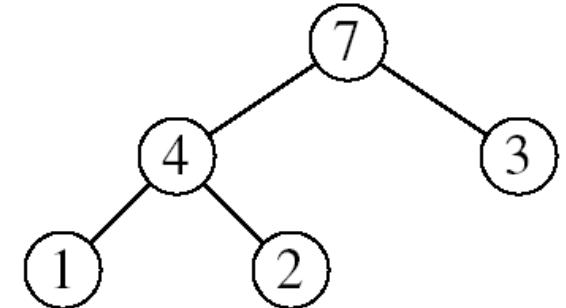
$$\leq n \sum_{k=0}^{\infty} \frac{k}{2^k} \quad \text{The sum above is smaller than the sum of all elements to } \infty \text{ and } h = \lg n$$

$$= O(n) \quad \text{The sum above is 2}$$

Running time of BUILD-MAX-HEAP: $T(n) = O(n)$

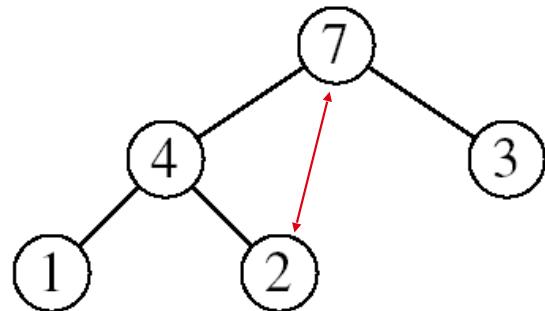
Heapsort

- ▶ **Goal:**
 - ▶ Sort an array using heap representations
- ▶ **Idea:**
 - ▶ Build a **max-heap** from the array
 - ▶ Swap the root (the maximum element) with the last element in the array
 - ▶ “Discard” this last node by decreasing the heap size
 - ▶ Call MAX-HEAPIFY on the new root
 - ▶ Repeat this process until only one node remains

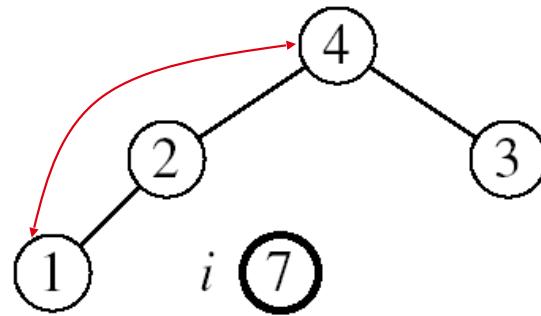


Example

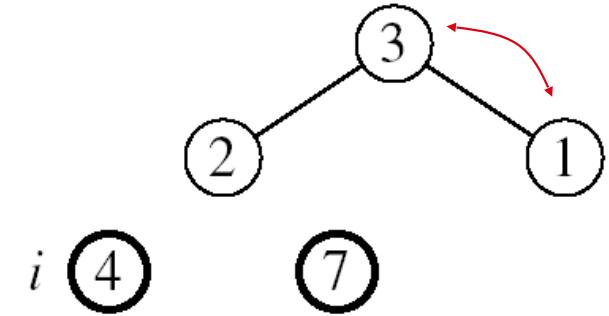
$A=[7, 4, 3, 1, 2]$



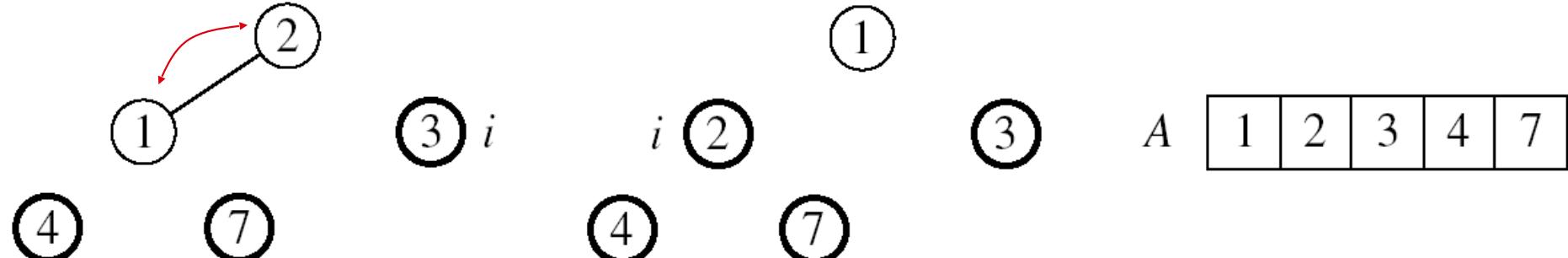
MAX-HEAPIFY($A, 1, 4$)



MAX-HEAPIFY($A, 1, 3$)



MAX-HEAPIFY($A, 1, 2$)



MAX-HEAPIFY($A, 1, 1$)

Analyzing Heapsort

1. **BUILD-MAX-HEAP(A)**

$O(n)$

2. **for** $i \leftarrow \text{length}[A]$ **downto** 2

3. **do** exchange $A[1] \leftrightarrow A[i]$

4. **MAX-HEAPIFY(A, 1, i - 1)**

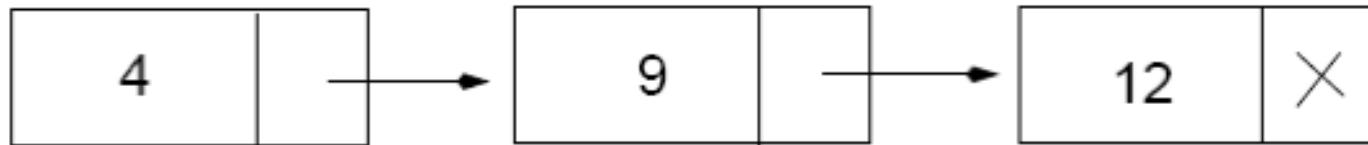
$O(\lg n)$

$\brace{n-1 \text{ times}}$

- ▶ The call to **BUILD-MAX-HEAP** takes $O(n)$ time
- ▶ Each of the $n - 1$ calls to **MAX-HEAPIFY** takes $O(\lg n)$ time
- ▶ Thus the total time taken by **HeapSort**
 $= O(n) + (n - 1) O(\lg n) = O(n) + O(n \lg n) = O(n \lg n)$

Priority Queues

- ▶ The heap data structure is incredibly useful for implementing *priority queues*, which maintains a set of elements.
- ▶ Properties of priority queues
 - ▶ Each element is associated with a value (priority)
 - ▶ The key with the highest (or lowest) priority is extracted first



Operations on Priority Queues

- ▶ Max-priority queues support the following operations:
 - ▶ **INSERT(S, x)**: inserts element x into set S
 - ▶ **EXTRACT-MAX(S)**: removes and returns element of S with largest key
 - ▶ **MAXIMUM(S)**: returns element of S with largest key
 - ▶ **INCREASE-KEY(S, x, k)**: increases value of element x's key to k (Assume $k \geq$ x's current key value)

HEAP-MAXIMUM

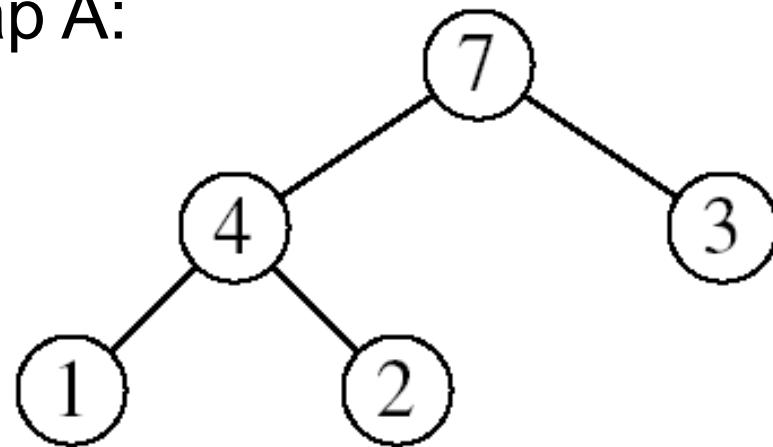
Goal: Return the largest element of the heap

Alg: HEAP-MAXIMUM(A)

