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Quadrature rules of Gaussian type for trigonometric polynomials with preassigned nodes ☆

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ABSTRACT

In this paper we consider Gaussian type quadrature rules for trigonometric polynomials where an even number of nodes is fixed in advance. For an integrable and nonnegative weight function w on the interval $E=[a,a+2\pi)$, $a\in\mathbb{R}$, these quadrature rules have the following form

$$\int_E t(x)w(x)\mathrm{d}x=\sum_{i=1}^{2k}a_it(y_i)+\sum_{i=1}^{2(n+\gamma)}A_it(x_i),\quad t\in\mathcal{T}_{2(n+\gamma)+k-1},$$

where the nodes $y_i\in E$, $i=1,2,\dots,2k$, are fixed and prescribed in advance, $\gamma\in\{0,1/2\}$ and $\mathcal{T}_n=\{\cos kx,\sin kx\mid k=0,1,\dots,n\}$, $n\in\mathbb{N}$.
Also, for $\gamma=1/2$, i.e., for the case of quadrature rules for trigonometric polynomials with odd number of nodes, we consider the optimal sets of quadrature rules in the sense of Borges (see [1,13]) for trigonometric polynomials with even number of fixed nodes. Let $\mathbf{n}=(n_1,n_2,\dots,n_r)$, $r\in\mathbb{N}$, be a multi-index and let $W=(w_1,w_2,\dots,w_r)$ be a system of weight functions on the interval $E=[a,a+2\pi)$, $a\in\mathbb{R}$. The optimal set of quadrature rules with respect to $(W,\mathbf{n})$, with even number of fixed nodes, have the form

$$\int_E f(x)w_m(x)\mathrm{d}x\approx\sum_{i=1}^{2k}a_{m,i}f(y_i)+\sum_{i=1}^{2|\mathbf{n}|+1}A_{m,i}f(x_i),\quad m=1,2,\dots r,$$

where $|\mathbf{n}|=n_1+n_2+\cdots+n_r$ and the nodes $y_i\in E$, $i=1,2,\dots,2k$, are fixed and prescribed in advance. For $r=1$ the optimal set of quadrature rules reduces to Gaussian quadrature rule for trigonometric polynomials with odd number of nodes.
For all mentioned quadrature rules, in addition to the theoretical results, we will present the method for construction and give appropriate numerical examples.

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1. Introduction

One generalization of the Gaussian quadrature rules which have maximal algebraic degree of exactness are quadrature rules with maximal degree of exactness in some linear space different from the space of algebraic polynomials. Quadrature rules of Gaussian type for trigonometric polynomials are an example of those extensions. Such quadrature rules have application in numerical integration of 2π-periodic functions, as well as in different fields of pure and applied mathematics and other sciences. They were considered by many authors (see [2,3,5,6,8–11,14,15,17,21]).

Let m be a nonnegative integer, $\mathcal{T}_m^0 = \mathcal{T}_m$ is the linear space of all trigonometric polynomials of degree less than or equal to m and $\mathcal{T}_m^{1/2}$ is the linear space of all trigonometric polynomials of semi–integer degree less than or equal to $m+1/2$. For the simplicity of the notation, we will denote by $\mathcal{T}_m^{\gamma}$ the linear span of the set $\{\cos(k+\gamma)x, \sin(k+\gamma)x \mid k = 0, 1, \ldots, m\}$, $\gamma \in \{0, 1/2\}$. Hence, $\dim(\mathcal{T}_m^{\gamma}) = 2(m+\gamma)+1$.

Remark 1.1. We say the trigonometric polynomial of degree $k+\gamma$, $k \in \mathbb{N}$, $\gamma \in \{0, 1/2\}$, regardless of whether it is trigonometric polynomial of integer degree ($\gamma = 0$) or semi–integer degree ($\gamma = 1/2$).

Let w be a given weight function, integrable and nonnegative on the interval $E = [a, a+2\pi)$, $a \in \mathbb{R}$, vanishing there only on a set of a measure zero.

Quadrature rule of the following form

$$\int_E f(x)w(x)\mathrm{d}x = \sum_{k=1}^{n} A_k f(x_k) + R_n(f),$$

where $a \leqslant x_0 < x_1 < \cdots < x_n < a+2\pi$, has trigonometric degree of exactness equal to d if $R_n(f) = 0$ for all $f \in \mathcal{T}_d$ and there exists $g \in \mathcal{T}_{d+1}$ such that $R_n(g) \neq 0$. Maximal trigonometric degree of exactness for quadrature rule with n nodes is $n-1$.

