Dynamic Programming

Cheng-Hung Wu

@NTUIE

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What is DP ?

- Dynamic Programming is an optimization procedure to solve problems requiring a sequence of inter-related decisions.
- Each decision transforms the current situation into a new situation.

What do we want to cover ?

- Finite horizon deterministic dynamic programming
- Finite horizon stochastic dynamic programming

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Canonical Example :

Single item inventory problems : order a number of item each period in order to minimize expected (discounted) cost.

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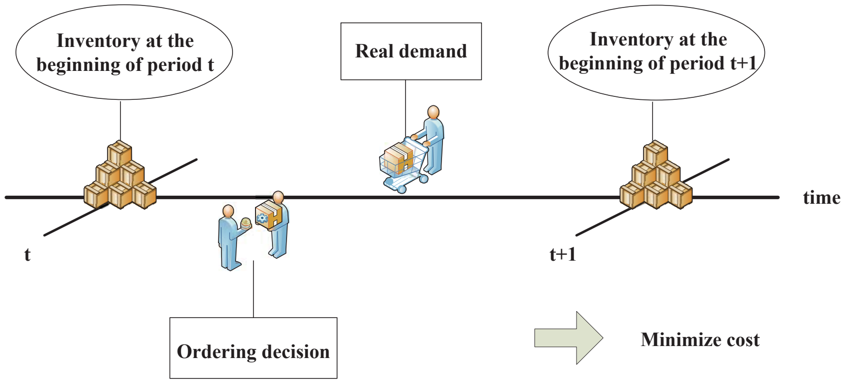


Characteristic of our example :

(1) Must make ordering decision sequentially.

(2) Decision are inter-related.

(3) If I do not know exactly demand in the coming period, there are uncertainties.

We will see how to write equations for relationships between decisions.

For our single item inventory example :
Let X_t : inventory position at the beginning of period t.
D_t : demand during period t.
A_t : order placed at the beginning of period t.(order the item from supplier)

$\Rightarrow X_{t+1} = X_t + A_t - D_t$

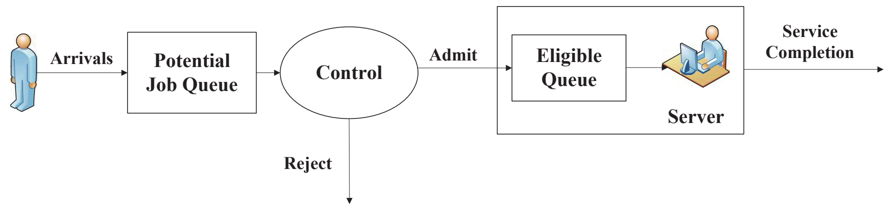
Can you write a state transition equation ?

$$X_{t+1} = f(X_t, A_t, D_t)$$
$$= X_t + A_t - D_t.$$

(Deterministic case $D_t = d_t$ as constant)
(Stochastic case $D_t \sim F$)

Another example : queueing system

- Single server queue, batch arrivals
- Must decide how many (if any), to allow in your system.
  - Consider the web server
  - Holding cost $c(i)$, when there are i customer in the system.
  - Each served customer yields a reward of R.



For the queueing example :

Let

$X_t = \#$ of customers in the system at time t.

$Y_t = \#$ of customers seeking admittance at time t.

$S_t = \#$ of customers served during period t.

$A_t = \#$ of customers admitted.

$X_{t+1} = X_t - S_t + A_t$

Same equation as before ?

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$X_{t+1} = X_t - S_t + A_t$

Same equation as before ? NO.

A_t is depend on Y_t.

However,

- decisions are still sequential.
- decisions are still inter-related.
- the interesting case is the stochastic case.

Reward Function :

$r(Z_n, A_n)$: reward function

$\bar{r}(Z_N)$: terminal reward

(Z_n : state, A_n : action)

Optimality Criterion :

For this course, we will consider for $0 < \alpha \le 1$

$$V_\alpha^N(i) = \sum_{n=0}^{N-1} \alpha^n \, r(Z_n, A_n) + \bar{r}(Z_n)$$

(Starting at $Z_0 = i$)

($V_\alpha^N(i)$ is the total (discounted) reward over a N period planning horizon.)

If $\alpha = 1$. total reward problem

$\alpha < 1$. total discounted reward

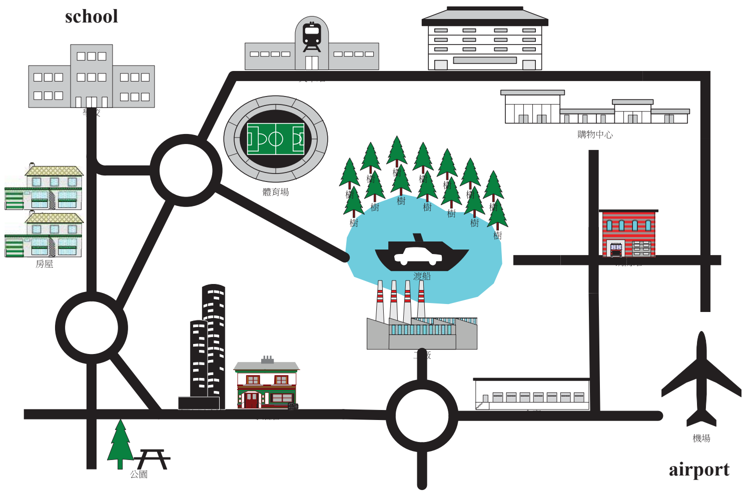
In the stochastic case,

$$\hat{V}_\alpha^N(i) = E_i\{[\sum_{n=0}^{N-1} \alpha^n \, r(Z_n, A_n)] + \alpha^N \bar{r}(Z_N)\}$$

$$\text{Where } E_i = E(\cdot | Z_0 = i)$$

Return to our queuing example, so we needed to know Y_t before we could decide what A_t to use ...
$Z_t = (X_t, Y_t)$ (2-dimensioned state space)

EX (Routing) : Choose the time a vehicle leaves an origin and the route the vehicle takes to destination at a particular time (JIT).
(Kim,Lewis,White (2003))



Routing :

Choose the time a vehicle leaves and the route it takes to get the destination.

