Introduction to Machine Learning

Linear Models for Regression

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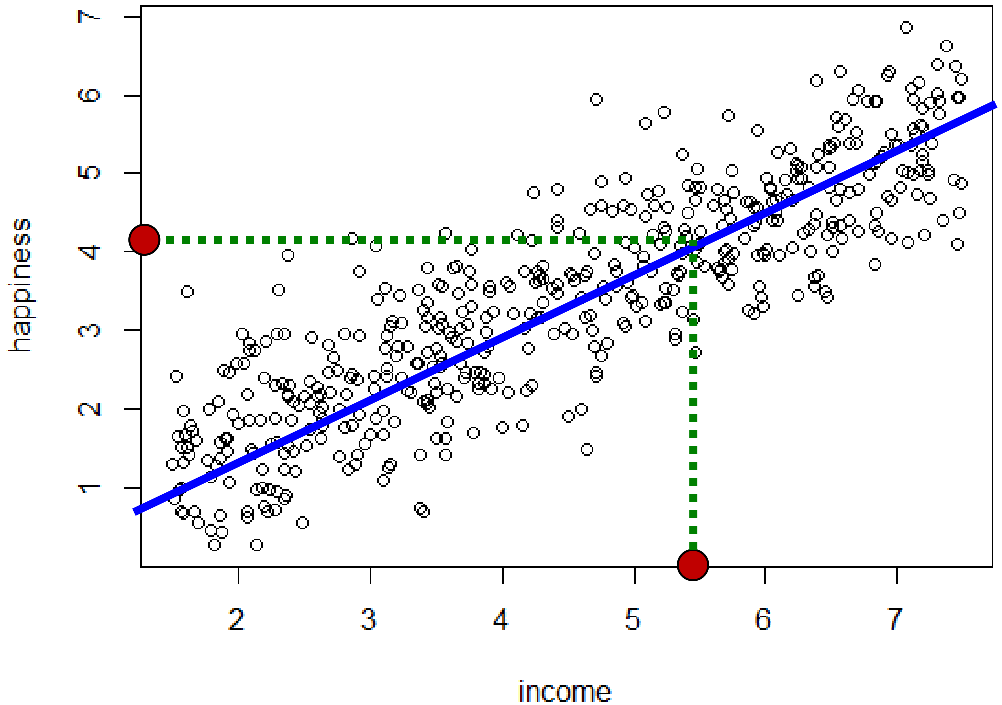
Some slides are modified from Prof. Sheng-Jyh Wang and Prof. Hwang-Tzong Chen

Regression

- Given a training data set comprising N observations $\{\mathbf{x}_n\}_{n=1}^{N}$ and the corresponding target values $\{t_n\}_{n=1}^{N}$, the goal of regression is to predict the value of t for a new value of $\mathbf{x}$



https://www.scribbr.com/statistics/linear-regression-in-r/

A simple regression model

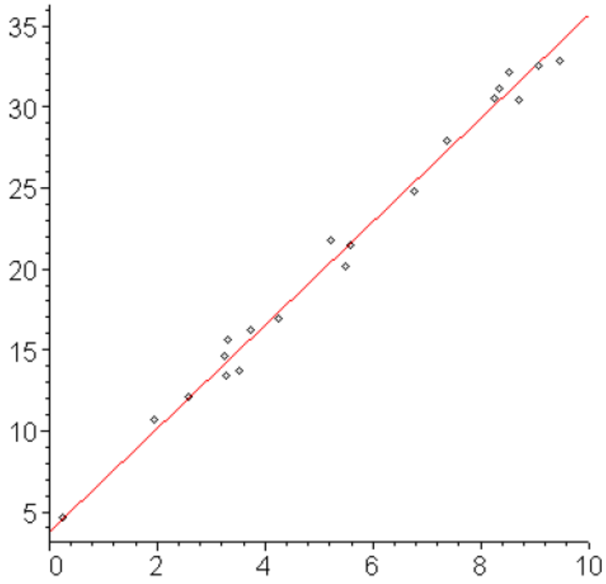
- A simple linear model:

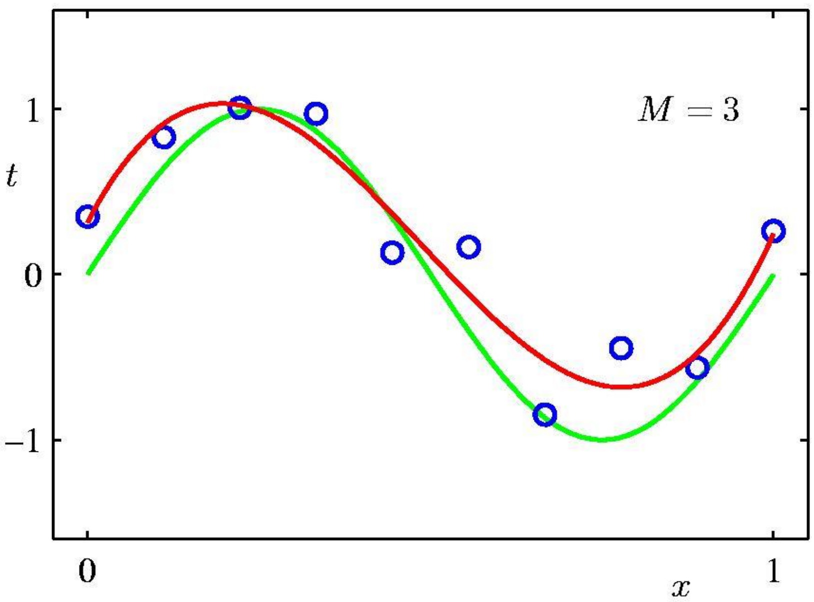
$$y(\mathbf{x}, \mathbf{w}) = w_0 + w_1 x_1 + \ldots + w_D x_D$$

  - Each observation is in a D–dimensional space $\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$
  - y is a regression model parametrized by $\mathbf{w} = (w_0, \ldots, w_D)^{\mathrm{T}}$
  - The output is a linear combination of the input variables
  - It is a linear function of parameters
  - The fitting power is quite limited. Seeking a nonlinear extension for the input variables

An example

- A regressor in the form of $y(\mathbf{x}, \mathbf{w}) = w_0 + w_1 x_1 + \ldots + w_D x_D$
  - A straight line in this case -> Insufficient fitting power
  - Nonlinear feature transforms before linear regression





Linear regression with nonlinear basis functions

- Simple linear model:

$$y(\mathbf{x}, \mathbf{w}) = w_0 + w_1 x_1 + \ldots + w_D x_D$$

- A linear model with nonlinear basis functions

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{j=1}^{M-1} w_j \phi_j(\mathbf{x})$$

where $\{\phi_j\}_{j=1}^{M-1}$: nonlinear basis functions

M: the number of parameters

w_0: the bias parameter allowing a fixed offset

- The regression output is a linear combination of nonlinear basis functions of the inputs

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Linear regression with nonlinear basis functions

- A linear model with nonlinear basis functions

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{j=1}^{M-1} w_j \phi_j(\mathbf{x})$$

