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Asymmetric Fixed-effects Models for Panel Data

Paul D. Allison[1] iD

Abstract
Standard fixed-effects methods presume that effects of variables are symmetric: The effect of increasing a variable is the same as the effect of decreasing that variable but in the opposite direction. This is implausible for many social phenomena. York and Light showed how to estimate asymmetric models by estimating first-difference regressions in which the difference scores for the predictors are decomposed into positive and negative changes. In this article, I show that there are several aspects of their method that need improvement. I also develop a data-generating model that justifies the first-difference method but can be applied in more general settings. In particular, it can be used to construct asymmetric logistic regression models.

Keywords
causal inference, first difference, fixed effects, generalized least squares, panel data

Introduction

In his 1987 book *Making It Count*, Stanley Lieberson devoted a whole chapter to arguing against the nearly universal presumption that causal effects are symmetric. What he meant is that for both theorists and data analysts, there is usually an implicit assumption that if a one-unit increase in variable X produces a change of B units in variable Y, then a one-unit *decrease* in X will result in a change of $-B$ units in Y. Essentially, this is an assumption that causal effects are exactly reversible.

Lieberson (1987) persuasively argued that there are strong reasons to suspect that many social and psychological phenomena do not work this way. Is the increase in happiness when a person gets married exactly matched by the decrease in happiness when that same person gets divorced? Is the effect of imprisonment on physical health canceled out by the effect of release from prison? Does a 10 percent increase in income have the same effect on savings as a 10 percent decrease in income (in the opposite direction)? On the face of it, most people would probably answer no. But these are empirical questions, and unfortunately, good methodological tools for answering such questions have generally been lacking.

There have been several efforts to develop methods for testing asymmetric hypotheses, but most of those efforts have focused on the distinction between necessary and sufficient conditions and have relied primarily on cross-sectional data. Rosenberg, Knuppe, and Braumoeller (2017) provide a fairly comprehensive review of this literature. In contrast, I take it as axiomatic that longitudinal data are essential to capture Lieberson's (1987) notion of asymmetry. We need to see what happens when X goes up and compare that with what happens when X goes down. Chamlin and Cochran (1998) did address asymmetry with an autoregressive integrated moving average model applied to bivariate time series data on oil prices and burglaries. However, their approach was simply to separate the time series into one long period in which the price of oil was increasing followed by another long period in which the price was decreasing. That method might be adequate for some applications, but it has little generality.

A more general solution was proposed by York and Light (2017), who showed how to estimate asymmetric models for panel data by

(1) calculating first differences for all time varying variables, both dependent and independent;
(2) decomposing each difference score into positive and negative components; and
(3) estimating a first-difference regression model using the decomposed variables instead of the original ones.

[1]Statistical Horizons LLC

Corresponding Author:
Paul D. Allison, Statistical Horizons LLC, P.O. Box 282, Ardmore, PA 19003, USA.
Email: allison@statisticalhorizons.com

Their method has since been applied to panel data on countries to investigate the relationship between economic development and carbon dioxide emissions (Huang and Jorgenson 2018; McGee and York 2018) and to panel data on U.S. counties to study the effects of population change on land development (Clement and York 2017).

I believe that York and Light (2017) have hit on a method that is essentially correct as a way to implement Lieberson's (1987) vision. However, their proposal is marred by a few problems:

1. They don't explain how their method is related to standard methods for analyzing panel data,
2. Their estimation method has some unnecessary peculiarities whose implications are not fully understood or explained,
3. Their method does not satisfactorily address the correlation structure that is typical of multiperiod data that are first differenced, and
4. They do not present a data-generating model whose parameters are estimable by their method.

The first goal of this article is to fix these four problems. The second goal is to extend the method to logistic regression for categorical dependent variables. Computations for the examples were done using SAS, Stata, and R. Online supplementary files contain all the code used to produce the tables. Data sets are available at https://statisticalhorizons.com/resources/data-sets.

Symmetric Fixed Effects and First Differencing for Two-period Data

Let's first consider traditional symmetric models for panel data. I begin with the case in which individuals are observed at only two points in time, $t = 1$ and $t = 2$. Things are much clearer and less complicated in this situation. For each individual i at each time t, we observe only two variables, Y_{it} and X_{it}. The extension to multiple Xs is straightforward.

I'll begin with a linear model that is a common starting point for the analysis of panel data:

$$\begin{aligned}Y_{i1} &= \mu_1 + \beta X_{i1} + \alpha_i + \varepsilon_{i1}\\ Y_{i2} &= \mu_2 + \beta X_{i2} + \alpha_i + \varepsilon_{i2}.\end{aligned} \tag{1}$$

In this model, μ_1 and μ_2 are different intercepts that allow for a change over time that is unrelated to X. The effect of X on Y is β, which is assumed to be the same at both time points. The ε_{it}s are random error terms that are specific to each time point and assumed to be independent of anything else on the right-hand side. Finally, α_i is an unobserved variable that represents the combined effects on Y of all variables that are specific to the individual but that do not change over time.

The α_i term can be treated as a set of random variables (a random-effects model) or as a set of fixed constants (a fixed-effects model). I'll treat them here as constants because then there is no need to make strong assumptions about them. In particular, the αs may be correlated with the Xs, which implies that estimates of β control, in a fundamental way, for all time-invariant predictors both observed and unobserved. In a fixed-effects model, only within-individual variation is used to estimate the parameters, which makes them particularly appropriate for examining the differential consequences of increases and decreases in X. For further discussion of this point, see Allison (2009) or Halaby (2004).