N : Set of Nodes. (in a street network)
$A \subseteq N \times N$ (Directed Arcs)
$i \in N$ starting nodes.
$r \subseteq N$ Goal set (maybe a singleton)
$c(n, n') =$ cost function for arc (n, n')
$c(n, t, n') =$ cost function depends on time
$\bar{c}(t) =$ terminal cost

If we are interested in minimizing the total cost

$$E_{i,t}\{\sum_{k=1}^{K} c(n_k, t_k, n_{k+1}) + \bar{c}(t_K)\}$$

(t_k is the arrival time to Γ, and i is the initial node.)

So what do we need to know to make DP work ?

- state or current node
- current time (or stage, or period)
- planning horizon (how long do we look into the future)
- available decision (action, possibly state dependent)
- optimality criterion

For us,

(1) Temporally additive cost functions (no products/non-after effect)
(2) Finite planning horizons
(3) Initially assume no uncertainties
(4) Mathematically describable

What constitutes a solution ?

What constitutes a solution ?

We need to know:

- the optimal cost/profit
- the policy that achieves it.

EX 1 Inventory

(a) Ordering quantity (inventory to hold) given current inventory position

(b) Total cost

EX 2 Queuing Problem :

(a) How many customers to admit (Given the current in system and the current batch size)

(b) Total cost

Ex 3 Routing Problem

(a) Optimal route to take and the departure time

(b) Minimal cost from initial node i to Γ given the departure time

Techniques used for "sequential decision problems"

(1) In some cases , exhaustive search

(2) Heuristic method

(3) Dynamic Programming (DP)

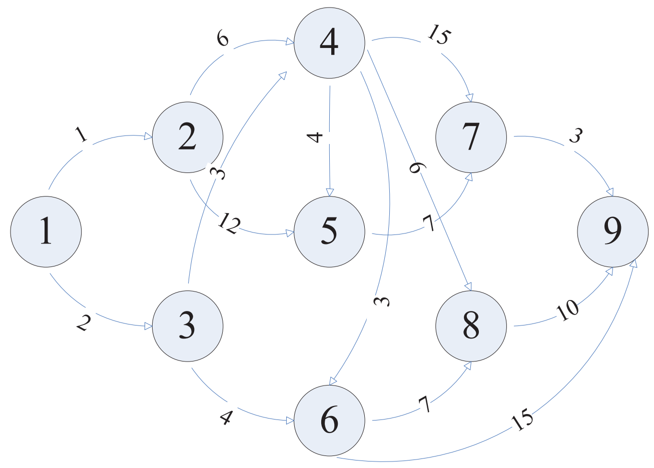
Summary :

We will

(1) Examine decision problems
    (a) Deterministic and Stochastic

(2) We must
    (a) Model
    (b) Define what constitute a solution
    (c) Develop solution procedures

Min. path problem



The prototypical sequential decision problem

- Shortest path problem
    (1) In some cases , exhaustive search
    (2) Heuristic method
    (3) Dynamic Programming (DP)

a How many different routes do we have in this shortest path problem?

b How much calculation effort does it take?

It turns out exhaustive search require 55 adds and 12 comparisons, and there are 13 different paths.

The optimal routes 1 , 3 , 4 , 5 , 7 , 9 , cost 19.

Let's compare how to calculate the length of a path

(1) Suppose we compute the length of

1 , 2 , 4 , 6 , 8 , 9.

(2) Suppose we want to compute the length of

1 , 3 , 4 , 6 , 8 , 9.

We can reuse calculation of 4 , 6 , 8 , 9

(3) Similarly for

1 , 2 , 4 , 7 , 9
1 , 2 , 4 , 6 , 8 , 9

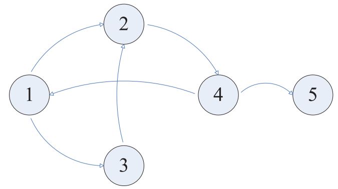
Backward and forward induction exploit these facts

some notation :

S: set of nodes
$T :\subseteq S \times S$, directed arcs
$SCS(i) \equiv \{j : (i,j) \in T\}$
($SCS(i)$: successive set)

EX1 :

$S = \{1, 2, \ldots, 5\}$
$T = \{(1,2), (1,3), (2,4), (3,2), (4,1), (4,5)\}$



Let t_{ij} be the length of (i,j)

Theorem

For path $(i_1, i_2, \ldots, i_n)$, *the path length* $= \sum_{k=1}^{n-1} t_{i_k, i_{k+1}}$

Let's start with finite, acyclic networks

Fact : One can label finite , acyclic network s.t. $(i,j) \in T$ is s.t. $i < j$

1. cyclic : There is a cycle in the graph.
   SCS(4) = {1,5}

2. acyclic : There is no cycles in the graph.

Backward Induction (Backward DP)

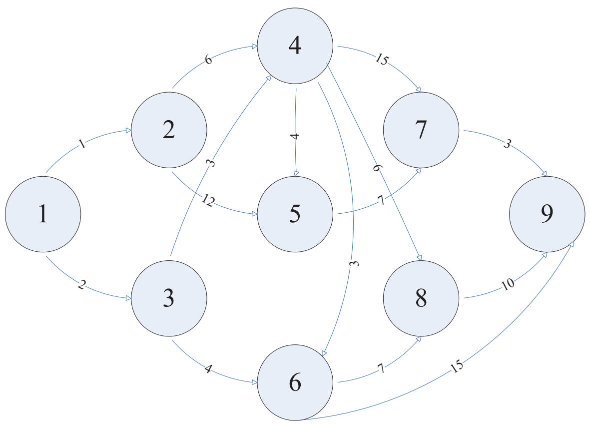
suppose S $= \{1,\ldots,k\}$ and arcs are s.t. $i < j$

if $(i,j) \in T$
(k : terminal node)

