Multiple Regression - Matrices

SDS6105 - Statistics For Data Science

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In multiple regression, we want to estimate the coefficients β_0, β_1, and β_2 in the equation:

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

For a least squares solution, we can set up the problem in matrix form and solve it using linear algebra.
1. Set Up the Matrix Equation
Let:
  y be the vector of observed y values.
  X be the matrix of inputs (including a column of ones for the intercept term).

The model can be represented as:

$[\mathbf{y} = \mathbf{X} \cdot \boldsymbol{\beta} + \varepsilon]$
where:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} \\ 1 & X_{1,2} & X_{2,2} \\ \vdots & \vdots & \vdots \\ 1 & X_{1,n} & X_{2,n} \end{bmatrix}$$ (with a leading column of ones for the intercept),

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix},$$

ε represents the residuals.

To estimate $\boldsymbol{\beta}$, we minimize the sum of squared residuals by solving:

$$[\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}]$$

2. Construct the Matrices with Your Data

Given:

$X_1 = [0, 1, 2, 3, 4, 5]$

$X_2 = [0, 1, 4, 9, 16, 25]$

$y = [9.1, 7.3, 3.2, 4.6, 4.8, 2.9]$

We'll create the matrix $\mathbf{X}$ and vector $\mathbf{y}$, and then solve for $\boldsymbol{\beta}$.

3. Construct the matrix $\mathbf{X}$:
    - Add a column of ones for the intercept.
    - This results in:

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}$$

4. Construct the matrix equation:

$$\text{We have } \mathbf{y} = \begin{bmatrix} 9.1 \\ 7.3 \\ 3.2 \\ 4.6 \\ 4.8 \\ 2.9 \end{bmatrix}.$$

5. Calculate (β) using the formula: [$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$]

Step 4: Calculate $\mathbf{X}^T\mathbf{X}$

1. Transpose the $\mathbf{X}matrix(\mathbf{X}^T)$ by swapping rows with columns.

2. Multiply $\mathbf{X}^T$ by $\mathbf{X}$ to get a 3x3 matrix.

Step 5: Calculate $\mathbf{X}^T\mathbf{y}$
1. Multiply the transposed matrix $\mathbf{X}^T$ with $\mathbf{y}$ to get a 3x1 matrix.

Step 6: Calculate $(\mathbf{X}^T\mathbf{X})^{-1}$

1. Invert the 3x3 matrix obtained in Step 2 using matrix inversion methods (you can do this in Python, R, or Excel if manual inversion is too complex).

Step 7: Calculate $\hat{\boldsymbol{\beta}}$

Multiply $(\mathbf{X}^T\mathbf{X})^{-1} by \mathbf{X}^T\mathbf{y}$

to solve for the coefficient vector ($\hat{\boldsymbol{\beta}}$).

This will give you values for β_0, β_1, and β_2.

Step 6: Predict y when $x_1 = 2$

To predict y when $x_1 = 2$:

Plug $x_1 = 2$ and $x_2 = 2^2 = 4$ into the regression equation:

$\hat{y} = \beta_0 + \beta_1 \cdot 2 + \beta_2 \cdot 4$

Using the estimated values for β_0, β_1, and β_2, compute $\hat{y}$.