For the two-period case, there are several equivalent ways to estimate the fixed-effects model, all producing identical estimates. Here are the three most common methods:

1. *Least squares dummy variables (LSDV)*. In this method, the data are arranged in "long form," with two records per person, one for each time point. Dummy (indicator) variables are created for all individuals (less one). Using ordinary least squares (OLS), Y is regressed on X plus the dummy variables for individuals and a dummy for time. LSDV produces "best linear unbiased" estimates, but it can be computationally intensive when the number of individuals is large.
2. *Mean deviation*. This is the classic fixed-effects method. Again the data are in long form, but for each individual, the mean of the two Y values is calculated, as is the mean of the two X values. These individual-specific means are subtracted from the original values of X and Y for each record to create the mean-deviation variables Y^* and X^*. Then Y^* is regressed on X^* (using OLS), along with a dummy variable for time. This produces exactly the same estimate of β as LSDV, but the computations are much faster. However, it does require an adjustment of the standard errors and degrees of freedom. Many statistical packages can automate all these steps (e.g., the **xtreg** command in Stata, the GLM procedure in SAS, or the **plm** function in R).
3. *First difference*. This is the simplest method in the two-period case. For equation 1, subtract the time 1 equation from the time 2 equation to produce

$$Y_{i2} - Y_{i1} = (\mu_2 - \mu_1) + \beta(X_{i2} - X_{i1}) + (\varepsilon_{i2} - \varepsilon_{i1}). \tag{2}$$

Notice that the α_is now have been "differenced out" of the equation. To estimate β, simply create the two difference scores (using data in the "wide form"), and then regress the Y difference on the X difference using OLS. No standard error adjustment is needed.

For example, 581 children were studied in 1990 and 1992 as part of the National Longitudinal Survey of Youth (Center for Human Resource Research 2002). I'll use three variables that were measured at each of the time points:

ANTI   antisocial behavior, measured on a scale from 0 to 6.

SELF   self-esteem, measured on a scale from 6 to 24.

POV   poverty status of family, coded 1 for family in poverty, otherwise 0.

The goal is to estimate the regression of ANTI on SELF and POV, corresponding to equation 1. Although all three methods described above produce exactly the same results, I'll focus on the first-difference method. Thus, using data in the wide form, I created new variables:

ANTIDIFF = ANTI2 − ANTI1

SELFDIFF = SELF2 − SELF1

POVDIFF = POV2 − POV1

Let's take a look at the distributions of these difference scores. Figures 1 and 2 display histograms for ANTIDIFF and SELFDIFF. Both graphs roughly resemble normal distributions centered on 0. This is nice but a little surprising given that the original variables are highly skewed—ANTI to the right (most children are low on antisocial behavior) and SELF to the left (most children are high on self-esteem).

Because POV is a dummy variable, POVDIFF has only three values, with the distribution shown in Table 1. During the two-year period, 45 children moved into poverty, 49 children moved out of poverty, and the remainder (83 percent) stayed the same.

Table 2 shows the results of a least squares regression of ANTIDIFF on SELFDIFF and POVDIFF. Exactly the same results can be obtained with the LSDV method or the mean-deviation method, so there's no need to display them separately. The POVDIFF coefficient of .197 is clearly not statistically significant by any conventional criterion. On the other hand, the SELFDIFF coefficient of −.039 is highly significant. For both coefficients, the interpretation is symmetric. Thus, a 1-unit increase in self-esteem is associated with a .039 decrease in antisocial behavior, while a 1-unit decrease in self-esteem is associated with a .039 increase in antisocial behavior. For poverty, moving into poverty increases antisocial behavior by .197 units, while moving out of poverty decreases antisocial behavior by .197 units.

An Asymmetric Model for Two-period Data

Now let's modify the first-difference model to allow effects to be asymmetric. The key insight of York and Light (2017) was that difference scores for predictor variables can be
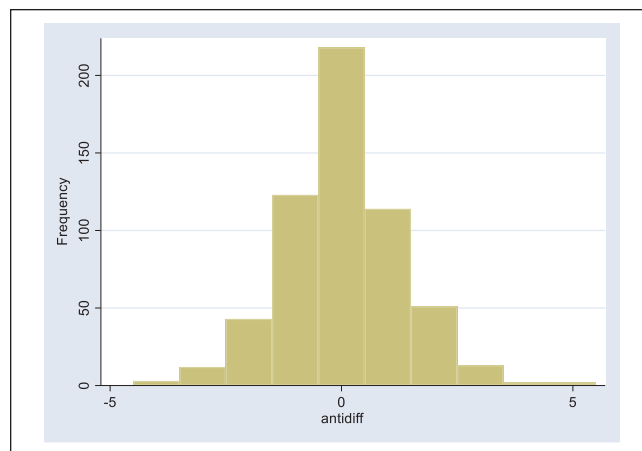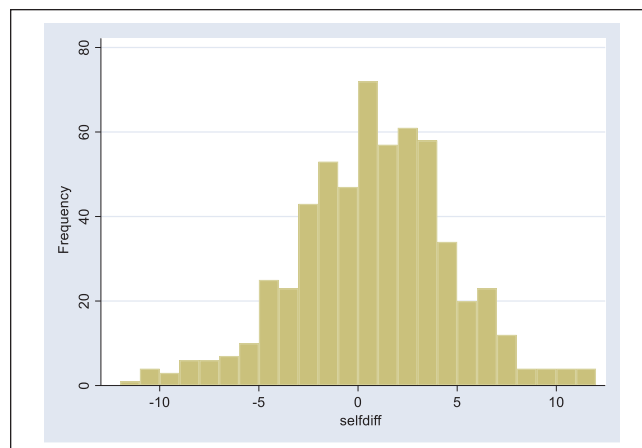

Figure 1. Histogram for ANTIDIFF.

Figure 2. Histogram for SELFDIFF.

Table 1. Frequency Distribution for POVDIFF.

Value	Frequency	Percentage
−1	49	8.4
0	487	83.8
1	45	7.7
Total	581	100.0

Table 2. Symmetric Regression of Antisocial Behavior on Self-esteem and Poverty.