- Let $\phi_0(\mathbf{x}) = 1$, a dummy basis function. The regression function is equivalently expressed as

$$y(\mathbf{x}, \mathbf{w}) = \sum_{j=0}^{M-1} w_j \phi_j(\mathbf{x}) = \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x})$$

where $\mathbf{w} = (w_0, \ldots, w_{M-1})^{\mathrm{T}}$ and $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_{M-1})^{\mathrm{T}}$

Examples of basis functions

- Polynomial basis function: taking the form of powers of x

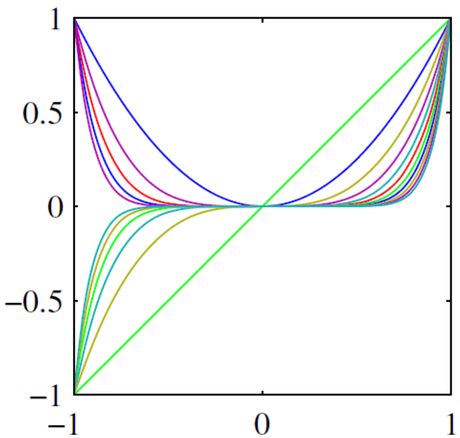
$$\phi_j(x) = x^j$$

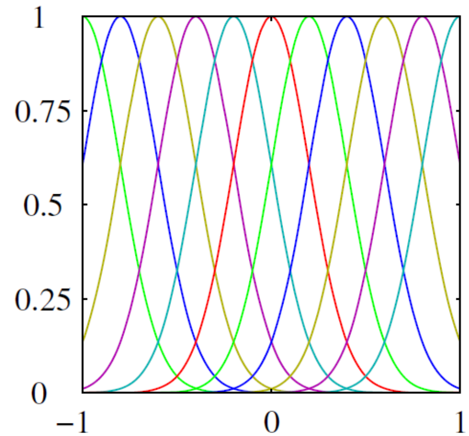
- Gaussian basis function: governed by μ_j and s
  - μ_j governs the location while s governs the scale

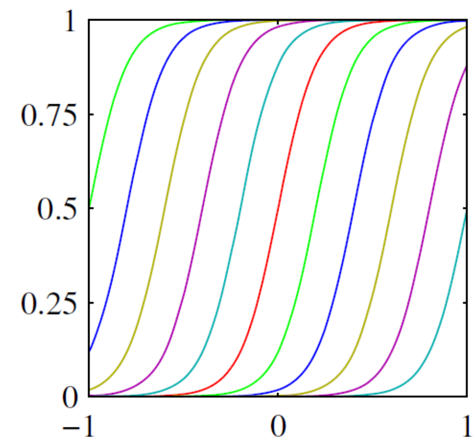
$$\phi_j(x) = \exp\left\{-\frac{(x-\mu_j)^2}{2s^2}\right\}$$

- Sigmoidal basis function: governed by μ_j and s

$$\phi_j(x) = \sigma\left(\frac{x-\mu_j}{s}\right) \quad \text{where} \quad \sigma(a) = \frac{1}{1+\exp(-a)}$$



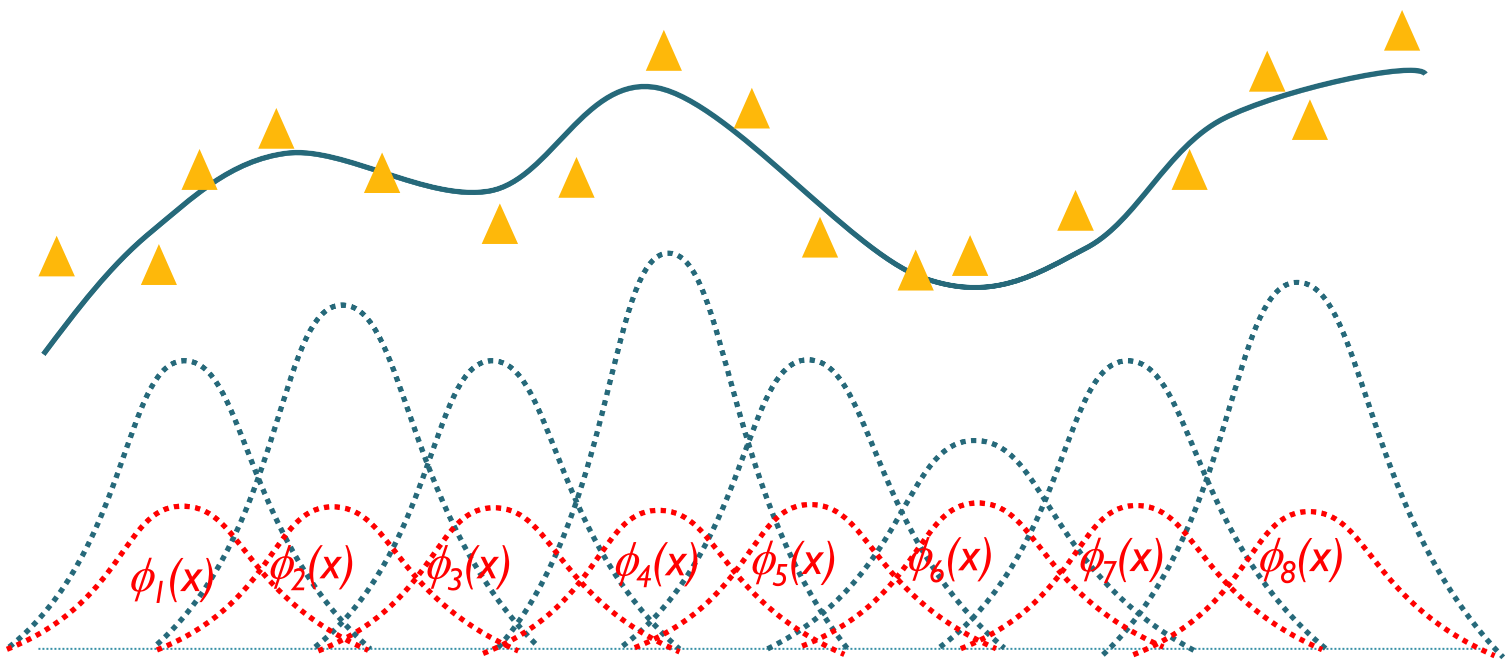




How basis functions work

- Take Gaussian basis functions as an example

$$y = w_0 + w_1\phi_1(\mathbf{x}) + w_2\phi_2(\mathbf{x}) + \ldots + w_{M-1}\phi_{M-1}(\mathbf{x})$$



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Maximum likelihood and least squares