Let $f_i =$ min travel cost form i to k

$f_k = 0$

Return to the Example

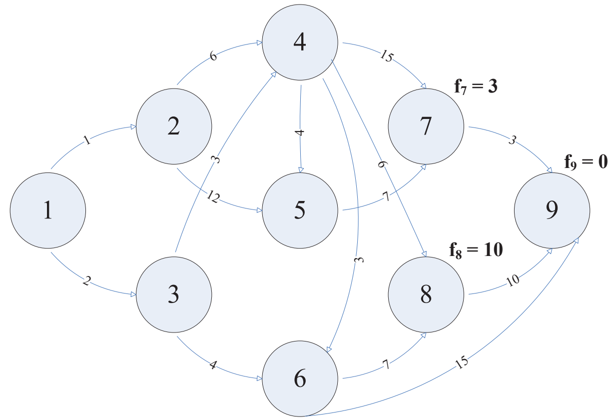


$f_9 = 0$

$f_8 =$

$f_7 =$

Return to the Example

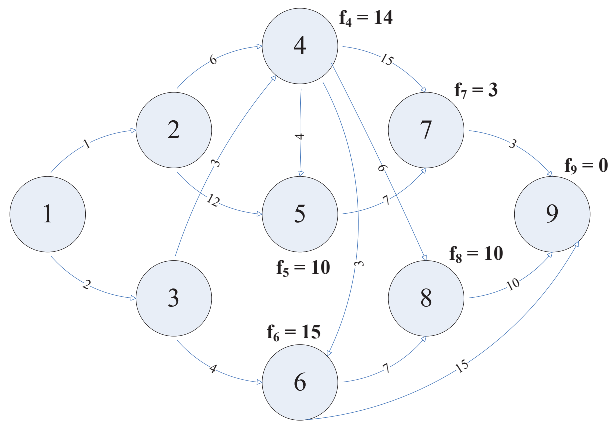


$2_+, 1 compare$ $\quad f_6 = 15 \quad min(7+10, 15+0)$

$f_5 =$

$f_4 =$

Return to the Example



$f_3 =$

$f_2 =$

$f_1 =$

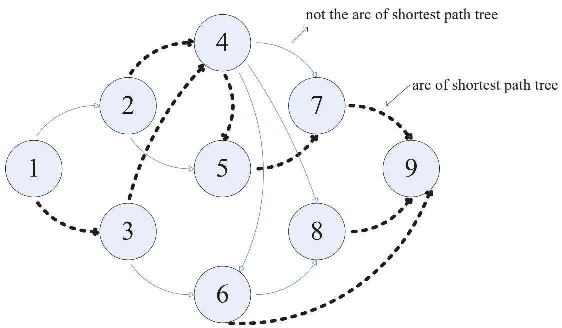
Optimal Strategy is
$= 1 \to 3 \to 4 \to 5 \to 7 \to 9$
Total computational effort : 12 adds , 7 compare
Recall exhaustive search : 55 adds, 12 compare

In general, optimality equation

$f_i = min\{t_{ij} + f_j : (i,j) \in T\}, f_k = 0$

($(i,j) \in T$: reduce each sub. problem to the successive set SCS(i) of i)

(Shortest Path Tree)



How would we construct (dynamically) the shortest path tree ?

- As we compute the optimal costs , delete arcs that are not on the shortest path from each node to k.

Ex :

$$\begin{aligned} f_4 &= min(4 + f_5, 3 + f_6, 15 + f_7, 7 + f_8) \\ &= 4 + f_5 \end{aligned}$$

$\Rightarrow$ delete (4,6) , (4,7) , (4,8) from the original network to form the shortest tree from 4 to 9

(What if the cost is equal ?)

Theorem

A policy tells you what to do when you are at a particular node or state

Example : Here is a "good" policy

(If at node i , go to node j)

i	j
1	3
2	4
3	4
4	5
5	7
6	9
7	9
8	9

(shortest path tree is a good policy)

For acyclic graphic with finite nodes :

Theorem

f_i is the minimum distance from i to sink

Optimality equation $f_i = min\{t_{ij} + f_j : (i,j) \in T\}$
$((i,j) \in T$ or $j \in$ SCS(i))

Boundary condition $f_j = 0, \forall j \in \Gamma$
(This algorithm will lead to a "policy", which is now a set of if-then statement.)

A few notes :

(1) D.P. offers a significant reduction in the # of operations v.s. exhaustive search
(2) Sub-paths of optimal paths are optimal
(principle of optimality)
(3) D.P.gives both optimal cost and policy
(4) Problem is decomposed into finding the solution of a set of smaller problem f_i

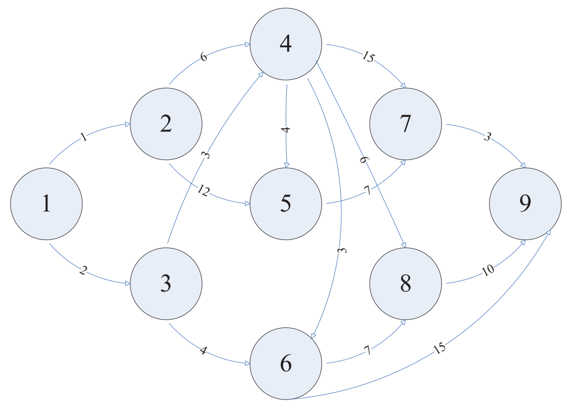
Two things,

(1) We will use backward induction more often
(Because in stochastic case, forward induction won't work)
(2) D.P. produces an optimal path and a minimum path cost (guaranteed)

It's possible , we could reduced the amount of work using heuristics

- possible at the loss of optimality

Ex:
A heuristic rule : go to the node with the lowest immediate cost (Myopic !)
forward :



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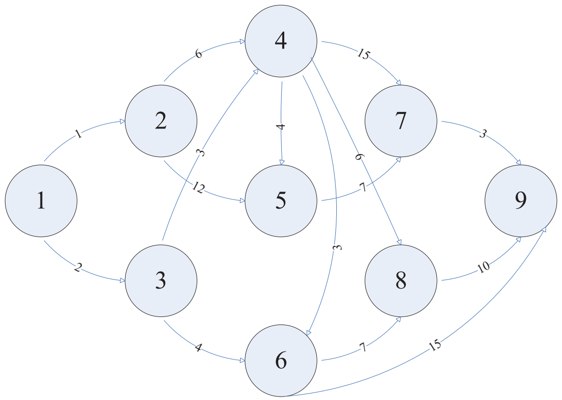
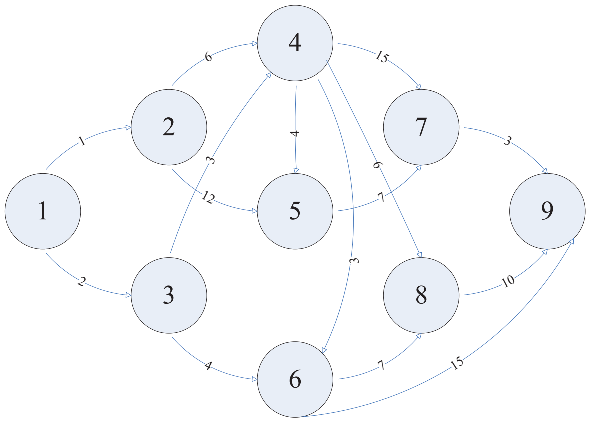


$1 \to 2 \to 4 \to 6 \to 8 \to 9$

total 27

this is the max length

backward :



$1 \to 3 \to 4 \to 5 \to 7 \to 9$

total 19

this is the min length

Both Forward and Backward Induction use something called "Recursive Fixing"

Consider how they work in Longest Route Problem

(1) Set $V_1 = 0$,$V_j = -\infty$,j = 2,. . . ,N

Do for j = 2,. . . ,N

Do for i = 1,. . . ,j-1

$V_j = Max\{V_j, V_i + t_{ij}\}$

(If $(i, j) \notin T$ let $t_{ij} = -\infty$)

Note : Recursive Fixing examines each node and "fixes" the value at that node based on the "Ancestor" nodes.