Predictor	Estimate	*SE*	*z*
SELFDIFF	−.039	.014	−2.87
POVDIFF	.197	.133	1.48

decomposed into positive and negative components. Specifically, I define

$$X_i^+ = X_{i2} - X_{i1} \text{ if } (X_{i2} - X_{i1}) > 0, \text{ otherwise } 0,$$
$$X_i^- = -(X_{i2} - X_{i1}) \text{ if } (X_{i2} - X_{i1}) < 0, \text{ otherwise } 0.$$

Both of these variables are never negative, but the first represents an increase, and the second represents a decrease. It easily can be shown that

$$X_i^+ - X_i^- = X_{i2} - X_{i1}, \tag{3}$$

a result that will be used in Appendix B. The working model then becomes

$$Y_{i2} - Y_{i1} = (\mu_2 - \mu_1) + \beta^+ X_i^+ + \beta^- X_i^- + (\varepsilon_{i2} - \varepsilon_{i1}). \tag{4}$$

Thus, β^+ represents the change in Y for a one-unit increase in X, and β^- represents the change in Y for a one-unit decrease in X. Typically, β^+ and β^- will have opposite signs, although that won't always happen. When $\beta^+ = -\beta^-$, then we are back to the original symmetric model.

Note that when X is a dummy variable, like POV, the decomposed model has two dummy variables:

$X^+ = 1$ if X changed from 0 to 1, otherwise 0.

$X^- = 1$ if X changed from 1 to 0, otherwise 0.

The implicit reference category is no change from time 1 to time 2.

Now let's estimate this model for the National Longitudinal Survey of Youth data using OLS with decomposed changes in self-esteem and poverty. Creating the necessary variables is reasonably straightforward in most software. Here, for example, is the Stata code to read in the data and generate the needed variables:

```
use https://statisticalhorizons.com/wp-content/uploads/nlsy.dta,
clear

generate antidiff = anti92 − anti90
generate selfdiff = self92 − self90
generate povdiff = pov92− pov90
generate selfpos = selfdiff * (selfdiff>0)
generate selfneg = –selfdiff * (selfdiff<0)
generate povpos = povdiff * (povdiff>0)
generate povneg = –povdiff * (povdiff<0)
```

Table 3 displays the results for regressing the difference score for antisocial behavior on the decomposed difference scores for self-esteem and poverty. The coefficient for SELFPOS is small and far from statistically significant, indicating that increases in self-esteem have no detectable effect on antisocial behavior. On the other hand, a decrease in self-esteem produces an increase in antisocial behavior that is much larger in magnitude (.074) and highly significant. Neither of the poverty coefficients is statistically significant, which is consistent with the overall lack of effect in the symmetric model. Nevertheless, the coefficients are in the expected direction: moving into

Table 3. Asymmetric Regression of Antisocial Behavior on Self-esteem and Poverty.

Predictor	Estimate	SE	z
SELFPOS	−.005	.025	−0.19
SELFNEG	.074	.026	2.90
POVPOS	.250	.200	1.25
POVNEG	−.126	.192	−0.66

poverty increases antisocial behavior while moving out of poverty reduces antisocial behavior.

The numbers in Table 3 suggest that the effect of a decrease in self-esteem is much larger than the effect of an increase in that variable. But is the difference statistically significant? To answer that question, one must test whether $\beta^+ = -\beta^-$. Most statistical packages have simple commands for doing this. For example, in Stata, one can simply enter

```
test selfpos = –selfneg
```

That produced an F statistic with a p value of .106, not quite statistically significant. So there's no compelling evidence that the effects of self-esteem are asymmetric.

Symmetric Estimation for Multiperiod Data

As we shall see in the next section, the decomposition of first-difference predictors into positive and negative components can be easily extended to panel data with more than two periods. Before going there, however, it's essential to address some critical issues regarding the estimation of *symmetric* models with the first-difference approach. This section is a rather extended digression from the topic of asymmetry but, unfortunately, a necessary one.

To explain these issues, a new example is needed. I'll use a subsample of the data described in Firebaugh, Warner, and Massoglia (2013). For 1,134 respondents, five waves of survey data were collected annually from 1983 to 1987, when the respondents were ages 18 through 22. Here are the variables measured in each of the five years:

LWAGE   logged hourly wage for current or most recent job

MAR   1 if married, otherwise 0

EDU   years of education

URB   1 if living in an urban area, otherwise 0

Logged hourly wage rate will be the dependent variable for the example. For simplicity, observations with any missing data on these four variables have been excluded.

For the two-period case, we saw in the Symmetric Fixed Effects and First Differencing for Two-period Data section that there are three methods for estimating a symmetric fixed-effects model that produce identical results: LSDV, mean deviation, and first differencing. These three methods also can be used when there are more than two time points, but now there is a potential divergence. LSDV and mean deviation still produce identical results. With first differencing, on the other hand, there are several possible estimation methods. Those that are most commonly used will not produce the same estimates as LSDV and mean deviation. However, I will describe and demonstrate a first-difference estimation method that *does* produce estimates that are identical to those produced by mean deviation and LSDV.

The basic model for multiperiod data is a straightforward extension of the model for two-period data.

$$Y_{it} = \mu_t + \beta X_{it} + \alpha_i + \varepsilon_{it} \quad i = 1,\dots,n; t = 1,\dots,T. \quad (5)$$

The intercept, μ_t, is allowed to differ at each point in time. X_{it} can be regarded either as a single variable or as a vector of variables. As before, α_i represents the combined effect of all variables that do not vary with time, regardless of whether they are observed or unobserved. To make this a fixed-effects model, α_i is treated as a set of constants. Finally, the ε_{it} are random errors that are assumed to be independent of X and α, to be independent of each other, and to have a common variance.