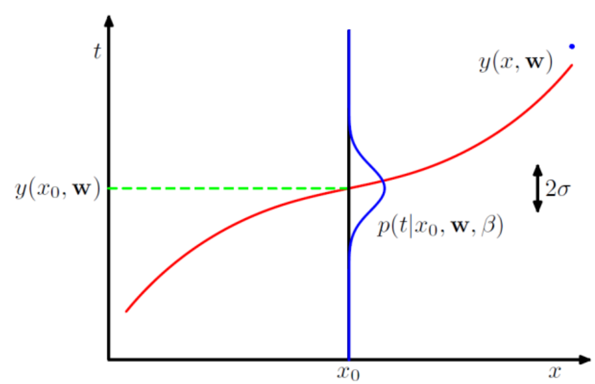
- Assume each observation is sampled from a deterministic function with an added Gaussian noise

$$t = y(\mathbf{x}, \mathbf{w}) + \epsilon$$

where ε is a zero mean Gaussian and precision (inverse variance) is β

- Thus, we have the conditional probability

$$p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$$



Maximum likelihood and least squares

- Given a data set of inputs $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ with corresponding target values $t_1, \ldots, t_N$, we have the likelihood function

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1})$$

- The log likelihood function is

$$\begin{aligned} \ln p(\mathbf{t}|\mathbf{w}, \beta) &= \sum_{n=1}^{N} \ln \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}) \\ &= \frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi) - \beta E_D(\mathbf{w}) \end{aligned}$$

where $E_D(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\}^2.$

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Maximum likelihood and least squares

- Given a data set of inputs **X** = {$\mathbf{x}_1, \ldots, \mathbf{x}_N$} with corresponding target values $t_1, \ldots, t_N$, we have the likelihood function

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n}^{N} \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1})$$

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{\frac{1}{-2\sigma^2}(x-\mu)^2\right\}$$

$$= \frac{\sqrt{\beta}}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}\beta(x-\mu)^2\}$$

- The log likelihood function

$$\begin{aligned} \ln p(\mathbf{t}|\mathbf{w}, \beta) &= \sum_{n=1}^{N} \ln \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}) \\ &= \frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi) - \beta E_D(\mathbf{w}) \end{aligned}$$

How?

where $E_D(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\}^2$.

Maximum likelihood and least squares

- Gaussian noise likelihood ⇔ sum-of-squares error function
- Maximum likelihood solution: Optimize $\mathbf{w}$ by maximizing the log likelihood function
- Step 1: Compute the gradient of log likelihood w.r.t. $\mathbf{w}$

$$\nabla \ln p(\mathbf{t}|\mathbf{w}, \beta) = \sum_{n=1}^{N} \left\{ t_n - \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \right\} \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}}$$

- Step 2: Set the gradient to zero, which gives

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

Maximum likelihood and least squares

- Define the design matrix in this task

$$\mathbf{\Phi} = \begin{pmatrix} \phi_0(\mathbf{x}_1) & \phi_1(\mathbf{x}_1) & \cdots & \phi_{M-1}(\mathbf{x}_1) \\ \phi_0(\mathbf{x}_2) & \phi_1(\mathbf{x}_2) & \cdots & \phi_{M-1}(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}_N) & \phi_1(\mathbf{x}_N) & \cdots & \phi_{M-1}(\mathbf{x}_N) \end{pmatrix}$$

  - It has N rows, one for each training sample
  - It has M columns, one for each basis function

Maximum likelihood and least squares

- Setting the gradient to zero

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

we have

$$\mathbf{w}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi} \right)^{-1} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t}$$

- How to derive?
  - Hint 1: $\sum_{n=1}^{N} t_n \phi^{\mathrm{T}}(\mathbf{x}_n) = \left(\boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t} \right)^{\mathrm{T}}$
  - Hint 2:

$$\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} = \sum_{n=1}^{N} \begin{pmatrix} \phi_0(\mathbf{x}_n) \\ \phi_1(\mathbf{x}_n) \\ \vdots \end{pmatrix} (\phi_0(\mathbf{x}_n), \phi_0(\mathbf{x}_n), \ldots) = \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi}$$

Maximum likelihood and least squares

- The ML solution

$$\mathbf{w}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{t}$$

- $\boldsymbol{\Phi}^{\dagger} \equiv \left(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{T}}$ is known as the Moore-Penrose pseudo-inverse of the design matrix $\boldsymbol{\Phi}$

- $\boldsymbol{\Phi}$ has linearly independent columns. Why is $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}$ invertible?

Suppose that $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}$ is not invertible. $\exists \mathbf{v} \neq 0$ such that $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\mathbf{v} = 0$.

$$\begin{aligned} \mathbf{v}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\mathbf{v} &= 0 \\ \|\boldsymbol{\Phi}\mathbf{v}\|^2 &= 0 \\ \boldsymbol{\Phi}\mathbf{v} &= 0 \end{aligned}$$

$\Rightarrow$ $\boldsymbol{\Phi}$ columns linearly dependent. A contradiction.

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Maximum likelihood and least squares

- Similarly, β is optimized by maximizing the log likelihood

$$\begin{aligned}\ln p(\mathbf{t}|\mathbf{w},\beta) &= \sum_{n=1}^{N} \ln \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n),\beta^{-1}) \\ &= \frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi) - \beta E_D(\mathbf{w})\end{aligned}$$

where $E_D(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\}^2.$

- We get

$$\frac{1}{\beta_{\mathrm{ML}}} = \frac{1}{N}\sum_{n=1}^{N}\{t_n - \mathbf{w}_{\mathrm{ML}}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\}^2$$

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Regression for a new data point

- The conditional probability (likelihood function)

$$p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$$

- After learning, we get $\mathbf{w} \leftarrow \mathbf{w}_{\mathrm{ML}}$ and $\beta \leftarrow \beta_{\mathrm{ML}}$

- Specify the prediction of a data point $\mathbf{x}$ in the form of a Gaussian distribution with mean $y(\mathbf{x}, \mathbf{w}_{\mathrm{ML}})$ and variance β_{ML}^{-1}

Regularized least squares

- Add a regularization term helps alleviate over-fitting

$$E_D(\mathbf{w}) + \lambda E_W(\mathbf{w})$$

- The simplest form of the regularization term

$$E_W(\mathbf{w}) = \frac{1}{2}\mathbf{w}^{\mathrm{T}}\mathbf{w}$$

- The total error function becomes

$$\frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}_n)\}^2 + \frac{\lambda}{2}\mathbf{w}^{\mathrm{T}}\mathbf{w}$$

