Time Domain Analysis

RL Circuits, RC Circuits, Sinusoidal Forcing Functions

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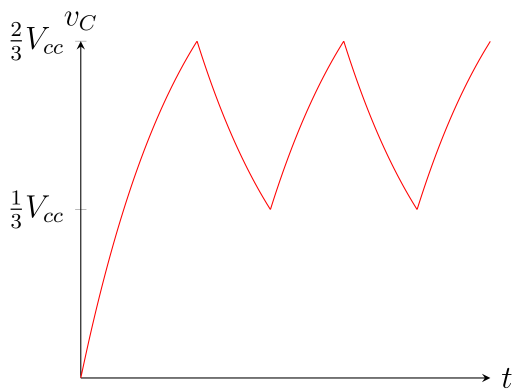

The famous astable 555 waveform

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1 Current And Voltage Conventions

The conventions we shall follow, called ***passive sign conventions***, simply state that when current "flows into" the positive terminal of the capacitor/inductor, as indicated by the polarity of v in Fig.1.1, it is taken as positive. We may then write the current-voltage relations:

$$\boxed{\begin{aligned} i &= C\frac{dv}{dt} \\ v &= L\frac{di}{dt} \end{aligned}}$$

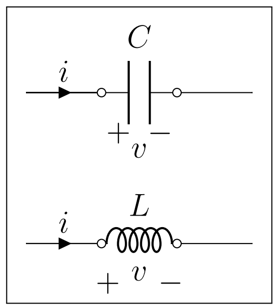


Figure 1.1: The voltage conventions

2 First Order Circuits: RL and RC

A first order circuit is one that is governed by a first order differential equation. Often, it is incorrectly stated that a first order circuit is one that contains only one energy storage element (capacitor/inductor). This is wrong, as there are certain arrangements of $R - L$ and $R - C$ circuits which can be simplified to obtain a first order equation, as we shall see later. So, there being only one energy storage element in a circuit is a *sufficient* but not *necessary* condition for it to be first order.

2.1 The Natural Response

2.1.1 The Source-Free RL Circuit

A ***natural response*** is one that is free of any external voltage/current sources, which are also known as *forcing functions*. It depends on the "general nature" of the circuit (types of elements, sizes and interconnections). It is also known as the ***transient response***, as without any external sources, it must eventually die out. Consider the simple series RL circuit shown in Fig.2.1:

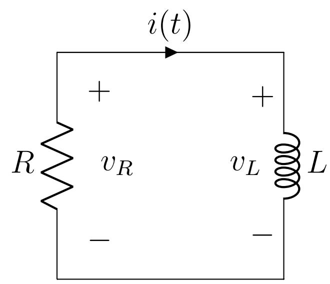


Figure 2.1: Simple R-L circuit for which $i(t)$ is to be determined

Let the time varying current be designated $i(t)$, and the value of i at $t = 0$ be I_0. It may seem counter-intuitive to have a non-zero current initially in a circuit with no sources. In order for a current to be flowing, a source had to be present at some point of time, but we are only bothered with the response of the circuit after $t = 0$, ie. after the source has been removed. Applying Kirchhoff's voltage law, we have

$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \tag{2.1}$$

We can separate variables and integrate easily, giving

$$\boxed{i(t) = I_0 e^{-\frac{R}{L}t}} \tag{2.2}$$

where we have incorporated the intitial codition $i(0) = I_0$. The general form of the natural response will be

$$\boxed{i(t) = Ke^{-\frac{R}{L}t}} \tag{2.3}$$

where K is a constant, and is nothing but the value of i at $t = 0$.

Accounting For The Energy

The power dissipated through the resistor at any time t

$$p_R = i^2 R = I_0^2 R e^{-\frac{2Rt}{L}}$$

and the total energy turned into heat by the resistor till the response has died out

$$
\begin{aligned}
w_R &= \int_0^\infty p_R \, dt \\
&= I_0^2 R \int_0^\infty e^{-\frac{2Rt}{L}} \, dt \\
&= I_0^2 R \left(\frac{-L}{2R} \right) e^{-\frac{2Rt}{L}} \Bigg|_0^\infty \\
\boxed{w_R} &\boxed{= \frac{1}{2} L I_0^2}
\end{aligned}
$$

which is the total energy stored initially in the inductor, as expected.