Running time: $O(1)$

1. **return** $A[1]$

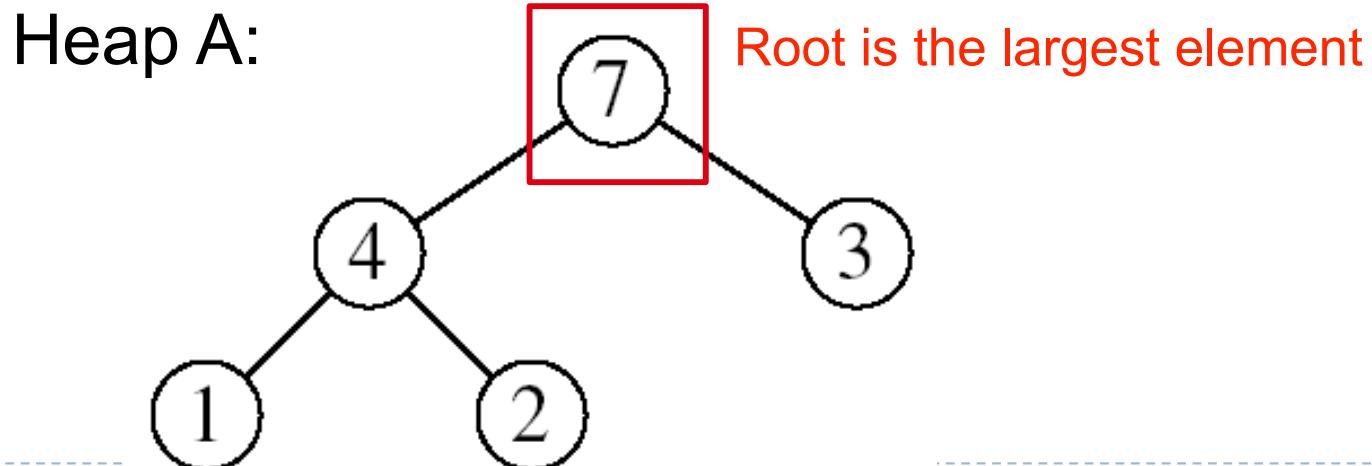
Heap A:



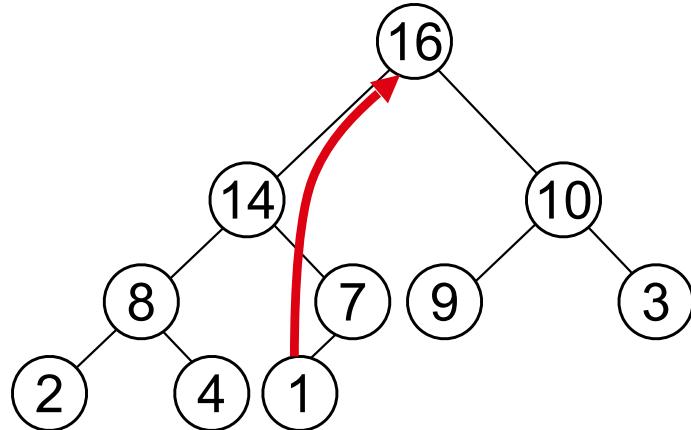
Heap-Maximum(A) returns 7

HEAP-EXTRACT-MAX

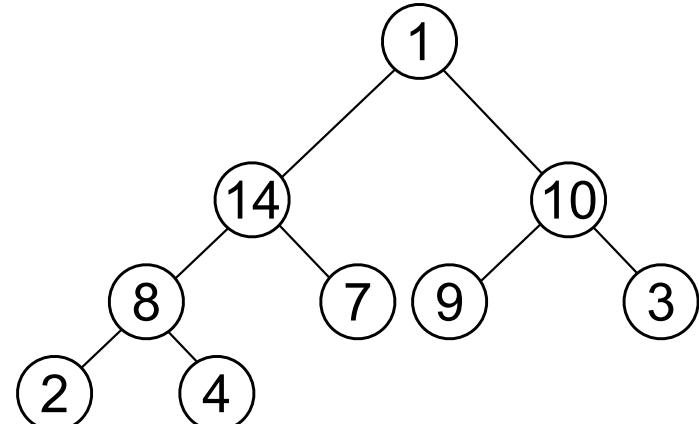
- ▶ Goal
 - ▶ Extract the largest element of the heap (i.e., return the max value and also remove that element from the heap)
- ▶ Idea
 - ▶ Exchange the root element with the last
 - ▶ Decrease the size of the heap by 1 element
 - ▶ Call MAX-HEAPIFY on the new root, on a heap of size $n-1$



Example: HEAP-EXTRACT-MAX

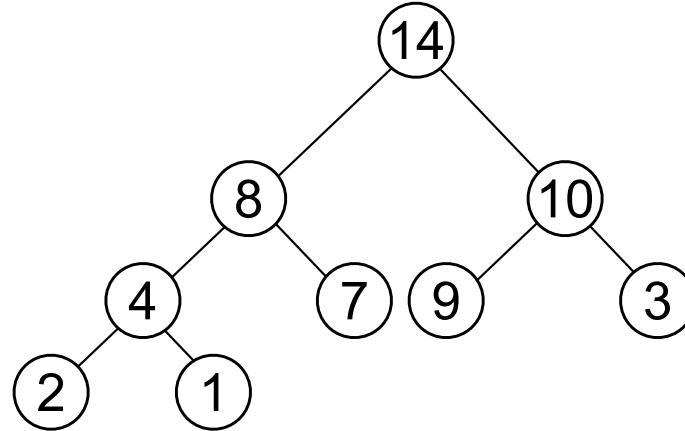


max = 16



Heap size decreased with 1

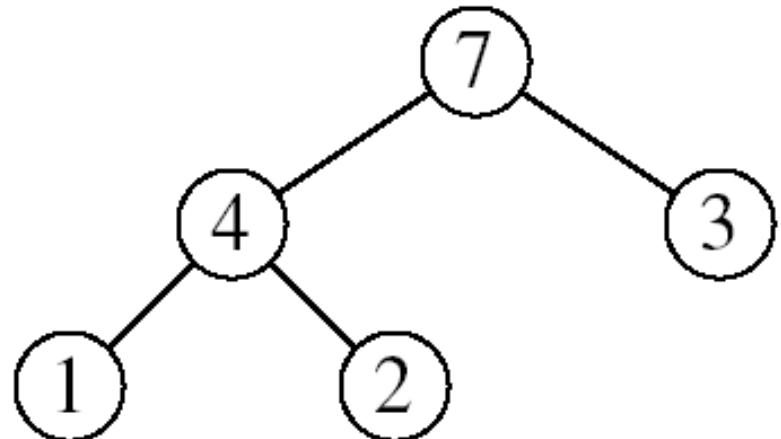
Call MAX-HEAPIFY(A, 1, n-1)



HEAP-EXTRACT-MAX

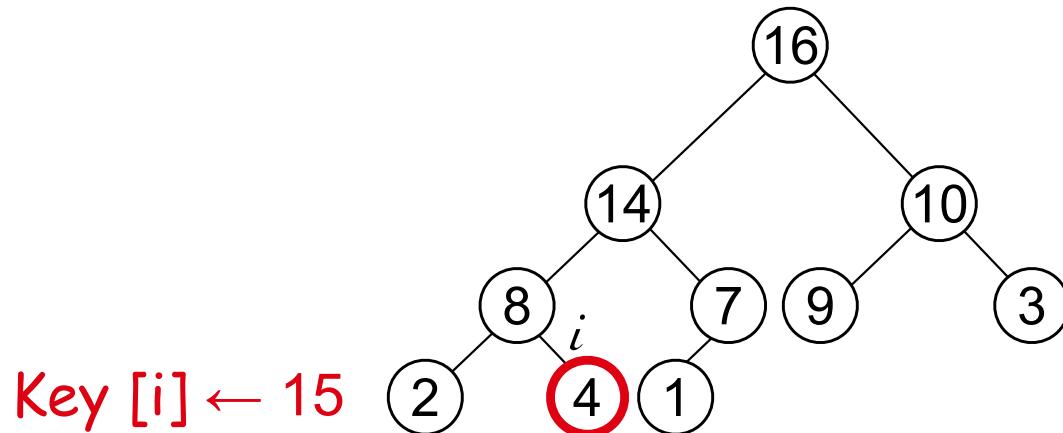
Alg: HEAP-EXTRACT-MAX(A, n)

1. **if** $n < 1$
2. **then error** “heap underflow”
3. $\max \leftarrow A[1]$
4. $A[1] \leftarrow A[n]$
5. MAX-HEAPIFY($A, 1, n-1$) remakes heap
6. **return** \max

