Algorithmic Game Theory Summative Assignment – Individual Component

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1 Exercise 1

1.1 Part a

All task weights are the same and all machines have equal speed.

$$cost(A) = \frac{1}{4}(1 \cdot 2 + 2 \cdot 2) = \frac{3}{2}$$

since there are 2^2 possible assignments, 2 of these are the cases where the makespan is 1 and 2 are where the makespan is 2 (both tasks on one machine).

$$cost(OPT) = 1$$

since the optimal assignment is when all tasks are evenly distributed. As such,

$$cost(A)/cost(OPT) = \frac{3}{2}$$

1.2 Part b

All task weights are the same and all machines have equal speed.

$$cost(A) = \frac{1}{27}(1 \cdot 6 + 2 \cdot 18 + 3 \cdot 3) = \frac{17}{9}$$

since there are 3^3 possible assignments, 6 of these are the cases where the makespan is 1 (3! ways to arrange 3 tasks across 3 machines), 18 of these are where the makespan is 2 (i.e. one machine has 2 tasks, 1 has one and the other has zero - 6 ways to arrange these, then 3 different pairs so $6 \cdot 3 = 18$) and 3 are where makespan is 3 (all tasks one one machine).

$$cost(OPT) = 1$$

since the optimal assignment is when all tasks are evenly distributed. As such,

$$cost(A)/cost(OPT) = \frac{17}{9}$$

1.3 Part c

Generally,

$$cost(A) \leq \frac{1}{m^m}(m \cdot m^m)$$

since the number of tasks and machines are the same which means there are m^m possible ways to arrange m tasks across m machines. The maximum possible makespan always happens when all tasks are assigned to the same machine hence the m in the brackets. We can assume that this makespan were to occur for all possible combinations (unrealistic, especially since we know there is a $1 \cdot m$ for makespan 1 but it does give a mathematical upper bound) which then means that the $cost(A) \leq m$. Since the $cost(OPT)$ will always be when all tasks are assigned to a unique machine, it is always 1. This means that $cost(A)/cost(OPT) \leq m$ and hence the Price of Anarchy on identical machines for mixed Nash equilibria is $\leq m$.

2 Exercise 2

2.1 Part a

R tends towards r as n becomes very large

2.2 Part b

The second highest value can be 1 of 2 values, either r or 1 for each player. Given this binary choice, there are 2^{n-1} possible ways the second highest bid can be decided. Since we are already evaluating the second highest bid, we are only interested in the highest value possible here. In any set of bids, if there is an r bid by any player, this will be the second highest bid. If not, then 1 will be the second highest bid. Clearly there is only one case where 1 is the second highest bid: where all bids are 1. Drawing from this, we can create the equation:

$$R = (\frac{1}{2})^{n-1} \cdot 2^{n-1} \cdot r - (\frac{1}{2})^{n-1} \cdot r + (\frac{1}{2})^{n-1} \cdot 1$$

The first multiplication is the expectation for all cases where there exists an r as well as where there is not so the second multiplication removes one instance of these to account for the all 1 instance. The last multiplication calculates the expectation of all 1.

$$(\frac{1}{2})^{n-1} \cdot 2^{n-1} \cdot r$$

can be rearranged to

$$(\frac{1^{n-1} \cdot 2^{n-1}}{2^{n-1}}) \cdot r =$$
$$r$$

Taking the limit of the second multiplication:

$$\lim_{n\to\infty} ((\frac{1}{2})^{n-1} \cdot r) = 0$$

since

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Taking the limit of the third multiplication:

$$\lim_{n\to\infty} ((\frac{1}{2})^{n-1} \cdot 1) = 0$$

Therefore,

$$\lim_{n\to\infty} R = r - 0 + 0$$
$$= r$$

3 Exercise 3

3.1 Part a

$u_M(B,W) = 2, u_M(C,J) = 1, u_M(C,W) = 0, u_M(B,J) = 0, u_A(C,J) = 2, u_A(B,W) = 1, u_A(C,W) = 0, u_A(B,J) = 0$

3.2 Part b

The bimatrix game can be expressed as a table:

	J	W
C	(1, 2)	(0, 0)
B	(0, 0)	(2, 1)

or using the matrices where M_M is Mary's matrix and M_A is Alice's matrix.

$$M_M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \; M_A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