- Setting the gradient of the function w.r.t. $\mathbf{w}$ to 0, we have

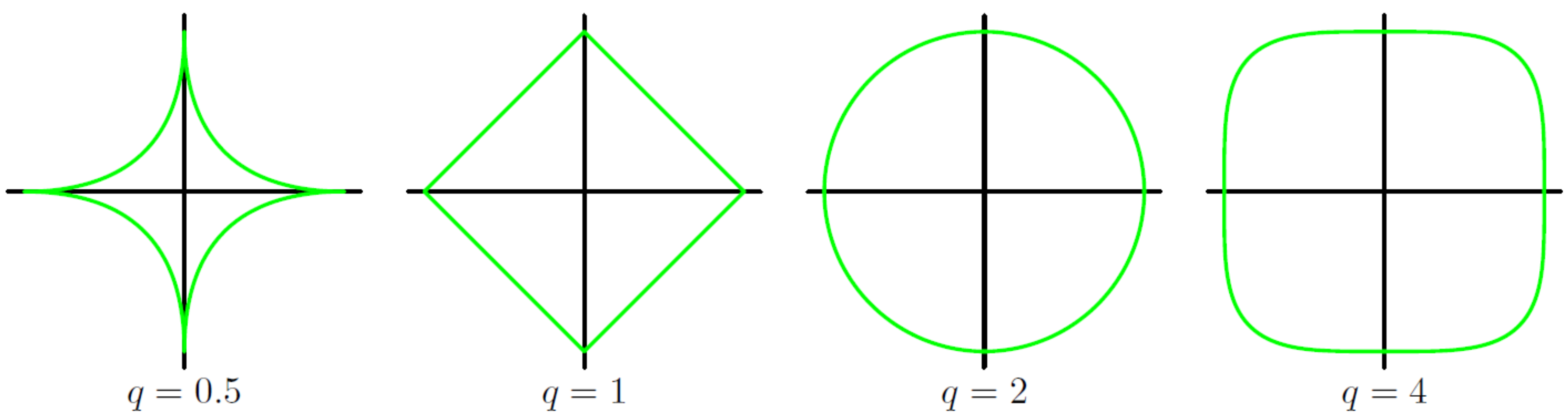
$$\mathbf{w} = \left(\lambda\mathbf{I} + \mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^{\mathrm{T}}\mathbf{t}.$$

Regularized least squares

- A more general regularizer

$$\frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}_n)\}^2 + \frac{\lambda}{2}\sum_{j=1}^{M}|w_j|^q$$

- q=2 → quadratic regularizer
- q=1 → the lasso in the statistics literature
- Contours of the regularization term



Multiple outputs

- In some applications, we wish to predict $K > 1$ target values
  - One target value: Income -> Happiness
  - Multiple target values: Income -> Happiness, Hours of duty, Health
- Recall the one-dimensional case

$$y(\mathbf{x}, \mathbf{w}) = \sum_{j=0}^{M-1} w_j \phi_j(\mathbf{x}) = \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x})$$

- With the same basis functions, the regression approach becomes

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{W}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x})$$

where **W** is a $M \times K$ matrix, M is the number of basis functions, and K is the number of target values

Multiple outputs

- The conditional probability of a single observation is

$$p(\mathbf{t}|\mathbf{x}, \mathbf{W}, \beta) = \mathcal{N}(\mathbf{t}|\mathbf{W}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}), \beta^{-1}\mathbf{I})$$

  - An isotropic Gaussian, i.e., with a diagonal covariance matrix
  - Each pair of variables are independent

- The log likelihood function is

$$\begin{aligned} \ln p(\mathbf{T}|\mathbf{X}, \mathbf{W}, \beta) &= \sum_{n=1}^{N} \ln \mathcal{N}(\mathbf{t}_n|\mathbf{W}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}\mathbf{I}) \\ &= \frac{NK}{2} \ln\left(\frac{\beta}{2\pi}\right) - \frac{\beta}{2}\sum_{n=1}^{N} \left\|\mathbf{t}_n - \mathbf{W}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\right\|^2 \end{aligned}$$

Multiple outputs: Maximum likelihood solution

- Setting the gradient of the log likelihood function w.r.t. $\mathbf{W}$ to 0, we have

$$\mathbf{W}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{T}$$

- Consider the *k*th column of $\mathbf{W}_{\mathrm{ML}}$

$$\mathbf{w}_k = \left(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{t}_k = \boldsymbol{\Phi}^{\dagger}\mathbf{t}_k$$

  where $\mathbf{t}_k$ is a N-dimensional vector with components $[t_{nk}]$

- It leads to K independent regression problems

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Sequential learning

- The maximum likelihood derivation is a batch technique
  - It takes all training data into account at the same time
  - Case 1: The training data set is sufficiently large
  - Case 2: Data points are arriving in a continuous stream

- For the two cases, it is worthwhile to use sequential algorithms, or on-line algorithms, in which the data points are considered one by one, and the model parameters are updated incrementally

Sequential learning

- Stochastic gradient descent
  - Error function comprises a sum over data points $E = \sum_n E_n$

$$E_D(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}_n)\}^2.$$

  - Given data point $\mathbf{x}_n$, the parameter vector $\mathbf{w}$ is updated by

$$\mathbf{w}^{(\tau+1)} = \mathbf{w}^{(\tau)} - \eta \nabla E_n$$

where τ is the iteration number and η is the learning rate

  - In the case of sum-of-squares error, it is

$$\mathbf{w}^{(\tau+1)} = \mathbf{w}^{(\tau)} + \eta(t_n - \mathbf{w}^{(\tau)\mathrm{T}}\phi_n)\phi_n$$

Maximum a posterior

- Likelihood function

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n|\mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}_n), \beta^{-1})$$

- Let's consider a prior function, which is a Gaussian

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

where $\mathbf{m}_0$ is the mean and S_0 is the covariance matrix

- The posterior function is also a Gaussian

$$p(\mathbf{w}|\mathbf{t}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N)$$

where $\mathbf{m}_N = \mathbf{S}_N \left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{t}\right)$ is the mean

and $\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}$ is the covariance

How to derive the mean and covariance in posterior

- According to the marginal and conditional Gaussians on page 93 of the PRML textbook

$$
\begin{aligned}
p(\mathbf{x}) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}) \\
p(\mathbf{y}|\mathbf{x}) &= \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}) \\
p(\mathbf{x}|\mathbf{y}) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\Sigma}\{\mathbf{A}^{\mathrm{T}}\mathbf{L}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Lambda}\boldsymbol{\mu}\}, \boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= (\boldsymbol{\Lambda} + \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})^{-1}
\end{aligned}
$$

A zero-mean isotropic Gaussian prior

- A general Gaussian prior function

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

where $\mathbf{m}_0$ is the mean and S_0 is the covariance matrix

- A widely used Gaussian prior

$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|0, \alpha^{-1}\mathbf{I})$$

- Mean and covariance of the resulting posterior function

$$\begin{aligned} \mathbf{m}_N &= \mathbf{S}_N\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{t}\right) \\ \mathbf{S}_N^{-1} &= \mathbf{S}_0^{-1} + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi} \end{aligned} \Rightarrow \begin{aligned} \mathbf{m}_N &= \beta\mathbf{S}_N\mathbf{\Phi}^{\mathrm{T}}\mathbf{t} \\ \mathbf{S}_N^{-1} &= \alpha\mathbf{I} + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi} \end{aligned}$$

Sequential Bayesian learning: An example

- Data, including observations and target values, are given one-by-one
- Data are in a one-dimensional space
- Data are sampled from the function $f(x, \mathbf{a}) = a_0 + a_1 x$, where $a_0 = -0.3$ and $a_1 = 0.5$, and added by a Gaussian noise
  - Note that the function is unknown
  - We have just the observations and the target values

