1. 证法很多，提供其中一种，参考 https://math.stackexchange.com/questions/3373870/prove-ra-times-b-ra-times-rb-given-r-in-mathcalso3-and-a-b-in

Recall that the cross product $a \times b$ is characterized by the property that

$$\det(x, a, b) = \langle x, a \times b \rangle, \qquad \forall x \in \mathbb{R}^3.$$

Now let $R \in \mathcal{SO}(3)$. Then by using the fact that $R^\top = R^{-1}$, we get

$$\langle x, R(a \times b) \rangle = \langle R^\top x, a \times b \rangle = \langle R^{-1} x, a \times b \rangle = \det(R^{-1} x, a, b).$$

Then, utilizing the assumption $\det(R) = 1$,

$$= \det(R) \det(R^{-1} x, a, b) = \det(x, Ra, Rb) = \langle x, Ra \times Rb \rangle.$$

Finally, since $\langle x, R(a \times b) \rangle = \langle x, Ra \times Rb \rangle$ holds for any $x \in \mathbb{R}^3$, the desired identity follows.

5.3) First, velocity analysis:

$$^{1}W_1 = {}^{1}_{0}R\,{}^{0}W_0 + \dot{\theta}_1\,{}^{1}\hat{z}_1 = \dot{\theta}_1\hat{z}$$

$$^{1}V_1 = {}^{1}_{0}R({}^{0}V_0 + {}^{0}W_0 \times {}^{0}P_1) = 0$$

$$^{2}W_2 = {}^{2}_{1}R\,{}^{1}W_1 + \dot{\theta}_2\,{}^{2}\hat{Z}_2$$

$$^{2}W_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^{2}V_2 = {}^{2}_{1}R({}^{1}V_1 + {}^{1}W_1 \times {}^{1}P_2)$$

$$^{2}V_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$^{2}V_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ L_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}$$

5.3) (*Continued*)

$$^{3}W_3 = {}^{3}_{2}R\,{}^{2}W_2 + \dot{\theta}_3\,{}^{3}\hat{Z}_3$$

$$^{3}W_3 = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} S_2C_3\dot{\theta}_1 + C_2S_3\dot{\theta}_1 \\ -S_2S_3\dot{\theta}_1 + C_2C_3\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$^{3}W_3 = \begin{bmatrix} S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$^{3}V_3 = {}^{3}_{2}R({}^{2}V_2 + {}^{2}W_2 \times {}^{2}P_3)$$

$$^{3}V_3 = {}^{3}_{2}R\left(\begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$^{3}V_3 = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ L_2\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} L_2S_3\dot{\theta}_2 \\ L_2C_3\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix}$$

$$^{4}W_4 = {}^{4}_{3}R\,{}^{3}W_3 + O; \quad {}^{4}_{3}R = I; \quad {}^{4}W_4 = {}^{3}W_3$$

$$^{4}V_4 = {}^{4}_{3}R({}^{3}V_3 + {}^{3}W_3 \times {}^{3}P_4)$$

$$^{4}V_4 = {}^{4}_{3}R\left(\begin{bmatrix} L_2S_3\dot{\theta}_2 \\ L_2C_3\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \times \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$^{4}V_4 = \left(\begin{bmatrix} L_2S_3\dot{\theta}_2 \\ L_2C_3\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ L_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -L_3C_{23}\dot{\theta}_1 \end{bmatrix}\right)$$

$$^4V_4 = \begin{bmatrix} L_2 S_3 \dot\theta_2 \\ L_2 C_3 \dot\theta_2 + L_3(\dot\theta_2 + \dot\theta_3) \\ -L_1\dot\theta_1 - L_2C_2\dot\theta_1 - L_3C_{23}\dot\theta_1 \end{bmatrix}$$

$$^4V_4 = {}^4J(\underline{\theta})\underline{\dot\theta}$$

$$\therefore \quad {}^4J(\underline{\theta}) = \begin{bmatrix} 0 & L_2S_3 & 0 \\ 0 & L_2C_3 + L_3 & L_3 \\ -L_1 - L_2C_2 - L_3C_{23} & 0 & 0 \end{bmatrix}$$

