1.

1. Parallel-axis theorem: ${}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$

$${}^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$

Prove: ① ${}^{A}I_{zz} = \iiint_V ({}^{A}x^2 + {}^{A}y^2)\rho dv = \iiint_V [({}^{C}x + x_c)^2 + ({}^{C}y + y_c)^2]\rho dv$

$$= \iiint_V ({}^{C}x^2 + {}^{C}y^2)\rho dv + \iiint_V (x_c^2 + y_c^2)\rho dv$$

$$+ \iiint_V (2\,{}^{C}x\, y_c + 2\,{}^{C}y\, x_c)\rho dv.$$

$$2y_c \iiint_V {}^{C}x \rho dv = 2 y_c m \iiint_V \frac{{}^{C}x}{m}\rho dv = 2 y_c m \cdot {}^{C}x_c = 0.$$

$$\therefore {}^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$$

②. Similarly. ${}^{A}I_{xy} = -\iiint_V {}^{A}x\,{}^{A}y\, \rho dv$

$$= -\iiint_V ({}^{C}x + x_c)({}^{C}y + y_c)\rho dv$$

$$= -\iiint_V {}^{C}x\,{}^{C}y\,\rho dv - \iiint_V x_c y_c \rho dv$$

$$= {}^{C}I_{xy} - m x_c y_c.$$

If you use the definition $I_{xy} = \iiint_V xy\rho dv$.
then the relationship is ${}^{A}I_{xy} = {}^{C}I_{xy} + m x_c y_c$.
The negative sign appears in the inertia matrix

$${}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ & \cdots & \\ & \cdots & \end{bmatrix}.$$

2.

6.5)

α_{i-1}	a_{i-1}	d_i	θ_i
0	0	0	θ_1
90°	L_1	0	θ_2
0	L_2	0	0

$$ {}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} {}^1P_2 = L_1\hat{X}_1 \\ {}^2P_3 = L_2\hat{X}_2 \end{matrix} $$

$$ {}^1P_{C_1} = L_1\hat{X}_1 \, {}^2P_{C_2} = L_2\hat{X}_2 \, {}^{C_1}I_1 = 0 \, {}^{C_2}I_2 = 0 $$

$$ {}^0\dot{V}_0 = g\hat{Z}_0. \quad \text{(since gravity points in } -\hat{Z}_0 \text{ Dir.)} $$

$$ W_0 = \dot{W}_0 = 0 \quad \text{(base stationary)} $$

$$ F_3 = x_3 = 0 \quad \text{(no forces on hand)} $$

Forward Velocity & Acceleration Iterations:

Link 1

$$ {}^1W_1 = {}^1_0R\,{}^0W_0 + \dot{\theta}_1\,{}^1\hat{Z}_1 = \dot{\theta}_1\,{}^1\hat{Z}_1 = [0 \quad 0 \quad \dot{\theta}_1\,]^T $$

$$ {}^1\dot{W}_1 = {}^1_0R\,{}^0\dot{W}_0 + {}^1_0R\,{}^0W_0 \otimes \dot{\theta}_1\,{}^1\hat{Z}_1 + \ddot{\theta}_1\,{}^1\hat{Z}_1 $$

$$ {}^1\dot{W}_1 = \ddot{\theta}_1\,{}^1\hat{Z}_1 = [0 \quad 0 \quad \ddot{\theta}_1\,]^T $$

$$ {}^1\dot{V}_1 = {}^1_0R({}^0\dot{W}_0 \otimes {}^0P_1 + {}^0W_0 \otimes ({}^0W_0 \otimes {}^1P_2) + {}^0\dot{V}_0) $$

$$ {}^1\dot{V}_1 = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} $$

$$ {}^1\dot{V}_{C_1} = {}^1\dot{W}_1 \otimes {}^1P_{C_1} + {}^1W_1 \otimes ({}^1W_1 \otimes {}^1P_{C_1}) + {}^1\dot{V}_1 $$

$$ {}^1\dot{V}_{C_1} = \ddot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1 + \dot{\theta}_1\hat{Z}_1 \otimes (\dot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1) + g\hat{Z}_1 $$

$$ {}^1\dot{V}_{C_1} = L_1\ddot{\theta}_1\hat{Y}_1 + \dot{\theta}_1\hat{Z}_1 \otimes L_1\dot{\theta}_1\hat{Y}_1 + g\hat{Z}_1 $$

$$ {}^1\dot{V}_{C_1} = L_1\ddot{\theta}_1\hat{Y}_1 + (-L_1\dot{\theta}_1^2)\hat{X}_1 + g\hat{Z}_1 = \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ L_1\ddot{\theta}_1 \\ g \end{bmatrix} $$