2.1.2 Properties Of The Natural Response

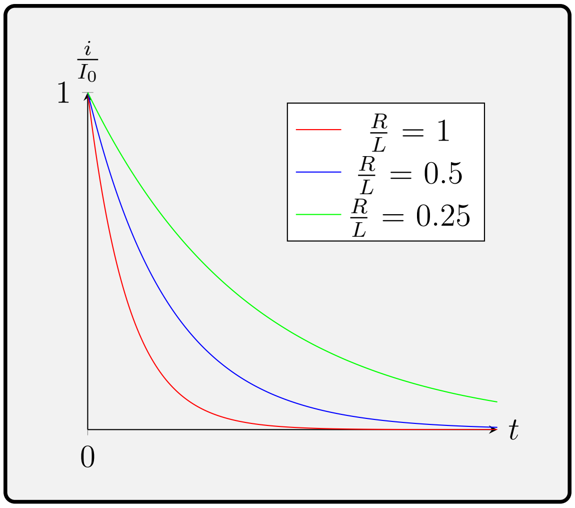


Figure 2.2: Some possible plots for the natural response

From the natural response of the series RL circuit given by Eq.2.2, it is clear that that the current decays exponentially to zero from an initial value of I_0. The graph of $\frac{i}{I_0}$ is plotted in figure 2.2 for three different values of $\frac{R}{L}$, as indicated in the legend.
Consider now the initial rate of decay, which is found by differentiating equation 2.2:

$$
\frac{d}{dt} \frac{i}{I_0} \bigg|_{t=0} = -\frac{R}{L} e^{-Rt/L} \bigg|_{t=0} = -\frac{R}{L}
$$

Assuming that the decay continues at this rate, the time τ taken by $\frac{i}{I_0}$ to drop from 1 to 0 is given by

$$
\left(\frac{R}{L} \right) \tau = 1
$$

or

$$\boxed{\tau = \frac{L}{R}} \tag{2.4}$$

The constant τ has the units of seconds and is called the **time constant** of the circuit. It may be found graphically from the response curve by drawing the tangent to the curve at $t = 0$ and finding where it intersects the x axis, as shown below:

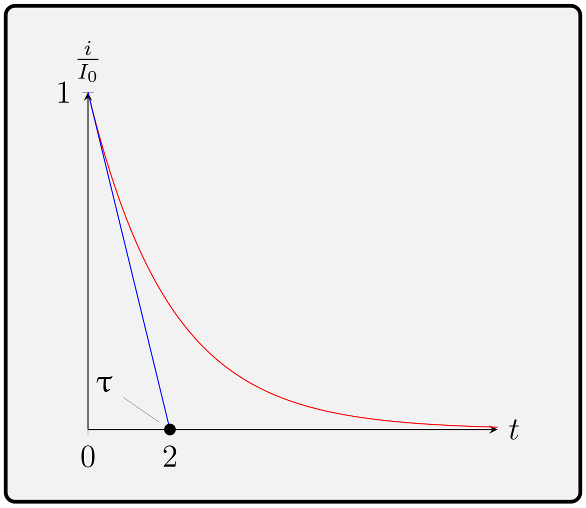
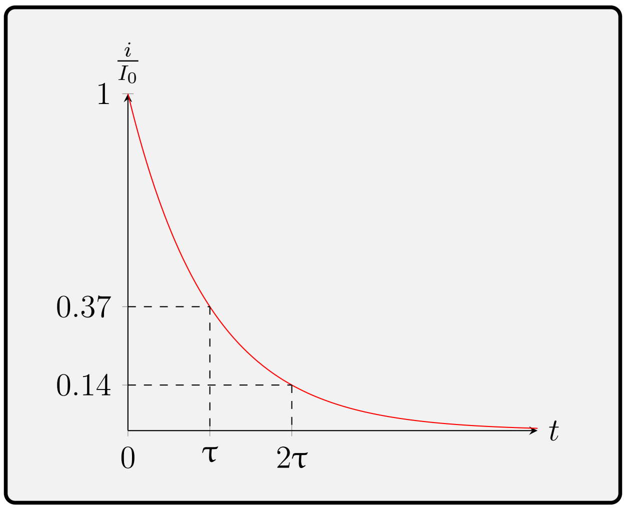


