Time Domain Analysis

RL Circuits, RC Circuits, Sinusoidal Forcing Functions

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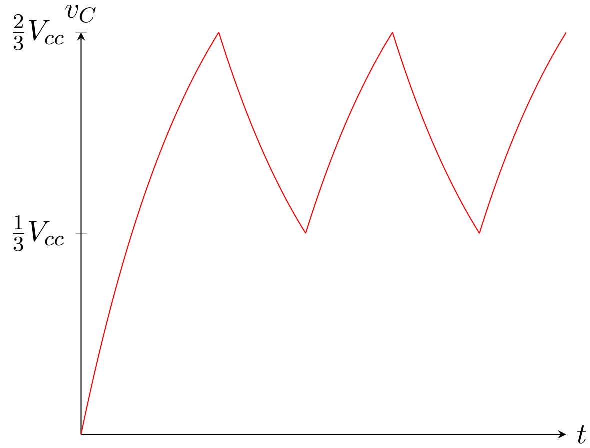

The astable 555 timer IC waveform

January 18, 2021

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1 Current And Voltage Conventions

The conventions we shall follow, called ***passive sign conventions***, simply state that when current "flows into" the positive terminal of the capacitor/inductor, as indicated by the polarity of v in Fig.1.1, it is taken as positive. We may then write the current-voltage relations:

$$\boxed{\begin{aligned} i &= C\frac{dv}{dt} \\ v &= L\frac{di}{dt} \end{aligned}}$$

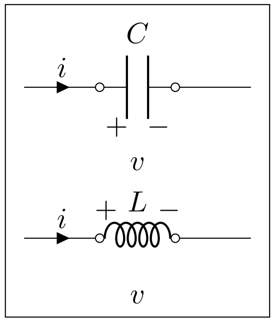


Figure 1.1: The voltage conventions

2 First Order Circuits: RL and RC

A first order circuit is one that is governed by a first order differential equation. Often, it is incorrectly stated that a first order circuit is one that contains only one energy storage element (capacitor/inductor). This is wrong, as there are certain arrangements of $R-L$ and $R-C$ circuits which can be simplified to obtain a first order equation, as we shall see later. So, there being only one energy storage element in a circuit is a *sufficient* but not *necessary* condition for it to be first order.

2.1 The Natural Response

2.1.1 The Source-Free RL Circuit

A ***natural response*** is one that is free of any external voltage/current sources, which are also known as *forcing functions*. It depends on the "general nature" of the circuit (types of elements, sizes and interconnections). It is also known as the ***transient response***, as without any external sources, it must eventually die out. Consider the simple series RL circuit shown in Fig.2.1:

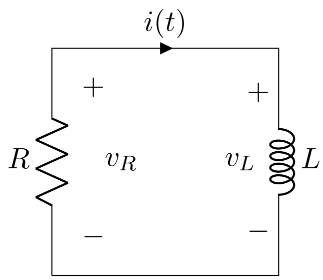


Figure 2.1: Simple R-L circuit for which $i(t)$ is to be determined

Let the time varying current be designated $i(t)$, and the value of i at $t = 0$ be I_0. It may seem counter-intuitive to have a non-zero current initially in a circuit with no sources. In order for a current to be flowing, a source had to be present at some point of time, but we are only bothered with the response of the circuit after $t = 0$, ie. after the source has been removed. Applying Kirchhoff's voltage law, we have

$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \tag{2.1}$$

We can separate variables and integrate easily, giving

$$\boxed{i(t) = I_0 e^{-\frac{R}{L}t}} \tag{2.2}$$

where we have incorporated the intitial codition $i(0) = I_0$. The general form of the natural response will be

$$\boxed{i(t) = K e^{-\frac{R}{L}t}} \tag{2.3}$$

where K is a constant, and is nothing but the value of i at $t = 0$.

Accounting For The Energy

The power dissipated through the resistor at any time t

$$p_R = i^2 R = I_0^2 R e^{-\frac{2Rt}{L}}$$

and the total energy turned into heat by the resistor till the response has died out

$$\begin{aligned}
w_R &= \int_0^\infty p_R \, dt \\
&= I_0^2 R \int_0^\infty e^{-\frac{2Rt}{L}} \, dt \\
&= I_0^2 R \left(\frac{-L}{2R}\right) e^{-\frac{2Rt}{L}} \Bigg|_0^\infty
\end{aligned}$$

$$\boxed{w_R = \frac{1}{2} L I_0^2}$$

which is the total energy stored initially in the inductor, as expected.