3.3 Part c

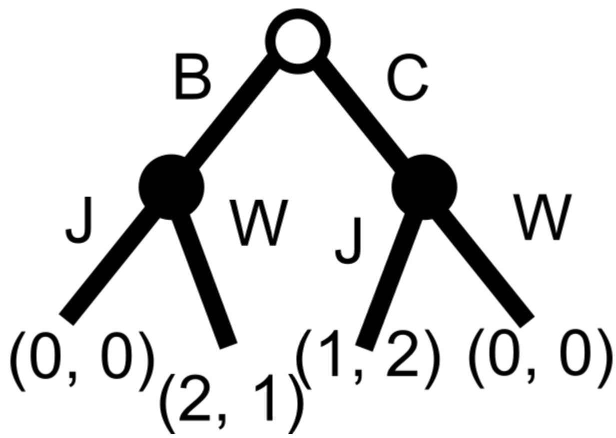

Figure 1: Extended Game

3.4 Part d

At step 1, see Figure 2, the choice of Juice/Wine by Alice is dominated by the choice Wine with the outcome (2, 1). At step 2, the other branch in Figure 3, the choice of Juice/Wine by Alice is also dominated by the choice Wine with outcome (1, 2). At step 3 in Figure 4, the choice is made by Mary and this is dominated by choosing Beef with outcome (2, 1). Therefore, the solution is (2, 1) which is a Nash Equilibrium.

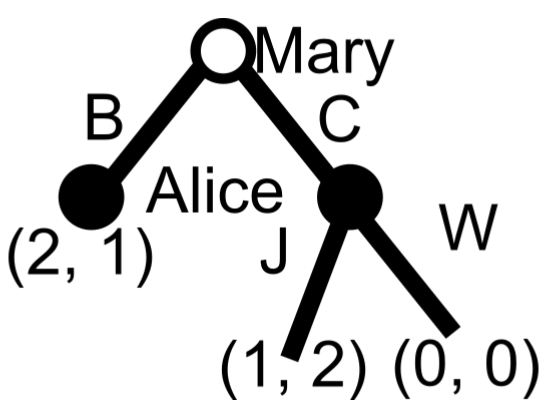

Figure 2: Backward Induction Step 1

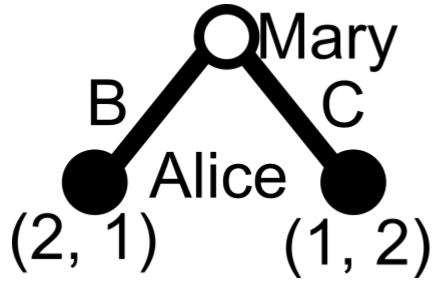

Figure 3: Backward Induction Step 2

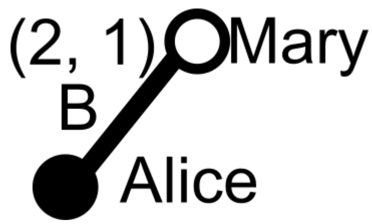

Figure 4: Backward Induction Step 3

4 Exercise 4

4.1 Part a

The price for item 1 is $k-1$ and all other items have price 0 when the market clears. Buyer 1 gets the first item at the price of $k-1$ which is buyer 2's valuation of the item.

4.2 Part b

There are k buyers. Initially all prices are 0 but each buyer i has valuation $k-i+1$. This intuitively means that each buyer can maximise their payoff by choosing the first item for a payoff equal to their valuation. This isn't a perfect matching so forms a constricted set for all buyers containing only the first item as a neighbour since all buyers are demanding the same item. The price of this first item is increased by 1 which removes the lowest item 1 valuation buyer from the constricted set since they can achieve a payoff of 0 with all items. All other buyers still remain choosing item 1 so item 1 is still the only neighbour of buyers in a constricted set consisting of all buyers except the lowest valuation buyer. On the next iteration, the lowest valuation buyer will no longer be interested in item 1 since their payoff for all items will be 0 except for item 1 where it becomes negative. At the same time, the next lowest valuation buyer becomes no longer part of the constricted set. This process repeats until a stage where the first and second highest valuation buyers are the only ones interested in item 1 but the second highest valuation buyer isn't in the constricted set and hence is interested in all other items too. This obviously occurs when the second highest valuation buyer's valuation equals the price of item 1 such that the second highest valuation buyer's payoff is 0 in all cases. As such there is now a perfect matching and the market clearing ends with the buyer 1 taking item 1 for buyer 2's valuation.

4.3 Part c

In this case, the market clearing implements a second-price sealed bid auction since we can assume that each player (buyer) bids their valuation but buyer 1 wins with the valuation of buyer 2.