Figure 2.3: Finding the time constant graphically

Yet another way to define the time constant is to note that

$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679$$

ie, in one time constant, the response falls to 36.8% of it's initial value. The value of τ may also be determined from this fact:



Example 2.1

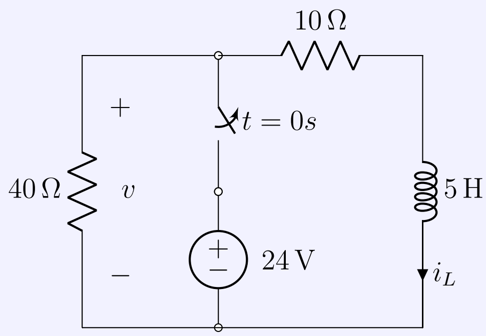
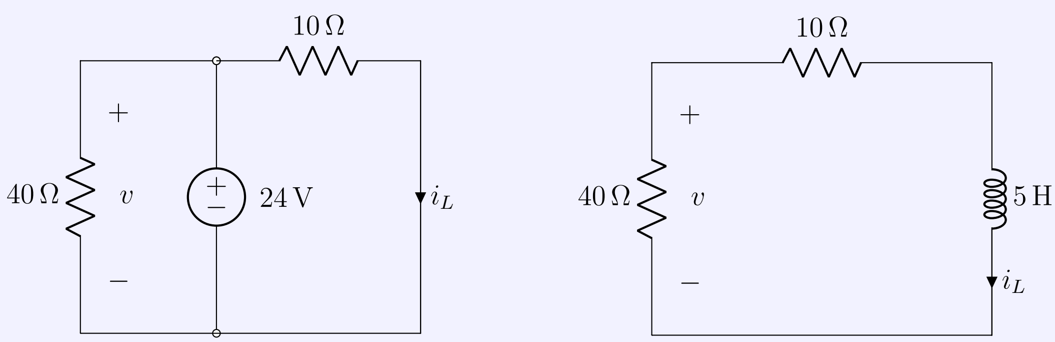


Figure 2.4: A simple RL circuit with a switch thrown at $t = 0$

For the circuit given in figure 2.4, let us try to find the voltage at $t = 200ms$.

Figures 2.5a and 2.5b show the state of the circuit prior to and post the throwing of the switch. In 2.5a, the inductor acts like a short to DC current after all transients have died down.



(a) The circuit prior to $t = 0$

(b) The circuit post $t = 0$

Figure 2.5: The simplified circuit before and after the switch is opened

Applying Kirchhoff's Law in 2.5b, we may write

$$
\begin{aligned}
-v + 10i_L + 5\frac{di_l}{dt} &= 0 \\
\Rightarrow \frac{5}{40}\frac{dv}{dt} + \left(\frac{10}{40} + 1\right)v &= 0 \\
\Rightarrow \frac{dv}{dt} + 10v &= 0
\end{aligned}
$$

where we have taken $i_l = -v/40$ by our conventions. The solution to this differential equation is

$$v(t) = v(0^+)e^{-10t}$$

Note that the initial value of voltage here is denoted by $v(0^+)$, as the voltage across the resistor may change discontinuously when the switch is thrown. We have to use the fact that i_L cannot change discontinuously, and remains the same just before and just after the switch is thrown. From 2.5a, $i_l(0) = 24/10 = 2.4$A. Hence, $v(0^+) = (40) \cdot (-2.4) = -96$V, and so

$$\boxed{v(t) = -96e^{-10t}}$$

which gives $v(0.2) = -12.99$V.