Once we fix the nodes , they are fixed forever

Embedding : we embedded the problem of interest (Find f_1 or g_9) into a set of problems (finding $f_i = 1, \ldots, 9$)

This is called the Recursive Feature.

Functional Equation (optimality equation)

Example: $f_i = \max_j \{t_{ij} + f_j, (i,j) \in T\}, f_\alpha = 0 \quad \forall \alpha \in Destination$

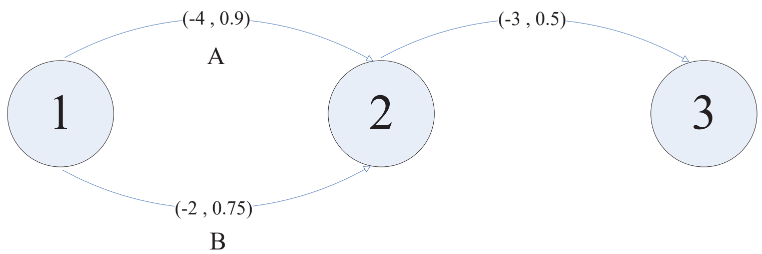
Recursive Fixing : The process of recursively determining the solution of the optimality equation

Principle of Optimality

(1) All sub-paths of the optimal path are optimal
(2) There exists an optimal policy (that is optimal for every state)
(3) An optimal policy is s.t. whatever the initial state (node) and decision the remaining state and decisions must be optimal as well

What if the objective functions or costs are not additive? Suppose that there are 2 routes

If they arrive on time $\Rightarrow$ Then you have \$20 bouns



First element $=$ cost of travel

Second element $=$ prob. of arriving on time

$g_2 = max\{-4 + (0.9) \times 20, -2 + (0.75) \times 20\}$
$\quad = max\{14, 13\} = 14$

By route A $g_3 \equiv -4 - 3 + (0.9)(0.5)(20) = 2$

However, by route B, $-2 - 3 + (0.75)(0.5)(20) = 2.5$

Why? It turns out the objective is not additive , it is a mixture of add and multiplies.

(Note: we assume temporally additive cost/reward structure from the very beginning.)

Pseudo code for Recursive Fixing

Pseudo code for Recursive Fixing (Backward / Min. Path)

(1) Set $V_N = 0$, $V_j = \infty$, j = 1, . . . ,N-1

(2) Do for i = N-1, . . . ,1
    Do for j = i,. . . ,N
        $V_i = Min\{V_i, t_{ij} + V_j\}$

Now , Resource Allocation

Example:

Deployment of Fighters
Bandwidth to various information classes
Worker to Jobs
Money to Investments
Capacity to Different Products (Product-Mix)

General Problem
Single resource allocation problem

- Suppose we have K units of a resource available
- This resource is allocated to N different commodities
  (for now assume allocation in integer quantities)

What case do I need to know?
Producing/Using x_n units of/at commodity

(1) Consumes $C_n(x_n)$ units of resource

(2) Yields profit $P_n(x_n)$

(3) Assume $x_n \leq B, \forall n \quad \Rightarrow \quad$ (upper bound)

Problem: Maximize Profit subject to resource constraints
Mathematical Programming Formulation

$$
\begin{aligned}
\Rightarrow Max \quad & \{P_1(x_1) + \cdots + P_n(x_N)\} \\
s.t. \quad & C_1(x_1) + \cdots + C_N(x_N) \leq K \\
& x_n, integer \quad \forall n \\
& 0 \leq x_n \leq B \quad \forall n
\end{aligned}
$$

(If cost and constraints are linear and integer constraints, this is an LP or MILP)

The LP formulation

$$
\begin{aligned}
Max \quad & \{\sum_{n=0}^{N} P_n(x_n)\} \\
s.t. \quad & \sum_{n=0}^{N} C_n(x_n) = K \\
& x_n \geq 0, n = 0, \ldots, N
\end{aligned}
$$

I have made a few small change

(1) A 0th commodity akin to empty space

This is called the "Knapsack Problem" like placing objects in a Knapsack with limited space

Examples:
Allocate 5 medical teams to 3 countries to improve health education and training programs
Scenario: the world health council needs to determine how many (if any) to allocate to each country to maximize total effectiveness of the 5 teams
Measure of Effectiveness : MOE , additional person-years of life

Estimated additional person-year of life (in multiple of 1000) for each country

	countries		
# of teams	1	2	3
0	0	0	0
1	50	20	45
2	70	45	70
3	80	75	90
4	100	110	105
5	130	150	120

Theorem

$f_n(y)$ = max total "profit" obtained from allocating y units of your resource to the "production" of commodities 1 through n (n: stage)

Objective Find $f_N(K)$ and an allocation policy $(x_1^*, x_2^*, \ldots, x_N^*)$

Boundary Conditional

$$\begin{aligned} f_1(y) = \max_{x_1} \quad & P_1(x_1) \\ & 0 \le x_1 \le B \\ & 0 \le x_1 \le y \end{aligned}$$