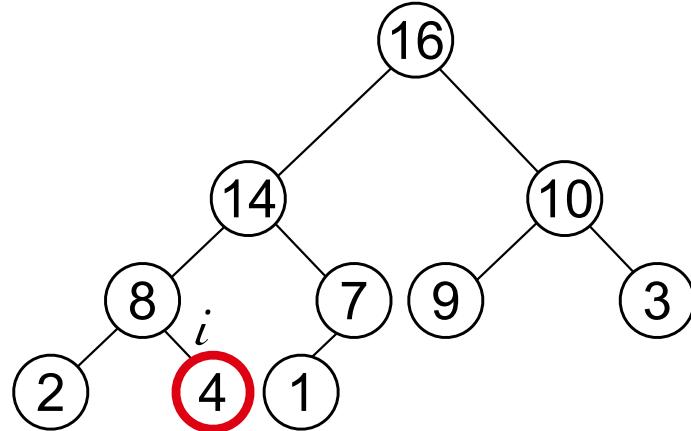


HEAP-INCREASE-KEY

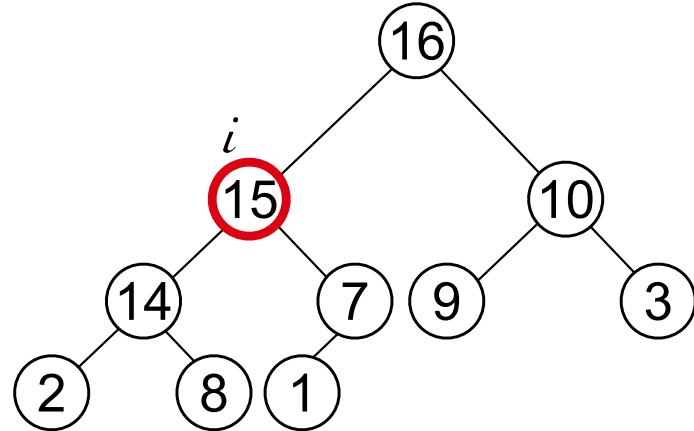
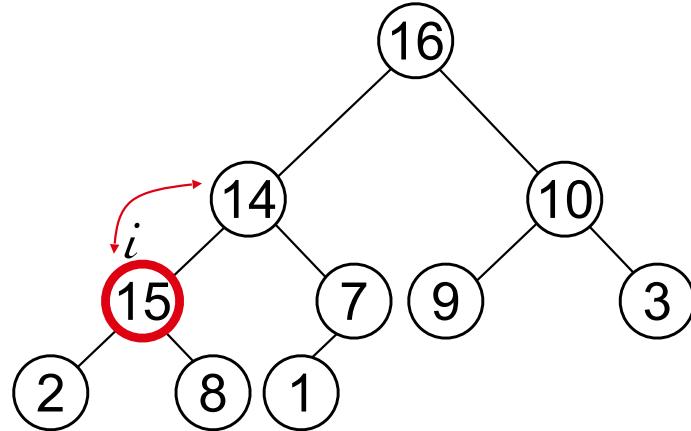
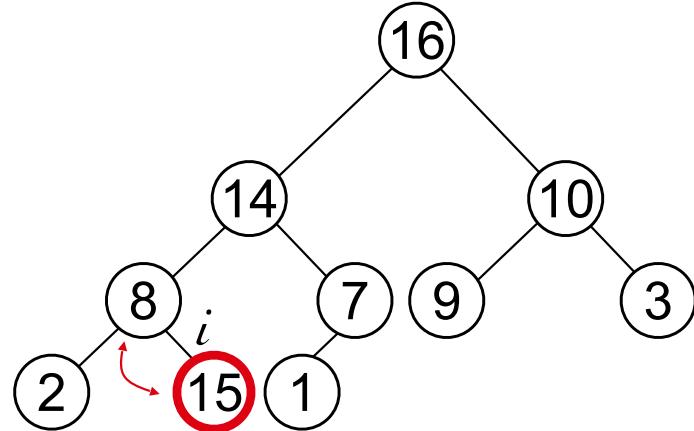
- ▶ Goal
 - ▶ Increases the key of an element i in the heap
- ▶ Idea
 - ▶ Increment the key of $A[i]$ to its new value
 - ▶ If the max-heap property does not hold anymore:
traverse a path toward the root to find the proper place
for the newly increased key



Example: HEAP-INCREASE-KEY



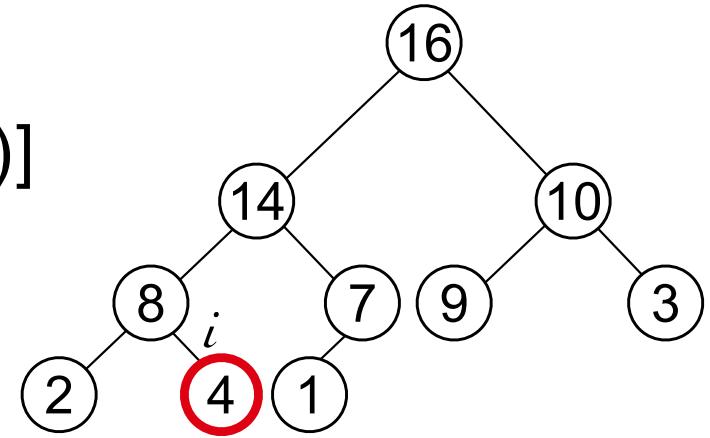
Key [i] $\leftarrow 15$



Analyzing HEAP-INCREASE-KEY

Alg: HEAP-INCREASE-KEY(A, i, key)

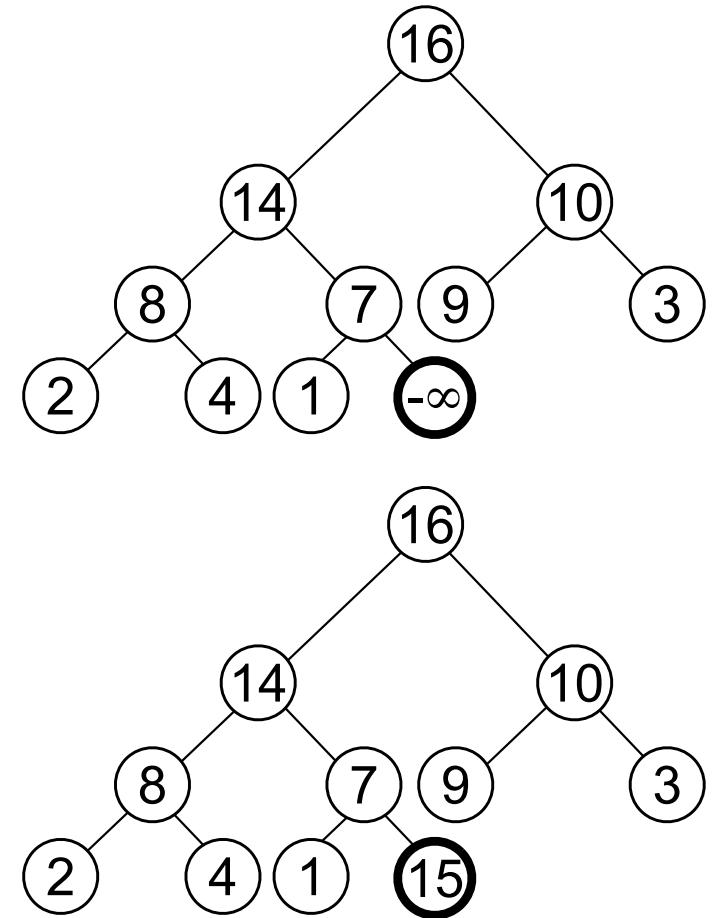
1. **if** $\text{key} < A[i]$
2. **then error** “new key is smaller than current key”
3. $A[i] \leftarrow \text{key}$
4. **while** $i > 1$ and $A[\text{PARENT}(i)] < A[i]$
5. **do exchange** $A[i] \leftrightarrow A[\text{PARENT}(i)]$
6. $i \leftarrow \text{PARENT}(i)$



► Running time: $O(\lg n)$

MAX-HEAP-INSERT

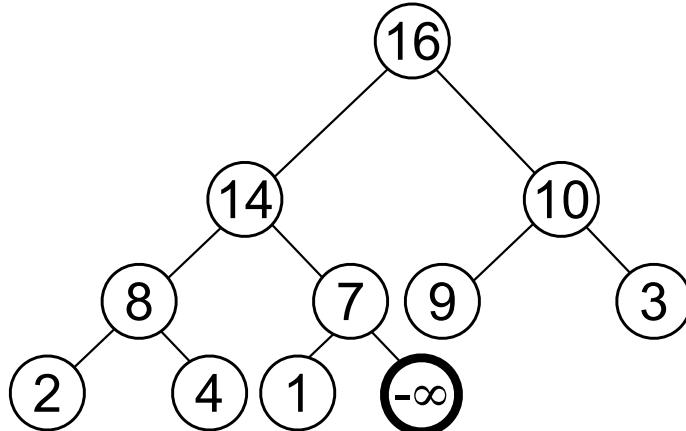
- ▶ Goal
 - ▶ Inserts a new element into a max-heap
- ▶ Idea
 - ▶ Expand the max-heap with a new element whose key is $-\infty$
 - ▶ Calls HEAP-INCREASE-KEY to set the key of the new node to its correct value and maintain the max-heap property