For a given $n \in \mathbb{N}$ and $\gamma \in \{0, 1/2\}$, quadrature rule

$$\int_E f(x)w(x)\mathrm{d}x = \sum_{i=1}^{2(n+\gamma)} A_i f(x_i) + R_n(f),$$

is of Gaussian type for trigonometric polynomials, i.e., is exact for $t \in \mathcal{T}_{2(n+\gamma)-1}$, if and only if the nodes $x_i \in E$, $i = 1, 2, \ldots, 2(n+\gamma)$, are the zeros of trigonometric polynomial $T_n^{\gamma} \in \mathcal{T}_n^{\gamma}$, which is orthogonal on E with respect to the weight function w to all trigonometric polynomials from $\mathcal{T}_{n-1}^{\gamma}$ (see [2,5,8,21]). In this case explicit formulas for the weights A_i, $i = 1, \ldots, 2(n+\gamma)$, are given in [2] and [8], for $\gamma = 0$ and $\gamma = 1/2$, respectively.

Remark 1.2. Let us notice that trigonometric degree of exactness for Gaussian type quadrature rules for trigonometric polynomials is $2n-1$ (for $\gamma = 0$) or $2n$ (for $\gamma = 1/2$), i.e., in the both case such quadrature rules integrate exactly trigonometric polynomials of integer degree up to certain degree. However, the number of nodes is even ($2n$ for $\gamma = 0$) or odd ($2n+1$ for $\gamma = 1/2$). Since the nodes are the zeros of the corresponding orthogonal trigonometric polynomials it is necessary to consider orthogonal trigonometric polynomial of integer degree T_n (for $\gamma = 0$), which has an even number of zeros, or orthogonal trigonometric polynomial of semi–integer degree $T_n^{1/2}$ (for $\gamma = 1/2$), which has an odd number of zeros.

For the problem of numerically evaluating a set of r definite integrals taken with respect to r distinct weight functions, but related to a common integrand and the same interval of integration it is not efficient to use a set of r quadrature rules of Gaussian type, because valuable information is wasted. For example, such problem arises in the evaluation of computer graphics illumination models (see [1]). We restrict our attention to the quadrature rules with odd number of nodes. By simulating the development of the quadrature rules of Gaussian type, and choosing a set of distinct nodes common for all quadrature rules, the optimal set of quadrature rules for trigonometric polynomials are obtained in [13]. The construction of such quadrature rules is related to the type II multiple orthogonal polynomials of semi-integer degree, defined by using orthogonality conditions spread out over r different measures [13,18].

Let $W = (w_1, w_2, \ldots, w_r)$, $r \in \mathbb{N}$, be a system of weight functions, which are integrable and nonnegative on the interval $E = [a, a+2\pi)$, $a \in \mathbb{R}$. Let $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, where $n_1, n_2, \ldots, n_r$ are nonnegative integers, and $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$ is its length. We introduce a partial order on multi-indices in the following way: $\mathbf{m} \preceq \mathbf{n} \Leftrightarrow m_\nu \le n_\nu$ for every $\nu = 1, 2, \ldots, r$.

There are two types of multiple orthogonal trigonometric polynomials of semi-integer degree [13].

• Type I multiple orthogonal trigonometric polynomials of semi-integer degree with respect to $(W, \mathbf{n})$ are collected in a vector $(T_{\mathbf{n},1}^{1/2}, T_{\mathbf{n},2}^{1/2}, \ldots, T_{\mathbf{n},r}^{1/2})$, where $T_{\mathbf{n},m}^{1/2}$ are trigonometric polynomials of semi-integer degree $n_m - 1/2$, $m = 1, 2, \ldots, r$, such that:

$$\sum_{m=1}^{r} \int_E T_{\mathbf{n},m}^{1/2}(x) \cos\left(k+\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 0,$$
$$\sum_{m=1}^{r} \int_E T_{\mathbf{n},m}^{1/2}(x) \sin\left(k+\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 0, \quad k = 0, 1, 2, \ldots, |\mathbf{n}| - 2,$$

with the normalizations:

$$\sum_{m=1}^{r} \int_E T_{\mathbf{n},m}^{1/2}(x) \cos\left(|\mathbf{n}|-\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 1,$$
$$\sum_{m=1}^{r} \int_E T_{\mathbf{n},m}^{1/2}(x) \sin\left(|\mathbf{n}|-\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 1.$$

• Type II multiple orthogonal trigonometric polynomial of semi-integer degree with respect to $(W, \mathbf{n})$ is trigonometric polynomial $T_{\mathbf{n}}^{1/2}$ of semi-integer degree $|\mathbf{n}| + 1/2$ which satisfies the following orthogonality conditions

$$\int_E T_{\mathbf{n}}^{1/2}(x) \cos\left(k_m+\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 0,$$
$$\int_E T_{\mathbf{n}}^{1/2}(x) \sin\left(k_m+\frac{1}{2}\right) x\, w_m(x) \mathrm{d}x = 0, \quad k_m = 0, 1, \ldots, n_m - 1,$$

for $m = 1, 2, \ldots, r$.

In the sequel, our attention is paid on the type II multiple orthogonal trigonometric polynomials of semi-integer degree. The uniqueness of such polynomials can only be guaranteed under additional assumptions on the r weights, i.e., if the set of functions

$$\{w_m \cos(k_m + 1/2)x,\, w_m \sin(k_m + 1/2)x : k_m = 0, 1, \ldots, n_m - 1, m = 1, 2, \ldots, r\},$$

form a Chebyshev system on E for the multi-index $\mathbf{n}$. Such system of weight functions $W = (w_1, w_2, \ldots, w_r)$ is called trigonometric T system (TT system) for multi-index $\mathbf{n}$ (see [13]).

