1.8~2.1 solution

EX1.

$$[A \quad b]=\begin{bmatrix}1&-3&2&1\\3&-8&8&6\\0&1&2&3\\1&0&8&10\end{bmatrix}\sim\begin{bmatrix}1&-3&2&1\\0&1&2&3\\0&1&2&3\\0&3&6&9\end{bmatrix}\sim\begin{bmatrix}1&-3&2&1\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&8&10\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\begin{matrix}x_1+8x_3=10\\x_2+2x_3=3\end{matrix}\bullet\begin{cases}x_1=10-8x_3\\x_2=3-2x_3\\x_3 \quad is \quad free\end{cases}$$

General solution: $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10-8x_3\\3-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}10\\3\\0\end{bmatrix}+x_3\begin{bmatrix}-8\\-2\\1\end{bmatrix}$,one choice: $\begin{bmatrix}10\\3\\0\end{bmatrix}$

EX2.

Solve Ax=0.

$$\begin{bmatrix}3&2&10&-6&0\\1&0&2&-4&0\\0&1&2&3&0\\1&4&10&8&0\end{bmatrix}\sim\begin{bmatrix}1&0&2&-4&0\\3&2&10&-6&0\\0&1&2&3&0\\1&4&10&8&0\end{bmatrix}\sim\begin{bmatrix}1&0&2&-4&0\\0&2&4&6&0\\0&1&2&3&0\\0&4&8&12&0\end{bmatrix}\sim\begin{bmatrix}1&0&2&-4&0\\0&1&2&3&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$x_1+2x_3-4x_4=0$$
$$x_2+2x_3+3x_4=0$$

$$\begin{cases}x_1=-2x_3+x_4\\x_2=-2x_3-3x_4\\x_3 \quad is \quad free\\x_4 \quad is \quad free\end{cases}\quad x=\begin{bmatrix}-2x_3+x_4\\-2x_3-3x_4\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-2\\-2\\1\\0\end{bmatrix}+x_4\begin{bmatrix}1\\-3\\0\\1\end{bmatrix}$$

EX3.

Use the basic definition of Ax to construct A.Write

$$T(x)=x_1v_1+x_2v_2=\begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-3&7\\5&-2\end{bmatrix}x, A=\begin{bmatrix}-3&7\\5&-2\end{bmatrix}$$

EX4.

a. True. See the paragraph following the definition of a linear transformation.

b. False. If A is an $m \times n$ matrix, the codomain is $\mathbf{R}^m$. See the paragraph before Example 1.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. See the paragraph following equation (5).

EX5.

a. $T(e_1)=e_1, T(e_2)=e_2+2e_1=\begin{bmatrix}2\\1\end{bmatrix}, \mathrm{A}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$

b. $T(e_1)=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}, T(e_2)=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, A=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$

EX6.

$T(x)=\begin{bmatrix}2x_1-x_2\\-3x_1+x_2\\2x_1-3x_2\end{bmatrix}=[A]\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2&-1\\-3&1\\2&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$. To solve $T(x)=\begin{bmatrix}0\\-1\\-4\end{bmatrix}$, row reduce the augmented matrix:

$$\begin{bmatrix}2&-1&0\\-3&1&-1\\2&-3&-4\end{bmatrix}\sim\begin{bmatrix}1&-1/2&0\\-3&1&-1\\2&-3&-4\end{bmatrix}\sim\begin{bmatrix}1&-1/2&0\\0&-1/2&-1\\0&-2&-4\end{bmatrix}\sim\begin{bmatrix}1&-1/2&0\\0&1&2\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&2\\0&0&0\end{bmatrix}, x=\begin{bmatrix}1\\2\end{bmatrix}$$

EX7.

a. False. See the paragraph preceding Example 2.

b. True. See Theorem 10.

c. True. See Table 1.

d. False. See the definition of one-to-one. Any *function* from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps a vector onto a single (unique) vector.

e. True. See the solution of Example 5.

EX8.

$$T(x)=\begin{bmatrix}x_1-5x_2+4x_3\\x_2-6x_3\end{bmatrix}=[A]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&-5&4\\0&1&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

The standard matrix A of the transformation T is $\begin{bmatrix}1&-5&4\\0&1&-6\end{bmatrix}$. The columns of A are linearly dependent because A has more columns than rows. So T is not one-to-one, by Theorem 12. Also, A has a pivot in each row, so the rows of A span R^2. By theorem 12, T maps R^3 onto R^2.

EX9.

$$A-5I_3=\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}0&-1&3\\-4&-2&-6\\-3&1&-3\end{bmatrix}$$

$$(5I_3)A=5(I_3A)=5A=5\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}=\begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix},or$$

$$(5I_3)A=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}=\begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix}$$

EX10.

a. False. AB must be a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.

b. True. See the box after Example 6.

c. False. The left-to-right order of B and C cannot be changed, in general.

d. False. See Theorem 3(d).

e. True. This general statement follows from Theorem 3(b).

EX11.

If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x}=\mathbf{0}$. From this, $A(B\mathbf{x})=A\mathbf{0}$ and $(AB)\mathbf{x}=\mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

EX12.

The product u^Tv is a 1×1 matrix,which usually is identified with a real number and is written without the matrix brackets.

$$u^Tv=\begin{bmatrix}-3&2&-5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=-3a+2b-5c$$

$$v^Tu=\begin{bmatrix}a&b&c\end{bmatrix}\begin{bmatrix}-3\\2\\-5\end{bmatrix}=-3a+2b-5c$$

$$uv^T=\begin{bmatrix}-3\\2\\-5\end{bmatrix}\begin{bmatrix}a&b&c\end{bmatrix}=\begin{bmatrix}-3a&-3b&-3c\\2a&2b&2c\\-5a&-5b&-5c\end{bmatrix}$$

$$vu^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -3 & 2 & -5 \end{bmatrix} = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix}$$

Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n\times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.