Continuous Function

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Introduction

Generally speaking in Calculus class, if x goes and infinitely close to a, then we say function $f(x)$ converges to $f(a)$. Is it understandable? It might make sense; however, is it possible to logically prove that the value can be infinitely close to a certain value? To be clear this, we will talk about cool Mathematical theorem in this chapter.

§ 1.1 Continuity

Definition 1.1.1

Let $D \subseteq \mathbb{R}$. A function $f: D \to \mathbb{R}$ is **continuous** at $c \in D$ if, for every sequence x_n in *D* which converges to *c*, the sequence $f\ (x_n)$ converges to $f(c)$.

In symbols, assuming $x_n \in D$ for all *n,*

$$x_n \to c \Rightarrow f(x_n) \to f(c).$$

If *f* is continuous at every point in its domain, we simply say *f is continuous.*

However**,** a function $f: D \to \mathbb{R}$ is **discontinuous** at a point $a \in D$ if, there exists sequence $\{x_n\}$ in *D* that converges to *a*, for which $f(x_n)$ does not converge to $f(a)$.

Example 1.1.2

(a) $f(x) = 3x - 1$is continuous at *x = 2.*

Let $x_n \to 2$.

Now need to show $f(x_n) \to f(2)$,

for any ε > 0, there exists $N\ such\ that\ n > N \Rightarrow |f(x_n) - f(2)| < \varepsilon$.

Then, $|f(x_n) - f(2)| = |3x_n - 1 - \ (3(2) - 1)|$

$= |3x_n - 1 - (6 - 1)|$

$= |3x_n - 1 - (5)|$

$= |3x_n - 6|$

$= 3(|x_n - 2|)$

< 3ε, since $x_n - 2 < \varepsilon$ by definition

≤ ε', as required.

So given ε' > 0 there exists N ∈ ℕ such that for all n ∈ ℕ, n ≥ N implies $|a_n^2 - 2| < \varepsilon$'

(b) $f(x) = \begin{cases} \frac{1}{x}\ if\ x \neq 0 \\ 0\ if\ x = 0 \end{cases}$

This is not continuous at 0 since $\lim_{x \to 0} f(x)$ does not exists.

Also, $\lim_{x \to -0} f(x)$ does not exist at 0.

$\lim_{x \to +0} f(x)$ does not exist at 0.

Thus, *f* has a discontinuity at 0.

Theorem 1.1.3

Let *f* and *g* be functions from $D \subseteq \mathbb{R}$ to ℝ and let $c \in D$.

Suppose that *f* and *g* are continuous at *c.* Then:

(a) $f + g$ and fg are continuous at *c.*

(b) $k \bullet f$ is continuous for any $k \in \mathbb{R}$.

(c) f/g is continuous at *c* if $f(c) \neq 0$.

Proof

(a) Let *w* be a point in the intersection of the domains of *f* and *g*.

Then $|(f+g)(w)-(f+g)(c)| = |\big(f(w)+g(w)\big) - \ (f(c)-g\ (c)|$
$\leq |f(w)-f(c)| + |g(w)-g(c)|$

Now, given any ε > 0, since $f(w) \to f(c)$, there exists $n \geq N_1$ implies that $|f(w)-f(c)| < \varepsilon/2$. Similarly, since $g(w) \to g\ (c)$, there exists $N_2 \in \mathbb{N}$, such that $n \geq N_2$ implies that $|g(w)-g(c)| < \varepsilon/2$.

$$|(f+g)(w)-(f+g)(c) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon|$$

(b) Let *w* be a point in the intersection of the domains of *f* and *g*.

$|(fg)(w)-(fg)(c)|$
$= |\big(f(w)g(w)\big) - \big(f(c)g(c)\big)|$
$= |f(w)g(w)-f(w)g(c)| + |f(w)g(c)-f(c)g(c)|$
$\leq |f(w)||g(w)-g(c)| + |g(c)||f(w)-f(c)|$
$= |f(w)-f(c)+f(c)||g(w)-g(c)| + |g(c)||f(w)-f(c)|$
$\leq (|f(w)-f(c)|+|f(c)|)|g(w)-g(c)| + |g(c)||f(w)-g(c)|$
$= |f(w)-f(c)||g(w)-g(c)| + |f(c)||g(w)-g(c)| + |g(c)||f(w)-f(c)|$

Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

By the continuity of *f* at *c*, we know that there exists $\varepsilon' > 0$ such that $|f(w)-f(c)| < \varepsilon_2$ for all *w* in the domain of *f* with $|w-c| < \varepsilon_1'$.

Similarly, by the continuity of *g* at *c*, we know that there exists $\varepsilon_2' > 0$ such that $|g(w)-g(c)| < \varepsilon_2$ for all *w* in the domain of *g* with $|w-c| < \varepsilon_2'$.

Now choose any $\varepsilon' > 0$ smaller than both $\varepsilon_1'$and ε_2'.

So we get for all *w* in the domains of both *f* and *g* with $|w-z| < \varepsilon'$.

$< \varepsilon_2 \bullet \varepsilon_2 + |f(c)|\varepsilon_2 + |g(c)|\varepsilon_2$
$= \varepsilon_2(\varepsilon_2 + |f(c)| + |g(c)|$
$< \varepsilon_2(1 + |f(c)| + |g(c)|$
$< \varepsilon$

(c) Let *w* be a point in the intersection of the domains of *f* and *g*.

Assume that *c* is in the domains of *f* and *g* as well as $g(c) \neq 0$.

$\left|\left(\frac{f}{g}\right)(w) - \left(\frac{f}{g}\right)(c)\right| = \left|\frac{f(w)}{g(w)} - \frac{f(c)}{g(c)}\right|$
$= |\frac{f(w)g(c)-f(c)g(w)}{g(w)g(c)}|$
$= |\frac{\big(f(w)g(c)-f(c)g(c)\big)+\big(f(c)g(c)-f(c)g(w)\big)}{g(w)g(c)}|$
$= \frac{|\big(f(w)g(c)-f(c)g(c)\big)|+|\big(f(c)g(c)-f(c)g(w)\big)|}{|g(w)||g(c)|}$
$= \frac{|\big(f(w)-f(c)||g(c)\big)|+|\big(f(c)||g(w)-g(c)|}{|g(w)||g(c)|}$

Let $\varepsilon_2 > 0$.

Now choose $|\varepsilon_2' > 0|$, so $|f(w)-f(c)| < \varepsilon_2$for all *w* in the domains of *f* and *g* such that $|w-z| < \varepsilon_2'$. Thus,

$< \frac{\varepsilon_2|g(c)|+|f(c)|\varepsilon_2}{\left(\frac{1}{2}\right)|g(c)^2|}$

$$= \varepsilon_2 \bullet \frac{|g(c)|+|f(c)|}{\left(\frac{1}{2}\right)|g(c)^2|}$$

$$< \varepsilon$$

Theorem 1.1.4

Let f and g be continuous real-valued functions such that the range of f is contained in the domain of g. Then the composition $g \circ f$ is continuous.

Proof

Consider that for any sequence (a_n) such that $a_n \to a$,
$g \circ f(a_n) \to g \circ f(a)$
Since f is continuous at a, $f(a_n) \to f(a)$.
Thus, the sequence $(f(a_n))$ is a convergent sequence converging to $f(a)$.
Therefore, since g is continuous at $f(a)$, $g\big(f(a_n)\big) \to g\big(f(a)\big)$.
Hence, $g \circ f(a_n) = g\big(f(a_n)\big) \to g\big(f(a)\big) = g \circ f(a)$.
Thus, $g \circ f$ is continuous at a.