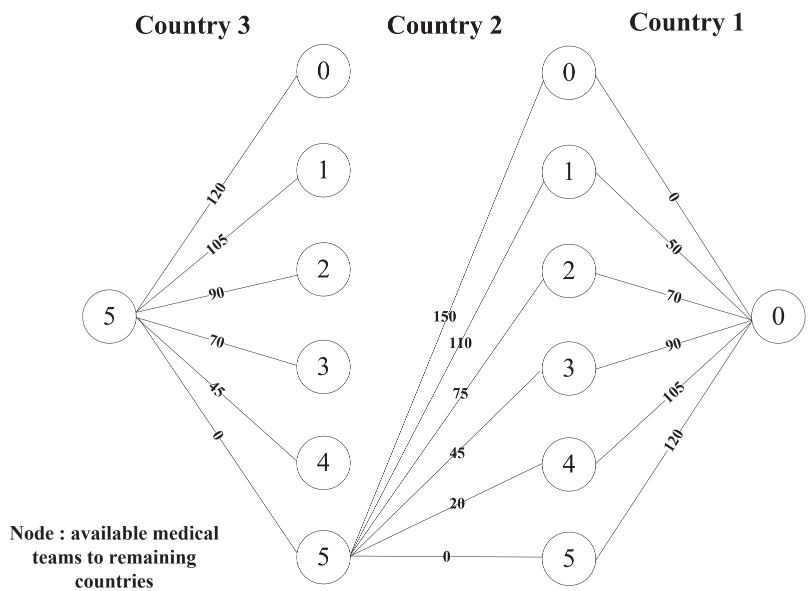
Optimality Equation's

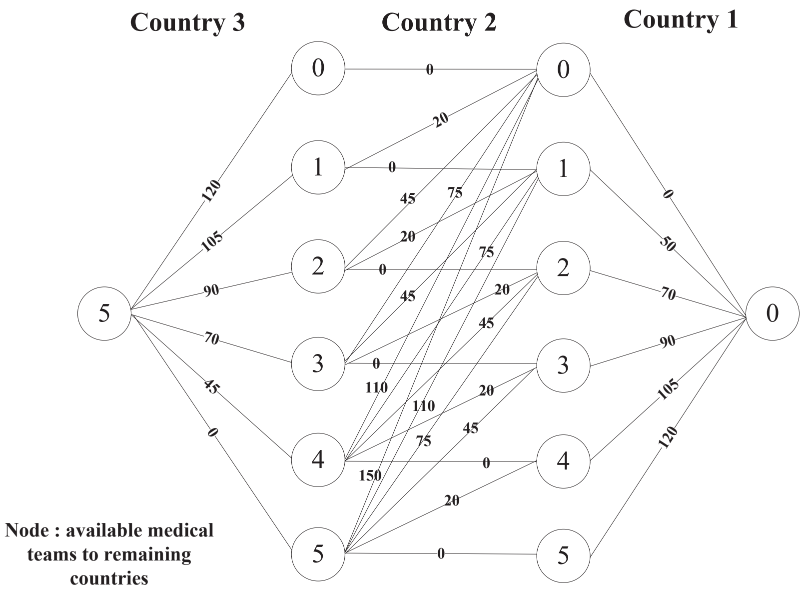
$$\begin{aligned} f_n(y) = \max_{x_n} \quad & \{P_n(x_n) + f_{n-1}(y - c_n(x_n))\} \\ & 0 \le x_n \le B \\ & 0 \le c_n(x_n) \le y \\ & x_n : integer \end{aligned}$$

Solution Procedure

Recursive Fixing —Backward Induction

$$\begin{aligned} \textit{Procedure : determine} \quad & f_1(y) \quad \forall y \\ \textit{then} \quad & f_2(y) \quad \forall y \\ & . \\ & . \\ & . \\ & f_N(y) \quad \forall y \end{aligned}$$





Recall our example:

Let $x_n = \#$ of medical teams allocated to country n

$$Max \quad \sum_{n=1}^{3} P_n(x_n)$$

$$s.t. \quad \sum_{n=1}^{3} x_n = 5 \quad , x_n \geq 0, int.$$

$$(c_n(x_n) = x_n)$$

$f_n(y) =$ max estimated additional person years of life achieved by allocation y team to country 1,. . . ,n

we know $y \leq 5$ and y is integer, Then

$f_n(y) = max\{P_n(x_n) + f_{n-1}(y - x_n); x_n \leq y, x_n \in Z^+\}$

$f_1(y) = max\{P_1(x_1); x_1 \leq y, x_1 \in Z^+\}$

We start with $f_1(y)$

y	$f_1(y)$	x_1^*
0	0	0
1		1
2		2
3		3
4		4
5		5

$f_1(y) = max\{P_1(x_1); x_1 \leq y, x_1 \in Z^+\}$

We start with $f_1(y)$

y	$f_1(y)$	x_1^*
0	0	0
1	50	1
2	70	2
3	80	3
4	100	4
5	130	5

$$f_2(y) = max\{P_2(x_2) + f_1(y - x_2)\}$$

$$f_2(y) = max\{P_2(x_2) + f_1(y - x_2)\}$$

y \ x_2	0	1	2	3	4	5	$f_2(y)$	x_2^*
0								
1								
2								
3								
4								
5								

$$f_2(y) = max\{P_2(x_2) + f_1(y - x_2)\}$$

y \ x_2	0	1	2	3	4	5	$f_2(y)$	x_2^*
0	0						0	0
1	50	20					50	0
2	70	70	45				70	0 or 1
3	80	90	95	75			95	2
4	100	100	115	125	110		125	3
5	130	120	125	145	160	150	160	4

$$f_3(y) = max\{P_3(x_3) + f_2(y - x_3)\}$$

y \ x_3	0	1	2	3	4	5	$f_3(y)$	x_3^*
0								
1								
2								
3								
4								
5								

$f_3(y) = max\{P_3(x_3) + f_2(y - x_3)\}$

y \ x_3	0	1	2	3	4	5	$f_3(y)$	x_3^*
0								
1								
2								
3								
4								
5	160	170	165	160	155	120	170	1

More resource allocation examples:

- An airline needs to allocate space to various customer classes
  - Make it easy ,all seats are the same size

To formulate the problem as DP what information do we need?

- $ earned from seat of each class
- Available space in a plane (total number of seats)
- Number of seat types

$f_n(y)$ = max profit from a y-seat plane and classes 1,. . . ,n of customers.

Suppose you have some stock and some prospective buyers . . .

- How much stocks do you have -K units
- How many buyer do you have -N Buyers
- How much is earned for each buyer? -$P_n(x_n)$ Mamer("1984")
  $f_n(y)$ = max total profit from y units of stock to n buyers

Multiple Resources,

Assume 2 resources
K units of Resource 1
L units of Resource 2

Allocation x_n units to commodity n consumes $c_n(x_n)$ of resource 1 and $d_n(x_n)$ of resource 2

Can you write math program to solve the max total reward problem ?
Yes, but difficult to solve. Knapsack problems with integer decision variables are known to be NP-complete problems. Dynamic programming is a possible way to reduce the computational complexity.