The following important theorem was proved in [13].

Theorem 1.1. *Let $\mathbf{n}$ be a multi-index and $W = (w_1, w_2, \ldots, w_r)$ be a TT system of weight functions for all multi-indices $\mathbf{m}$ such that $\mathbf{m} \preceq \mathbf{n}$. Type II multiple orthogonal trigonometric polynomial of semi-integer degree $T_{\mathbf{n}}^{1/2}(x)$ with respect to $(W, \mathbf{n})$ has exactly $2|\mathbf{n}| + 1$ simple zeros on E.*

The following definition introduces the optimal set of quadrature rules for trigonometric polynomials [13, Definition 3.2].

Definition 1.1. Let $\mathbf{n}$ be a multi-index and let $W = (w_1, w_2, \ldots, w_r)$ be a TT system for $\mathbf{n}$ on interval E. A set of quadrature rules of the form

$$\int_E f(x) w_m(x) \mathrm{d}x \approx \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} f(x_i), \quad m = 1, 2, \ldots, r, \tag{1}$$

is an optimal set for trigonometric polynomials with respect to $(W, \mathbf{n})$ if and only if the weight coefficients, $A_{m,i}$, $m = 1, 2, \ldots, r$, $i = 1, 2, \ldots, 2|\mathbf{n}| + 1$, and the nodes, x_i, $i = 1, 2, \ldots, 2|\mathbf{n}| + 1$, satisfy the following equations:

$$\begin{aligned}
&\sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} = \int_E w_m(x) \mathrm{d}x,\\
&\sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} \cos(k_m x_i) = \int_E \cos(k_m x)\, w_m(x) \mathrm{d}x,\\
&\sum_{k=1}^{2|\mathbf{n}|+1} A_{m,i} \sin(k_m x_i) = \int_E \sin(k_m x)\, w_m(x) \mathrm{d}x, \quad k_m = 1, 2, \ldots, |\mathbf{n}| + n_m,
\end{aligned} \tag{2}$$

for $m = 1, 2, \ldots, r$.

For $r=1$ the optimal set of quadrature rules reduces to quadrature rule of Gaussian type for trigonometric polynomials with odd number of nodes.

The next theorem is counterpart to the fundamental theorem of Gaussian type quadrature rule (see [13] for the proof) and gives a characterization of the optimal set of the quadrature rule for trigonometric polynomials.

Theorem 1.2. *Let $\mathbf{n}$ be a multi-index and let $W=(w_1,w_2,\dots,w_r)$ be a TT system for $\mathbf{n}$ on an interval E. A set of quadrature rules (1) is the optimal set for trigonometric polynomials with respect to $(W,\mathbf{n})$ if and only if:*

(a) *All rules are exact for all polynomials from $\mathcal{T}_{|\mathbf{n}|}$;*
(b) *$T_{\mathbf{n}}^{1/2}(x)=\prod_{i=1}^{2|\mathbf{n}|+1}\sin\left(\frac{x-x_i}{2}\right)$ is the type II multiple orthogonal trigonometric polynomial of semi-integer degree $|\mathbf{n}|+1/2$ with respect to $(W,\mathbf{n})$.*

Motivated by the quadrature rules of Gaussian type for algebraic polynomials with preassigned nodes (see for example [7, Subsection 2.2.1], [12]), we consider quadrature formulas of Gaussian type for trigonometric polynomials with preassigned nodes.

The paper is organized as follows. In Section 2 we consider quadrature rules of Gaussian type for trigonometric polynomials, when we have an even number of preassigned nodes. Section 3 is devoted to the optimal set of quadrature rules for trigonometric polynomials with an odd number of nodes (for $\gamma=1/2$), where an even number of nodes are preassigned. Finally, some numerical examples are given in Section 4.

2. Quadrature rules of Gaussian type for trigonometric polynomials with preassigned nodes

As the title of the section indicates, we will observe quadrature rules with preassigned nodes. However, it is important to note that we must have an even number of preassigned nodes to remain in the linear space of trigonometric polynomials.

Definition 2.1. Let w be a weight function on E. Suppose that $y_i\in E$, $i=1,2,\dots,2k$. Quadrature rule

$$\int_E f(x)w(x)\mathrm{d}x=\sum_{i=1}^{2k}a_i f(y_i)+\sum_{i=1}^{2(n+\gamma)}A_i f(x_i)+R_n(f) \tag{3}$$

is of Gaussian type for trigonometric polynomials with preassigned nodes $y_i\in E$, $i=1,2,\dots,2k$, if it is exact for all $f\in\mathcal{T}_{2(n+\gamma)+k-1}$, i.e., $R_n(f)=0$ for all $f\in\mathcal{T}_{2(n+\gamma)+k-1}$.

The following theorem refers to the construction of Gaussian quadrature rules for trigonometric polynomials with preassigned nodes.