Link 2

$$ {}^2W_2 = {}^2_1R\,{}^1W_1 + \dot{\theta}_2\,{}^2\hat{Z}_2 $$

$$ {}^2W_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

6.5) *(Continued)*

$$^2\dot{W}_2 = {}^2_1R\,{}^1\dot{W}_1 + {}^2_1R\,{}^1W_1 \otimes \dot{\theta}_2\hat{Z}_2 + \ddot{\theta}_2\hat{Z}_2$$

$$^2\dot{W}_2 = \begin{bmatrix} S_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$^2\dot{W}_2 = \begin{bmatrix} S_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} C_2\dot{\theta}_1\dot{\theta}_2 \\ -S_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} S_2\ddot{\theta}_1 + C_2\dot{\theta}_1\dot{\theta}_2 \\ C_2\ddot{\theta}_1 - S_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$^2\dot{V}_2 = {}^2_1R({}^1\dot{W}_1 \times {}^1P_2 + {}^1W_1 \otimes ({}^1W_1 \otimes {}^1P_2) + {}^1\dot{V}_1)$$

$$^2\dot{V}_2 = {}^2_1R(\ddot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1 + \dot{\theta}_1\hat{Z}_1 \otimes (\dot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1) + g\hat{Z}_1)$$

$$^2\dot{V}_2 = {}^2_1R(L_1\ddot{\theta}_1\hat{Y}_1 - L_1\dot{\theta}_1^2\hat{X}_1 + g\hat{Z}_1)$$

$$= \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ L_1\ddot{\theta}_1 \\ g \end{bmatrix}$$

$$^2\dot{V}_2 = [-L_1C_2\dot{\theta}_1^2 + S_2g,\ L_1S_2\dot{\theta}_1^2 + C_2g,\ -L_1\ddot{\theta}_1\,]^T$$

$$^2\dot{V}_{C_2} = {}^2\dot{W}_2 \otimes {}^2P_{C_2} + {}^2W_2 \otimes ({}^2W_2 \otimes {}^2P_{C_2}) + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} S_2\ddot{\theta}_1 + C_2\dot{\theta}_1\dot{\theta}_2 \\ C_2\ddot{\theta}_1 - S_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\otimes \left(\begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \right) + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} 0 \\ L_2\ddot{\theta}_2 \\ -L_2C_2\ddot{\theta}_1 + L_2S_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ L_2\dot{\theta}_2 \\ -L_2C_2\dot{\theta}_1 \end{bmatrix} + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} -(L_1C_2 + L_2C_2^2)\dot{\theta}_1^2 - L_2\dot{\theta}_2^2 + S_2g \\ (L_1S_2 + L_2S_2C_2)\dot{\theta}_1^2 + L_2\ddot{\theta}_2 + C_2g \\ 2L_2S_2\dot{\theta}_1\dot{\theta}_2 - L_1\ddot{\theta}_1 - L_2C_2\ddot{\theta}_1 \end{bmatrix}$$

Inertial Equations:

Link 1

$$^1F_1 = M_1\,{}^1\dot{V}_{C_1}$$

$$^1F_1 = \begin{bmatrix} -M_1L_1\dot{\theta}_1^2 \\ M_1L_1\ddot{\theta}_1 \\ M_1g \end{bmatrix}$$

$$^1N_1 = {}^{C_1}I_1\,{}^1\dot{W}_1 + {}^1W_1 \otimes {}^{C_1}I_1\,{}^1W_1$$

$$^1N_1 = [0 \quad 0 \quad 0]^T$$

Link 2

$$^2F_2 = M_2\,{}^2\dot{V}_{C_2}$$

6.5) *(Continued)*

$$
{}^{2}F_{2} = \begin{bmatrix} -M_2(L_1 + L_2C_2)C_2\dot{\theta}_1^2 - M_2L_2\dot{\theta}_2^2 + M_2S_2g \\ M_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 + M_2L_2\ddot{\theta}_2 + M_2C_2g \\ 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 - M_2L_1\ddot{\theta}_1 - M_2L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^{2}N_2 = {}^{C_2}I_2\,{}^{2}\dot{W}_2 + {}^{2}W_2 \otimes {}^{C_2}I_2\,{}^{2}W_2 = [0 \quad 0 \quad 0]^T
$$

