EX1.

$$\begin{bmatrix} 2 & 4 & 1 & -9 & 1 \\ 1 & 0 & 0 & -4 & 2 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 1 & -9 & 1 \\ 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 2 & 4 & 1 & -9 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -9 & 1 \\ 0 & 0 & 3 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 2 & 4 & 1 & -9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 4 & 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -4 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is now in triangular form and has a solution. The next section discusses how to continue with this type of system.

EX2.

a. True. See the box preceding the subsection titled Existence and Uniqueness Questions.
b. False. The definition of row equivalent requires that there exist a sequence of row operations that transforms one matrix into the other.
c. False. By definition, an inconsistent system has no solution.
d. True. This definition of equivalent systems is in the second paragraph after equation (2).

EX3.

Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace $R2$ by $R2+(-c)R1$.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d-c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g-c(f/a)$ is nonzero (which can certainly happen). The condition that $d-c(b/a)\neq 0$ can also be written as $ad-bc\neq 0$, or $ad\neq bc$.

EX4.

$$\begin{bmatrix} -1 & -1 & 3 & 1 & -1 \\ 3 & 3 & -8 & -5 & 7 \\ 1 & 1 & -2 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 3 & 1 & -1 \\ 1 & 1 & -2 & -3 & 5 \\ 3 & 3 & -8 & -5 & 7 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 3 & 1 & -1 \\ 0 & 0 & 1 & -2 & 4 \\ 3 & 3 & -8 & -5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -1 & 3 & 1 & -1 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 3 & 1 & -1 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -7 & 13 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution:

$$\begin{cases} x_1 = 13 - x_2 + 7x_4 \\ x_3 = 4 + 2x_4 \\ x_2 \quad is \quad free \\ x_4 \quad is \quad free \end{cases}$$

EX5.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$$

a. When $h = -6$ and $k \neq 2$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq -6$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = -6$ and $k = 2$, the system is consistent and has many solutions.

EX6.

a. False. See the statement preceding Theorem 1. Only the reduced echelon form is unique.

b. False. See the beginning of the subsection Pivot Positions. The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.

c. True. See the paragraph after Example 3.

d. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.

e. True. See the paragraph just before Example 4.

EX7.

The system is inconsistent because the pivot in column 5 means that there is a row of the form $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has no solution.