The Frenet-Serret Frame

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Contents

1 The TNB/Frenet-Serret Frame

1.1 The Basis Vectors

The TNB Frame, or the Frenet-Serret Frame, is defined by the following vectors:

- $\hat{T}$, the tangent unit vector, which points in the direction of motion.
- $\hat{N}$, the normal unit vector.
- $\hat{B}$, the binormal unit vector, the cross product of the tangent and normal vectors.

In further equations, we shall drop the unit vector notation and represent vectors in boldface, it is implied that these vectors are of unit magnitude.

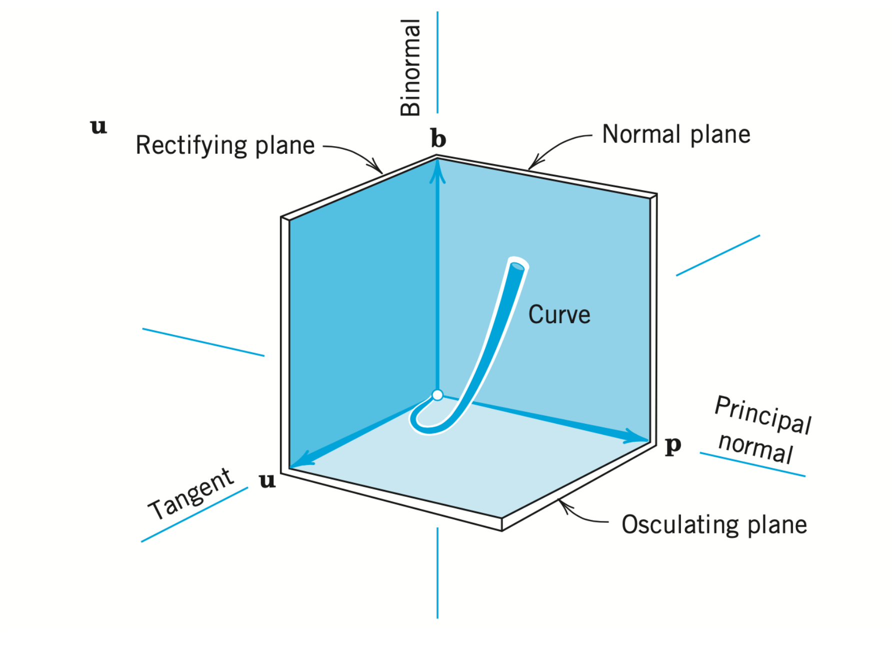

Figure 1: The Frenet-Serret Frame

The three unit vectors are defined as:

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} \tag{1}$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} \tag{2}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \tag{3}$$

Here, the variable s is the arclength parameter, given by

$$\boxed{\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|} \tag{4}$$

Henceforth, we shall denote differentiation wrt. the arbitrary parameter t as $\mathbf{r}'$ and differentiation wrt. the arclength s as $\dot{\mathbf{r}}$.

2 The Curvature κ

The curvature of a space curve $\mathbf{r}(t)$, κ, gives a measure of how much it deviates from a straight line at a particular point. It is defined as

$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = |\dot{\mathbf{T}}|} \tag{5}$$

An analytical expression for κ in terms of $\mathbf{r}$ and it's derivatives is obtained as follows:
Using the chain rule,

$$\begin{aligned} \mathbf{r}' &= s'\dot{\mathbf{r}} \\ \Rightarrow \mathbf{r}'' &= s''\mathbf{T} + s'^2\dot{\mathbf{T}} \end{aligned} \tag{6}$$

Using $\kappa = |\dot{\mathbf{T}}|$ and $\mathbf{N} = \frac{\dot{\mathbf{T}}}{|\dot{\mathbf{T}}|}$,

$$\mathbf{r}'' = s''\mathbf{T} + s'^2\kappa\mathbf{N} \tag{7}$$

Taking the cross product of (6) and (7),

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= s'\mathbf{T} \times s''\mathbf{T} + s'\kappa s'^2\mathbf{T} \times \mathbf{N} \\ \Rightarrow \mathbf{r}' \times \mathbf{r}'' &= \kappa s'^3\mathbf{B} \\ &\boxed{\Rightarrow \kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}} \end{aligned} \tag{8}$$

Specifically, consider the curve $\mathbf{r} = x\hat{\mathbf{i}} + f(x)\hat{\mathbf{j}}$. Then,

$$\begin{aligned}
\mathbf{r}' &= \hat{\mathbf{i}} + f'(x)\hat{\mathbf{j}} \\
\Rightarrow \mathbf{r}'' &= f''(x)\hat{\mathbf{j}} \\
\Rightarrow \mathbf{r}' \times \mathbf{r}'' &= f''(x)\hat{\mathbf{k}}
\end{aligned}$$

So, the expression for κ reduces to

$$\boxed{\kappa = \frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}}}$$

Which is the familiar expression for the curvature of a planar curve.