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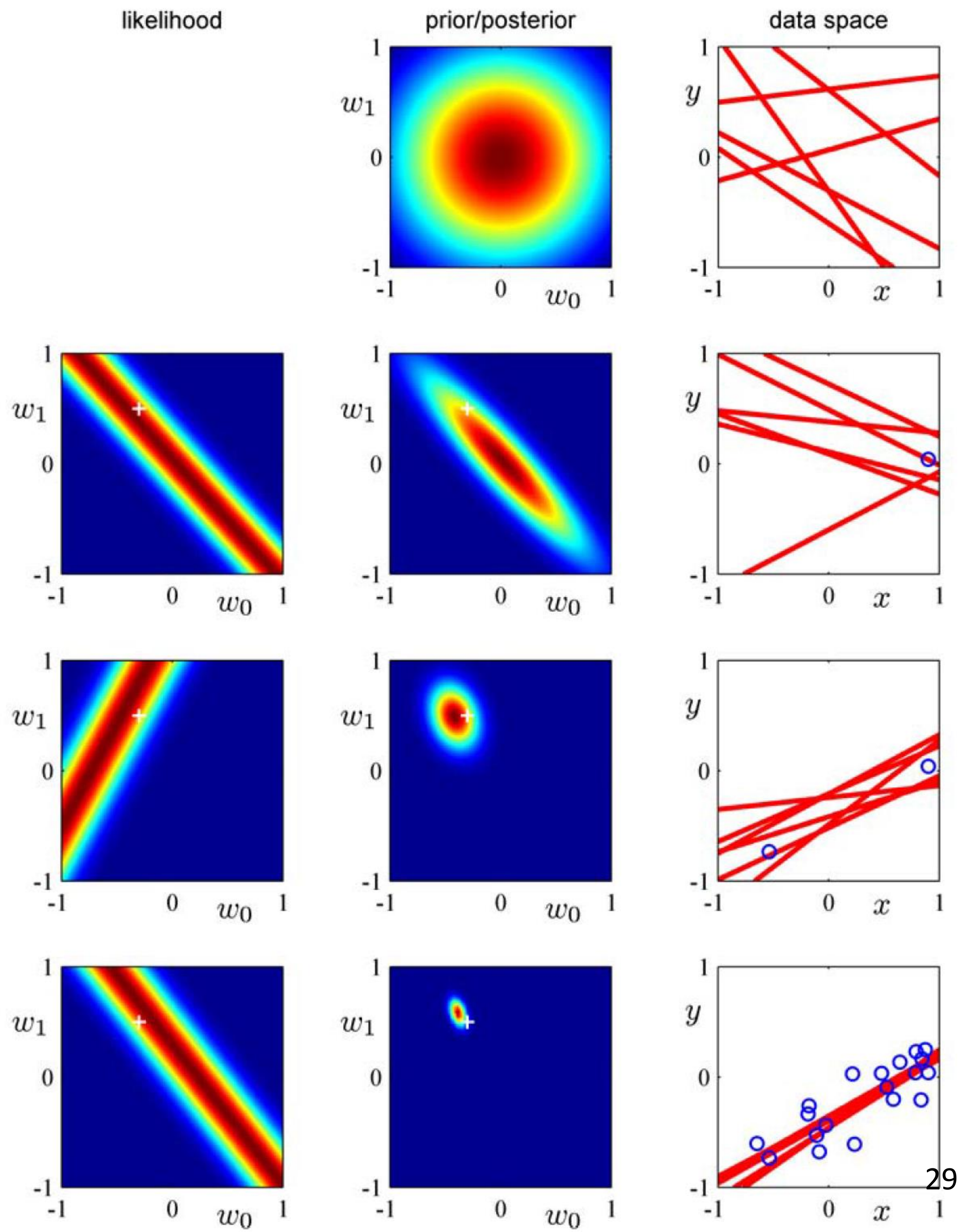
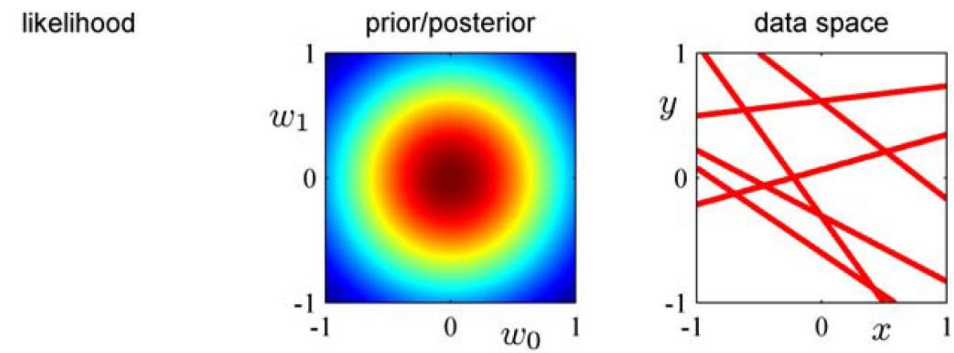
An example

- Regression function

$$y(x, \mathbf{w}) = w_0 + w_1 x$$





An example

- Regression function

$$y(x, \mathbf{w}) = w_0 + w_1 x$$

- In the beginning, no data are available

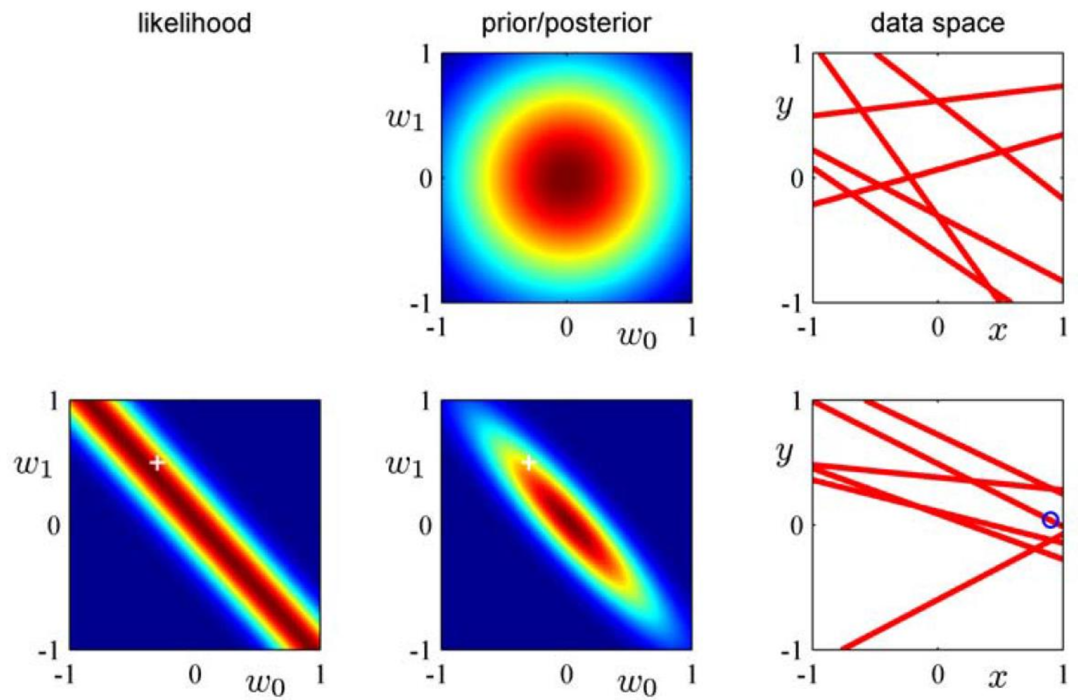
- Constant likelihood

- Prior = posterior $$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|0, \alpha^{-1}\mathbf{I})$$