Example: MAX-HEAP-INSERT

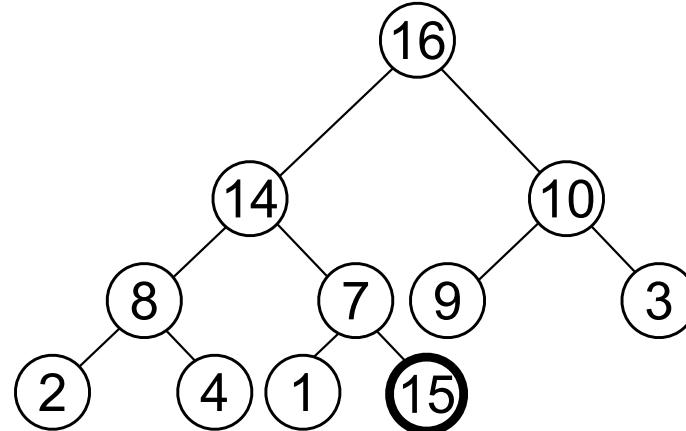
Insert value 15:

- Start by inserting $-\infty$

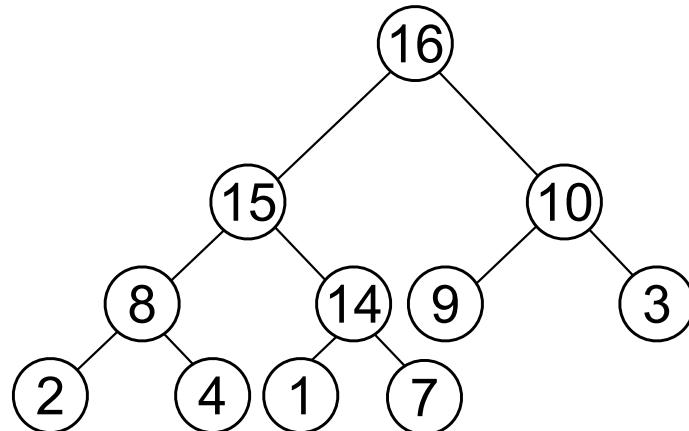
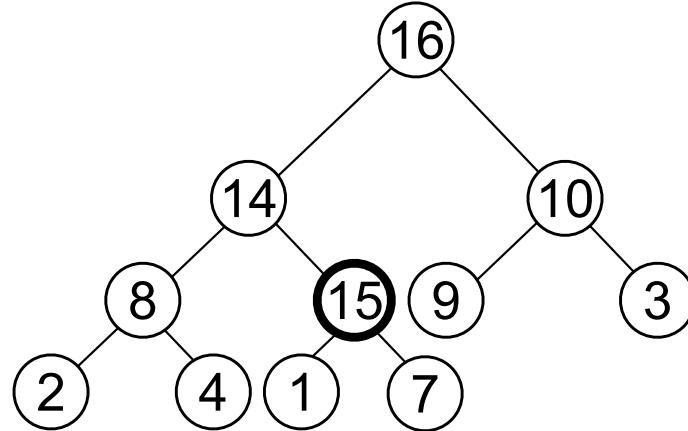


Increase the key to 15

Call HEAP-INCREASE-KEY on $A[11] = 15$



The restored heap containing
the newly added element



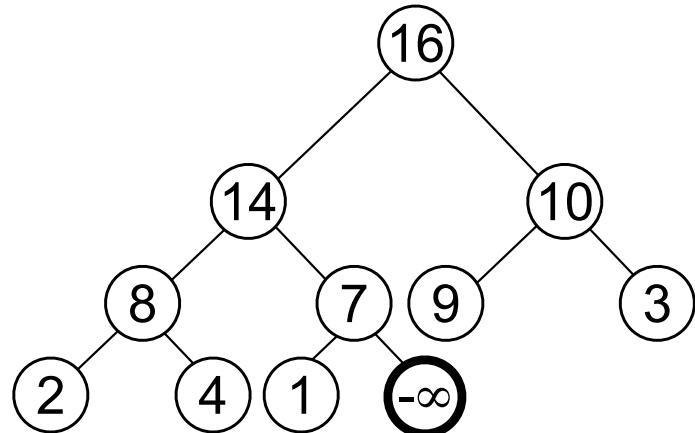
Analyzing MAX-HEAP-INSERT

Alg: MAX-HEAP-INSERT(A , key, n)

1. $\text{heap-size}[A] \leftarrow n + 1$

2. $A[n + 1] \leftarrow -\infty$

3. HEAP-INCREASE-KEY(A , $n + 1$, key)



Running time: $O(\lg n)$

Summary

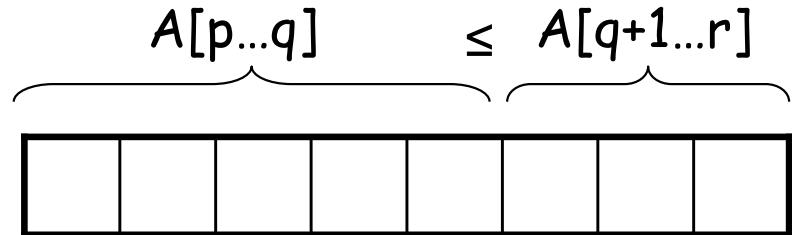
- ▶ We can perform the following operations on heaps:
 - ▶ MAX-HEAPIFY $O(lgn)$
 - ▶ BUILD-MAX-HEAP $O(n)$
 - ▶ HEAP-SORT $O(nlgn)$
 - ▶ MAX-HEAP-INSERT $O(lgn)$
 - ▶ HEAP-EXTRACT-MAX $O(lgn)$
 - ▶ HEAP-INCREASE-KEY $O(lgn)$
 - ▶ HEAP-MAXIMUM $O(1)$

Quicksort: Brief Review

- ▶ Sorts in place
- ▶ Sorts $O(n \lg n)$ in the average case
- ▶ Sorts $O(n^2)$ in the worst case
 - ▶ But the worst case doesn't happen often (more on this later...)

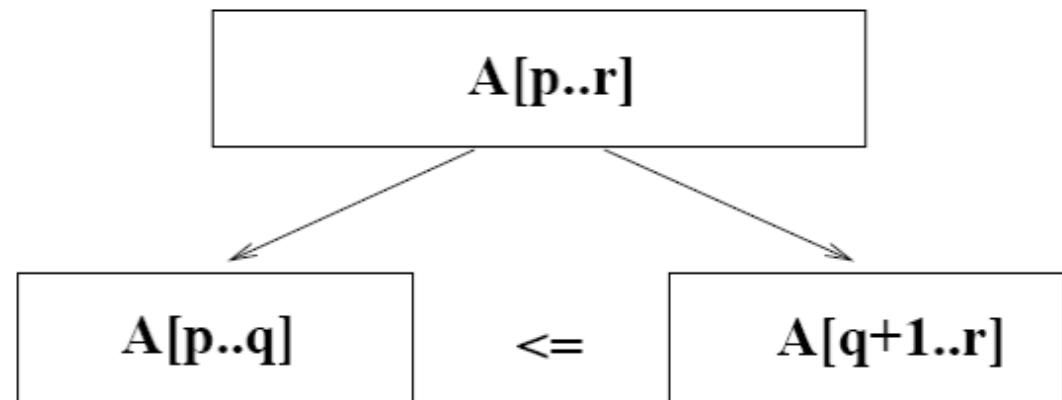
Quicksort

Sort an array $A[p \dots r]$

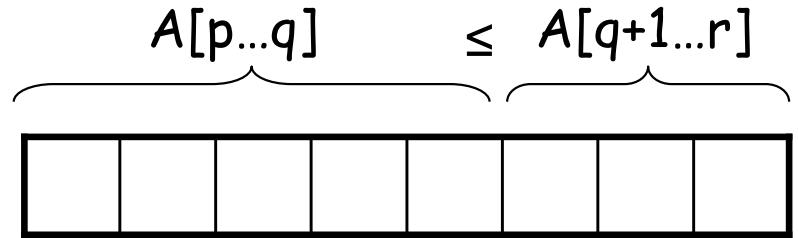


► Divide

- ▶ Partition the array A into 2 subarrays $A[p..q]$ and $A[q+1..r]$, such that each element of $A[p..q]$ is smaller than or equal to each element in $A[q+1..r]$
- ▶ Need to find index q to partition the array



Quicksort



- ▶ **Conquer**
 - ▶ Recursively sort $A[p..q]$ and $A[q+1..r]$ by calls to Quicksort
- ▶ **Combine (unlike merge sort)**
 - ▶ Trivial: the arrays are sorted in place
 - ▶ No additional work is required to combine them
 - ▶ The entire array is now sorted

Quicksort

Alg.: QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{PARTITION}(A, p, r)$

 QUICKSORT (A, p, q)