Theorem 2.1. *Let w be a weight function on E, $\gamma\in\{0,1/2\}$ and $y_i\in E$, $i=1,2,\dots,2k$. Quadrature rule (3) is of Gaussian type for trigonometric polynomials with preassigned nodes $y_i\in E$, $i=1,2,\dots,2k$, if and only if*

(a) *It is exact for all polynomials of degree less than or equal to $n+\gamma+k-1$.*
(b) *$T_n^{\gamma}(x)=\prod_{i=1}^{2(n+\gamma)}\sin\left(\frac{x-x_i}{2}\right)\in\mathcal{T}_n^{\gamma}$ is orthogonal trigonometric polynomial of degree $n+\gamma$, with respect to $w^G(x)=\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)$.*

Proof. Let us first assume that the quadrature rule (3) is of Gaussian type for trigonometric polynomials with preassigned nodes $y_i\in E$, $i=1,2,\dots,2k$. Hence, quadrature rule (3) is exact for all $t\in\mathcal{T}_{2(n+\gamma)+k-1}$, then it is exact for all $t\in\mathcal{T}_{n+k}$ and thus (a) is proved. Further, assume that $t_{n+\gamma-1}$ is a trigonometric polynomial of degree less than or equal to $n+\gamma-1$. Then the trigonometric polynomial $T_n^{\gamma}(x)t_{n+\gamma-1}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)$ has degree less than or equal to $2(n+\gamma)+k-1$. Since the corresponding quadrature rule is exact for all such polynomials, we have

$$\begin{aligned}&\int_E T_n^{\gamma}(x)t_{n+\gamma-1}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)\mathrm{d}x\\ &=\sum_{i=1}^{2k}a_iT_n^{\gamma}(y_i)t_{n+\gamma-1}(y_i)\prod_{j=1}^{2k}\sin\left(\frac{y_i-y_j}{2}\right)\\ &\quad+\sum_{i=1}^{2(n+\gamma)}A_iT_n^{\gamma}(x_i)t_{n+\gamma-1}(x_i)\prod_{j=1}^{2k}\sin\left(\frac{x_i-y_j}{2}\right).\end{aligned} \tag{4}$$

Since the both sums on the right hand side in (4) are identical to zero (in the first sum $\prod$ is equal to zero, and in the second $T_n^{\gamma}(x_i)=0$ for all $i=1,2,\ldots,2(n+\gamma)$), we have

$$\int_E T_n^{\gamma}(x)t_{n+\gamma-1}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)\mathrm{d}x=0,$$

which means that $T_n^{\gamma}\in\mathcal{T}_n^{\gamma}$ is an orthogonal trigonometric polynomial with respect to $w^G(x)=\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)$ and thus (b) is proved.

Now suppose that assumptions (a) and (b) are satisfied. The nodes $x_i\in E$, $i=1,2,\ldots,2(n+\gamma)$, for the quadrature rule (3) are the zeros of trigonometric polynomial $T_n^{\gamma}\in\mathcal{T}_n^{\gamma}$, which is orthogonal with respect to $w^G(x)=\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)$ to all trigonometric polynomials of degree less than or equal to $n-1$.

Let t be a trigonometric polynomial of degree less than or equal to $2(n+\gamma)+k-1$. We can write $t(x)=u(x)T_n^{\gamma}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)+v(x)$, where u is a trigonometric polynomial of degree less than or equal to $n+\gamma-1$ and v is a trigonometric polynomial of degree less than or equal to $n+k$ (see [13, Lemma 3.1]. It is easy to see that

$$\begin{aligned} t(y_i)&=v(y_i),\quad i=1,2,\ldots,2k,\\ t(x_i)&=v(x_i),\quad i=1,2,\ldots,2(n+\gamma). \end{aligned}\tag{5}$$

Thus, we obtain

$$\begin{aligned}\int_E t(x)w(x)\mathrm{d}x&=\int_E\left(u(x)T_n^{\gamma}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)+v(x)\right)w(x)\mathrm{d}x\\ &=\int_E u(x)T_n^{\gamma}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)\mathrm{d}x+\int_E v(x)w(x)\mathrm{d}x.\end{aligned}$$

By the second assumption, we have $\int_E u(x)T_n^{\gamma}(x)\prod_{j=1}^{2k}\sin\left(\frac{x-y_j}{2}\right)\cdot w(x)\mathrm{d}x=0$, and therefore we obtain

$$\int_E t(x)w(x)\mathrm{d}x=\int_E v(x)w(x)\mathrm{d}x.$$

Since v is a polynomial of degree less than or equal to $n+k$, it follows that, by the first assumption,

$$\int_E v(x)w(x)\mathrm{d}x=\sum_{i=1}^{2k}a_i v(y_i)+\sum_{i=1}^{2(n+\gamma)}A_i v(x_i)$$

and hence, by (5), we obtain

$$\int_E t(x)w(x)\mathrm{d}x=\sum_{i=1}^{2k}a_i t(y_i)+\sum_{i=1}^{2(n+\gamma)}A_i t(x_i).$$