Under these assumptions, the mean-deviation method (or the equivalent LSDV method) yields estimates that are best linear unbiased. As a baseline for comparison, I'll apply the mean-deviation method to the wage example. The first step is to put the data into long form, yielding 5,670 records, 5 per person. As in the two-period case, the mean is calculated for each variable for each person. These person-specific means are subtracted from the original values of each variable to create deviation scores. The final step is to do OLS regression with the deviation scores. As previously noted, these steps are automated in many statistical packages. Results are shown in Table 4, with robust standard errors to compensate for any specification errors.

Because the dependent variable is logged and the coefficients are all less than .10 in magnitude, they can be interpreted in terms of percentage changes. For example, the .04 coefficient for marital status says that the effect of getting married is a 4 percent increase in wages. And since the model is symmetric, the effect of getting divorced is a 4 percent *decrease* in wages. Each additional year of schooling produces a 5.7 percent increase in wages. The effect of moving from a nonurban area to an urban area is an 8 percent increase in wages, while moving from a nonurban to an urban area yields an 8 percent decrease.

Now let's estimate the same symmetric model using first differences. Equation 5 implies the working model,

$$Y_{it} - Y_{it-1} = (\mu_t - \mu_{t-1}) + \beta(X_{i1} - X_{it-1}) + (\varepsilon_{it} - \varepsilon_{it-1}) \\ t = 2,\dots,T. \quad (6)$$

Again, we see that the α_is have been "differenced out" of the equation. Estimation of this equation is a fairly natural extension of the difference score method for two-period data, except that there are now multiple records for each person. With five-period data, there are four records for each person. The first record contains difference scores for periods 1 and 2. The second record contains difference scores for periods 2 and 3, and so on. Pooling all these records into a single data set, we then estimate the regression of the *y* difference score on the *x* difference scores.

Unfortunately, there are differing opinions about exactly how to estimate the difference score regression. Wooldridge (2010) proposed using OLS, with robust standard errors to correct for heteroscedasticity and for correlations among the errors. Under the assumptions stated for equation 5, he showed that OLS produces unbiased estimators of the regression coefficients. However, OLS estimators are less efficient than the ones produced by LSDV or mean deviation and, therefore, have larger true standard errors.

York and Light (2017), on the other hand, applied a fixed-effects regression to the difference scores (using the mean-deviation method), along with robust standard errors to correct for heteroscedasticity and correlated errors. Their use of a fixed-effects estimator is peculiar because the effects of any time-invariant variables are already removed by first differencing. A fixed-effects model for the difference scores is equivalent to a model that says that the effect of time is linear with a slope that is unique to each individual. Although there is nothing intrinsically wrong with such a model, it goes well beyond what most people want to achieve when they do fixed effects. Moreover, the robustness that may be gained by this approach comes at a cost of larger standard errors. Finally, any attractions of this method are completely unrelated to the estimation of asymmetric effects.

As a possible alternative, York and Light (2017) suggested the use of a random-effects estimator. But a random-effects model would allow for only positive correlations among the multiple records for each individual. In actuality, the most salient feature of first-difference data is a *negative* correlation between the error terms of adjacent difference scores.

Why is there a negative correlation? Consider the error terms in the difference regressions, $(\varepsilon_3 - \varepsilon_2)$ and $(\varepsilon_2 - \varepsilon_1)$. Clearly, whenever ε_2 is high, the first error term will tend to be low while the second will tend to be high. More specifically, it can be shown that, under the assumption that the ε_{it} are uncorrelated with each other and are homoscedastic, the correlation between any two adjacent error terms is $-.50$

Table 4. Symmetric Fixed-effects Model with Five-period Data Using Mean Deviation.

Predictor	Estimate	SE	z
MAR	.042	.023	1.83
EDU	.057	.016	3.61
URB	.081	.030	2.74
Year			
2	.031	.014	2.25
3	.103	.014	7.19
4	.166	.016	10.23
5	.231	.018	12.58

Table 5. Symmetric First-difference Model Estimated with Ordinary Least Squares and with Fixed Effects.

Predictor	Ordinary Least Squares			Fixed Effects		
	Estimate	SE	z	Estimate	SE	z
MARDIFF	.019	.026	0.74	.013	.029	0.46
EDUDIFF	−.021	.025	−0.83	−.126	.043	−2.91
URBDIFF	.076	.032	2.37	.075	.035	2.11
Year						
3	.037	.023	1.64	.032	.023	1.42
4	.025	.019	1.31	.016	.019	0.82
5	.026	.019	1.38	.014	.019	0.74

(see Appendix A). Although robust standard errors adjust for such correlations, they do nothing to address statistical inefficiency of the coefficient estimates themselves.

To accommodate this, I propose the use of generalized least squares (GLS) to estimate equation 6. Unlike the other difference score methods, GLS can produce estimates that are unbiased and efficient. As we will see, the choice among these methods can make a big difference.

I now apply these three first-difference methods to estimate a symmetric model for the wage example. The left panel in Table 5 displays results from an OLS regression with robust standard errors (Wooldridge's 2010 method). Ideally, we would expect these results to be similar to those obtained with the classic fixed-effects results in Table 4.

Although the coefficient for urban residence is pretty close to the one in Table 4, marital status and education have coefficients that are much smaller than those in Table 4 and *p* values that are much larger. (There's no reason to expect the year coefficients to be similar because the coefficients in Table 4 estimate differences between years while the coefficients in Table 5 estimate differences in differences between years.)

The right panel of Table 5 shows the results from applying the fixed-effects method used by York and Light (2017). Now the results are even more dramatically different from those produced by classic fixed effects in Table 4. Although the urban coefficient is still about the same (but with a

substantially larger standard error), the education coefficient is now −.126, which says that each additional year of schooling yields a 12.6 percent *decrease* in wages. That's a hard estimate to swallow.

