The Coupled Pendulum

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1 Two Coupled Pendulums

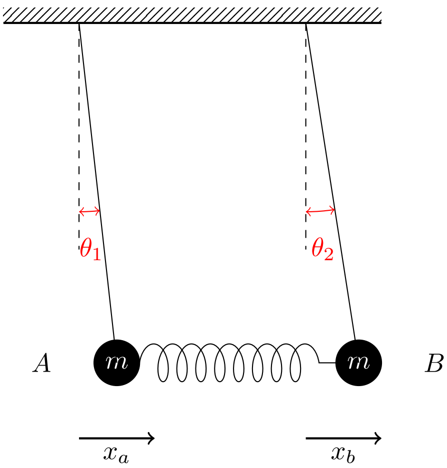

Figure 1: Two pendulums coupled by an ideal spring

Consider two pendulums of equal mass m, A and B, coupled by an ideal massless spring of spring constant k, as shown in Figure 1. The string may be considered sufficiently light so that it's mass may be neglected compared to the bobs. The equations of motion, considering small angle approximations ($\sin\theta \approx \theta, \ddot{y} \approx 0$), for the two pendulums are

$$m\frac{d^2x_a}{dt^2} = -mg\frac{x_a}{l} + k(x_b - x_a)$$
$$m\frac{d^2x_b}{dt^2} = -mg\frac{x_b}{l} - k(x_b - x_a)$$

or

$$\ddot{x}_a + (\omega_0^2 + \omega_c^2)x_a - \omega_c^2 x_b = 0 \tag{1}$$
$$\ddot{x}_b + (\omega_0^2 + \omega_c^2)x_b - \omega_c^2 x_a = 0 \tag{2}$$

where we have let $\omega_0^2 = \frac{g}{l}$ and $\omega_c^2 = \frac{k}{m}$

2 Normal Modes

Before we try to solve equations 1 and 2 for the most general motion of the system, let us consider what happens when we draw both A and B aside by equal amounts and release them. The spring remains relaxed and exerts no force on either masses. Both of them then oscillate with the same natural frequency ω_0:

$$x_a = C\cos\omega_0 t$$
$$x_b = C\cos\omega_0 t$$

This is known as a *normal mode of oscillation*, where all masses oscillate with the same frequency. For this system, there is yet another normal mode: pull A and B aside by equal amounts but in opposite direction. The equation of motion for A is

$$\ddot{x}_a + (\omega_0^2 + 2\omega_c^2)x_a = 0$$

which is readily identified as a simple harmonic motion of frequency $\omega\prime = (\omega_0^2 + 2\omega_c^2)^{1/2}$, so that

$$x_a = C\cos\omega^{'} t$$
$$x_b = -C\cos\omega^{'} t$$

The two normal modes are illustrated in figure 2.

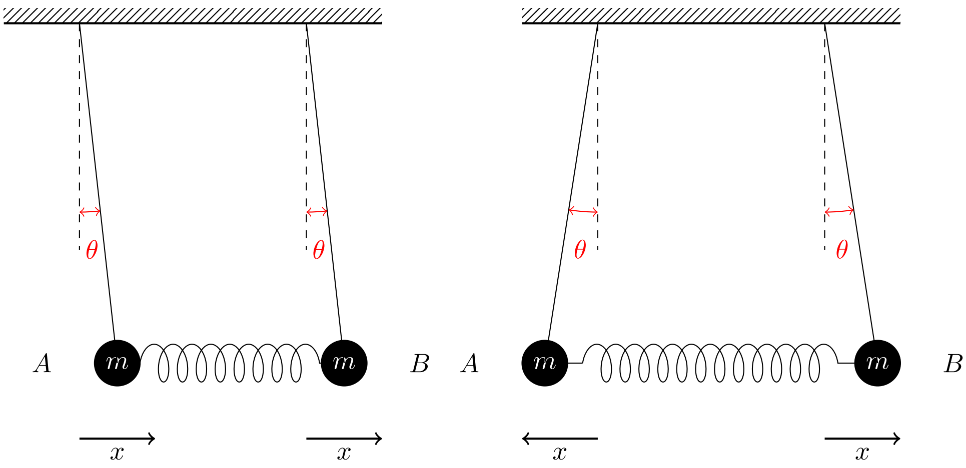

Figure 2: The two normal modes

3 Superposition of the normal modes

Let us now try to solve equations 1 and 2 for general initial conditions. The symmetry of the two equations suggests that adding and subtracting them might give some joy:

$$\begin{aligned}
\ddot{x}_a + \ddot{x}_b + \omega_0^2(x_a + x_b) &= 0 \\
\ddot{x}_a - \ddot{x}_b + (\omega_0^2 + 2\omega_c^2)(x_a - x_b) &= 0
\end{aligned}$$