This proves that the corresponding quadrature rule is exact for all polynomials of degree less than or equal to $2(n+\gamma)+k-1$. □

Remark 2.1. Weight function $w^G(x)$ is a variable-sign function, and the standard theory for orthogonal trigonometric polynomials cannot be applied. Existence of Gaussian type quadrature rules for trigonometric polynomials with preassigned nodes (3) and internity of the nodes depends on the behavior of the orthogonal trigonometric polynomial T_n^{γ}, which is orthogonal with respect to variable-sign weight function $w^G(x)$. So, as in the case of Gauss-Kronrod quadrature rules for algebraic polynomials (see [16,20]), results can be obtained for specific weight functions. For example, if we have even weight function $w(x)=(\sin^2 x)^{\alpha}$, $0<\alpha\le 2$, on interval $[-\pi,\pi)$, and integral $\int_{-\pi}^{\pi}f(x)w(x)\mathrm{d}x$, then by using $x:=\arccos x$, we obtain integral $\int_{-1}^{1}f_1(x)(1-x^2)^{\alpha-1/2}\mathrm{d}x$, $0<\alpha\le 2$, $f_1(x)=f(-\arccos x)+f(\arccos x)$ (see [19, Lemma 2.3 and 2.4]). The obtained weight function is Gegenbauer weight function, for which was proved existence of Gauss-Kronrod quadrature rules for algebraic polynomials and internity of the nodes (see [16]).

Although the quadrature formulas from Theorem 2.1 do not always exist, they are still useful because they give higher trigonometric degree of exactness, with a smaller number of calculations. Also, such rules could be applied to estimate the error of some classes of quadrature rules. We will continue to research the existence and application of such quadrature rules.

3. Optimal set of quadrature rules for trigonometric polynomials with preassigned nodes

In this section we consider the optimal set of quadrature rules for trigonometric polynomials with an odd number of nodes (for $\gamma = 1/2$) in the case when an even number of nodes is preassigned.

Definition 3.1. Let $\mathbf{n}$ be a multi–index and let $W = (w_1, w_2, \dots, w_r)$ be a TT system for $\mathbf{n}$ on interval E. A set of quadrature rules of the form

$$\int_E f(x) w_m(x) \mathrm{d}x \approx \sum_{i=1}^{2k} a_{m,i} f(y_i) + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} f(x_i), \quad m = 1, 2, \dots, r, \tag{6}$$

where the nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, are fixed and prescribed in advance, will be called an optimal set of quadrature rules for trigonometric polynomials with preassigned nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, with respect to $(W, \mathbf{n})$ if and only if the weight coefficients, $a_{m,i}$, $A_{m,i}$, $m = 1, 2, \dots, r$, $i = 1, 2, \dots, 2|\mathbf{n}| + 1$, and the nodes x_i, $i = 1, 2, \dots, 2|\mathbf{n}| + 1$, satisfy the following equations:

$$\begin{aligned}
&\sum_{i=1}^{2k} a_{m,i} + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} = \int_E w_m(x) \mathrm{d}x, \\
&\sum_{i=1}^{2k} a_{m,i} \cos(\ell_m y_i) + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} \cos(\ell_m x_i) = \int_E \cos(\ell_m x) w_m(x) \mathrm{d}x, \\
&\sum_{i=1}^{2k} a_{m,i} \sin(\ell_m y_i) + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} \sin(\ell_m x_i) = \int_E \sin(\ell_m x) w_m(x) \mathrm{d}x, \\
&\quad \ell_m = 1, 2, \dots, |\mathbf{n}| + n_m + k,
\end{aligned}$$

for $m = 1, 2, \dots, r$.

Denote $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \dots, \widetilde{w}_r)$, $\widetilde{w}_m(x) = \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) \cdot w_m(x)$, $m = 1, 2, \dots, r$, where the nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, are fixed and prescribed in advance.

Theorem 3.1. *Let $W = (w_1, w_2, \dots, w_r)$ be a TT system for $\mathbf{n}$ on interval E. Suppose that for preassigned nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2, \dots, \widetilde{w}_r)$ is also a TT system for $\mathbf{n}$ on interval E. The set of quadrature rules (6) form the optimal set of quadrature rules for trigonometric polynomials with preassigned nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, with respect to $(W, \mathbf{n})$ if and only if:*

(a) *They are exact for all trigonometric polynomials $t \in \mathcal{T}_{|\mathbf{n}|+k}$;*
(b) *$T_{\mathbf{n}}^{1/2}(x) = \prod_{k=1}^{2|\mathbf{n}|+1} \sin\left(\frac{x - x_k}{2}\right)$ is the type II multiple orthogonal trigonometric polynomial of semi-integer degree $|\mathbf{n}| + 1/2$ with respect to $(\widetilde{W}, \mathbf{n})$.*

Proof. Let us first assume that the quadrature rules (6) form the optimal set of quadrature rules for trigonometric polynomials with preassigned nodes $y_i \in E$, $i = 1, 2, \dots, 2k$, with respect to $(W, \mathbf{n})$.

