1. Velocity is a free vector and will only be affected by rotation.

$${}^{A}V = {}^{A}_{B}R\,{}^{B}V = [-1.34 \quad 22.32 \quad 30]^T$$

2. $${}^{B}_{C}T = {}^{B}_{A}T\,{}^{U}_{A}T^{-1}\,{}^{C}_{U}T^{-1}$$

3. Compute $-{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \Rightarrow -6.4$

4. $R \triangleq [r_1 \ r_2 \ r_3]$ From the definition, $r_1 \times r_2 = r_3$

$$\det(R) = (r_1 \times r_2)\cdot r_3 = r_3 \cdot r_3 = 1$$

5. $$\|P\| = \sqrt{P^T P} = \sqrt{(RP)^T(RP)} = \|RP\|$$

$|\cdot| \rightarrow$ determinant

$\|\cdot\| \rightarrow$ norm

6. $$R_{Y'X'Z'}(\alpha, \beta, \gamma) = R_Y(\alpha)\,R_X(\beta)\,R_Z(\gamma)$$

$$R_{X'Z'X'}(\alpha, \beta, \gamma) = R_X(\alpha)\,R_Z(\beta)\,R_X(\gamma)$$

7. $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \times \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \\ -\frac{\sqrt{3}}{4} \end{bmatrix}$$

$$R_{ZYX}(\gamma, \beta, \alpha) = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \vdots & \vdots & -\sin\alpha\cos\beta \\ & & \cos\alpha\cos\beta \end{bmatrix}$$

$$\sin\beta = \frac{1}{2} \Rightarrow \beta = \frac{\pi}{6} \text{ or } \beta = \frac{5}{6}\pi$$

$$\alpha = \mathrm{Atan2}\left(-\frac{R_{23}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right) \Rightarrow \alpha = -\frac{2}{3}\pi \text{ or } \alpha = \frac{\pi}{3}$$

$$\gamma = \mathrm{Atan2}\left(\frac{-R_{12}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right) \Rightarrow \gamma = -\frac{\pi}{2} \text{ or } \gamma = \frac{\pi}{2}$$

$$\therefore \begin{cases} \alpha = -\frac{2}{3}\pi \\ \beta = \frac{\pi}{6} \\ \gamma = -\frac{\pi}{2} \end{cases} \text{ or } \begin{cases} \alpha = \frac{\pi}{3} \\ \beta = \frac{5}{6}\pi \\ \gamma = \frac{\pi}{2} \end{cases}$$

8. Proposition: $R_Z(\pm\pi + \alpha)\,\underline{\underline{R_Y(-\beta)}}\,R_Z(\pm\pi + \gamma) = R_Z(\alpha)\,R_Y(\beta)\,R_Z(\gamma)$

9. After the transformation, the pose of B is

$$T_1 \, {}^{A}_{B}T \, T_2 = \begin{bmatrix} I_3 & {}^{A}Q_A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_K(\theta) & 0 \\ 0 & 1 \end{bmatrix} {}^{A}_{B}T \begin{bmatrix} I_3 & {}^{B}P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_L(\phi) & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_K(\theta) & {}^{A}Q_A \\ 0 & 1 \end{bmatrix} {}^{A}_{B}T \begin{bmatrix} R_L(\phi) & {}^{B}P \\ 0 & 1 \end{bmatrix}$$

$\because {}^{A}Q = [-3, -3, 3]^T$, ${}^{B}P = [2, -2, 1]^T$

For $R_K(\theta)$: ① Add up the diagnal elements

$$(k_x^2 + k_y^2 + k_z)^2 V\theta + 3c\theta = 1 + 2\cos\theta = 0.866 \times 2 + 0.75$$

$$\therefore \theta = 0.736 \text{ rad}$$

② $\cancel{R_{32} - R}$ $R_{32} - R_{23} = 2k_x \sin\theta$ $\quad \therefore k_x = \dfrac{R_{32} - R_{23}}{2\sin\theta}$

Likewise, we can compute k_y and k_z $\Rightarrow {}^{A}K = [0.695 \;\; -0.186 \;\; 0.695]^T$