Next, using force analysis:

$$^4F_4 = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} {}^4N_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^3F_3 = {}^3_4R\,{}^4F_4 = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}$$

$$^3N_3 = {}^3_4R\,{}^4N_4 + {}^3P_4 \times {}^3F_3 = \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \begin{bmatrix} 0 \\ -L_3F_Z \\ L_3F_Y \end{bmatrix}$$

$$^2F_2 = {}^2_3R\,{}^3F_3 = \begin{bmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \begin{bmatrix} C_3F_X - S_3F_Y \\ S_3F_X + C_3F_Y \\ F_Z \end{bmatrix}$$

5.3) (*Continued*)

$$^2N_2 = {}^2_3R\,{}^3N_3 + {}^2P_3 \times {}^2F_2$$

$$^2N_2 = \begin{bmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -L_3F_Z \\ L_3F_Y \end{bmatrix} + \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} C_3F_X - S_3F_Y \\ S_3F_X + C_3F_Y \\ F_Z \end{bmatrix}$$

$$^2N_2 = \begin{bmatrix} L_3S_3F_Z \\ -L_2F_Z - L_3C_3F_Z \\ L_2(S_3F_X + C_3F_Y) + L_3F_Y \end{bmatrix}$$

$$^1F_1 = {}^1_2R\,{}^2F_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} C_3F_X - S_3F_Y \\ S_3F_X + C_3F_Y \\ F_Z \end{bmatrix}$$

$$^1F_1 = \begin{bmatrix} C_2(C_3F_X - S_3F_Y) - S_2(S_3F_X + C_3F_Y) \\ -F_Z \\ S_2(C_3F_X - S_3F_Y) + C_2(S_3F_X + C_3F_Y) \end{bmatrix}$$

$$^1N_1 = {}^1_2R\,{}^2N_2 + {}^1P_2 \times {}^1F_1$$

$$^1N_1 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} L_3S_3F_Z \\ -L_2F_Z - L_3C_3F_Z \\ L_2(S_3F_x + C_3F_Y) + L_3F_Y \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times {}^1F_1$$

$$^1N_1 = \begin{bmatrix} C_2L_3S_3F_Z + S_2L_2F_Z + L_2S_2C_3F_Z \\ -L_2(S_3F_X + C_3F_Y) - L_3F_Y \\ L_3S_2S_3F_Z - L_2C_2F_Z - L_3C_2C_3F_Z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -L_1S_2(C_3F_X - S_3F_Y) - L_2C_2(S_3F_X + C_3F_Y) \\ -L_1F_Z \end{bmatrix}$$

To compute torques, take the z-component of the ${}^{i}N_i$:

$$\tau_1 = [-L_1 - L_2C_2 + L_3(S_2S_3 - C_2C_3)]F_Z$$

$$\tau_2 = L_2S_3F_X + (L_2C_3 + L_3)F_Y$$

$$\tau_3 = L_3F_Y$$

$$\underline{\tau} = {}^{4}J^{T}(\underline{\theta})\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}$$

Which leads to same expression as before for ${}^{4}J(\underline{\theta})$.

Finally, by differentiation of kinematic equations:

$${}^{\circ}P_{4ORG} = \begin{bmatrix} L_1C_1 + L_2C_1C_2 + L_3C_1C_{23} \\ L_1S_1 + L_2S_1C_2 + L_3S_1C_{23} \\ L_2S_2 + L_3S_{23} \end{bmatrix} \triangleq P$$

$${}^{\circ}J(\underline{\theta}) = \begin{bmatrix} \dfrac{\partial\rho_X}{\partial\theta_1} & \dfrac{\partial\rho_X}{\partial\theta_2} & \dfrac{\partial\rho_X}{\partial\theta_3} \\ \dfrac{\partial\rho_Y}{\partial\theta_1} & \dfrac{\partial\rho_Y}{\partial\theta_2} & \dfrac{\partial\rho_Y}{\partial\theta_3} \\ \dfrac{\partial\rho_Z}{\partial\theta_1} & \dfrac{\partial\rho_Z}{\partial\theta_2} & \dfrac{\partial\rho_Z}{\partial\theta_3} \end{bmatrix}$$