About the first statement, we note that for each $m = 1, 2, \dots, r$, the corresponding quadrature rule (6) is exact for trigonometric polynomials of degree less than or equal to $|\mathbf{n}| + k + n_{min}$, where $n_{min} = \min\{n_1, n_2, \dots, n_r\}$, and then it is exact for those of degree less than or equal to $|\mathbf{n}| + k$.

Concerning the second claim, for $m = 1, 2, \dots, r$, assume that $t_m^{1/2}$ is a trigonometric polynomial of semi-integer degree less than or equal to $n_m - 1/2$. Then the trigonometric polynomial $T_{\mathbf{n}}^{1/2}(x) t_m^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right)$ has degree less than or equal to $|\mathbf{n}| + n_m + k$. Since the corresponding quadrature rule (6) is exact for all such polynomials, it follows that

$$\begin{aligned}
&\int_E T_{\mathbf{n}}^{1/2}(x) t_m^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) \cdot w_m(x) \mathrm{d}x \\
&= \sum_{i=1}^{2k} a_{m,i} T_{\mathbf{n}}^{1/2}(y_i) t_m^{1/2}(y_i) \prod_{j=1}^{2k} \sin\left(\frac{y_i - y_j}{2}\right)
\end{aligned} \tag{7}$$

$$+\sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} T_{\mathbf{n}}^{1/2}(x_i) t_m^{1/2}(x_i) \prod_{j=1}^{2k} \sin\left(\frac{x_i - y_j}{2}\right), \quad m = 1, 2, \ldots, r.$$

Since the both sums on the right hand side in (7) are identical to zero (in the first sum $\prod$ is equal to zero, and in the second $T_{\mathbf{n}}^{1/2}(x_i) = 0$ for all $i = 1, 2, \ldots, 2|\mathbf{n}| + 1$), we have

$$\int_E T_{\mathbf{n}}^{1/2}(x) t_m^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) \cdot w_m(x) \mathrm{d}x = 0,$$

and (b) follows.

Suppose now that for the quadrature rules (6) the conditions (a) and (b) hold.

For $m = 1, 2, \ldots, r$, let $\widetilde{t}$ be a trigonometric polynomial of degree less than or equal to $|\mathbf{n}| + n_m + k$. We can write

$$\widetilde{t}(x) = u_m^{1/2}(x) T_{\mathbf{n}}^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) + v(x),$$

where $u_m^{1/2}$ is a trigonometric polynomial of degree less than or equal to $n_m - 1/2$ and v is a trigonometric polynomial of degree less than or equal to $|\mathbf{n}| + k$. It is easy to see that

$$\begin{aligned} &\widetilde{t}(y_i) = v(y_i), \quad i = 1, 2, \ldots, 2k, \\ &\widetilde{t}(x_i) = v(x_i), \quad i = 1, 2, \ldots, 2|\mathbf{n}| + 1. \end{aligned} \tag{8}$$

Then, we obtain

$$\begin{aligned} \int_E \widetilde{t}(x) w_m(x) &= \int_E \left(u_m^{1/2}(x) T_{\mathbf{n}}^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) + v(x) \right) w_m(x) \mathrm{d}x \\ &= \int_E u_m^{1/2}(x) T_{\mathbf{n}}^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x - y_j}{2}\right) \cdot w_m(x) \mathrm{d}x + \int_E v(x) w_m(x) \mathrm{d}x. \end{aligned}$$

According to (b) we have $\int_E u_m^{1/2}(x) T_{\mathbf{n}}^{1/2}(x) \prod_{j=1}^{2k} \sin\left(\frac{x-y_j}{2}\right) \cdot w_m(x) \mathrm{d}x = 0$ and therefore we obtain

$$\int_E \widetilde{t}(x) w_m(x) \mathrm{d}x = \int_E v(x) w_m(x) \mathrm{d}x.$$

Since v is a trigonometric polynomial of degree less than or equal to $|\mathbf{n}| + k$, it follows from (a) that

$$\int_E v(x) w_m(x) \mathrm{d}x = \sum_{i=1}^{2k} a_{m,i} v(y_i) + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} v(x_i)$$

and hence, by using (8), we obtain

$$\int_E \widetilde{t}(x) w_m(x) \mathrm{d}x = \sum_{i=1}^{2k} a_{m,i} \widetilde{t}(y_i) + \sum_{i=1}^{2|\mathbf{n}|+1} A_{m,i} \widetilde{t}(x_i).$$

This proves that for each $m = 1, 2, \ldots, r$, the corresponding quadrature rule is exact for all trigonometric polynomials of degree less than or equal to $|\mathbf{n}| + n_m + k$. □

Remark 3.1. Since the functions $\widetilde{W}$ are not nonnegative for arbitrary preassigned nodes $y_i \in E$, $i = 1, 2, \ldots, 2k$, the existence of the type II multiple orthogonal trigonometric polynomial of semi-integer degree $T_{\mathbf{n}}^{1/2}(x)$ is guaranteed by the condition that $\widetilde{W}$ is a TT system for $\mathbf{n}$ on interval E. Once we have a TT system, the number of zeros, that is the nodes of the corresponding quadrature rules, was given by Theorem 1.1.