I believe that GLS is a better way to go than either of these two methods. With GLS, it's necessary to specify or estimate the variances of the error terms and the correlations among them. With four records per person, there are six correlations and four variances. We can estimate an unstructured model that estimates all of these additional parameters—and there's something to be said for that. Or we can impose some appropriate structure on the variances and correlations.

Let's first try an unstructured model, estimated by maximum likelihood with robust standard errors (using Stata's **mixed** command or PROC MIXED in SAS). Results in the left panel of Table 6 are reasonably close to those for classic fixed effects in Table 4, although the coefficients are a little smaller and the *p* values are a little larger.

In the lower portion of this panel, we see estimates for the standard deviations of the four error terms (which are all quite similar), along with estimates of the six correlations. The three correlations between error terms that are adjacent in time are negative and quite large: −.50, −.33, and −.38. On the other hand, the correlations between error terms that are not adjacent in time (and therefore do not share a common component) are quite small, and their 95 percent confidence intervals (not shown) all include 0.

The good thing about the unstructured model is that it allows for departures from the standard assumptions. In particular, it allows error variances that change over time as well as for serial correlations among the ε_{it} in the original model of equation 5. On the other hand, if we want a method that's equivalent to the mean-deviation method, we should impose the following constraints: (1) all error standard deviations are the same, (2) all correlations among adjacent error terms are −.50, and (3) all correlations among nonadjacent error terms are 0. These constraints can be imposed with PROC MIXED in SAS and with the gls function in R (as shown in the online supplements). Results are identical to those produced by the mean-deviation method in Table 4.

Stata can't fully achieve these constraints. Its Toeplitz 1 model constrains the error standard deviations to be the same and the correlations between nonadjacent error terms to be 0. It also constrains all the correlations between adjacent error terms to be the same, but it can't constrain them to be exactly −.50. The right panel of Table 6 displays results from fitting this model to the wage data using the **mixed** command in Stata. These estimates are quite similar to those for the unstructured model. The estimated correlation between adjacent error terms is −.42, not quite the −.50 that was specified in SAS or in R.

Keep in mind that all three of these GLS models should produce estimates that are consistent (and, hence, approximately unbiased). The attraction of the more parsimonious

Table 6. Symmetric First Difference with Generalized Least Squares, Unstructured and Toeplitz 1.

	Unstructured				Toeplitz 1		
Parameter	Estimate	SE	z	Parameter	Estimate	SE	z
MARDIFF	.035	.022	1.57	MARDIFF	.037	.022	1.63
EDUDIFF	.037	.016	2.33	EDUDIFF	.038	.016	2.29
URBDIFF	.075	.028	2.67	URBDIFF	.082	.029	2.82
Year				Year			
3	.040	.023	1.75	3	.040	.023	1.76
4	.029	.019	1.56	4	.030	.019	1.57
5	.032	.020	1.71	5	.032	.019	1.69
sd(e2)	.456	.022		corr(e_t, e_{t+1})	−.418	.015	
sd(e3)	.441	.023		sd(e)	.446	.016	
sd(e4)	.424	.025					
sd(e5)	.456	.029					
corr(e2, e3)	−.496	.046					
corr(e2, e4)	−.043	.035					
corr(e2, e5)	.022	.036					
corr(e3, e4)	−.326	.052					
corr(e3, e5)	−.060	.039					
corr(e4, e5)	−.376	.057					

models is that they also should have smaller true standard errors.

The decision to use an unstructured or a structured model will depend in part on *T*, the number of time points. In an unstructured model, the number of error variances and correlations that must be estimated is $T(T+1)/2$. If there are 20 time points, that's 210 extra parameters! On the other hand, the Toeplitz 1 structure only has one error variance and one correlation, and the fully constrained model estimates only a single error variance. So when the number of time points is small, the unstructured model will be fine, especially when the number of individuals is large. On the other hand, when there are many time points, the Toeplitz 1 model or the fully constrained model will be strongly preferred.

Asymmetric Estimation for Multiperiod Data

We are finally ready to estimate an asymmetric first-difference model for multiperiod data using GLS. Once the data are in the long form, the decomposed difference scores for each data record can be defined in exactly the same way that we did it for two-period data. That is, for $t = 2, \ldots, T$.

$$X_{it}^+ = X_{it} - X_{it-1} \text{ if } (X_{it} - X_{it-1}) > 0, \text{ otherwise } 0,$$
$$X_{it}^- = -(X_{it} - X_{it-1}) \text{ if } (X_{it} - X_{it-1}) < 0, \text{ otherwise } 0.$$

The working model then becomes

$$Y_{it} - Y_{it-1} = (\mu_t - \mu_{t-1}) + \beta^+ X_{it}^+ + \beta^- X_{it}^- + (\varepsilon_{it} - \varepsilon_{it-1}), \quad (7)$$

which can be estimated by GLS. One important issue is that it's not obvious what sort of model for Y_{it} would imply this model for the difference scores. I will address that issue in the next section.

For the wage data, we can't decompose the difference scores for education because that variable only increases—you can't experience a decrease in years of schooling. We'll allow for asymmetry in the effect of marital status because there are definitely changes in both directions, as shown in Table 7. Thus, there were 324 entries into marriage and 117 exits from marriage. Similarly, for URBDIFF, there were 190 moves from nonurban areas to urban areas and 158 moves from urban areas to nonurban areas.

Table 8 displays results for a fully constrained GLS model (estimated with PROC MIXED in SAS). The coefficients are all in the expected direction. Getting married increases wages by 3.2 percent. Leaving a marriage decreases wages by 6.7 percent. But neither of these coefficients is significantly different from 0, and they are not significantly different from each other. Moving to an urban area increases wages by 8 percent. Leaving an urban area reduces wages by 8 percent. Both of these effects are significantly different from 0, but they are clearly not different from each other. Thus, there is no evidence for asymmetry in this example.