 QUICKSORT (A, q+1, r)

Initially: $p=1, r=n$

Recurrence: $T(n) = T(q) + T(n - q) + f(n)$

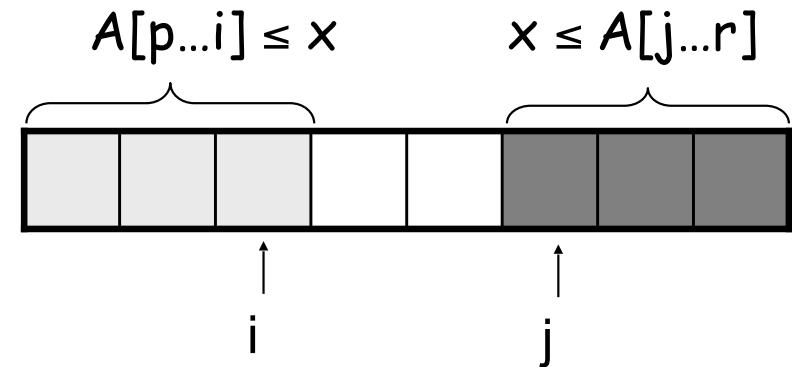

f(n) depends on partition()

Partition

- ▶ Clearly, all the action takes place in the **PARTITION()** function
 - ▶ Rearranges the subarray in place
 - ▶ End result:
 - ▶ Two subarrays
 - ▶ All values in first subarray \leq all values in second
 - ▶ Returns the index of the “pivot” element separating the two subarrays
- ▶ *How should you implement this?*

Partition

- ▶ Choosing PARTITION()
 - ▶ There are different ways to do this
 - ▶ Each has its own advantages/disadvantages
- ▶ Hoare partition
 - ▶ Select a pivot element x around which to partition
 - ▶ Starts from both ends
 - ▶ Grows two regions
 - ▶ $A[p \dots i] \leq x$
 - ▶ $x \leq A[j \dots r]$



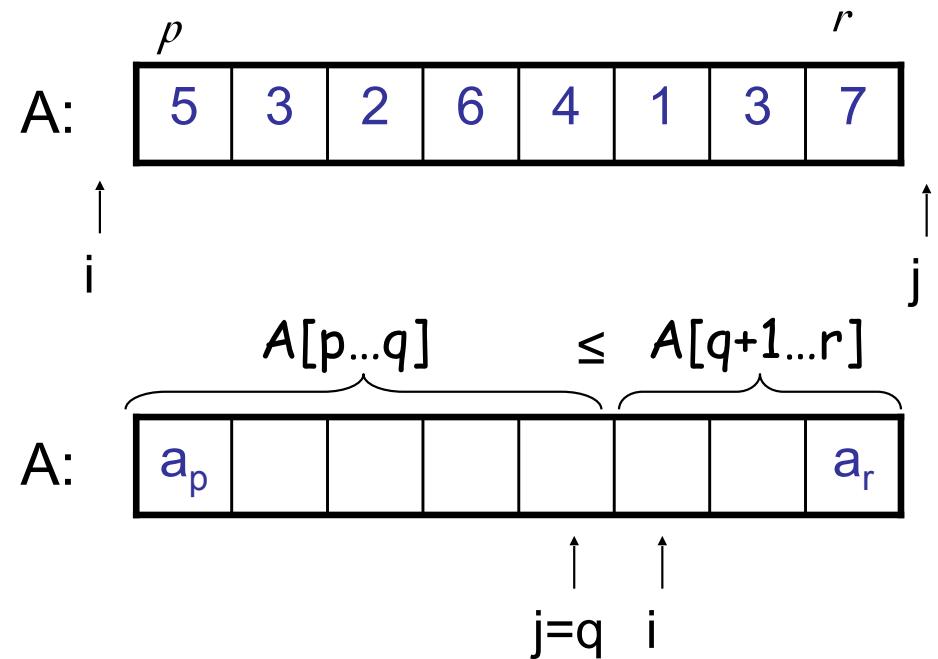
Partition in Words

- ▶ Partition(A, p, r):
 - ▶ Select an element to act as the “pivot” (*which?*)
 - ▶ Grow two regions, $A[p..i]$ and $A[j..r]$
 - ▶ All elements in $A[p..i] \leq \text{pivot}$
 - ▶ All elements in $A[j..r] \geq \text{pivot}$
 - ▶ Increment i until $A[i] \geq \text{pivot}$
 - ▶ Decrement j until $A[j] \leq \text{pivot}$
 - ▶ Swap $A[i]$ and $A[j]$
 - ▶ Repeat until $i \geq j$
 - ▶ Return j

Partition Code

Alg. PARTITION (A, p, r)

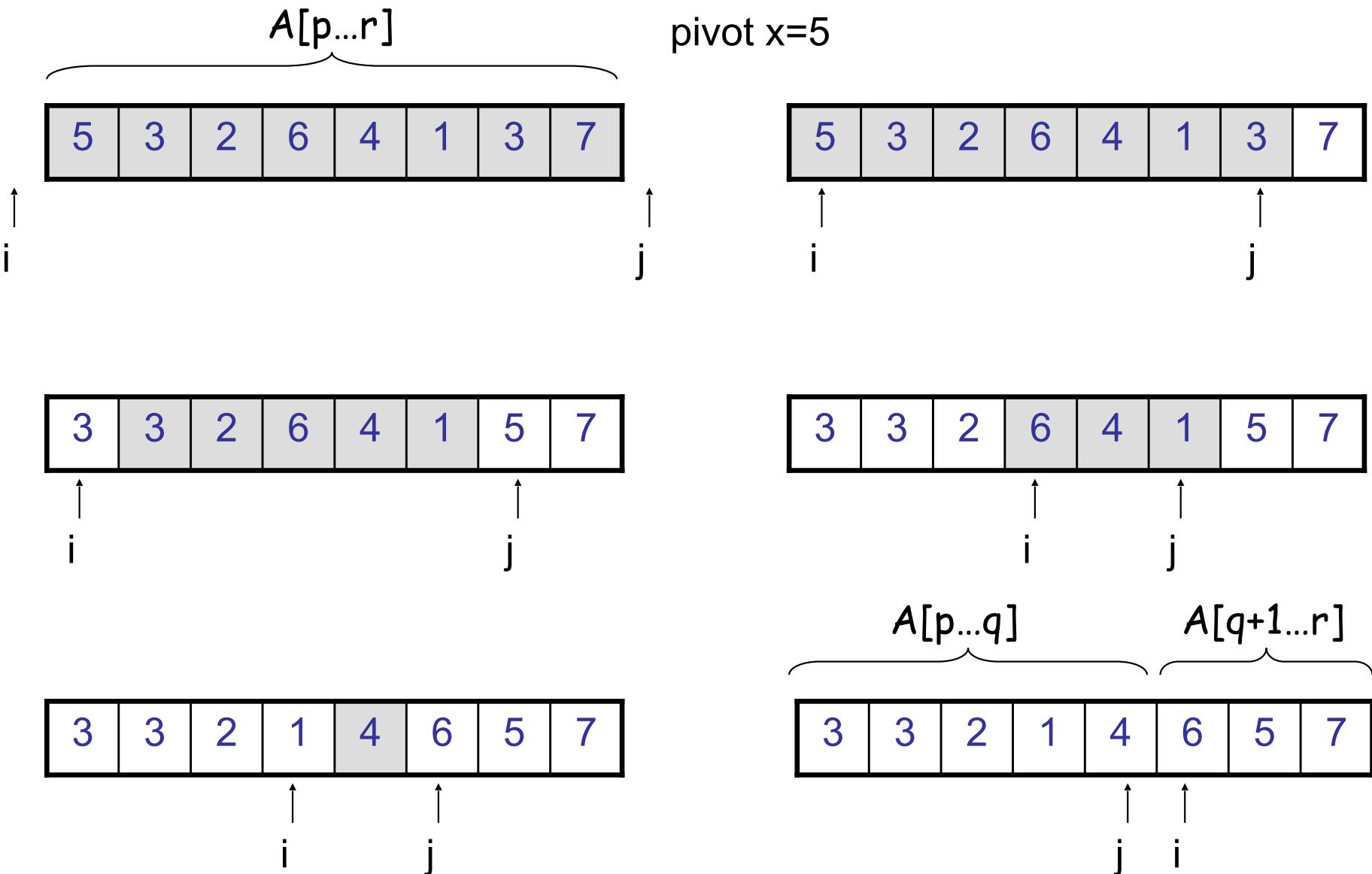
1. $x \leftarrow A[p]$
2. $i \leftarrow p - 1$
3. $j \leftarrow r + 1$
4. **while** TRUE
 do repeat $j \leftarrow j - 1$
 until $A[j] \leq x$
 do repeat $i \leftarrow i + 1$
 until $A[i] \geq x$
 if $i < j$
 then exchange $A[i] \leftrightarrow A[j]$
else return j