2.1.3 The Source-Free RC Circuit

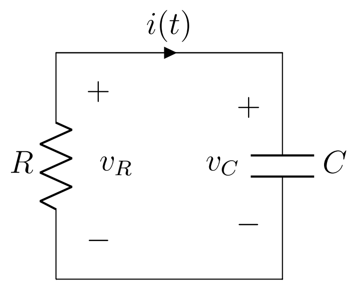


Figure 2.6: Source-Free RC Circuit

The source free response for RC circuits is nearly the same as with LC, but with slightly different expressions for the time constant. In this case, the general form of the response is given by

$$\boxed{v(t) = v(0)e^{-\frac{t}{\tau}}} \tag{2.5}$$

and the expression for the time constant is

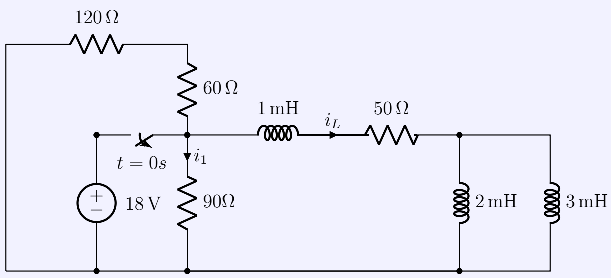
$$\boxed{\tau = R \cdot C} \tag{2.6}$$

It may also be noted that the voltage across a capacitor cannot change discontinuously, as opposed to the current for an inductor.

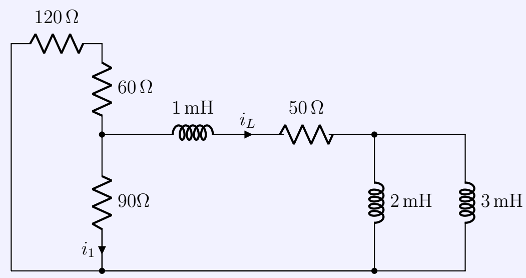
2.1.4 General RL and RC Circuits

For circuits containing more than one capacitor/inductor and more than one resistor, it is possible to arrive at a single time constant and reduce it to a simple circuit with one resistor and one energy storage element, as long as the Thévenin resistance "seen" by all the elements is the same. In such a case, we can write $\tau = \frac{L_{eq}}{R_{eq}}$ or $\tau = R_{eq} \cdot C_{eq}$. The general form of the natural response would then be $Ae^{-t/\tau}$, where the constant A can be determined using the initial conditions, and would be different for different elements. Let us illustrate this using the following example:

Example 2.2



(a) The circuit with multiple inductors and resistors



(b) The simplified circuit after $t = 0$

For the circuit given in figure 2.7a find i_1 and i_L for $t > 0$.

Firstly, note that the equivalent/Thévenin resistance seen by all three inductances is equal, and equal to $50 + 90||(120 + 60) = 110\Omega$. So, we can reduce it to a simple LR circuit, with

$$L_{eq} = \frac{2 \cdot 3}{2 + 3} + 1 = 2.2mH$$
$$\Rightarrow \tau = \frac{L_{eq}}{R_{eq}} = 20\mu s$$

And so every current and voltage in the network must have the form $Ke^{-t/\tau} = Ke^{-50,000t}$. If we consider the circuit prior to opening the switch and after all transients have died down, the inductors act as shorts to DC current and so i_L is easily found to be $i_L = 18/50 = 360mA$. This must also be the value of i_L just after the switch is opened. So,

$$i_L = \begin{cases} 360mA & t < 0 \\ 360e^{-50,000t}mA & 0 \le t \end{cases}$$

Now, i_1 may change discontiuously at $t = 0$ so we have to find it's value at 0^+ using the value of $i_L(0^+)$. Using current division,

$$i_1(0^+) = -i_L(0^+) \cdot \frac{120 + 60}{120 + 60 + 90} = -240mA$$

Hence,

$$i_1 = \begin{cases} 200mA & t < 0 \\ -240e^{-50,000t}mA & 0 \leq t \end{cases}$$

