

## Transfer Matrix Equations

The Transfer Matrix Equations for an  $L$ -layer structure with variations only along the  $z$ -axis can be written as

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \begin{pmatrix} E_L^+ \\ E_L^- \end{pmatrix}, \quad (S1)$$

where the elements  $M_{1,1}$  depend on the material properties (i.e. the refractive index  $n$ ) and the geometry of each layer, as well as on the frequency and polarization of incident light.

This formalism assumes that layer 1 and  $L$  are semi-infinite materials with real refractive indices; however, all intermediate layers have finite thickness and may consist of materials with complex refractive indices. The 2x2  $\mathbf{M}$  matrix can be computed from the following product of matrices:

$$\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} = \mathbf{D}_1^{-1} \left( \prod_{l=2}^{L-1} \mathbf{D}_l \mathbf{P}_l \mathbf{D}_l^{-1} \right) \mathbf{D}_L. \quad (S2)$$

The  $\mathbf{P}$  matrix is defined for each finite-thickness layer as

$$\mathbf{P}_l = \begin{pmatrix} \exp(-i \phi_l) & 0 \\ 0 & \exp(i \phi_l) \end{pmatrix}, \quad (S3)$$

where  $\phi_l = k_{z,l} d_l$  where  $d_l$  is the thickness of the  $l^{th}$  layer of the structure,

$$k_{z,l} = \sqrt{\left(n_l \frac{\omega}{c}\right)^2 - \left(n_l \sin(\theta_1) \frac{\omega}{c}\right)^2}, \quad (S4)$$

$\theta_1$  is the angle of incidence of light at frequency  $\omega$  upon the structure and  $n_l$  is the refractive index of the  $l^{th}$  layer at frequency  $\omega$ . If the incident light is s-polarized, then the  $\mathbf{D}$  matrix for the  $l^{th}$  layer has the form

$$\mathbf{D}_l = \begin{pmatrix} 1 & 1 \\ n_l \cos(\theta_l) & -n_l \cos(\theta_l) \end{pmatrix}, \quad (S5)$$

while if the light is p-polarized, the  $\mathbf{D}$  matrix for the  $l^{th}$  layer has the form

$$\mathbf{D}_l = \begin{pmatrix} \cos(\theta_l) & \cos(\theta_l) \\ n_l & -n_l \end{pmatrix}, \quad (S6)$$

where  $\theta_l$  is the refraction angle in the  $l^{th}$  layer determined by Snell's law.

The amplitudes  $E_1^+$  and  $E_1^-$  are interpreted as the incoming and outgoing wave amplitudes on the incident side, respectively; similarly,  $E_L^-$  and  $E_L^+$  are interpreted as the incoming and outgoing wave amplitudes on the terminal side, respectively. With access to the field amplitudes and wavevectors, a number of useful quantities may be computed. For example, the Fresnel reflection and transmission amplitudes may be computed as  $r = \frac{E_1^-}{E_1^+}$  and  $t = \frac{E_L^+}{E_1^+}$ . For the analysis of reflection experiments,  $E_1^+ = 1$  and  $E_L^- = 0$  by convention. Consideration of Eq. (S1) under these conditions leads to expressions for the reflection and transmission amplitudes in terms of the elements of the  $\mathbf{M}$  matrix,

$$r = \frac{M_{2,1}}{M_{1,1}} \quad (S7)$$

$$t = \frac{1}{M_{1,1}}. \quad (S8)$$

The reflection can then be calculated as  $R = |r|^2$ , the transmission as  $T = |t|^2 \frac{n_L \cos(\theta_L)}{n_1 \cos(\theta_1)}$ . The absorption at a given frequency, which can be taken to be equal to the emissivity by Kirchoff's law, can simply be computed as  $\epsilon(\omega) = A(\omega) = 1 - T(\omega) - R(\omega)$ . The transfer matrix equations can be used to compute the stored energy of the Bragg Relector ( $P_{BR}(\omega)$ ) as well, which is a key quantity for assessing the degree of critical coupling. Following the discussion by Piper and Fan, the amplitude associated with the stored energy in a lossless resonant reflector ( $E_S$ ) can be related to the incoming and outgoing wave amplitudes according to

$$E_S = \left| \frac{E_1^- + E_1^+}{\sqrt{2\gamma_l}} \right|^2 \quad (S9)$$

where  $\gamma_l$  is the leakage rate associated with the Bragg reflector. From Eq. S1, the stored energy spectrum can be calculated for a given frequency as

$$E_s = \frac{(r + 1)(r^* + 1)}{2\gamma_l}. \quad (S10)$$