For $R_L(\phi)$. $\phi = 0.523$ rad. ${}^{B}L = [0.577, 0.577, 0.577]^T$

10. $\theta = 45°$. $\eta = \cos\frac{\theta}{2} = 0.924$ $\quad \varepsilon = [k_x \sin\frac{\theta}{2} \;\; k_y \sin\frac{\theta}{2} \;\; k_z \sin\frac{\theta}{2}]^T$

$$= [0.221, \; -0.081 \;\; 0.302]^T$$

$$\bar{i}x_2 + \bar{j}y_2 + kz_2 = (\eta + \bar{i}\varepsilon_1 + \bar{j}\varepsilon_2 + k\varepsilon_3)(\bar{i}x_1 + \bar{j}y_1 + kz_1)(\eta + \bar{i}\varepsilon_1 + \bar{j}\varepsilon_2 + k\varepsilon_3)^*$$

$$= -0.432\bar{i} + 0.591\bar{j} + 3.671k$$

The result is $[-0.432 \;\; 0.591 \;\; 3.671]^T$

11. Grassmann product: $\left[\frac{3\sqrt{2}}{8} - \frac{\sqrt{6}}{24} \;\; \frac{\sqrt{2}-\sqrt{6}}{8} \;\; -\frac{\sqrt{2}}{4} - \frac{5\sqrt{6}}{24} \;\; \frac{\sqrt{2}}{8} - \frac{\sqrt{6}}{6}\right]^T$

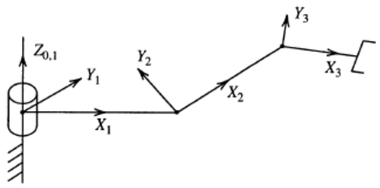
12.

This is easily derived if you work backwards. i.e, substitute into Rodriquez's formula wherever $\hat{K} \otimes \hat{Q}$ or $\hat{K} \cdot \hat{Q}$ occur, collect terms, and you'll get (2.80).

14.

3.3)

α_{i-1}	a_{i-1}	d_i
0	0	0
90°	L_1	0
0	L_2	0



$$
{}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3.3) (*Continued*)

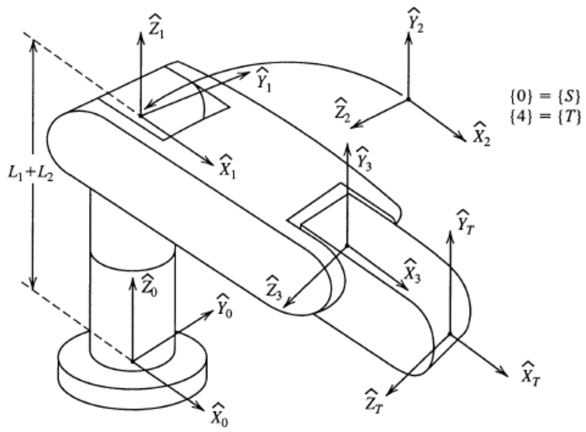
$$
{}^2_3T = \begin{bmatrix} C_3 & -S_3 & 0 & L_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^B_WT = {}^0_3T = {}^0_1T\,{}^1_2T\,{}^2_3T
$$

$$
{}^B_WT = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & L_1C_1 + L_2C_1C_2 \\ S_1C_{23} & -S_1S_{23} & -C_1 & L_1S_1 + L_2S_1C_2 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

15.

3.4)

α_{i-1}	a_{i-1}	d_i	θ_i
0	0	$L_1 + L_2$	θ_1
90°	0	0	θ_2
0	L_3	0	θ_3
0	L_4	0	0



$$ {}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} C_3 & -S_3 & 0 & L_3 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

4.2) This problem can have different solutions depending how it is interpreted. I intended that a goal is specified which includes a desired orientation of the last link. In this case, the solution is fairly easy.