Indeed, these are two simple harmonic motions. Introducing the *normal co-ordinates* $q_1 = x_a + x_b$ and $q_2 = x_a - x_b$,

$$\begin{aligned}
\ddot{q}_1 + \omega_0^2 q_1 &= 0 \\
\ddot{q}_2 + \omega'^2 q_2 &= 0
\end{aligned}$$

whose solutions are

$$\begin{aligned}
q_1 &= C \cos \omega_0 t \\
q_2 &= D \cos \omega' t
\end{aligned}$$

where we have already assumed two initial conditions (initial phases are zero) for simplicity. In terms of our original co-ordinates,

$$\begin{aligned}
x_a &= \frac{1}{2}(q_1 + q_2) = \frac{1}{2} C \cos \omega_0 t + \frac{1}{2} D \cos \omega' t \\
x_b &= \frac{1}{2}(q_1 - q_2) = \frac{1}{2} C \cos \omega_0 t - \frac{1}{2} D \cos \omega' t
\end{aligned}$$

We see that the general motion of the oscillator is a superposition of it's normal modes. This is a general result for any number of coupled oscillators. Let us now consider what happens when we pull aside pendulum A by a small amount while keeping B fixed, ie. the above equations subject to the initial conditions at $t = 0$

$$\begin{aligned}
x_a &= A_0 \\
\dot{x}_a &= 0 \\
x_b &= 0 \\
\dot{x}_b &= 0
\end{aligned}$$

We obtain

$$\begin{aligned}
x_a &= A_0 \cos \frac{\omega' - \omega_0}{2} t \cos \frac{\omega' + \omega_0}{2} t \\
x_b &= A_0 \sin \frac{\omega' - \omega_0}{2} t \sin \frac{\omega' + \omega_0}{2} t
\end{aligned}$$

These motions are plotted in figure 3. We see that A starts swinging initially, but it's amplitude continuously decreases. Pendulum B, initially at rest, starts oscillating and soon the amplitudes of A and B become equal. The amplitude of A then diminishes towards zero and the amplitude of B becomes that of A originally. The spring is therefore acting as a kind of energy transfer agent between the two pendulums.

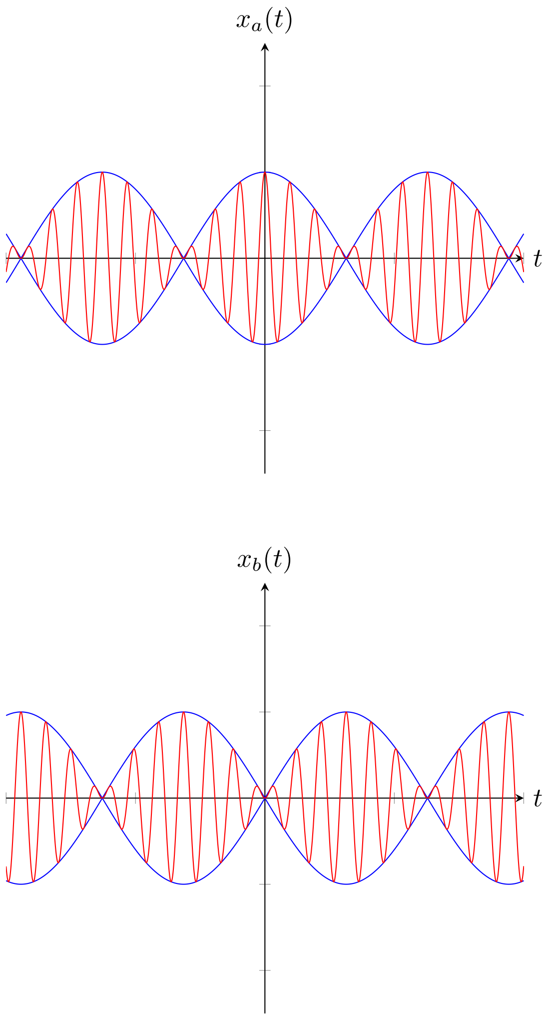

Figure 3: The plots of x_a and x_b

4 Normal Modes: The General Case

What if for a certain system it is not as simple to deduce what the normal mode frequencies are? In that case we have to assume solutions of the same frequency and work backwards to find for what values of that frequency and what values of amplitude those solutions are valid. If we try this for the coupled pendulum, ie. set $x_a = C\cos\omega t$ and $x_b = C'\cos\omega t$ and substitute into equations 1 and 2, we get

$$(-\omega^2 + \omega_0^2 + \omega_c^2)C - \omega_0^2 C' = 0 \quad (3)$$
$$-\omega_c^2 C + (-\omega^2 + \omega_0^2 + \omega_c^2)C' = 0 \quad (4)$$

If this system of linear equations in C and C' is to have a non trivial solution, it must be linearly dependent, ie.

$$\begin{vmatrix} -\omega^2 + \omega_0^2 + \omega_c^2 & \omega_0^2 \\ -\omega_c^2 & -\omega^2 + \omega_0^2 + \omega_c^2 \end{vmatrix}$$

This is a quadratic equation in ω^2, and after a bit of work we obtain

$$\omega = \omega_0$$
$$\omega = \sqrt{\omega_0^2 + 2\omega_c^2}$$

substituting these values into equations 3 and 4, we get

$$\frac{C}{C'} = 1 or \frac{C}{C'} = -1$$

which agrees with our original result.