2.2 The Unit Step Function

The unit step function, also called the **Heaviside step function** is a step function whose value is zero for negative arguments and one for positive arguments. This function is important as the switching action of a battery is equivalent to a forcing function which is zero up until the switch is thrown, and equal to the battery voltage thereafter. It is represented by u, θ or H.
If we shift the argument to some arbitrary time t_0, then $u(t - t_0)$ must be zero for all values of t less than t_0 and unity for values of t greater than t_0. The unit step function changes abruptly/ discontinuously from 0 to 1, and it's value at $t = t_0$ is not defined.
It can be represented analytically as

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

or graphically as

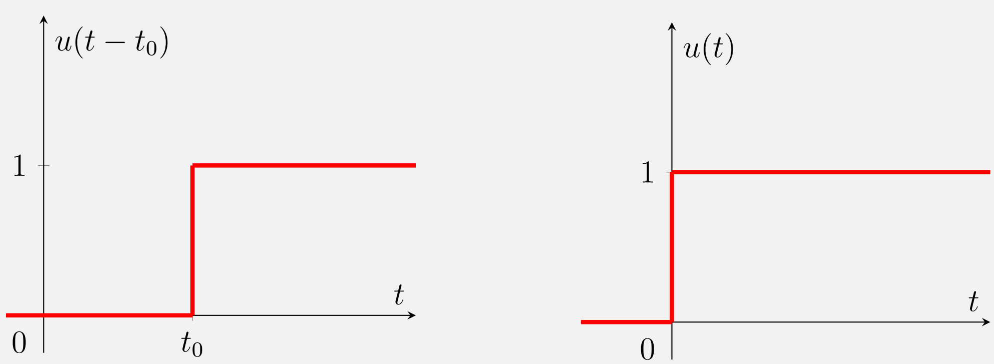


Figure 2.8: The unit step function

Note that the vertical line is not really a part of the unit step's definition but it is usally shown in each drawing as it looks nicer.
Interestingly, the physical equivalent of the unit step is not actually a simple voltage source in series with a switch. Consider the step voltage source shown in figure . Let us try to draw it's physical equivalent. On first thought, one might come up with figure . However, these two circuits are not equivalent for $t < 2s$: the voltage across the network in figure is completely unspecified in this time interval. The actual equivaent to the step voltage source is the single-pole double throw switch shown in figure . Clearly, the voltage is zero for $t < 2s$, and this is consistent with the step voltage function.

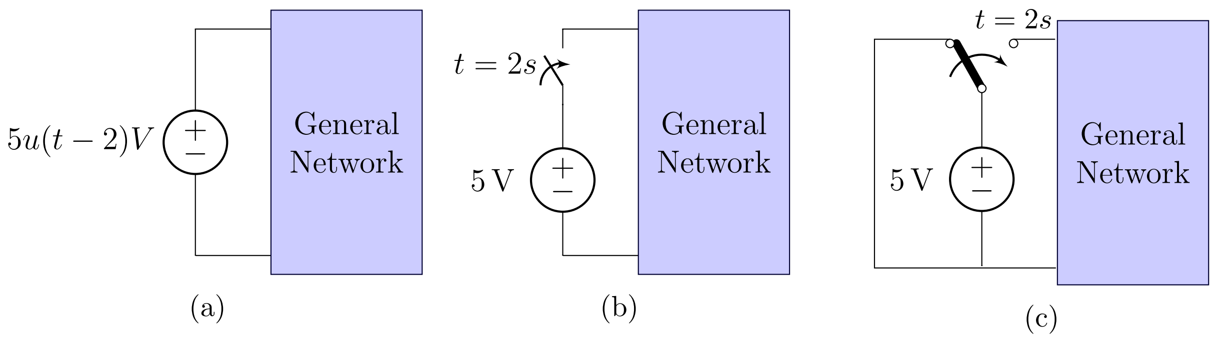


Figure 2.9: (a) The voltage-step forcing function. (b) A possible equivalent. (c) The exact equivalent.

2.2.1 The Rectangular Pulse Function

Various rectangular pulses may be obtained by manipulating the unit step function. Let us first consider the pulse given by

$$v(t) = \begin{cases} 0 & t < t_0 \\ V - 0 & t_0 < t < t_1 \\ 0 & t > t_1 \end{cases}$$