5.3) (*Continued*)

$${}^{\circ}J(\underline{\theta}) = \begin{bmatrix} -L_1S_1 - L_2S_1C_2 - L_3S_1C_{23} & -L_2C_1S_2 - L_3C_1S_{23} & -L_3C_1S_{23} \\ L_1C_1 + L_2C_1C_2 + L_3C_1C_{23} & L_2S_1S_2 - L_3S_1S_{23} & -L_3S_1S_{23} \\ 0 & L_2C_2 + L_3C_{23} & L_3C_{23} \end{bmatrix}$$

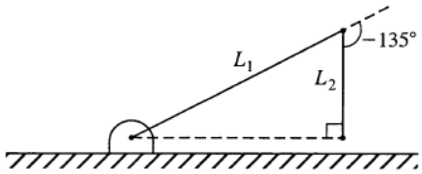
$${}^{\circ}V_4 = {}^{\circ}J(\underline{\theta})\underline{\dot{\theta}};\, {}^{4}V_4 = \underbrace{{}^{4}_{0}R^{\circ}J(\underline{\theta})}\, J(\underline{\theta})\underline{\dot{\theta}}$$

$${}^{4}_{0}R = \begin{bmatrix} C_1C_{23} & S_1C_{23} & S_{23} \\ -C_1S_{23} & -S_1S_{23} & C_{23} \\ S_1 & -C_1 & 0 \end{bmatrix}$$

Multiplying out ${}^{4}_{0}R^{\circ}J(\underline{\theta})$ is tedious, but sure enough, it leads again to the same expression for ${}^{4}J(\underline{\theta})$.

这里有排版错误

5.8) The Jacobian of this 2-link is:



$$^3J(\theta) = \begin{bmatrix} L_1 S_2 & 0 \\ L_1 C_2 + L_2 & L_2 \end{bmatrix}$$

An isotropic point exists if

$$^3J = \begin{bmatrix} L_2 & 0 \\ 0 & L_2 \end{bmatrix} \text{ so,}$$

$$L_1 S_2 = L_2$$

$$L_1 C_2 + L_2 = 0$$

$$\text{or, } S_2 = \frac{L_2}{L_1} C_2 = \frac{-L_2}{L_1}$$

Now $S_2^2 + C_2^2 = 1$,

$$\text{so } \left(\frac{L_2}{L_1}\right)^2 + \left(\frac{-L_2}{L_1}\right)^2 = 1$$

or $L_1^2 = 2L_2^2 \rightarrow L_1 = \sqrt{2} L_2$

Under this condition $S_2 = \dfrac{1}{\sqrt{2}} = \pm .707$

and $C_2 = -.707$

$\therefore$ An isotropic point exists if $L_1 = \sqrt{2} L_2$ and in that case it exists when $\theta_2 = \pm 135°$

In this configuration, the manipulator looks momentarily like a Cartesian manipulator.

5.15) The kinematics can be done easily to obtain:

$$ {}^{0}P_{\mathrm{YORG}} = \begin{bmatrix} (d_2 + L_2 + L_3)S_1 \\ -(d_2 + L_2 + L_3)C_1 \\ 0 \end{bmatrix} $$

$$ {}^{0}V = {}^{0}J\dot{\theta} $$

$$ {}^{0}J = \begin{bmatrix} \dfrac{\partial^{0}P_{\mathrm{YORGX}}}{\partial\theta_1} & \dfrac{\partial^{0}P_{\mathrm{YORGX}}}{\partial\theta_2} & \dfrac{\partial^{0}P_{\mathrm{YORGX}}}{\partial\theta_3} \\ \dfrac{\partial^{0}P_{\mathrm{YORGY}}}{\partial\theta_1} & \dfrac{\partial^{0}P_{\mathrm{YORGY}}}{\partial\theta_2} & \dfrac{\partial^{0}P_{\mathrm{YORGY}}}{\partial\theta_3} \\ \dfrac{\partial^{0}P_{\mathrm{YORGZ}}}{\partial\theta_1} & \dfrac{\partial^{0}P_{\mathrm{YORGZ}}}{\partial\theta_2} & \dfrac{\partial^{0}P_{\mathrm{YORGZ}}}{\partial\theta_3} \end{bmatrix} $$

So,

$$ {}^{0}J = \begin{bmatrix} (d_2 + L_2 + L_3)C_1 & S_1 & 0 \\ (d_2 + L_2 + L_3)S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} $$