${}^{S}_{T}T$ is given, so compute:

$${}^{B}_{W}T = {}^{B}_{S}T\,{}^{S}_{T}T\,{}^{W}_{T}T^{-1}$$

Now ${}^{B}_{W}T = {}^{0}_{3}T$ which we write out as:

$${}^{0}_{3}T = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the solution of exercise 3 from chapter 3 we have:

$${}^{0}_{3}T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(C_2L_2 + L_1) \\ S_1C_{23} & -S_1S_{23} & -C_1 & S_1(C_2L_2 + L_1) \\ S_{23} & C_{23} & 0 & S_2L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equate elements $(1, 3)$: $S_1 = R_{13}$

Equate elements $(2, 3)$: $-C_1 = R_{23}$

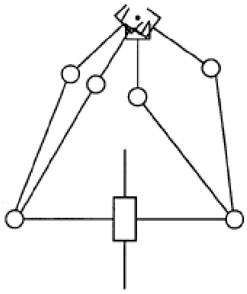
$$\therefore \quad \boxed{\theta_1 = \operatorname{atan2}(R_{13}, -R_{23})}$$

If both $R_{13} = 0$ and $R_{23} = 0$ the goal is unattainable.

Equate elements $(1, 4)$: $P_x = C_1(C_2L_2 + L_1)$

Equate elements $(2, 4)$: $P_y = S_1(C_2L_2 + L_1)$

4.2) (*Continued*)



If $C_1 \neq 0$ then $C_2 = \frac{1}{L_2}\left(\frac{P_x}{C_1} - L_1\right)$

$$\text{Else } C_2 = \frac{1}{L_2}\left(\frac{P_y}{S_1} - L_1\right)$$

Equate Elements (3,4): $P_z = S_2 L_2$

so, $\boxed{\theta_2 = \text{atan2}\left(\frac{P_z}{L_2}, C_2\right)}$

Equate elements (3, 1): $S_{23} = R_{31}$

Equate elements (3, 2): $C_{23} = R_{32}$

so, $\boxed{\boldsymbol{\theta_3 = \text{atan2}(R_{31}, R_{32}) - \theta_2}}$

If both R_{31} and R_{32} are zero, the goal is unattainable.

A second interpretation of the problem is that only a desired position is given (no orientation). In this there may be up to four solutions:

Assume ${}^3P_{\text{tool}} = L_3\hat{X}_3$, then

$$^0P_{\text{tool}} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} L_1C_1 + L_2C_1C_2 + L_3C_1C_{23} \\ L_1S_1 + L_2S_1C_2 + L_3S_1C_{23} \\ L_2S_2 + L_3S_{23} \end{bmatrix}$$

First,

$$S_1 = \frac{P_y}{L_1 + L_2C_2 + L_3C_{23}} \quad C_1 = \frac{P_x}{L_1 + L_2C_2 + L_3C_{23}}$$

so, $\boxed{\boldsymbol{\theta_1 = \text{atan2}(P_y, P_x) \text{ or } \text{atan2}(-P_y, -P_x)}}$

Since the sign of the "$L_1 + L_2C_2 + L_3C_{23}$" term may be $+$ or $-$.

Next, define:

$$\alpha = \begin{cases} \frac{P_x}{C_1} - L_1 & \text{if } C_1 \neq 0 \\ \frac{P_y}{S_1} - L_1 & \text{if } S_1 \neq 0 \end{cases}$$

And we have:

$$L_2C_2 + L_3C_{23} = \alpha$$

$$L_2S_2 + L_3S_{23} = P_z$$

Square and add these two equations to get:

$$L_2^2 + L_3^2 + 2L_2L_3C_3 = \alpha^2 + P_z^2$$

$$C_3 = \frac{1}{2L_2L_3}(\alpha^2 + P_z^2 - L_2^2 - L_3^2)$$

$S_3 = \pm\sqrt{1 - C_3^2}$; $\boxed{\theta_3 = \text{atan2}(S_3, C_3)}$

4.2) (*Continued*)

Finally,

$$L_3C_{23} = \alpha - L_2C_2$$

$$L_3S_{23} = P_z - L_2S_2$$

so, $$\boxed{\boldsymbol{\theta_2 = \mathrm{atan2}(P_z - L_2S_2, \alpha - L_2C_2) - \theta_3}}$$

17. NOTE: It's Example 3.4, NOT Exercise 3.4

4.4) For an arm like this it is reasonable to specify a goal by giving the desired x & y coordinates of the tip and a value for θ_3, the wrist roll. After transforming back to P_X & P_T of the wrist (i.e. remove offset due to L_3) the solution is simple and corresponds to the conversion between cartesian and polar coordinates.