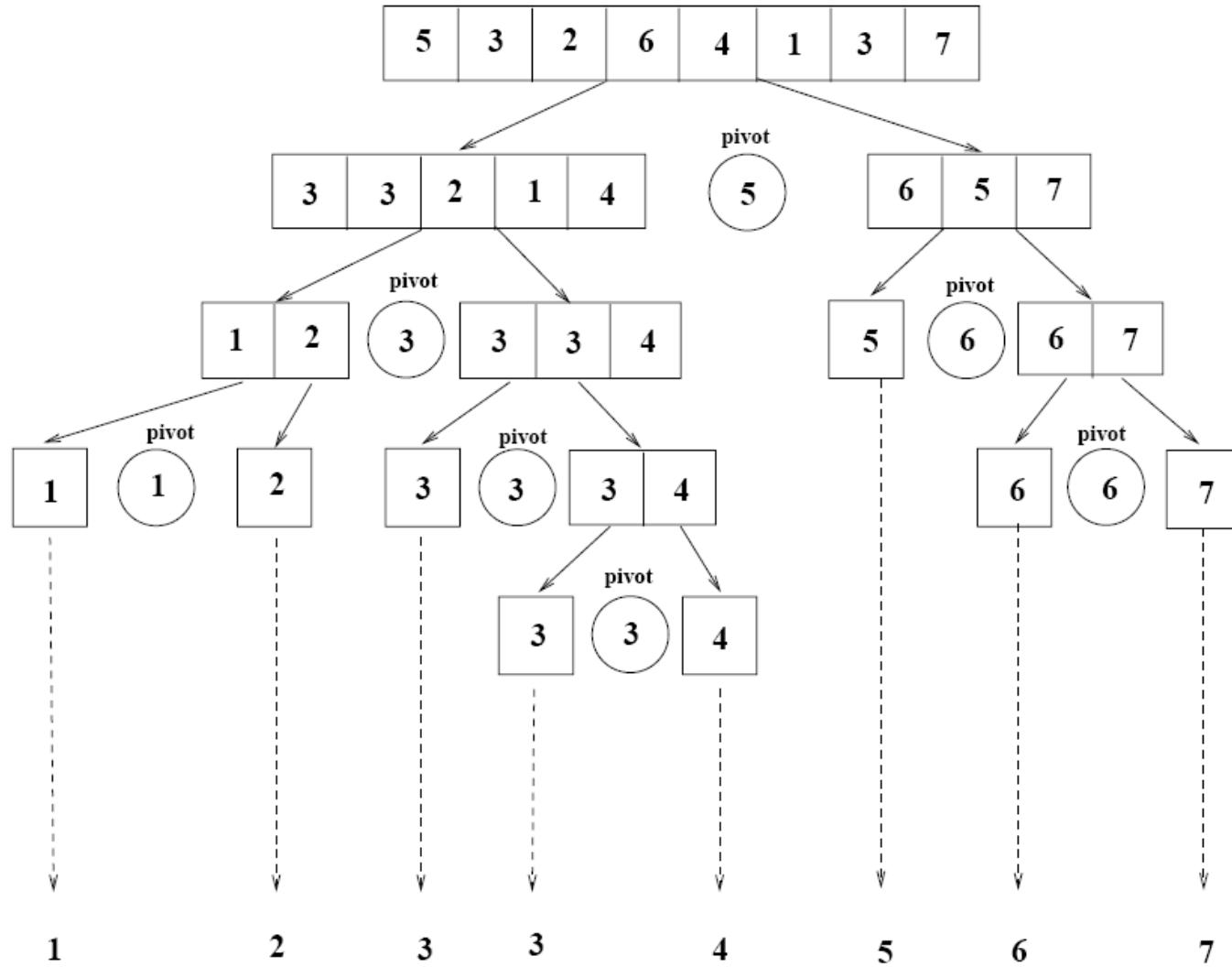
Each element is visited once!

Running time: $\Theta(n)$
 $n = r - p + 1$

Example



Example



Quicksort: Recurrence

Alg.: QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{PARTITION}(A, p, r)$

 QUICKSORT (A, p, q)

 QUICKSORT (A, q+1, r)

Initially: $p=1, r=n$

Recurrence: $T(n) = T(q) + T(n - q) + n$

Analyzing Quicksort

- ▶ *What will be the worst case for the algorithm?*
 - ▶ Partition is always unbalanced
- ▶ *What will be the best case for the algorithm?*
 - ▶ Partition is perfectly balanced
- ▶ *Which is more likely?*
 - ▶ The latter, except...
- ▶ *Will any particular input elicit the worst case?*
 - ▶ Yes: Already-sorted input

Analyzing Quicksort: Worst Case Partitioning

► Worst-case partitioning

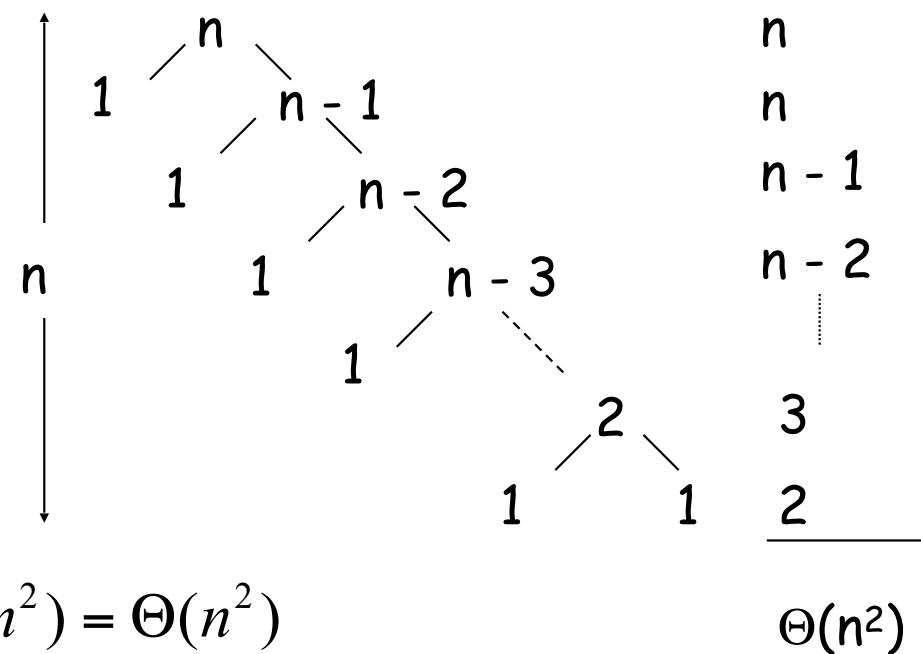
- One region has one element and the other has $n - 1$ elements
- Maximally unbalanced

► Recurrence: $q=1$

$$T(n) = T(1) + T(n - 1) + n,$$

- $T(1) = \Theta(1)$

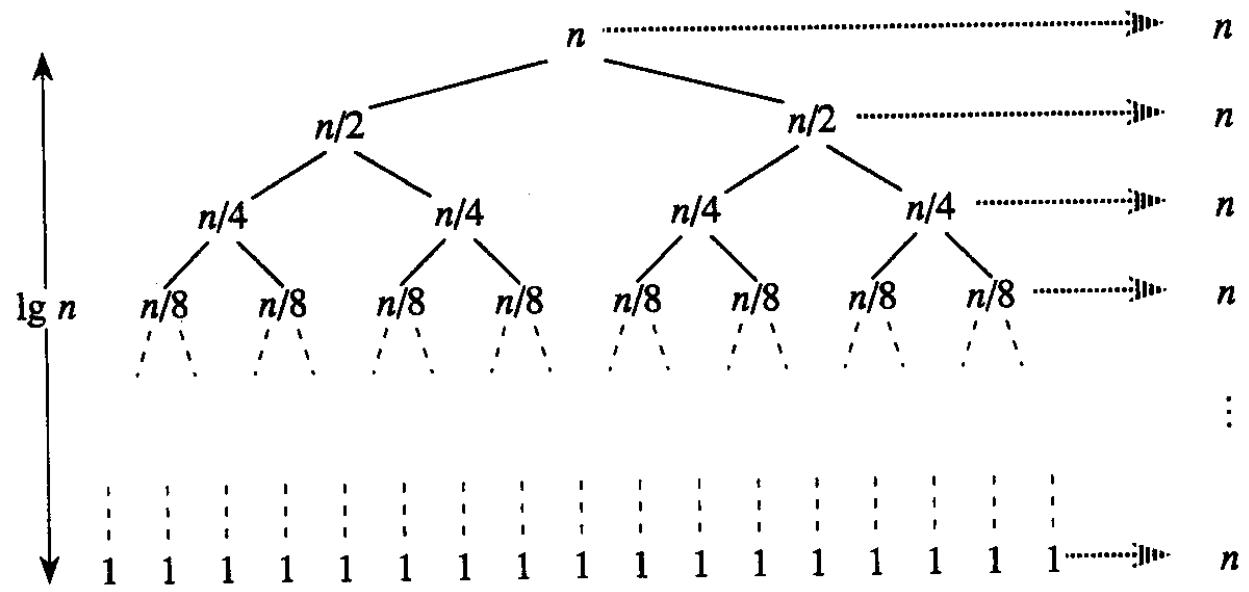
- $T(n) = n + \left(\sum_{k=1}^n k \right) - 1 = \Theta(n) + \Theta(n^2) = \Theta(n^2)$



When does the worst case happen?