And the optimality equation ?
$(c_n(x_n), d_n(x_n)) \in Z^+$, and $x_n \in Z^+$)
$f_n(y,z) = Max_{x_n \in A}\{P_n(x_n) + f_{n-1}(y - c_n(x_n), z - d_n(x_n))\}$

where,
A$\equiv \{x_n \mid c_n(x_n) \leq y, d_n(x_n) \leq z, x_n \leq B, x_n :$ non-negative integer$\}$

Boundary Condition: $f_1(y,z) = max_{x \in Z}(P_1(x_1))$
So in theory , we could solve this problem
But now our state space in larger, we might have dimensional problems.
$\{1,2,\dots\} \times \{0,1,\dots,K\} \times \{0,1,\dots,L\}$
($\{1,2,\dots\} \times \{0,1,\dots,K\} \times \{0,1,\dots,L\}$: # of states)

In general as additional resources are introduced ,we must add more state variables,

$$I \quad : \text{type of resource}$$
$$Y \quad : \text{is the \# of each resource}$$

There are Y^Istates to consider for each time period (decision epoch)
This is the so-called "curse of dimensionality"
(classic work Gallego and Van Ryzin)

A state is the amount of information that is sufficient to

(1) Describe transitions
(2) Enable induction
(3) Enough to make decisions

Examples:

(1) Med Team :

(2) Seat Allocation :

(3) Stock Example :

Examples:

(1) Med Team : must know # of countries left ,# of teams remaining "y"

(2) Seat Allocation : # of remaining customer classes, # of seats remaining

(3) Stock Example :

Examples:

(1) Med Team : must know # of countries left ,# of teams remaining "y"

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Examples:

(1) Med Team : must know # of countries left ,# of teams remaining "y"

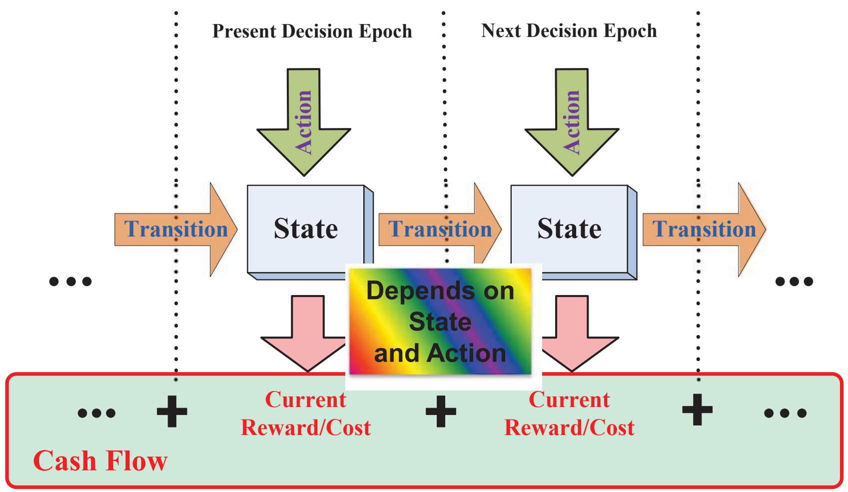
(2) Seat Allocation : # of remaining customer classes, # of seats remaining

(3) Stock Example : # of buyers,# of stocks remaining

(state:# of teams remaining y,# of seats remaining,# of stocks remaining)
(stage/time: # countries/ customer/ buyers)
Product-Mix Example:

With each state comes an action set, from which we make choice,

- Our choice cause a state change which we observe and continue



5 parts to make our dynamic programming model complete
$\{T, S, A_s, t_n(s_n, a_n), r_n(s_n, a_n)\}$

T : set of decision epochs, T = {1,...,N} the time horizon

S : state space,could divide S into S=$S_1 \times S_2 \times \cdots \times S_{n+1}$ (not usually done)

A $\equiv \bigcup_{s \in S} A_s \doteqdot$ action space

$t_n(s_n, a_n)$(How nodes gets from current state to next): transition structure, when in state s_n, choosing action $a_n \in A_{s_n}$, causes us to move to state $t_n(s_n, a_n) = s_{n+1}$

$r_n(s_n, a_n)$: reward earned or cost accrued from choosing action a_n in state s_n

Example: A company that makes fighter planes has one fighter on hand at the beginning of the current month. Orders for this month and for next three months are 1; 2; 1; 0 fighters, respectively. The company wants to have one fighter on hand at the beginning or the fifth month, which means that a total of 4 fighters must be manufactured over the next 4 months. Orders for a particular month may be filled from that months production or from inventory. The problem is to find a production schedule that satisfies demand and minimizes the total costs. The cost of producing 0,1, or 2 fighters in a give month is 10, 17, and 20, respectively. The cost of having 0, 1, or 2 machines in inventory at the start of a month is 0,3, and 7, respectively.

(a) Draw a network whose shortest path is the best production schedule.

(b) Find the best production schedule.

Now, let's move to stochastic dynamic decision problems.

Recall

Definition

A policy π that use the same decision rule, for each epochs is called stationary
Let π^{SD}, stationary deterministic

What is the smallest set of policies ?

(1) Stationary deterministic

Which one would you not expect to be enough for finite horizon problems?

(1) Stationary (can not always keep)
(2) Deterministic
(3) Markovian

Definition

Suppose we specify an action for every state at a particular decision epoch This is called a decision rule A decision rule "d_n" at epoch n, is then a function from state space to action set $d_n : S \to A$

Let's redefine our optimality equation:
Let $\pi = \{d_1, d_2, \ldots, d_N\}$ be a sequence of decision rules

$$f(s,\pi) = \sum_{n=1}^{N} r_n(s_n, d_n(s_n)) + \bar{r}_{N+1}(s_{N+1})$$

,where $s_1 = s$, $\bar{r}(s_{N+1})$: terminal reward function

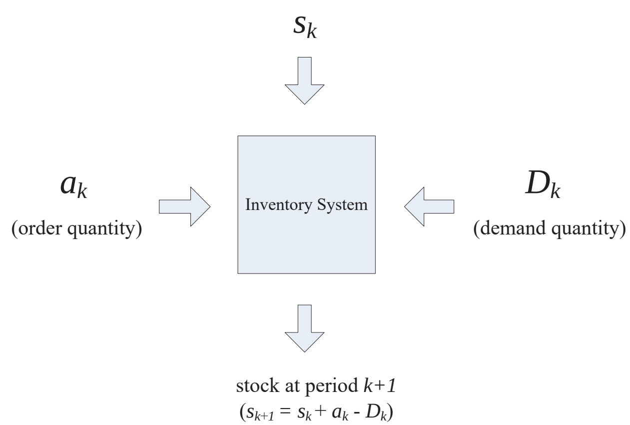
We are in search of $g(s) = sup_\pi\{f(s,\pi)\}$

