Let $\dot{\vec{r}}$ denote derivative wrt. s and $\vec{r}\,'$ denote derivative wrt. t. Let $\vec{u}$, $\vec{p}$ and $\vec{b}$ denote the tangent, normal and binormal vectors respectively. Firstly, note that

$$\vec{u} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\dot{\vec{r}}\, s'}{|\dot{\vec{r}}|\, s'}$$

Further, by the definition of the arclength s,

$$\frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right|$$

Instead of arbitrary t, put t = s:

$$\left| \frac{d\vec{r}}{ds} \right| = |\dot{\vec{r}}| = \frac{ds}{ds} = 1$$

So, $\vec{u} = \dot{\vec{r}}$. This greatly simplifies things.

Now, $$\vec{r}\,' = s'\dot{\vec{r}} = s'\vec{u} \qquad (1)$$
$$\vec{r}\,'' = s''\vec{u} + s'^2\dot{\vec{u}}$$

Also, $$\kappa = |\dot{\vec{u}}| \quad \text{and} \quad \vec{p} = \frac{\dot{\vec{u}}}{|\dot{\vec{u}}|} = \frac{\dot{\vec{u}}}{\kappa}$$

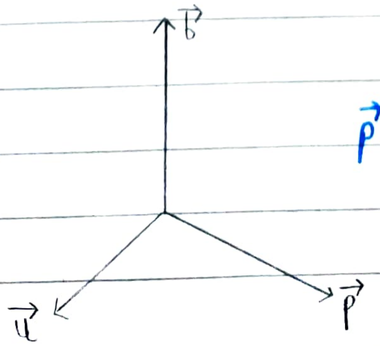
So, $$\vec{r}\,'' = s''\vec{u} + \kappa s'^2\vec{p} \qquad (2)$$

Crossing (1) × (2),
$$\vec{r}\,' \times \vec{r}\,'' = \underbrace{s'\vec{u} \times s''\vec{u}}_{0} + s'\kappa s'^2\, \vec{u} \times \vec{p}$$

But, $$\vec{b} = \vec{u} \times \vec{p} \text{ by definition.}$$
$$\Rightarrow \vec{r}\,' \times \vec{r}\,'' = \kappa s'^3\vec{b}$$
$$|\vec{r}\,' \times \vec{r}\,''| = \kappa s'^3$$
$$\Rightarrow \boxed{\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}}$$

Now, $\vec{b}$ is a is a vector of constant unit length. Hence, $\dot{\vec{b}}$ is perpendicular to $\vec{b}$. Using $\vec{b} = \vec{u} \times \vec{p}$, $\vec{b} \cdot \vec{u} = 0$ and $\vec{b} \cdot \vec{p} = \vec{b} \cdot \dot{\vec{u}} = 0$.
$$\Rightarrow (\vec{b} \cdot \vec{u})^{\cdot} = 0$$
$$\dot{\vec{b}} \cdot \vec{u} + \underbrace{\dot{\vec{u}} \cdot \vec{b}}_{0} = 0$$
$$\Rightarrow \dot{\vec{b}} \cdot \vec{u} = 0$$

So, $\dot{\vec{b}}$ is perpendicular to both $\vec{b}$ and $\vec{u}$, and hence
$$\dot{\vec{b}} = \lambda\vec{p}$$
By convention, $$\dot{\vec{b}} = -\tau\vec{p}, \text{ where } \tau \text{ is the torsion.}$$



$\vec{p}, \vec{u}, \vec{b}$ form a right-handed system.

Now, $\vec{p}' = (\vec{b} \times \vec{u})'$

$$\dot{\vec{p}} = \dot{\vec{b}} \times \vec{u} + (-\vec{u} \times \dot{\vec{b}})$$

$$= -\tau \vec{p} \times \vec{u} + \vec{b} \times \kappa \vec{p}$$

$$\dot{\vec{p}} = -\kappa \vec{u} + \tau \vec{b}$$

The above three relations

$$\boxed{\begin{aligned} \dot{\vec{u}} &= \kappa \vec{p} \\ \dot{\vec{p}} &= -\kappa \vec{u} + \tau \vec{b} \\ \dot{\vec{b}} &= -\tau \vec{p} \end{aligned}}$$

are called the Frenet formulas. In the differential geometry, it is shown that the whole differentio-geometric theory of curves is obtained from the Frenet formulas, whose solution shows that the "natural equations" $\kappa = \kappa(s)$, $\tau = \tau(s)$ determine a curve uniquely, except for it's position in space.

Now, $\tau = -(\vec{u} \times \vec{p})^{\cdot} \cdot \vec{p}$

$$= -\vec{p} \cdot (\underbrace{\dot{\vec{u}} \times \vec{p}}_{0} + \vec{u} \times \dot{\vec{p}})$$

$$= -\vec{p} \cdot (\vec{u} \times \dot{\vec{p}})$$

$$\boxed{\tau = (\vec{u} \; \vec{p} \; \dot{\vec{p}})}$$

And $\vec{u} = \dot{\vec{r}}$, $\vec{p} = \dfrac{\dot{\vec{u}}}{\kappa} = \dfrac{1}{\kappa} \ddot{\vec{r}}$

$$\Rightarrow \dot{\vec{p}} = \frac{1}{\kappa} \dddot{\vec{r}} + \left(\frac{1}{\kappa}\right)^{\cdot} \ddot{\vec{r}}$$

So, $\tau = (\vec{u} \; \vec{p} \; \dot{\vec{p}}) = \left(\dot{\vec{r}} \quad \dfrac{1}{\kappa}\ddot{\vec{r}} \quad \dfrac{1}{\kappa}\dddot{\vec{r}} + \left(\dfrac{1}{\kappa}\right)^{\cdot} \ddot{\vec{r}}\right)$

$$= \frac{1}{\kappa^2} (\dot{\vec{r}} \; \ddot{\vec{r}} \; \dddot{\vec{r}}) + \left(\dot{\vec{r}} \quad \frac{1}{\kappa}\ddot{\vec{r}} \quad \left(\frac{1}{\kappa}\right)^{\cdot} \ddot{\vec{r}}\right)$$

The second term is zero as $\ddot{\vec{r}}$ makes up two rows of the determinant.

$$\Rightarrow \boxed{\tau = \frac{1}{\kappa^2}\left(\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right) = \tau(s)}$$

To convert this to an arbitrary parameter 't', note that

$$\dot{\vec{r}} = \frac{1}{s'}\vec{r}\,'$$

$$\ddot{\vec{r}} = \frac{1}{s'}\left(\frac{\vec{r}\,''}{s'} - \underbrace{\frac{\vec{r}\,'}{s'^2}}\right)$$

something that makes two rows of a det. equal, hence discarded.

Similarly, $\dddot{\vec{r}} = \dfrac{\vec{r}\,'''}{s'^3} + \cdots\cdots$

So, $\tau = \dfrac{1}{\kappa^2}\left(\dfrac{1}{s'}\vec{r}\,'\ \ \dfrac{1}{s'^2}\vec{r}\,''\ \ \dfrac{1}{s'^3}\vec{r}\,'''\right)$

$$= \frac{1}{\kappa^2 s^6}\left(\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,'''\right)$$

$$\boxed{\tau(t) = \frac{\left(\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,'''\right)}{|\vec{r}\,' \times \vec{r}\,''|^2}}$$