Analyzing Quicksort: Best Case Partitioning

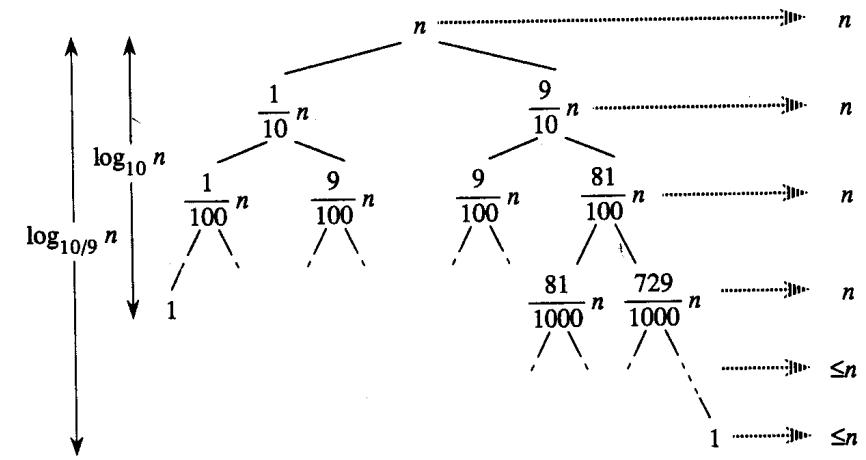
- ▶ Best-case partitioning
 - ▶ Partitioning produces two regions of size $n/2$
- ▶ Recurrence: $q=n/2$
 - ▶ $T(n) = 2T(n/2) + \Theta(n)$
 - ▶ $T(n) = \Theta(n \lg n)$ (Master theorem)



Case Between Worst and Best

- ▶ An intuitive explanation/example:
 - ▶ Suppose that partition() always produces a 9-to-1 split.
This looks quite unbalanced!
- ▶ The recurrence is thus:
 - ▶ $T(n) = T(9n/10) + T(n/10) + n$
 - ▶ *How deep will the recursion go?*

- Using the recursion tree:



$$\text{longest path: } Q(n) \leq n \sum_{i=0}^{\log_{10/9} n} 1 = n(\log_{10/9} n + 1) = c_2 n \lg n$$

$$\text{shortest path: } Q(n) \geq n \sum_{i=0}^{\log_{10} n} 1 = n \log_{10} n = c_1 n \lg n$$

$$\text{Thus, } Q(n) = \Theta(n \lg n)$$

How does partition affect performance?

- Any splitting of constant proportionality yields $\Theta(n \lg n)$ time !!!

- Consider the $(1 : n - 1)$ splitting:

ratio= $1/(n - 1)$ not a constant !!!

- Consider the $(n/2 : n/2)$ splitting:

ratio= $(n/2)/(n/2) = 1$ it is a constant !!

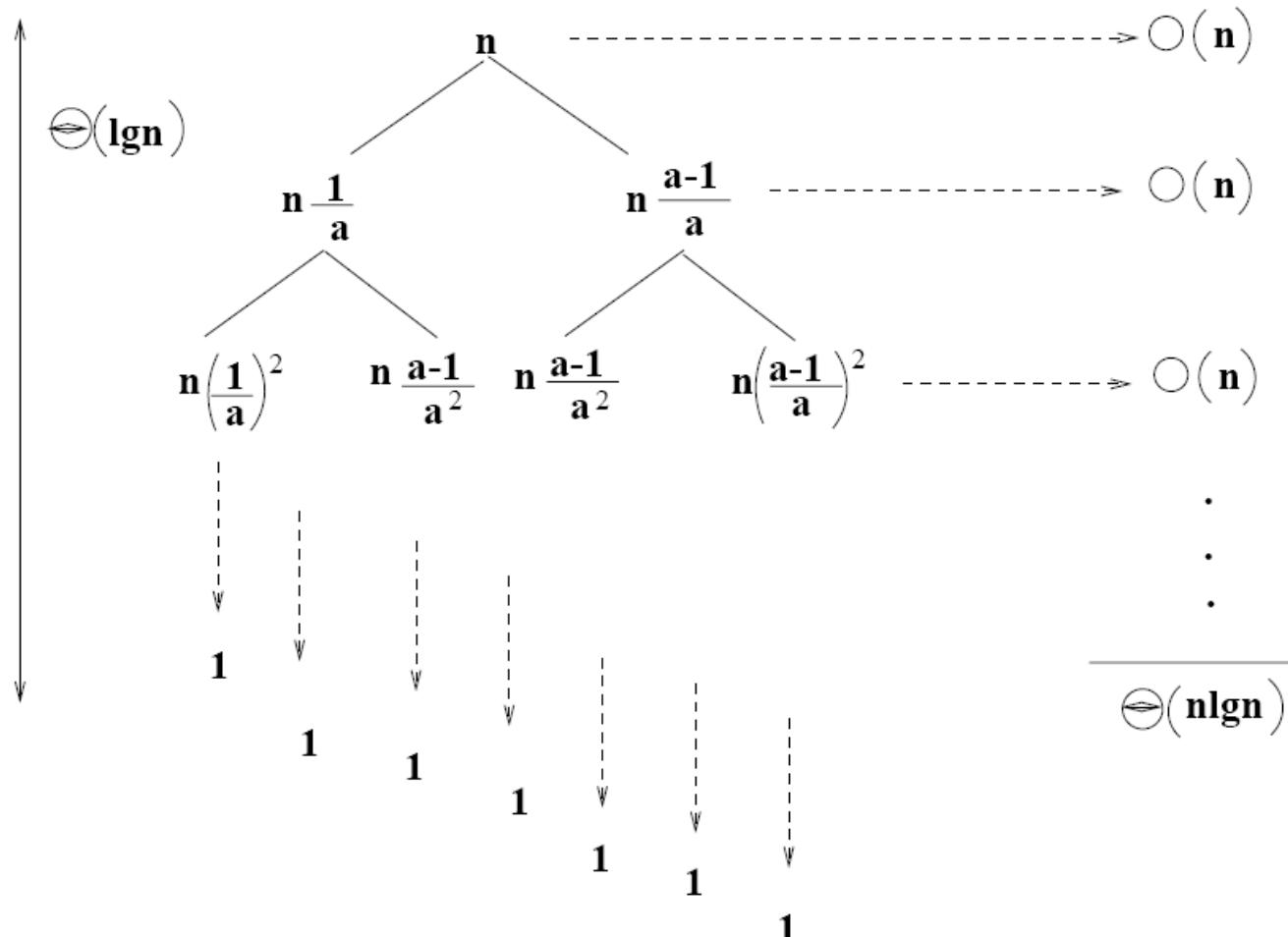
- Consider the $(9n/10 : n/10)$ splitting:

ratio= $(9n/10)/(n/10) = 9$ it is a constant !!

How does partition affect performance?