Backward Force Iterations:

Link 2

$$
{}^{2}F_2 = {}^{2}_{3}R\,{}^{3}F_3 + {}^{2}F_2 = {}^{2}F_2 \text{ (see above)}
$$

$$
{}^{2}n_2 = {}^{2}N_2 + {}^{2}_{3}R\,{}^{3}n_3 + {}^{2}P_{C_2} \otimes {}^{2}F_2 + {}^{2}P_3 \otimes {}^{2}_{3}R\,{}^{3}F_3
$$

$$
{}^{2}n_2 = {}^{2}P_{C_2} \otimes {}^{2}F_2 = L_2\hat{X}_2 \otimes {}^{2}F_2
$$

$$
{}^{2}n_2 = \begin{bmatrix} 0 \\ M_2L_1L_2\ddot{\theta}_1 - 2M_2L_2^2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2^2C_2\ddot{\theta}_1 \\ M_2L_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 - M_2gL_2C_2 + M_2L_2^2\ddot{\theta}_2 \end{bmatrix}
$$

Link 1

$$
{}^{1}F_1 = {}^{1}_{2}R\,{}^{2}F_2 + {}^{1}F_1 \text{ (not needed, so skip)}
$$

$$
{}^{1}n_1 = {}^{1}N_1 + {}^{1}_{2}R\,{}^{2}n_2 + {}^{1}P_{C_1} \otimes {}^{1}F_1 + {}^{1}P_2 \otimes {}^{1}_{2}R\,{}^{2}F_2
$$

$$
{}^{1}_{2}R\,{}^{2}n_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} {}^{2}n_2
$$

$$
{}^{1}_{2}R\,{}^{2}n_2 = \begin{bmatrix} * \\ * \\ M_2L_1L_2C_2\ddot{\theta}_1 - 2M_2L_2^2S_2C_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2^2C_2^2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^{1}P_{C_1} \otimes {}^{1}F_1 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -M_1L_1\dot{\theta}_1^2 \\ M_1L_1\ddot{\theta}_1 \\ M_1g \end{bmatrix} = \begin{bmatrix} 0 \\ -M_1gL_1 \\ M_1L_1^2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^{1}_{2}R\,{}^{2}F_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} -M_2(L_1 + L_2C_2)C_2\dot{\theta}_1^2 - M_2L_2\dot{\theta}_2^2 + M_2S_2g \\ M_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 + M_2gC_2 \\ 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 - M_2L_1\ddot{\theta}_1 - M_2L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^{1}_{2}R\,{}^{2}F_2 = \begin{bmatrix} * \\ M_2L_2C_2\ddot{\theta}_1 + M_2L_1\ddot{\theta}_1 - 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 \\ * \end{bmatrix}
$$

$$
{}^{1}P_2 \otimes {}^{1}_{2}R\,{}^{2}F_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} * \\ M_2L_1\ddot{\theta}_1 - 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2C_2\ddot{\theta}_1 \\ * \end{bmatrix}
$$

$$
{}^{1}P_2 \otimes {}^{1}_{2}R\,{}^{2}F_2 = \begin{bmatrix} * \\ * \\ M_2L_1^2\ddot{\theta}_1 - 2M_2L_1L_2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_1L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
\therefore\ {}^{1}n_1 = \begin{bmatrix} * \\ * \\ (M_1L_1^2 + M_2L_1^2 + M_2L_2^2C_2 + 2M_2L_1L_2C_2)\ddot{\theta}_1 - 2(L_1 + L_2C_2)M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}
$$

6.5) (*Continued*)

Joint Torque Equations

$\tau_i = {}^i n_i \cdot {}^i \hat{Z}_i + V_i \dot{\theta}_i$ Added viscous friction term.