3 The Frenet Formulas

The Frenet formulas are

$$\boxed{\begin{aligned}
\frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} && (9) \\
\frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N} && (10) \\
\frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B} && (11)
\end{aligned}}$$

The first formula simply follows from (2) and (5). To prove the second formula, note that $\mathbf{B}$ is a vector of constant unit length and hence $\dot{\mathbf{B}}$ is perpendicular to $\mathbf{B}$. By the definition of the cross product, $\mathbf{B}$ is perpendicular to $\mathbf{T}$, and so $\mathbf{B} \cdot \mathbf{T} = 0$. Also, $\mathbf{B} \cdot \dot{\mathbf{T}} = \mathbf{B} \cdot (\kappa\mathbf{N}) = 0$. Hence,

$$\begin{aligned}
\mathbf{B} \cdot \mathbf{T} &= 0 \\
\Rightarrow (\mathbf{B} \dot{\cdot} \mathbf{T}) &= 0 \\
\Rightarrow \dot{\mathbf{B}} \cdot \mathbf{T} + \mathbf{B} \cdot \dot{\mathbf{T}} &= 0 \\
\Rightarrow \dot{\mathbf{B}} \cdot \mathbf{T} &= 0
\end{aligned}$$

So $\dot{\mathbf{B}}$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, hence it is parallel to their cross product, $\mathbf{N}$.

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where the scalar is taken as $-\tau$by convention, and is called the torsion.
To prove the third formula, using $\mathbf{N} = \mathbf{B} \times \mathbf{T}$

$$
\begin{aligned}
\dot{\mathbf{N}} &= \mathbf{B} \times \dot{\mathbf{T}} + \dot{\mathbf{B}} \times \mathbf{T} \\
&= -\tau \mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa \mathbf{N} \\
\Rightarrow \frac{d\mathbf{N}}{ds} &= -\kappa \mathbf{T} + \tau \mathbf{B}
\end{aligned}
$$

which proves the third Frenet formula.
It can be shown that the whole differentio-geometric theory of curves is obtained from the Frenet formulas, whose solution shows that the "natural equations" $\kappa = \kappa(s)$ and $\tau = \tau(s)$ determine a curve uniquely, except for it's position in space.

4 The Torsion τ

The torsion of a curve, denoted by τ gives a measure of how much the curve is twisting out of the plane of curvature or osculating plane. To obtain an expression for torsion, use the second Frennet formula:

$$
\begin{aligned}
\frac{d\mathbf{B}}{ds} &= -\tau \mathbf{N} \\
\Rightarrow \tau &= -\dot{\mathbf{B}} \cdot \mathbf{N} \\
\Rightarrow \tau &= -(\mathbf{T} \dot{\times} \mathbf{N}) \cdot \mathbf{N} \\
&= -\mathbf{N} \cdot (\dot{\mathbf{T}} \times \mathbf{N} + \mathbf{T} \times \dot{\mathbf{N}}) \\
&= -\mathbf{N} \cdot (\mathbf{T} \times \dot{\mathbf{N}}) \\
\Rightarrow & \boxed{\tau = \begin{bmatrix} \mathbf{T} & \mathbf{N} & \dot{\mathbf{N}} \end{bmatrix}}
\end{aligned}
$$

where we have used the box notation for the scalar triple product. Further, using

$$
\begin{aligned}
\mathbf{T} &= \dot{\mathbf{r}} \\
\mathbf{N} &= \frac{\dot{\mathbf{T}}}{\kappa} = \frac{\ddot{\mathbf{r}}}{\kappa} \\
\Rightarrow \dot{\mathbf{N}} &= \frac{\dddot{\mathbf{r}}}{\kappa} + \dot{\left(\frac{1}{\kappa}\right)} \ddot{\mathbf{r}}
\end{aligned}
$$

So,

$$
\tau = \begin{bmatrix} \dot{\mathbf{r}} & \frac{\ddot{\mathbf{r}}}{\kappa} & \frac{\dddot{\mathbf{r}}}{\kappa} + \dot{\left(\frac{1}{\kappa}\right)} \ddot{\mathbf{r}} \end{bmatrix}
$$

The second term of the last column makes two columns of the determinant equal, and hence can be ignored using the properties of determinants.

$$\Rightarrow \boxed{\tau = \frac{1}{\kappa^2}\left[\dot{\mathbf{r}} \quad \ddot{\mathbf{r}} \quad \dddot{\mathbf{r}}\right]}$$

To convert this expression to an arbitrary parameter t, note that

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{\mathbf{r}'}{s'} \\
\ddot{\mathbf{r}} &= \frac{\mathbf{r}''}{s'^2} + ... \\
\dddot{\mathbf{r}} &= \frac{\mathbf{r}'''}{s'^3} + ...
\end{aligned}$$

where the ... denotes something that vanishes by properties of determinants. So,

$$\tau = \frac{1}{\kappa^2 s'^6}\left[\mathbf{r}' \quad \mathbf{r}'' \quad \mathbf{r}'''\right]$$

$$\boxed{\begin{aligned}
\tau &= \frac{\left[\mathbf{r}' \quad \mathbf{r}'' \quad \mathbf{r}'''\right]}{|\mathbf{r}' \times \mathbf{r}''|^2} \\
\tau &= \frac{\begin{vmatrix} \dot{x} & \ddot{x} & \dddot{x} \\ \dot{y} & \ddot{y} & \dddot{y} \\ \dot{z} & \ddot{z} & \dddot{z} \end{vmatrix}}{|\mathbf{r}' \times \mathbf{r}''|^2}
\end{aligned}}$$

where we have evaluated $\kappa^2 s'^6$ using (8).