2.1.2 Properties Of The Natural Response

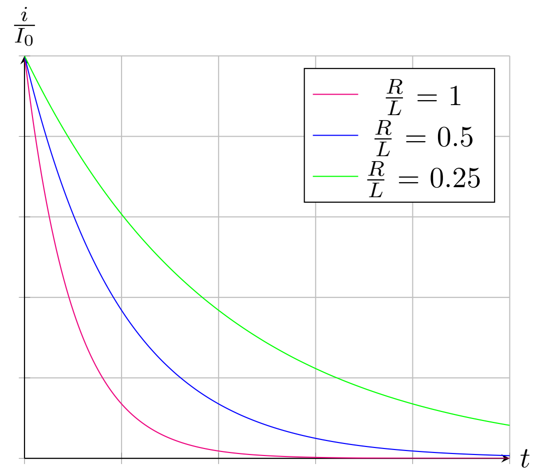


Figure 2.2: Some possible plots for the natural response

From the natural response of the series RL circuit given by Eq.2.2, it is clear that that the current decays exponentially to zero from an initial value of I_0. The graph of $\frac{i}{I_0}$ is plotted in figure 2.2 for three different values of $\frac{R}{L}$, as indicated in the legend.
Consider now the initial rate of decay, which is found by differentiating equation 2.2:

$$\left.\frac{d}{dt}\frac{i}{I_0}\right|_{t=0} = \left.-\frac{R}{L}e^{-Rt/L}\right|_{t=0} = -\frac{R}{L}$$

Assuming that the decay continues at this rate, the time τ taken by $\frac{i}{I_0}$ to drop from 1 to 0 is given by

$$\left(\frac{R}{L}\right)\tau = 1$$

or

$$\boxed{\tau = \frac{L}{R}} \tag{2.4}$$

The constant τ has the units of seconds and is called the **time constant** of the circuit. It may be found graphically from the response curve by drawing the tangent to the curve at $t = 0$ and finding where it intersects the x axis, as shown below:

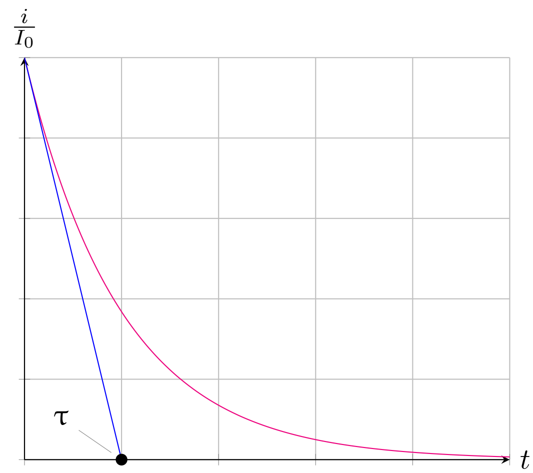


Figure 2.3: Finding the time constant graphically

Yet another way to define the time constant is to note that

$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679$$

ie, in one time constant, the response falls to 36.8% of it's initial value. The value of τ may also be determined from this fact:

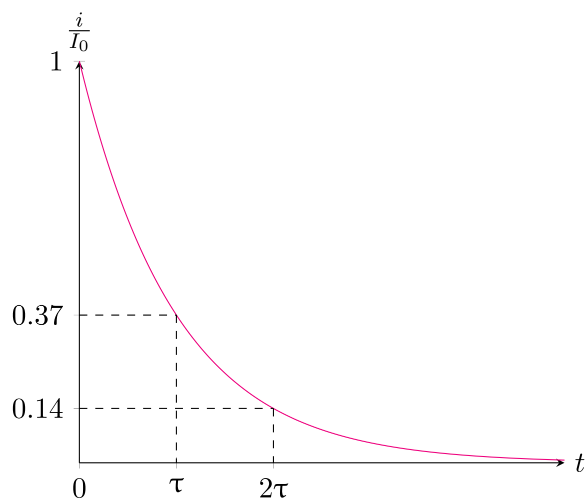


Figure 2.4: Yet another way to find the time constant graphically

Example 2.1

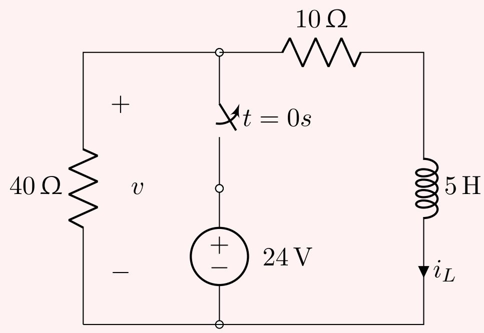
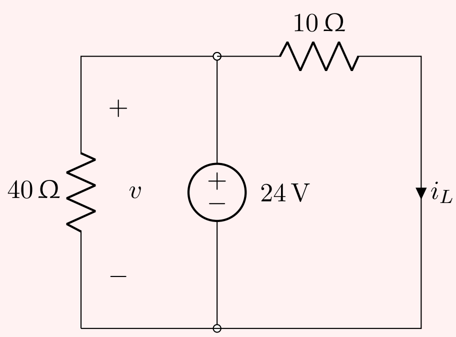