$$\tau_1 = (M_1 L_1^2 + M_2(L_1 + L_2 C_2)^2)\ddot{\theta}_1$$

$$- 2(L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 + V_1 \dot{\theta}_1$$

$$\tau_2 = M_2 L_2^2 \ddot{\theta}_2 + (L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1^2 + M_2 g L_2 C_2 + V_2 \dot{\theta}_2$$

or

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

where:

$$M(\theta) = \begin{bmatrix} (M_1 L_1^2 + M_2(L_1 + L_2 C_2)^2) & 0 \\ 0 & M_2 L_2^2 \end{bmatrix}$$

$$V(\theta_1 \theta) = \begin{bmatrix} -2(L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 \\ (L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ M_2 g L_2 C_2 \end{bmatrix}$$

3.

3. Lagrangian Formulation.

From the point mass assumption, we have ${}^{C_1}I_1 = {}^{C_2}I_2 = 0$.

① Kinetic energy: $k_1 = \frac{1}{2} m_1 V_{C_1}^T V_{C_1} = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2$

$$k_2 = \frac{1}{2} m_2 V_{C_2}^T V_{C_2}$$

$$V_{C_2} = \frac{d P_{C_2}}{dt} = \frac{d}{dt}\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 \dot\theta_1 - l_2 s_{12} (\dot\theta_1 + \dot\theta_2) \\ l_1 c_1 \dot\theta_1 + l_2 c_{12} (\dot\theta_1 + \dot\theta_2) \end{bmatrix}.$$

$$\therefore k_2 = \frac{1}{2} m_2 [(l_1^2 + l_2^2 + 2 l_1 l_2 c_2)\dot\theta_1^2 + l_2^2 \dot\theta_2^2 + (2 l_2^2 + 2 l_1 l_2 c_2)\dot\theta_1 \dot\theta_2].$$

$$k = k_1 + k_2$$

② Potential energy.

$$u_1 = m_1 g l_1 s_1 + m_1 g l_1$$

$$u_2 = m_2 g (l_1 s_1 + l_2 s_{12}) + m_2 g (l_1 + l_2).$$

$$u = u_1 + u_2$$

③ Lagrangian equation $\frac{d}{dt}\frac{\partial k}{\partial \dot\theta} - \frac{\partial k}{\partial \theta} + \frac{\partial u}{\partial \theta} = \tau$

$$\frac{\partial k}{\partial \dot\theta} = \begin{bmatrix} [(m_1 + m_2) l_1^2 + m_2 (l_2^2 + 2 l_1 l_2 c_2)]\dot\theta_1 + m_2 (l_2^2 + l_1 l_2 c_2)\dot\theta_2 \\ m_2 l_2^2 \dot\theta_2 + m_2 (l_2^2 + l_1 l_2 c_2)\dot\theta_1 \end{bmatrix}$$

$$\frac{\partial k}{\partial \theta} = \begin{bmatrix} 0 \\ - m_2 (l_1 l_2 \dot\theta_1^2 - l_1 l_2 \dot\theta_1 \dot\theta_2) s_2 \end{bmatrix}$$

$$\frac{\partial u}{\partial \theta} = \begin{bmatrix} (m_1 + m_2) g l_1 c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix}.$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot\theta} = \begin{bmatrix} [(m_1 + m_2) l_1^2 + m_2 (l_2^2 + 2 l_1 l_2 c_2)]\ddot\theta_1 - 2 m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 s_2 + m_2 l_2 (l_2 + l_1 c_2)\ddot\theta_2 - m_2 l_1 l_2 \dot\theta_2^2 s_2 \\ m_2 l_2^2 \ddot\theta_2 + m_2 l_2 (l_2 + l_1 c_2)\ddot\theta_1 - m_2 l_1 l_2 s_2 \dot\theta_1 \dot\theta_2 \end{bmatrix}$$

The computed model is the same as that in Section 6.7.

4.

4. By the symmetry of the inertia matrix we have:

$$\sum_{i,j} \left\{ \frac{\partial m_{kj}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{jk}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j$$

Since the summation runs over ~~all~~ all i, j, we can interchange i and j in the second term to obtain the result:

$$\sum_{i,j} \left\{ \frac{\partial m_{kj}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} \right\} \dot{q}_i \dot{q}_j$$