- Sample 6 curves for function according to posterior distribution

An example

- Regression function
  $y(x, \mathbf{w}) = w_0 + w_1 x$
- One data (blue circle) sample is given
- Likelihood for this sample $p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$
- White cross
- Posterior proportional to likelihood x prior
- Sample 6 curves according to posterior



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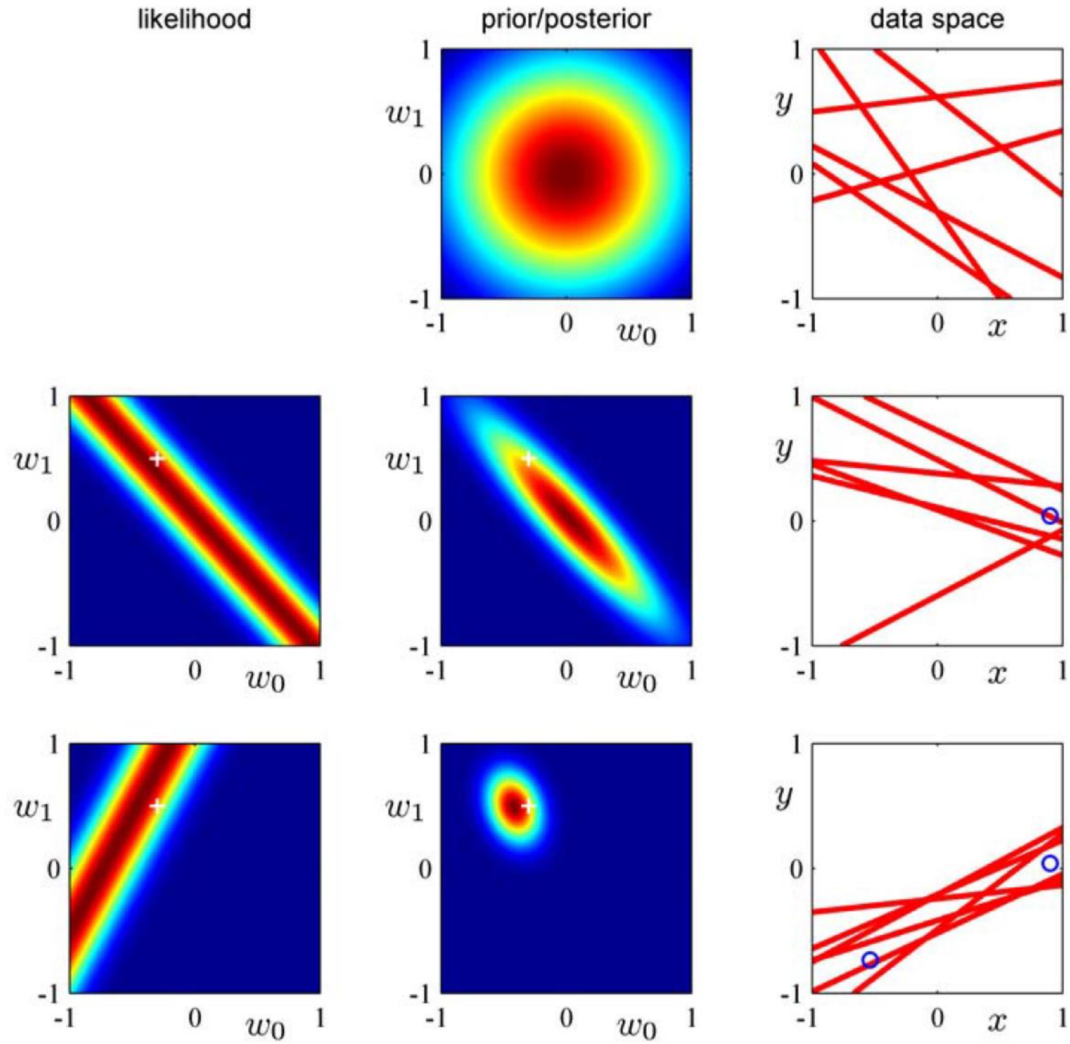
An example

- Regression function

  $y(x, \mathbf{w}) = w_0 + w_1 x$

- Second data (blue circle) sample is given
- Likelihood for the second sample
- White cross
- Posterior proportional to likelihood x prior
- Sample 6 curves according to posterior



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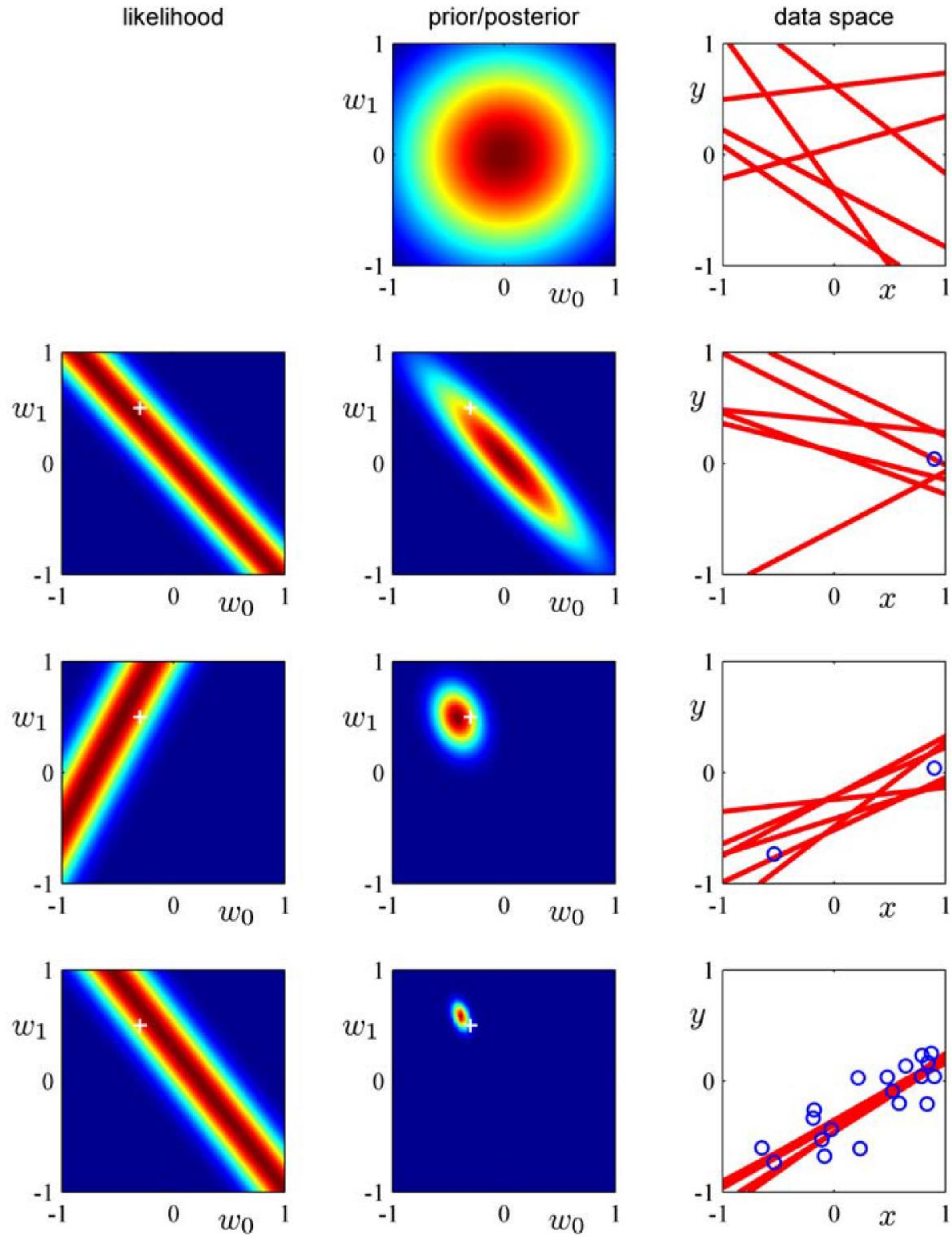
An example