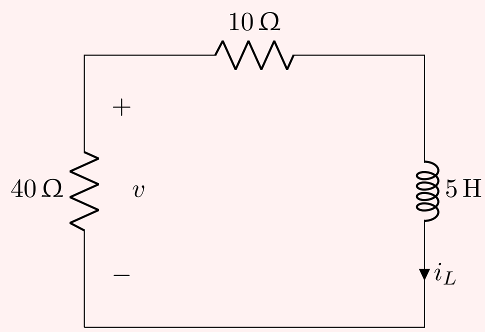
Figure 2.5: A simple RL circuit with a switch thrown at $t = 0$

For the circuit given in figure 2.5, let us try to find the voltage at $t = 200ms$.

Figures 2.6a and 2.6b show the state of the circuit prior to and post the throwing of the switch. In 2.6a, the inductor acts like a short to DC current after all transients have died down.



(a) The circuit prior to $t = 0$



(b) The circuit post $t = 0$

Figure 2.6: The simplified circuit before and after the switch is opened

Applying Kirchhoff's Law in 2.6b, we may write

$$
\begin{aligned}
-v + 10i_L + 5\frac{di_l}{dt} &= 0 \\
\Rightarrow \frac{5}{40}\frac{dv}{dt} + \left(\frac{10}{40} + 1\right) v &= 0 \\
\Rightarrow \frac{dv}{dt} + 10v &= 0
\end{aligned}
$$

where we have taken $i_l = -v/40$ by our conventions. The solution to this differential equation is

$$v(t) = v(0^+)e^{-10t}$$

Note that the initial value of voltage here is denoted by $v(0^+)$, as the voltage across the resistor may change discontinuously when the switch is thrown. We have to use the fact that i_L cannot change discontinuously, and remains the same just before and just after the switch is thrown. From 2.6a, $i_l(0) = 24/10 = 2.4$A. Hence, $v(0^+) = (40) \cdot (-2.4) = -96$V, and so

$$\boxed{v(t) = -96e^{-10t}}$$

which gives $v(0.2) = -12.99$V.

2.1.3 The Source-Free RC Circuit

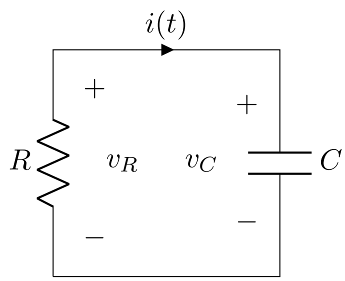


Figure 2.7: Source-Free RC Circuit

The source free response for RC circuits is nearly the same as with LC, but with slightly different expressions for the time constant. In this case, the general form of the response is given by

$$\boxed{v(t) = v(0)e^{-\frac{t}{\tau}}} \tag{2.5}$$

and the expression for the time constant is

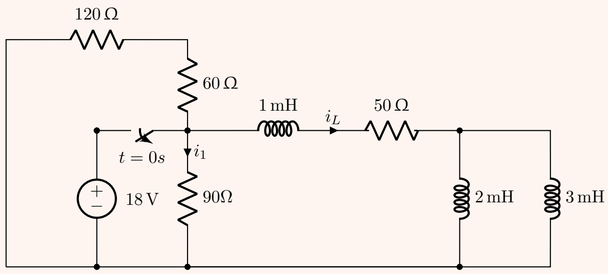
$$\boxed{\tau = R \cdot C} \tag{2.6}$$

It may also be noted that the voltage across a capacitor cannot change discontinuously, as opposed to the current for an inductor.