An Asymmetric Linear Model for Multiperiod Data

We now have a pretty good method for estimating asymmetric fixed-effects models using first differences. This method is reasonably easy to implement, is intuitively appealing, and seems to produce plausible results. What we don't have (yet) is a model for the data-generating process that would imply

Table 7. Frequency Distributions for MARDIFF and URBDIFF.

	MARDIFF		URBDIFF	
Value	Frequency	Percentage	Frequency	Percentage
−1	117	2.6	158	3.5
0	4,095	90.3	4,188	92.3
1	324	7.1	190	4.2
Total	4,536	100.0	4,536	100.0

Table 8. Asymmetric First-difference Model Estimated with Fully Constrained Generalized Least Squares.

Parameter	Estimate	SE	z
MARPOS	.032	.023	1.40
MARNEG	−.067	.048	−1.41
EDUDIFF	.056	.016	3.57
URBPOS	.081	.034	2.37
URBNEG	−.080	.036	−2.21
t2	−.035	.019	−1.81
t3	.007	.020	0.35
t4	−.003	.022	−0.15
t5	0		

the first-difference equations whose parameters we have been estimating. Specifically, we need a set of equations with individual Y_{it}s on the left-hand side instead of difference scores.

In this section I propose a simple model that leads to the difference equations used in previous sections. This model has several benefits:

1. It enriches our understanding of what sort of process would lead to asymmetric effects.
2. It provides a platform for generating simulated data that can be used evaluate the performance of alternative estimators.
3. It makes it possible to estimate an asymmetric model using the LSDV or mean-deviation methods rather than the first-difference method.
4. Most important, it makes it possible to generalize the model to handle categorical dependent variables.

Here is the model. We observe Y_{it} and X_{it} for $t = 1, \ldots, T$. Although X_{it} is a single variable, the extension to multiple variables should be straightforward. As in earlier sections, for $t = 2, \ldots, T$, define

$$X_{it}^{+} = X_{it} - X_{it-1} \text{ if } (X_{it} - X_{it-1}) > 0, \text{ otherwise } 0,$$
$$X_{it}^{-} = -(X_{it} - X_{it-1}) \text{ if } (X_{it} - X_{it-1}) < 0, \text{ otherwise } 0.$$

For $t = 1$, in which case X_{it-1} is not observed, both X_{it}^{+} and X_{it}^{-} are set to 0. Now define

$$Z_{it}^{+} = \sum_{s=1}^{t} X_{is}^{+} \tag{8}$$

$$Z_{it}^{-} = \sum_{s=1}^{t} X_{is}^{-}.$$

Thus, Z^{+} is the accumulation up to time t of all previous positive changes in X, and Z^{-} is the accumulation of all previous negative changes in X. When X is a dummy variable, Z^{+} is just the number of previous changes from 0 to 1, and Z^{-} is the number of previous changes from 1 to 0. For example, Z^{+} might be the number of previous marriages, and Z^{-} might be the number of previous divorces.

The model is simply

$$Y_{it} = \mu_{t} + \beta^{+} Z_{it}^{+} + \beta^{-} Z_{it}^{-} + \alpha_{i} + \varepsilon_{it}, \qquad t = 1, \ldots, T, \tag{9}$$

where α and ε satisfy the usual assumptions for a fixed-effects regression model.

I now show that this model implies the difference score model used in previous sections. If we construct difference scores based on this model, we have

$$Y_{it} - Y_{it-1} = (\mu_{t} - \mu_{t-1}) + \beta^{+}(Z_{it}^{+} - Z_{it-1}^{+}) + \\ \beta^{-}(Z_{it}^{-} - Z_{it-1}^{-}) + (\varepsilon_{it} - \varepsilon_{it-1}). \tag{10}$$

But by definition,

$$Z_{it}^{+} - Z_{it-1}^{+} = \sum_{s=1}^{t} X_{is}^{+} - \sum_{s=1}^{t-1} X_{is}^{+} \\ = X_{it}^{+}. \tag{11}$$

Similarly,

$$Z_{it}^{-} - Z_{it-1}^{-} = X_{it}^{-}.$$

So we end up with

$$Y_{it} - Y_{it-1} = (\mu_{t} - \mu_{t-1}) + \beta^{+} X_{it}^{+} + \mu^{-} X_{it}^{-} + (\varepsilon_{it} - \varepsilon_{it-1}),$$

which is just the working first-difference model in equation 7. I can't claim that this is the *only* data-generation model that would imply the difference-score model. But I have been unable to find any other.

One surprising implication of this model is that Y_{it} depends on the entire previous history of changes in X. That raises two issues:

1. We don't know the history of X prior to time 1. That's not a problem, however, because that history does not vary over the observed time periods. Therefore, it gets absorbed into α_{i}, which is removed by first differencing or otherwise adjusted for by standard fixed-effects methods.

2. What happens when the effects of *X* are symmetric, that is, $\beta^+ = -\beta^-$? In that case, it can be shown (see Appendix B) that

$$\beta^+ Z_{it}^+ + \beta^- Z_{it}^- = \beta^+ X_{it},$$

and we are back to a model in which *Y* depends only on the current value of *X*.

We are now in a position to estimate an asymmetric model using the mean-deviation method or the LSDV method rather than first differences. The only tricky part is the programming needed to compute the *Z* variables, that is, the accumulated positive and negative changes for each predictor variable. See the supplementary files for the details using SAS, Stata, or R. After constructing appropriate *Z* variables, I applied the mean-deviation method to the five-period example of the Asymmetric Estimation for Multiperiod Data section. As anticipated, results were *identical* to those in Table 8 that were obtained by applying the fully constrained GLS method to the first-difference data.

Is there any advantage to doing it this way rather than using the first-difference method? Well, the mean-deviation method is more familiar than first differencing to most researchers who use fixed effects. However, in my experience, the programming for asymmetric models is generally easier with the first-difference method.