Inventory Control :

- We must order Inventory for N periods to meet (stochastic demand)

  Let, s_k = inventory position at the beginning of k^{th} period
  a_k = order quantity in period k (immediately delivery)
  D_k : bounded demand during period k (Assume i.i.d. for all periods)
  c : per unit production cost
  $H(i)$: holding cost for $i > 0$, backlogging cost for $i < 0$

A picture, stock at period k



$s_{k+1} = s_k + a_k - D_k$
$c(a_k) + h(s_k + a_k - D_k)$
$(c(a_k)$: cost of order , $s_k + a_k - D_k$: holding cost)

5 Key Elements for any stochastic dynamic programming model:

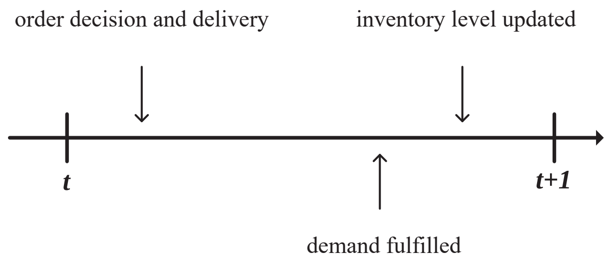
$\{T, S, A, t_n(s_n, a_n), r_n(s_n, a_n)\}$
T: time, epoch
S: state space
A: action space
$t_n(s_n, a_n)$: How nodes get from current state to next state
$r_n(s_n, a_n)$: reward/cost function

5 Key Elements for any stochastic dynamic programming model:

T: set of decision epochs = {1, 2, ..., N}
S: state space
A: $\bigcup_{s \in S} A_s$
$t_n(s_n, a_n)$: transition structure, in state s_n choosing a_n causes us to move to s_{n+1}
$r_n(s_n, a_n)$: reward earned or cost accrued from choosing a_n in state s_n

- Order decision and delivery. (beginning of each month)
- Demands are filled before the end of each month
- If demand exceeds inventory, sales are lost. (no backlogging)
- Revenue, cost and demand are stationary (do not vary from month to month)
- Warehouse capacity M

$\Rightarrow s_{t+1} = max\{s_t + a_t - D_t, 0\}$, (inventory level)
$= [s_t + a_t - D_t]^+$



$O(a_t)$: ordering cost
$r_t(s_t, a_t, s_{t+1}) = -O(a_t) - h(s_t + a_t) + f(s_t + a_t - s_{t+1})$
ordering cost

$$O(a_t) = \begin{cases} K + c(a_t) & , \quad if \quad a_t > 0 \\ 0 & , \quad if \quad a_t = 0 \end{cases}$$

$r_N(s) = g(s)$, (t=N, terminal reward)
$h(s_t + a_t)$: holding cost

A Numerical example :
p(demand $= j) = p_j$

$$p_j = \begin{cases} \frac{1}{4} & , \quad if \quad j=0 \\ \frac{1}{2} & , \quad if \quad j=1 \\ \frac{1}{4} & , \quad if \quad j=2 \end{cases}$$

$f(s_t + a_t - s_{t+1}) = 8(s_t + a_t - s_{t+1})$, revenue
M $= 3$, warehouse capacity
g(u) $= 0$, salvage value, after the final period
N $= 3$, # of periods
h(u) $=$ u, inventory cost
K $= 4$, ordering cost (fixed)
c(u) $=$ 2u, ordering cost (variable)

Let $u = s_t + a_t$, (inventory level after ordering decision)
Define

$$\begin{aligned} F(u) = & \ E[f(s_t + a_t - s_{t+1})] \\ = & \sum_{j=0}^{u-1} f(j)p_j + f(u)q_u, \quad where \quad q_u = \sum_{j=u}^{\infty} p_j \end{aligned}$$

$$\begin{aligned} F(u) = & \ E[f(s_t + a_t - s_{t+1})] \\ = & \sum_{j=0}^{u-1} f(j)p_j + f(u)q_u, \quad where \quad q_u = \sum_{j=u}^{\infty} p_j \end{aligned}$$

u	F(u)
0	
1	
2	
3	

$$\begin{aligned} F(u) = & \ E[f(s_t + a_t - s_{t+1})] \\ = & \sum_{j=0}^{u-1} f(j)p_j + f(u)q_u, \quad where \quad q_u = \sum_{j=u}^{\infty} p_j \end{aligned}$$

u	F(u)
0	0
1	$0 \times \frac{1}{4} + 8 \times \frac{3}{4} = 6$
2	$0 \times \frac{1}{4} + 8 \times \frac{1}{2} + 16 \times \frac{1}{4} = 8$
3	$0 \times \frac{1}{4} + 8 \times \frac{1}{2} + 16 \times \frac{1}{4} = 8$

$$
\begin{aligned}
E\{r_t(s_t,a_t,s_{t+1})\} &= E\{-O(a_t)-h(s_t+a_t)+f(s_t+a_t-s_{t+1})\}\\
&= -O(a_t)-h(s_t+a_t)+E[f(s_t+a_t-s_{t+1})]
\end{aligned}
$$

	$r_t(s,a)$			
	a=0	1	2	3
s_t=0				
1				
2				
3				

$$
\begin{aligned}
E\{r_t(s_t,a_t,s_{t+1})\} &= E\{-O(a_t)-h(s_t+a_t)+f(s_t+a_t-s_{t+1})\}\\
&= -O(a_t)-h(s_t+a_t)+E[f(s_t+a_t-s_{t+1})]
\end{aligned}
$$

	$r_t(s,a)$			
	a=0	1	2	3
s_t=0	0	-1	-2	-5
1	5	0	-3	×
2	6	-1	×	×
3	5	×	×	×

$p_j(j|s,a) = ?$ Can we create an transition probability table as well?

s_t+a_t \ s_{t+1}	0	1	2	3
0				
1				
2				
3				

$p_j(j|s,a) = ?$ Can we create an transition probability table as well?