Table 1
The nodes y_i, $i = 1, 2$, x_i, $i = 1, 2, \ldots, 5$, and the weight coefficients a_i, $i = 1, 2$, A_i, $i = 1, 2, \ldots, 5$, of the quadrature rule of Gaussian type for trigonometric polynomials (3) for $\gamma = 1/2$ with respect to $w(x) = 3 - 2\cos x$, $x \in [-\pi, \pi)$, and the preassigned nodes $\{y_1, y_2\} = \{-\pi/3, \pi/3\}$.

i	y_i	a_i
1	$-\pi/3$	1.878478081527918
2	$\pi/3$	1.878478081527918

i	x_i	A_i
1	−2.746128371299984	3.854013235999326
2	−1.930697986003116	3.118341831528616
3	0	1.147889623427040
4	1.930697986003116	3.118341831528616
5	2.746128371299984	3.854013235999326

4. Numerical examples

In this section we give some numerical examples as a demonstration of the obtained theoretical results. We use symbolic computations in Wolfram Mathematica and software package OrthogonalPolynomials described in [4].

In order to obtain the nodes and the weights for the quadrature rule of Gaussian type for trigonometric polynomials with preassigned nodes (3), first we have to check the existence of the orthogonal trigonometric polynomial of degree $n+\gamma$, with respect to $w^G(x) = \prod_{j=1}^{2k} \sin\left(\frac{x-y_j}{2}\right) \cdot w(x)$, where $\gamma \in \{0, 1/2\}$, w is given weight function on E and $y_i \in E$, $i = 1, 2, \ldots, 2k$, are the reassigned nodes. Then, the algorithm has two parts, dealing with the computation of the nodes and the weights. By introducing weight function $w^G(x)$, the computation of the nodes reduces to the classical quadrature rule of Gaussian type for trigonometric polynomials (without preassigned nodes), which was described in [8, Section 5]. The weight coefficients can be obtained by using the exactness of such quadrature rule from Theorem 2.1 (a).

For the construction of the optimal set of quadrature rules for trigonometric polynomials with the preassigned nodes (6) one has to assure that W and $\widetilde{W}$ are TT systems for $\mathbf{n}$ on interval E. By proving that $\widetilde{W}$ is a TT systems for $\mathbf{n}$ on interval E, the existence of the type II multiple orthogonal trigonometric polynomial of semi-integer degree $|\mathbf{n}| + 1/2$, with respect to $(\widetilde{W}, \mathbf{n})$ is guaranteed. The zeros of such trigonometric polynomial are the nodes of the corresponding optimal set of quadrature rules. The weight coefficients $a_{m,i}$, $i =, 1, 2, \ldots, k$, and $A_{m,i}$, $i = 1, 2, \ldots, 2|\mathbf{n}| + 1$, can be computed by requiring that each quadrature rule integrates exactly trigonometric polynomials from $\mathcal{T}_{|\mathbf{n}|+k}$. For more details see [13,18], where method for the construction of the optimal set of quadrature rules for trigonometric polynomials (without preassigned nodes) was given.

Example 4.1. Let us construct the quadrature rule of Gaussian type for trigonometric polynomials (3) on $E = [-\pi, \pi)$, for $n = 2$, $\gamma = 1/2$, with respect to the weight function $w(x) = 3 - 2\cos x$ and the preassigned nodes $\{y_1, y_2\} = \{-\pi/3, \pi/3\}$.

It is easy to check that the orthogonal trigonometric polynomial of semi–integer $T_2^{1/2}(x)$, with respect to the

$$w^G(x) = \sin\left(\frac{x + \pi/3}{2}\right) \sin\left(\frac{x - \pi/3}{2}\right) (3 - 2\cos x)$$

exists. By using procedure given in [8, Section 5], we obtain nodes x_i, $i = 1, 2, \ldots, 5$, and weight coefficients a_i, $i = 1, 2$, A_i, $i = 1, 2, \ldots, 5$, which are given in Table 1.

The obtained quadrature rule integrates exact all trigonometric polynomials from $\mathcal{T}_5$, as it is stated in Section 2. For the integrand $f(x) = \frac{1+\cos x}{\sqrt{9+\cos x}}$, the approximation error is -6.421449×10^{-8}.

Example 4.2. Let us construct the quadrature rule of Gaussian type for trigonometric polynomials (3) on $E = [-\pi, \pi)$, for $n = 3$, $\gamma = 1/2$, with respect to the weight function $w(x) = \sqrt{1 - (x/\pi)^2}$ (the Chebyshev weight function of the second kind scaled to the interval E), and the preassigned nodes $\{y_1, y_2\} = \{-\pi/4, \pi/2\}$.

By using the same procedure as in Example 4.1 we obtain nodes x_i, $i = 1, 2, \ldots, 7$, and weight coefficients a_i, $i = 1, 2$, A_i, $i = 1, 2, \ldots, 7$, which are given in Table 2.