One benefit of the mean-deviation method is that it may be better for certain patterns of missing data, specifically, if data are always observed for predictor variables but missing for some values of the dependent variable. For example, suppose that in the three-period case, some people have missing data on *Y* at time 2. With the first-difference method, those people would be lost entirely because the middle value of *Y* is needed to calculate both of the difference scores. With the mean-deviation method, however, they could still contribute observations for times 1 and 3.

Another big payoff from this alternative approach is that it is easily extended to categorical data. I'll show how to do that in the next section. A final attraction is that this approach can be incorporated into the structural equation model panel methods proposed by Allison, Williams, and Moral-Benito (2017), which make it possible to relax many of the hidden constraints of conventional fixed-effects models. For example, instead of assuming that the unobserved individual component α_i has the same effect at all points in time, one can allow it to have different effects at different time points.

An Asymmetric Logistic Model for Multiperiod Dichotomous Data

Suppose the dependent variable Y_{it} is dichotomous with values of 1 and 0. For many well-known reasons, a linear model for *Y* is not very attractive (Allison 2012). Instead, let's consider an asymmetric logistic regression model. Letting p_{it} be the probability that $Y_{it} = 1$, the model is

$$\log\left(\frac{p_{it}}{1-p_{it}}\right) = \mu_t + \beta^+ Z_{it}^+ + \beta^- Z_{it}^- + \alpha_i, \qquad t = 1,\dots,T,$$

where the *Z*s are the accumulated positive and negative difference scores as defined in the An Asymmetric Linear Model for Multiperiod Data section. As in previous sections, the α_is are treated as constants that represent the combined effects on *Y* of all time-invariant variables.

Clearly, computing difference scores for the dependent variable would not work for this model. The standard method for estimating fixed-effects logistic regression models is conditional maximum likelihood (Allison 2009), which removes the α_is by conditioning each individual's likelihood on the total number of 1s and 0s observed for that individual. We'll apply this method to the following example.

The sample consists of 1,151 teenage girls who were interviewed annually for five years beginning in 1979. The dependent variable is POV, which for each year equals 1 if the girl's household was in poverty, otherwise 0. The following predictor variables also are measured in each of the five years:

MOTHER 1 if respondent currently had a least one child, otherwise 0

SPOUSE 1 if respondent currently lived with a husband, otherwise 0

INSCHOOL 1 if respondent was currently enrolled in school, otherwise 0

HOURS hours worked during the week of the survey

I'll begin with a symmetric model that includes year as a categorical variable. Table 9 displays the estimates, which were obtained with the **clogit** command in Stata. All the predictors have coefficients that are significantly different from 0. When girls are currently enrolled in school or if they have children, they are more likely to be in poverty. When they are living with a husband or work more hours, they are less likely to be in poverty. Keep in mind that these are within-person estimates, so they represent the effects of changing from one condition to another.

Now we'll estimate an asymmetric model by doing conditional likelihood with the accumulated first differences of the predictors. Note that we cannot decompose the effect of MOTHER because it only increases from 0 to 1, never from 1 to 0. Results in Table 10 show a slightly larger effect of MOTHER than results in Table 9. For SPOUSE, we see a strong negative effect of getting married on the log-odds of

Table 9. Symmetric Logistic Regression via Conditional Likelihood.

Parameter	Estimate	SE	z
MOTHER	.582	.167	3.49
SPOUSE	−.748	.182	−4.10
INSCHOOL	.272	.120	2.26
HOURS	−.020	.003	−5.83
Year			
2	.332	.096	3.44
3	.335	.108	3.11
4	.433	.118	3.67
5	.402	.136	2.95

Table 10. Asymmetric Logistic Regression via Conditional Likelihood.

Parameter	Estimate	SE	z
MOTHER	.631	.168	3.76
SPOUSEPOS	−.830	.185	−4.48
SPOUSENEG	.071	.375	0.19
INSCHOOLPOS	−.082	.240	−0.34
INSCHOOLNEG	−.309	.127	−2.42
HOURSPOS	−.023	.004	−5.80
HOURSNEG	.015	.004	3.36
Year			
2	.369	.098	3.76
3	.424	.114	3.73
4	.596	.136	4.37
5	.633	.165	3.85

being in poverty but a rather small positive effect of becoming unmarried. Only 48 people got unmarried (the overwhelming majority presumably by divorce), so that might account for the high p value for this coefficient. Nevertheless, the difference in the magnitudes of the positive and negative coefficients is statistically significant at conventional levels ($p = .04$). This is our first example in which there is noteworthy evidence of asymmetry.

The INSCHOOL coefficients are a little surprising. Entering school reduces the odds of poverty by about 27 percent (calculated as $100[e^\beta - 1]$), but leaving school also reduces the odds, by about 8 percent. The latter figure is far from statistically significant, so the negative coefficient could be just random error. But it does serve to point out that there is no guarantee that changes in opposite directions for the predictors also have opposite sign.

For hours worked, the coefficients do have the expected opposite signs and are both highly significant. An increase of one hour reduces the odds of poverty by 2.3 percent, while a decrease of one hour increases the odds of poverty by about 1.5 percent. However, the difference between these two magnitudes is not quite statistically significant ($p = .12$).

There is one other unexpected finding. Under the asymmetric model, the YEAR effects show a clear ordering (with the odds of poverty steadily increasing) that was not nearly so apparent in the symmetric model.

Although I consider only the dichotomous case here, this method could easily be applied to estimate asymmetric fixed effects for ordered logistic regression, multinomial logistic regression, or negative binomial regression for count data. It's all a matter of appropriately coding the predictor variables.

Discussion

In this article, I have proposed methods to improve and extend the method of York and Light (2017) for estimating asymmetric fixed-effects models for panel data. Fixed-effects models are the natural way to go for asymmetric causal effects because they focus on within-individual change rather than between-individuals differences. That allows us to separate out the differential effects of increases and decreases of the predictor variables.

