Transfer Matrix Equations

The Transfer Matrix Equations for an *L*-layer structure with variations only along the *z*-axis can be written as

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \begin{pmatrix} E_L^+ \\ E_L^- \end{pmatrix}, \qquad (S1)$$

where the elements $M_{1,1}$depend on the material properties (i.e. the refractive index *n*) and the geometry of each layer, as well as on the frequency and polarization of incident light.

This formalism assumes that layer 1 and *L* are semi-infinte materials with real refractive indices; however, all intermediate layers have finite thickness and may consist of materials with complex refractive indices. The 2x2 **M** matrix can be computed from the following product of matrices:

$$\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} = \mathbf{D}_1^{-1} \left(\prod_{l=2}^{L-1} \mathbf{D}_l \, \mathbf{P}_l \, \mathbf{D}_l^{-1} \right) \mathbf{D}_{\mathrm{L}}. \qquad (S2)$$

The **P** matrix is defined for each finite-thickness layer as

$$\mathbf{P}_l = \begin{pmatrix} \exp(i\,\phi_l) & 0 \\ 0 & \exp(-i\,\phi_l) \end{pmatrix}, \quad (S3)$$

where $\phi_l = k_{z,l}\, d_l$ where d_l is the thickness of the l^{th} layer of the structure,

$$k_{z,l} = \sqrt{\left(n_l \frac{\omega}{c}\right)^2 - \left(n_l \sin(\theta_1) \frac{\omega}{c}\right)^2}, \qquad (S4)$$

θ_1 is the angle of incidence of light at frequency ω upon the structure and n_l is the refractive index of the l^{th} layer at frequency ω. If the incident light is s-polarized, then the **D** matrix for the l^{th} layer has the form

$$\mathbf{D}_l = \begin{pmatrix} 1 & 1 \\ n_l \cos(\theta_l) & -n_l \cos(\theta_l) \end{pmatrix}, \qquad (S5)$$

while if the light is p-polarized, the **D** matrix for the l^{th} layer has the form

$$\mathbf{D}_l = \begin{pmatrix} \cos(\theta_l) & \cos(\theta_l) \\ n_l & -n_l \end{pmatrix}, \qquad (S6)$$

where θ_l is the refraction angle in the l^{th} layer determined by Snell's law.

The amplitudes $E_1^+$and E_1^- are interpreted as the incoming and outgoing wave amplitudes on the incident side, respectively; similarly, $E_L^-$and E_L^+ are interpreted is the incoming and outgoing wave amplitudes on the terminal side, respectively. With access to the field amplitudes and wavevectors, a number of useful quantities may be computed. For example, the Fresnel reflection and transmission amplitudes may be computed as $r = \frac{E_1^-}{E_1^+}$ and $t = E_L^+$. For the analysis of reflection experiments, $E_1^+ = 1$ and $E_L^- = 0$ by convention. Consideration of Eq. (S1) under these conditions leads to expressions for the reflection and transmission amplitudes in terms of the elements of the **M** matrix,

$$r = \frac{M_{2,1}}{M_{1,1}} \qquad (S7)$$

$$t = \frac{1}{M_{1,1}}. \qquad (S8)$$

The reflection can then be calculated as $R = |r|^2$, the transmission as $T = |t|^2 \frac{n_L \cos(\theta_L)}{n_L \cos(\theta_1)}$. The absorption at a given frequency, which can be taken to be equal to the emissivity by Kirchoff's law, can simply be computed as $\epsilon(\omega) = A(\omega) = 1 - T(\omega) - R(\omega)$. The transfer matrix equations can be used to compute the stored energy of the Bragg Relector ($P_{BR}(\omega)$) as well, which is a key quantity for assessing the degree of critical coupling. Following the discussion by Piper and Fan, the amplitude associated with the stored energy in a lossless resonant reflector (E_S) can be related to the incoming and outgoing wave amplitudes according to

$$E_S = \left| \frac{E_1^- + E_1^+}{\sqrt{2\gamma_l}} \right|^2 \qquad (S9)$$

where γ_l is the leakage rate associated with the Bragg reflector. From Eq. S1, the stored energy spectrum can be calculated for a given frequency as

$$E_s = \frac{(r+1)(r^*+1)}{2\gamma_l}. \qquad (S10)$$