- Regression function
  $y(x, \mathbf{w}) = w_0 + w_1 x$
- 20 data (blue circle) sample are given
- Likelihood for the 20th sample
- White cross
- Posterior proportional to likelihood x prior
- Sample 6 curves according to posterior



Predictive distribution

- Recall the posterior function

$$p(\mathbf{w}|\mathbf{t}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N)$$

where $\mathbf{m}_N = \mathbf{S}_N \left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{t}\right)$ and $\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}$

- Given $\mathbf{w}$, we regress a data sample via

$$p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$$

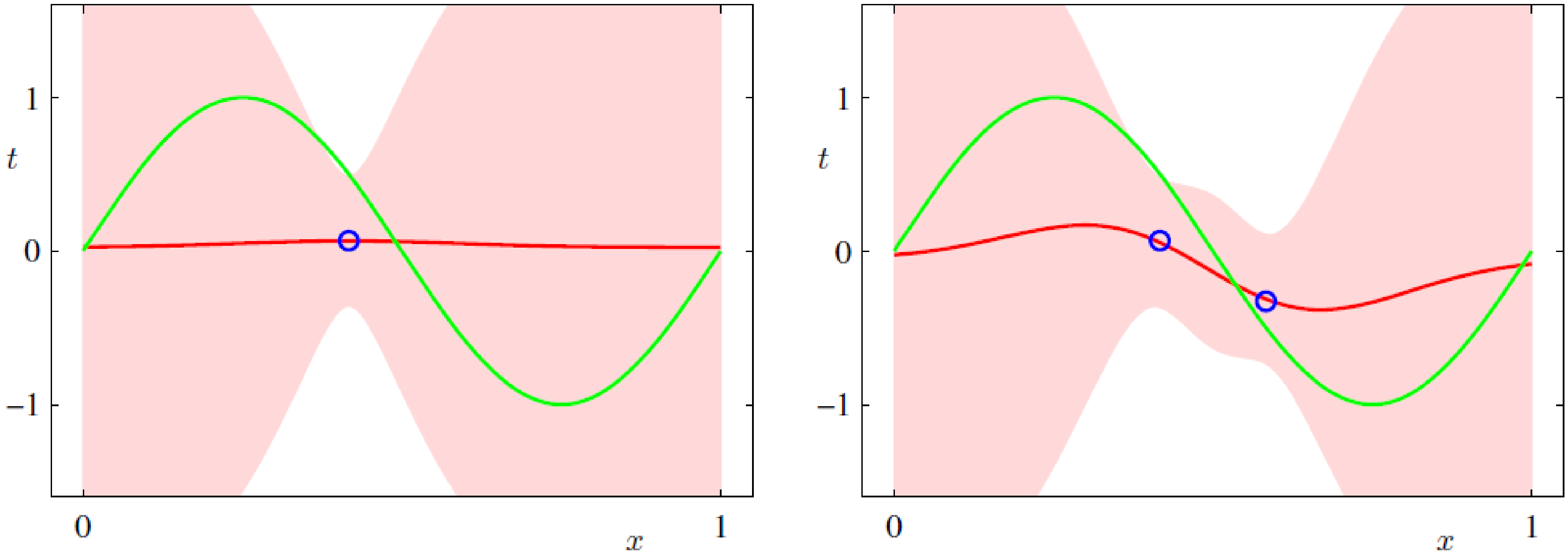
- In Bayesian treatment, the predictive distribution is

$$p(t|\mathbf{t}, \alpha, \beta) = \int p(t|\mathbf{w}, \beta)p(\mathbf{w}|\mathbf{t}, \alpha, \beta)\,\mathrm{d}\mathbf{w}$$

- Then we have

$$p(t|\mathbf{x}, \mathbf{t}, \alpha, \beta) = \mathcal{N}(t|\mathbf{m}_N^{\mathrm{T}}\phi(\mathbf{x}), \sigma_N^2(\mathbf{x}))$$

- where $\sigma_N^2(\mathbf{x}) = \frac{1}{\beta} + \phi(\mathbf{x})^{\mathrm{T}}\mathbf{S}_N\phi(\mathbf{x})$



- Green curve $\sin(2\pi x)$ is used to sample data. It is unknown
- Blue circle: a sampled data
- After learning, the predictive distribution

$$p(t|\mathbf{x}, \mathbf{t}, \alpha, \beta) = \mathcal{N}(t|\mathbf{m}_N^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}), \sigma_N^2(\mathbf{x}))$$

- Red curve: the mean of the Gaussian above
- Red shaded region: One standard deviation on either side of mean