For linear models, first differencing is the most intuitively appealing and computationally convenient method for estimating asymmetric fixed-effects models. However, for best results, GLS is necessary to handle the negative correlations that naturally arise between adjacent first-difference observations. GLS can produce efficient estimates under a variety of correlation structures that are appropriate both when there are and when there are not correlations among the original, undifferenced error terms.

In the An Asymmetric Linear Model for Multiperiod Data section, I proposed a data-generating model that implies the working equations of the first-difference approach. In this model, Y_{it} is a function of the accumulated positive changes in X and the accumulated negative changes in X during the observed history of the individual. This implies that any given change has an effect that persists over time, unless canceled out (fully or partially) by a change in the opposite direction. To better understand this, suppose that X is a dummy variable for employed versus unemployed. The model says that each entry into employment will raise the expected value of Y (say, income) by a fixed amount. And holding other factors constant, that expected value will persist until employment is terminated. At that point, the expected value of Y will be reduced by a fixed but different amount.

Is this a plausible model? Well, keep in mind that it's less restrictive than the standard fixed-effects model that assumes that the effect of becoming unemployed is just the reverse of the effect of becoming employed. However, there are many applications where it might be more plausible to suppose that the effect of a given change is temporary rather than persistent. Allison (1994) discusses some ways to model this kind

of process, but both the formulation and estimation of such models can become substantially more complicated.

The new data-generating model proposed here makes it possible to estimate asymmetric effects without using first differences of the dependent variable. That's only mildly attractive for linear models, but it's a great boon for estimating asymmetric models in situations where difference scores on the dependent variable are not appropriate. That includes logistic regression models of all kinds as well as models for count data.

Appendix A

Proof that under the assumption that the ε_{it} in equation 5 are uncorrelated with each other and are homoscedastic, the correlation between any two adjacent, first-difference error terms is –.50.

I will prove this for the first three error terms, ε_1, ε_2, and ε_3. The results also apply to any three consecutive error terms. Consider, first, the covariance between the adjacent errors in the difference equations:

$$
\begin{aligned}
\operatorname{cov}(\varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1) &= \operatorname{cov}(\varepsilon_3, \varepsilon_2) - \operatorname{cov}(\varepsilon_3, \varepsilon_1) \\
&\quad - \operatorname{cov}(\varepsilon_2, \varepsilon_2) + \operatorname{cov}(\varepsilon_2, \varepsilon_1) \\
&= -\operatorname{cov}(\varepsilon_2, \varepsilon_2) \\
&= -\operatorname{var}(\varepsilon_2).
\end{aligned}
$$

The correlation is then

$$
\begin{aligned}
\operatorname{corr}(\varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1) &= \frac{\operatorname{cov}(\varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1)}{\sqrt{\operatorname{var}(\varepsilon_3 - \varepsilon_2)\operatorname{var}(\varepsilon_2 - \varepsilon_1)}} \\
&= \frac{-\operatorname{var}(\varepsilon_2)}{\operatorname{var}(\varepsilon_3 - \varepsilon_2)} \\
&= \frac{-\operatorname{var}(\varepsilon_2)}{\operatorname{var}(\varepsilon_3) + \operatorname{var}(\varepsilon_2) - 2\operatorname{cov}(\varepsilon_3, \varepsilon_2)} \\
&= \frac{-\operatorname{var}(\varepsilon_2)}{2\operatorname{var}(\varepsilon_2)} = -\frac{1}{2}.
\end{aligned}
$$

Appendix B

Proof that if $\beta^+ = -\beta^-$, then $\beta^+ Z_{it}^+ + \beta^- Z_{it}^- = \beta^+ X_{it}$:

By definition and by assumption, we have

$$
\begin{aligned}
\beta^+ Z_{it}^+ + \beta^- Z_{it}^- &= \beta^+ Z_{it}^+ - \beta^+ Z_{it}^- \\
&= \beta^+ (Z_{it}^+ - Z_{it}^-) \\
&= \beta^+ (\sum_{s=2}^{t} X_{is}^+ - \sum_{s=2}^{t} X_{is}^-) \\
&= \beta^+ \sum_{s=2}^{t} (X_{is}^+ - X_{is}^-).
\end{aligned}
$$

The summations can start at time 2 because X_{is}^+ and X_{is}^- are defined to be 0 when $s = 1$. Using the result in equation 3 in the main text, we can rewrite this as

$$
\begin{aligned}
\beta^+ Z_{it}^+ + \beta^- Z_{it}^- &= \beta^+ \sum_{s=2}^{t} (X_{is} - X_{is-1}) \\
&= \beta^+ \left(\sum_{s=2}^{t} X_{is} - \sum_{s=2}^{t} X_{is-1} \right) \\
&= \beta^+ \left(\sum_{s=2}^{t} X_{is} - \sum_{s=1}^{t-1} X_{is} \right) \\
&= \beta^+ X_{it}.
\end{aligned}
$$

The final equality follows because the second summation exactly cancels all but the final term in the first summation.

ORCID iD

Paul D. Allison [iD] https://orcid.org/0000-0002-0646-5242

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Author Biography

Paul D. Allison is the president of Statistical Horizons LLC and a professor emeritus of the University of Pennsylvania. After completing his doctorate in sociology at the University of Wisconsin, he did postdoctoral study in statistics at the University of Chicago and the University of Pennsylvania. He has published eight books and more than 75 articles on topics that include linear regression, logistic regression, structural equation models, inequality measures, missing data, survival analysis, and scientific careers. At present, his principal methodological research is focused on the analysis of longitudinal data, especially with determining the causes and consequences of events, and on methods for handling missing data. A former Guggenheim Fellow, Allison received the 2001 Lazarsfeld Award for distinguished contributions to sociological methodology. In 2010, he was named a fellow of the American Statistical Association.