The obtained quadrature rule integrates exact all trigonometric polynomials from $\mathcal{T}_7$, as it is stated in Section 2. For the integrand $f(x) = \frac{1+\cos x}{\sqrt{5+\cos x}}$, the approximation error is $-3.4168555 \times 10^{-8}$.

Table 2
The nodes y_i, $i = 1, 2$, x_i, $i = 1, 2, \ldots, 7$, and the weight coefficients a_i, $i = 1, 2$, A_i, $i = 1, 2, \ldots, 7$, of the quadrature rule of Gaussian type for trigonometric polynomials (3) for $\gamma = 1/2$ with respect to the weight function $w(x) = \sqrt{1 - (x/\pi)^2}$, $x \in [-\pi, \pi)$, and the preassigned nodes $\{y_1, y_2\} = \{-\pi/4, \pi/2\}$.

i	y_i	a_i
1	$-\pi/4$	0.7302004920494538
2	$\pi/2$	0.6273402200502432

i	x_i	A_i
1	−3.135385513084978	0.2905157247063272
2	−2.303421145210531	0.5194617256444344
3	−1.542485834689003	0.6613144018791357
4	−0.0393589763330550	0.7312150005887421
5	0.614350240503910	0.4706384077889477
6	0.949619073471995	0.3924742651577427
7	2.315089501751868	0.5116419626796524

Table 3
The nodes y_i, $i = 1, 2$, x_i, $i = 1, 2, \ldots, 7$, and the weight coefficients $a_{m,i}$, $i = 1, 2$, $m = 1, 2$, $A_{m,i}$, $i = 1, 2, \ldots, 7$, $m = 1, 2$, of the optimal set of quadrature rules (6) with respect to $W = (1, 1 + \sin 2x)$, $\mathbf{n} = (2, 1)$ and the preassigned nodes $\{y_1, y_2\} = \{-\pi/5, \pi/4\}$.

i	y_i	$a_{1,i}$	$a_{2,i}$
1	$-\pi/5$	0.800736958704361	0.039190856290215
2	$\pi/4$	2.36665196898271	4.73330393796541

i	x_i	$A_{1,i}$	$A_{2,i}$
1	−3.005941674854158	0.790889182235539	1.002836409824271
2	−2.215286690772217	0.790702456658771	1.55021336470876
3	−1.423895857757946	0.792535990680536	0.563023609777931
4	0.199490138020088	0.917236116124360	1.273563004601619
5	0.701822851834176	−1.77450396222122	−3.52427636030426
6	1.687633606477333	0.805514084705667	0.618994260528472
7	2.485381300257828	0.793422511308864	0.0263362237871699

Example 4.3. Let us construct the optimal set of quadrature rules for trigonometric polynomials (6) on $E = [-\pi, \pi)$, for $r = 2$, multi–index $\mathbf{n} = (2, 1)$, with respect to the weight functions $w_1(x) = 1$ and $w_2(x) = 1 + \sin 2x$ and the preassigned nodes $\{y_1, y_2\} = \{-\pi/5, \pi/4\}$.

It is easy to see that $W = (w_1, w_2)$ is TT system for multi-index $\mathbf{n}$, as well as $\widetilde{W} = (\widetilde{w}_1, \widetilde{w}_2)$, where

$$\widetilde{w}_1(x) = \sin\left(\frac{x + \pi/5}{2}\right) \sin\left(\frac{x - \pi/4}{2}\right),$$

$$\widetilde{w}_2(x) = \sin\left(\frac{x + \pi/5}{2}\right) \sin\left(\frac{x - \pi/4}{2}\right) (1 + \sin 2x).$$

First, we obtain the type II multiple orthogonal trigonometric polynomial of semi-integer degree $T_3^{1/2}(x)$, with respect to the $\widetilde{W}$, whose zeros x_i, $i = 1, 2, \ldots, 7$, are nodes of the optimal set of quadrature rules (6). By requiring that quadrature rule integrates exactly trigonometric polynomials from $\mathcal{T}_4$, with respect to the weight functions w_m, $m = 1, 2$, we obtain the weight coefficients. The obtained nodes x_i, $i = 1, 2, \ldots, 7$, and the weight coefficients $a_{m,i}$, $i = 1, 2$, $m = 1, 2$, $A_{m,i}$, $i = 1, 2, \ldots, 7$, $m = 1, 2$, are given in Table 3.

Let us notice that the obtained optimal set of quadrature rules integrates exactly all trigonometric polynomials from $\mathcal{T}_6$, with respect to the weight function w_1, and all trigonometric polynomials from $\mathcal{T}_5$, with respect to the weight function w_2, as it is stated in Section 3. For the integrand $f(x) = \frac{1+\cos x}{\sqrt{3-\cos x}}$, the approximation error is $2.21773133 \times 10^{-6}$, with respect to the weight function w_1, and 0.00009692686982, with respect to the weight function w_2. The results are good, although the number of nodes is small.

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