	$p_j(j\|s,a)$			
s_t+a_t \ s_{t+1}	0	1	2	3
0	1	0	0	0
1	$\frac{3}{4}$	$\frac{1}{4}$	0	0
2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The Stochastic Inventory Model

Because the model parameters are stationary, we drop the index on the transition probability and the reward.

$r_t(s_t, a_t) \to r(s, a) \qquad p_t(j|s_t, a_t) \to p(j|s, a)$

$r(s, a)$: the current reward/cost from choosing action a in state s

The Stochastic Inventory Model

Define $v_t(s, a)$ by $v_t(s, a) = r(s, a) + \sum_{j \in S} p(j|s, a) v^*_{t+1}(j)$

$$v^*_t(s) = max_{a \in A_s}\{v_t(s, a)\}$$

or

$$v^*_t(s) = max_{a \in A_s}\{r(s, a) + \sum_{j \in S} p(j|s, a) v^*_{t+1}(j)\}$$

$\sum_{j \in S} p(j|s, a) v^*_{t+1}(j)$: the expected future reward from choosing action a in state s and change to state j next stage

The Stochastic Inventory Model

The algorithm finds: $v^*_t(s) = max_{a \in A_s}\{r(s, a) + \sum_{j \in S} p(j|s, a) v^*_{t+1}(s)\}$

Pseudo code of algorithm

1. Set $v^*_{N+1}(s) = 0,\ s = 0, 1, 2, \ldots, S$
2. Do for $t = N, \ldots, 1$

   Do for $s = 0, 1, \ldots, S$

   Do for $a = 0, 1, \ldots, A$

   $v^*_t(s, a) = max\{v^*_t(s, a), r(s, a) + \sum_{j \in S} p(j|s, a) v^*_{t+1}(j)\}$

We implement the backward induction algorithm as follows.

1. Set t=4 and $v^*_4(s) = r_4(s) = 0$, s = 0, 1, 2, 3
2. Since $t \neq 1$, continue. Set t = 3 and

$$v^*_3(s) = max_{a \in A_s}\{r(s, a) + \sum_{j \in S} p(j|s, a) v^*_4(s)\}, \quad s = 0, 1, 2, 3$$
$$= max_{a \in A_s}\{r(s, a)\}$$

The quantities $v^*_3(s, a)$, $v^*_3(s)$ and $A^*_{s,3}$ are summarized in the following table with "$\times$" denoting infeasible actions.

	$v^*_3(s, a)$					
s	a=0	a=1	a=2	a=3	$v^*_3(s)$	$A^*_{s,3}$
0	0	-1	-2	-5	0	0
1	5	0	-3	$\times$	5	0
2	6	-1	$\times$	$\times$	6	0
3	5	$\times$	$\times$	$\times$	5	0

3. Since $t \neq 1$, continue. Set t = 2 and
$v_2^*(s) = max_{a \in A_s}\{v_2^*(s,a)\}$, where, for example,

$$\begin{aligned} v_2^*(0,2) = & \; r(0,2) + p(0|0,2)v_3^*(0) + p(1|0,2)v_3^*(1) + p(2|0,2)v_3^*(2) \\ & + p(3|0,2)v_3^*(3) \\ = & -2 + (\frac{1}{4}) \times 0 + (\frac{1}{2}) \times 5 + (\frac{1}{4}) \times 6 + 0 \times 5 \\ = & \; 2 \end{aligned}$$

The quantities $v_2^*(s,a)$, $v_2^*(s)$ and $A^*_{s,2}$ are summarized in the following table.

s	$v_2^*(s,a)$ a=0	a=1	a=2	a=3	$v_2^*(s)$	$A^*_{s,2}$
0	0	$\frac{1}{4}$	2	$\frac{1}{2}$	2	2
1	$\frac{25}{4}$	4	$\frac{5}{2}$	$\times$	$\frac{25}{4}$	0
2	10	$\frac{9}{2}$	$\times$	$\times$	10	0
3	$\frac{21}{2}$	$\times$	$\times$	$\times$	$\frac{21}{2}$	0

4. Since $t \neq 1$, continue. Set t=1 and
$v_1^*(s) = max_{a \in A_s}\{v_1^*(s,a)\}$
The quantities $v_1^*(s,a)$, $v_1^*(s)$, and $A^*_{s,1}$ are summarized in the following table.

s	$v_1^*(s,a)$ a=0	a=1	a=2	a=3	$v_1^*(s)$	$A^*_{s,1}$
0	0	$\frac{33}{16}$	$\frac{66}{16}$	$\frac{67}{16}$	$\frac{67}{16}$	3
1	$\frac{129}{16}$	$\frac{98}{16}$	$\frac{99}{16}$	$\times$	$\frac{129}{16}$	0
2	$\frac{194}{16}$	$\frac{131}{16}$	$\times$	$\times$	$\frac{194}{16}$	0
3	$\frac{227}{16}$	$\times$	$\times$	$\times$	$\frac{227}{16}$	0

5. Since t = 1, stop.
This algorithm yields the optimal expected total reward function $v_4^*(s)$ and the optimal policy $\pi^* = (d_1^*(S), d_2^*(s), d_3^*(s))$, which is tabulated below. Note in this example that the optimal policy is unique.

s	$d_1^*(S)$	$d_2^*(S)$	$d_3^*(S)$	$v_4^*(s)$
0	3	2	0	$\frac{67}{16}$
1	0	0	0	$\frac{129}{16}$
2	0	0	0	$\frac{194}{16}$
3	0	0	0	$\frac{227}{16}$

The quantity $v_t^*(s)$ gives the expected total reward obtained using the optimal policy when the inventory at the start of the month t is s_t units.
The optimal policy has a particular simple form; if at the start of month 1 the inventory is 0 units, order three units, otherwise do not order;
If at the start of month 2 the inventory is two units, order two units, otherwise do not order.
And do not order in month 3 for any inventory level.

$$d_1^*(S_1) = \begin{cases} 0 & s > 0 \\ 3 & s = 0 \end{cases}$$

Policies of this form are optimal in inventory models in which the ordering cost has the form

$$O(u) = \begin{cases} 0 & u = 0 \\ K + cu & u > 0 \end{cases}$$

where $K > 0$ and $c > 0$, the shortage and holding cost $h(u)$ is convex, and backlogging of unfilled orders is permitted, A proof that this type of policies are optimal under these assumption may be based on backward induction.