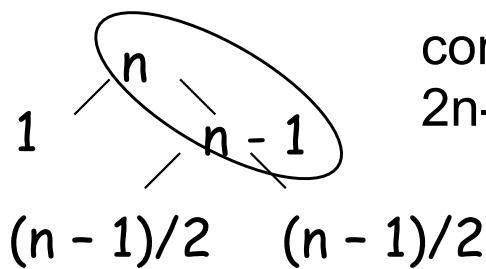
- Any $((a - 1)n/a : n/a)$ splitting:

$$\text{ratio} = ((a - 1)n/a)/(n/a) = a - 1 \text{ it is a constant !!}$$



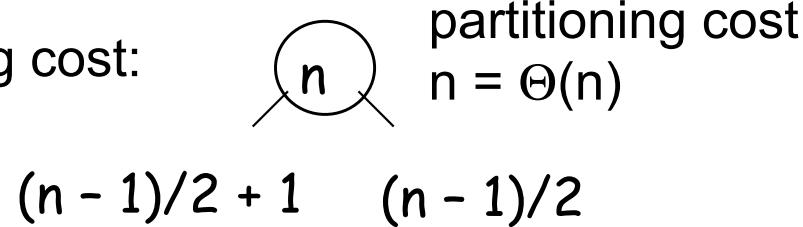
Analyzing Quicksort: Average Case Partitioning

- ▶ Average case
 - ▶ All permutations of the input numbers are equally likely
 - ▶ On a random input array, we will have a **mix** of well balanced and unbalanced splits
 - ▶ Good and bad splits are randomly distributed across throughout the tree



combined partitioning cost:
 $2n-1 = \Theta(n)$

Alternate of a good
and a bad split



partitioning cost:
 $n = \Theta(n)$

Nearly well
balanced split

Running time of Quicksort when levels alternate between good and bad splits is $O(nlgn)$

Randomizing Quicksort

- ▶ Randomly permute the elements of the input array before sorting
- ▶ OR ... modify the PARTITION procedure
 - ▶ At each step of the algorithm we exchange element $A[p]$ with an element chosen at random from $A[p \dots r]$
 - ▶ The pivot element $x = A[p]$ is equally likely to be any one of the $r - p + 1$ elements of the subarray

Randomizing PARTITION

Alg. : RANDOMIZED-PARTITION(A, p, r)

$i \leftarrow \text{RANDOM}(p, r)$

exchange $A[p] \leftrightarrow A[i]$

return PARTITION(A, p, r)

Randomizing Quicksort

Alg.:RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{RANDOMIZED-PARTITION}(A, p, r)$

 RANDOMIZED-QUICKSORT(A, p, q)

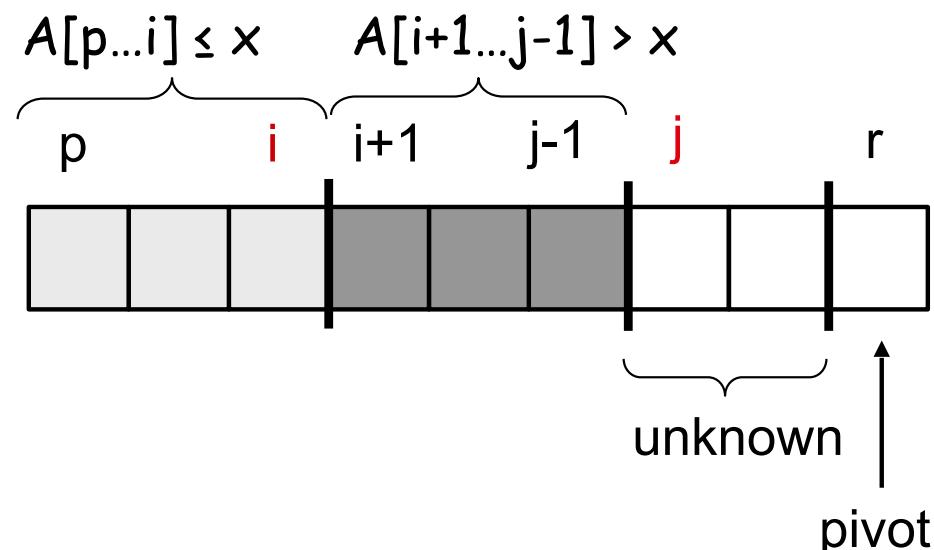
 RANDOMIZED-QUICKSORT($A, q + 1, r$)

Another Partitioning: Lomuto's Partition

- Given an array A, partition the array into the following subarrays:

- A pivot element $x = A[q]$
- Subarray $A[p..q-1]$ such that each element of $A[p..q-1]$ is smaller than or equal to x (the pivot)
- Subarray $A[q+1..r]$, such that each element of $A[p..q+1]$ is strictly greater than x (the pivot)

- The pivot element is not included in any of the two subarrays

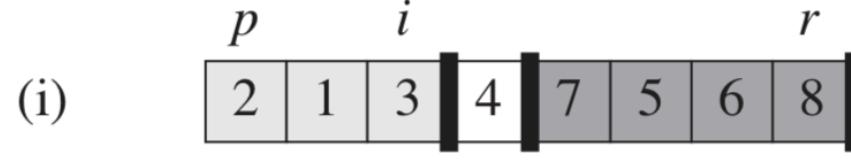
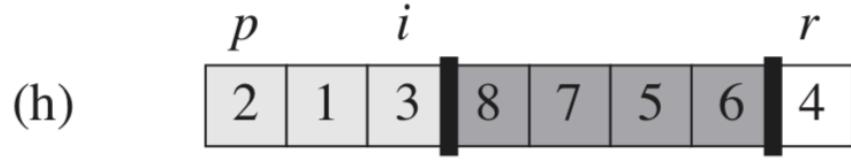
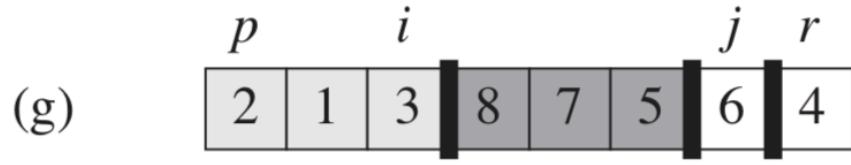
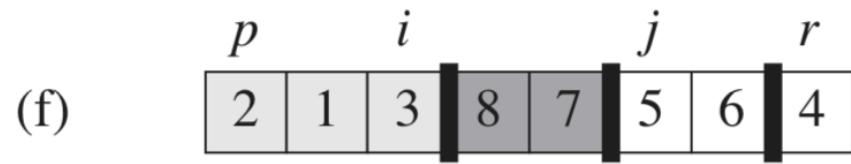
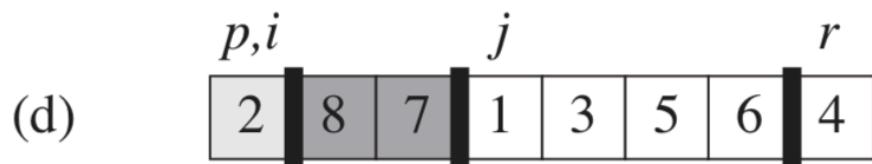
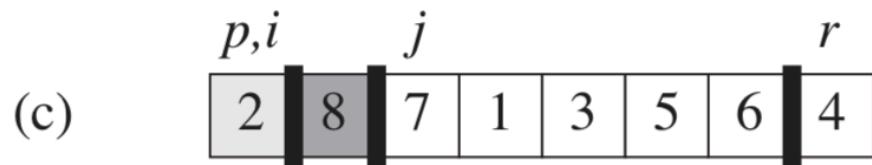
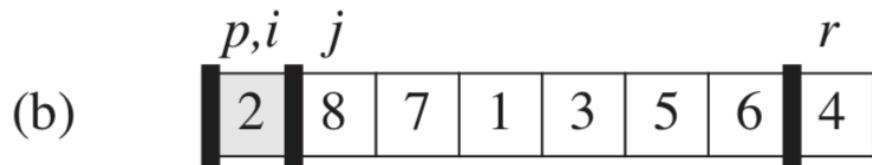
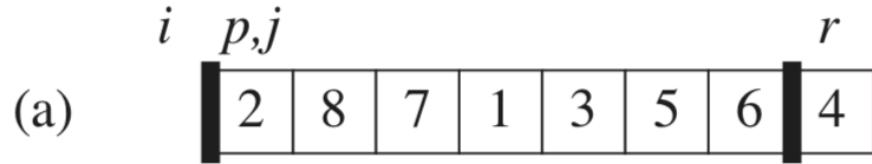


Partition Code

PARTITION(A, p, r)

```
1   $x = A[r]$ 
2   $i = p - 1$ 
3  for  $j = p$  to  $r - 1$ 
4      if  $A[j] \leq x$ 
5           $i = i + 1$ 
6          exchange  $A[i]$  with  $A[j]$ 
7  exchange  $A[i + 1]$  with  $A[r]$ 
8  return  $i + 1$ 
```

Example



Randomizing Quicksort using Lomuto's partition

Alg.: RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow$ RANDOMIZED-PARTITION(A, p, r)

RANDOMIZED-QUICKSORT(A, p, $q - 1$)

RANDOMIZED-QUICKSORT(A, $q + 1$, r)

The pivot is no longer included in any of the subarrays!!

Analysis of Randomized Quicksort

Alg.: RANDOMIZED-QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow \text{RANDOMIZED-PARTITION}(A, p, r)$

$\text{RANDOMIZED-QUICKSORT}(A, p, q - 1)$

$\text{RANDOMIZED-QUICKSORT}(A, q + 1, r)$

PARTITION is called at most n times

- ▶ (at each call a pivot is selected and never again included in future calls)

What's next...

- ▶ Analyze Quicksort
- ▶ Sorting Lower Bound
- ▶ Order Statistics & Selection