The Coupled Pendulum

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1 Two Coupled Pendulums

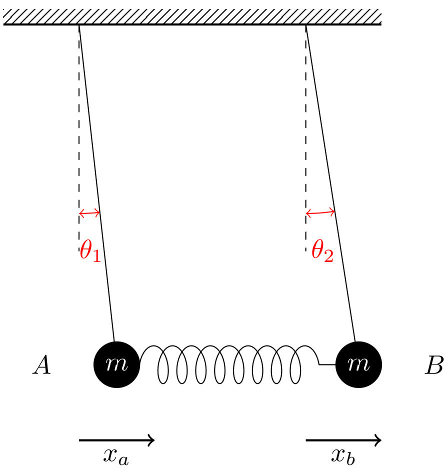

Figure 1: Two pendulums coupled by an ideal spring

Consider two pendulums of equal mass m, A and B, coupled by an ideal massless spring of spring constant k, as shown in Figure 1. The string may be considered sufficiently light so that it's mass may be neglected compared to the bobs. The equations of motion, considering small angle approximations ($\sin\theta \approx \theta, \ddot{y} \approx 0$), for the two pendulums are

$$m\frac{d^2x_a}{dt^2} = -mg\frac{x_a}{l} + k(x_b - x_a)$$
$$m\frac{d^2x_b}{dt^2} = -mg\frac{x_b}{l} - k(x_b - x_a)$$

or

$$\ddot{x}_a + (\omega_0^2 + \omega_c^2)x_a - \omega_c^2 x_b = 0 \tag{1}$$
$$\ddot{x}_b + (\omega_0^2 + \omega_c^2)x_b - \omega_c^2 x_a = 0 \tag{2}$$

where we have let $\omega_0^2 = \frac{g}{l}$ and $\omega_c^2 = \frac{k}{m}$

2 Normal Modes

Before we try to solve equations 1 and 2 for the most general motion of the system, let us consider what happens when we draw both A and B aside by equal amounts and release them. The spring remains relaxed and exerts no force on either masses. Both of them then oscillate with the same natural frequency ω_0:

$$x_a = C\cos\omega_0 t$$
$$x_b = C\cos\omega_0 t$$

This is known as a *normal mode of oscillation*, where all masses oscillate with the same frequency. For this system, there is yet another normal mode: pull A and B aside by equal amounts but in opposite direction. The equation of motion for A is

$$\ddot{x}_a + (\omega_0^2 + 2\omega_c^2)x_a = 0$$

which is readily identified as a simple harmonic motion of frequency $\omega^{'} = (\omega_0^2 + 2\omega_c^2)^{1/2}$ so that

$$x_a = C\cos\omega^{'} t$$
$$x_b = -C\cos\omega^{'} t$$