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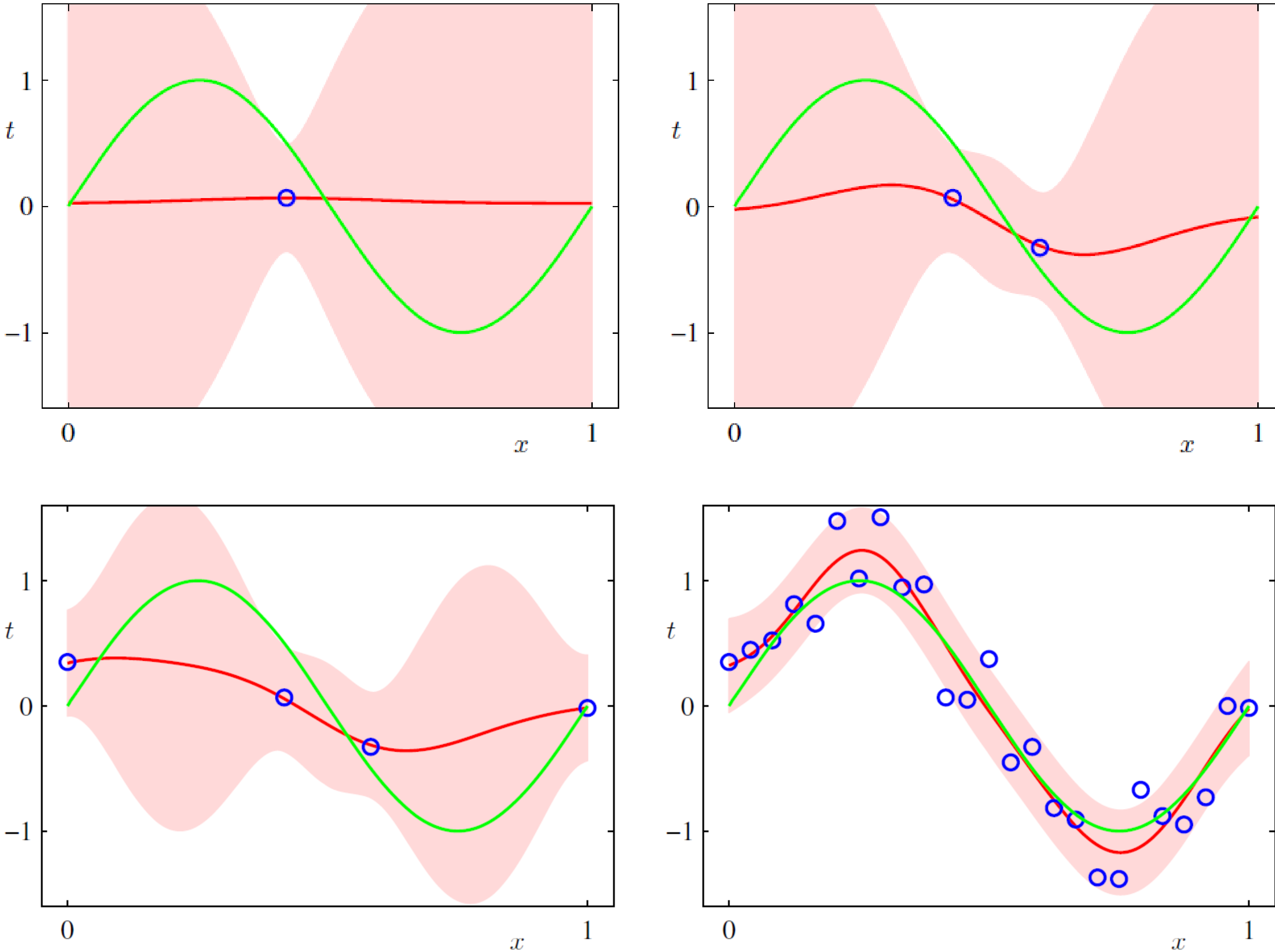
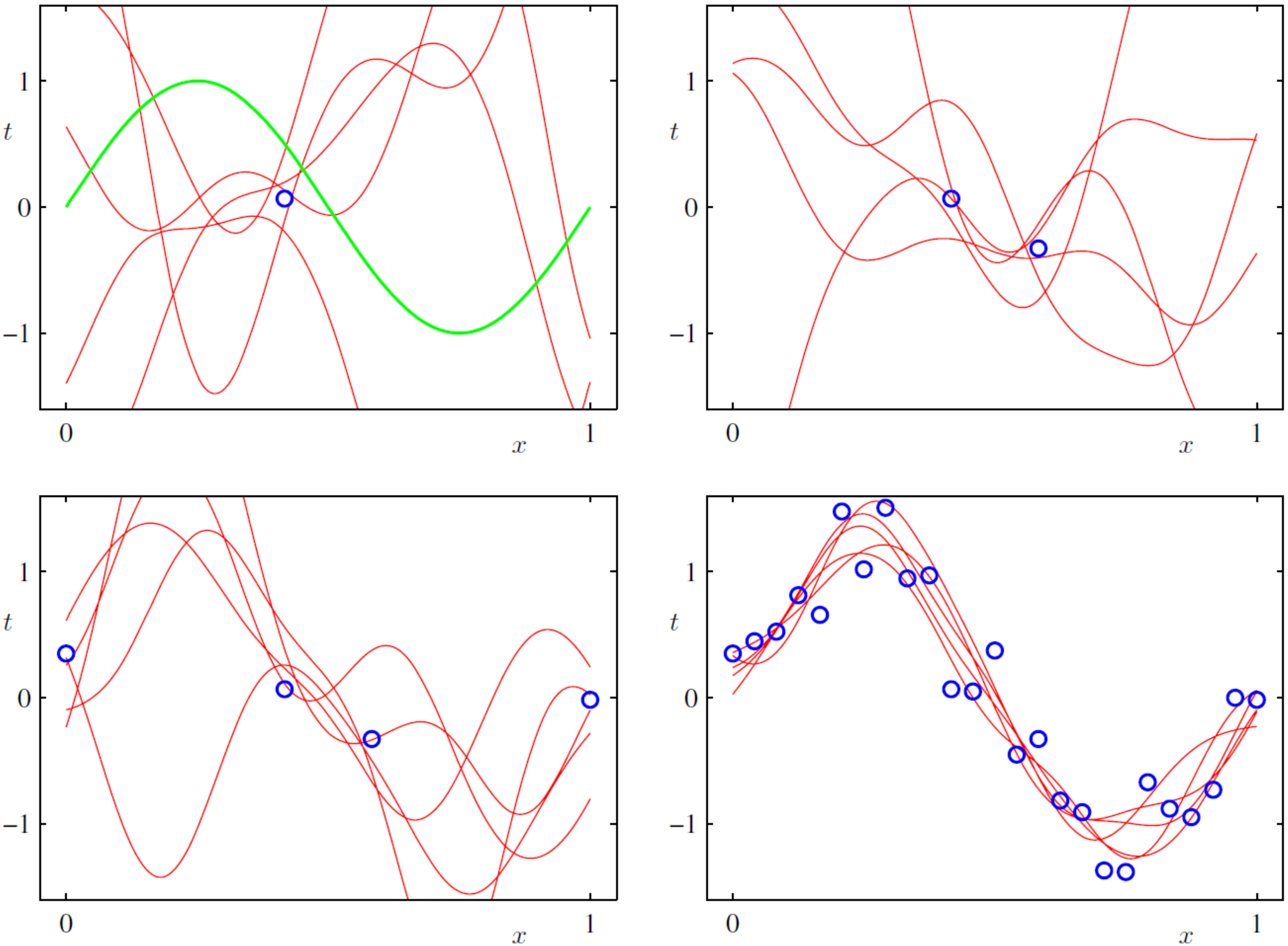


1
t
0
−1
0
x
1

- Sample 5 points of $\mathbf{w}$ according to the posterior function
- Plot the corresponding regression functions $y(x, \mathbf{w})$



References

- Chapters 3.1 and 3.3 in the PRML textbook

Thank You for Your Attention!

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