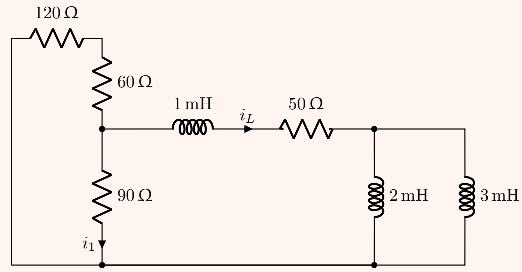
2.1.4 General RL and RC Circuits

For circuits containing more than one capacitor/inductor and more than one resistor, it is possible to arrive at a single time constant and reduce it to a simple circuit with one resistor and one energy storage element, as long as the Thévenin resistance "seen" by all the elements is the same. In such a case, we can write $\tau = \frac{L_{eq}}{R_{eq}}$ or $\tau = R_{eq} \cdot C_{eq}$. The general form of the natural response would then be $Ae^{-t/\tau}$, where the constant A can be determined using the initial conditions, and would be different for different elements. Let us illustrate this using the following example:

Example 2.2



(a) The circuit with multiple inductors and resistors



(b) The simplified circuit after $t = 0$

For the circuit given in figure 2.8a find i_1 and i_L for $t > 0$.

Firstly, note that the equivalent/Thévenin resistance seen by all three inductances is equal, and equal to $50 + 90||(120 + 60) = 110\text{V}$. So, we can reduce it to a simple LR circuit, with

$$L_{eq} = \frac{2 \cdot 3}{2+3} + 1 = 2.2\text{mH}$$
$$\Rightarrow \tau = \frac{L_{eq}}{R_{eq}} = 20\mu\text{s}$$

And so every current and voltage in the network must have the form $Ke^{-t/\tau} = Ke^{-50{,}000t}$. If we consider the circuit prior to opening the switch and after all transients have died down, the inductors act as shorts to DC current and so i_L is easily found to be $i_L = 18/50 = 360mA$. This must also be the value of i_L just after the switch is opened. So,

$$i_L = \begin{cases} 360\text{mA} & t < 0 \\ 360e^{-50{,}000t}\text{mA} & 0 \leq t \end{cases}$$

Now, i_1 may change discontiuously at $t = 0$ so we have to find it's value at 0^+ using the value of $i_L(0^+)$. Using current division,

$$i_1(0^+) = -i_L(0^+) \cdot \frac{120 + 60}{120 + 60 + 90} = -240mA$$

Hence,

$$i_1 = \begin{cases} 200\text{mA} & t < 0 \\ -240e^{-50{,}000t}\text{mA} & 0 \leq t \end{cases}$$

2.2 The Unit Step Function

The unit step function, also called the **Heaviside step function** is a step function whose value is zero for negative arguments and one for positive arguments. This function is important as the

switching action of a battery is equivalent to a forcing function which is zero up until the switch is thrown, and equal to the battery voltage thereafter. It is represented by u, θ or H.
If we shift the argument to some arbitrary time t_0, then $u(t-t_0)$ must be zero for all values of t less than t_0 and unity for values of t greater than t_0. The unit step function changes abruptly/ discontinuously from 0 to 1, and it's value at $t = t_0$ is not defined.
It can be represented analytically as

$$u(t-t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

or graphically as

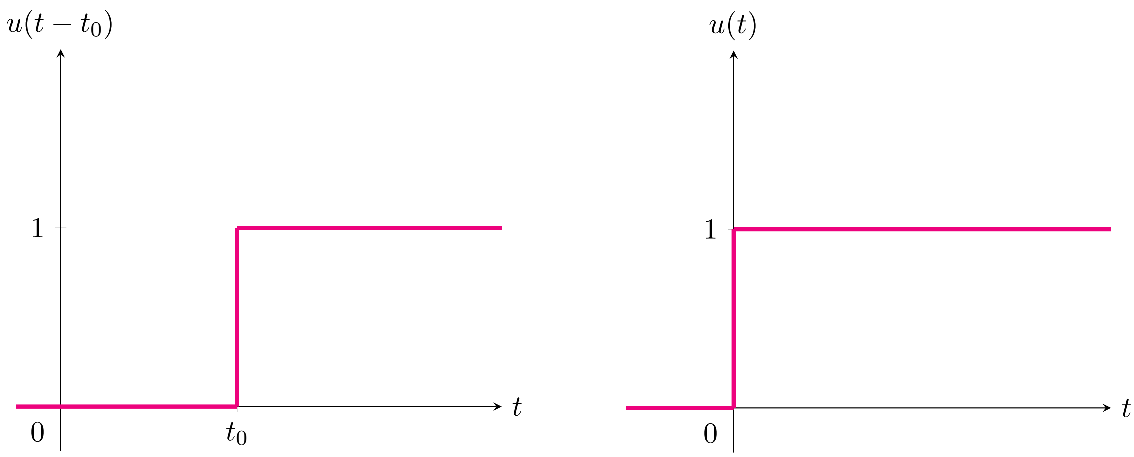


Figure 2.9: The unit step function

Note that the vertical line is not really a part of the unit step's definition but it is usally shown in each drawing as it looks nicer.

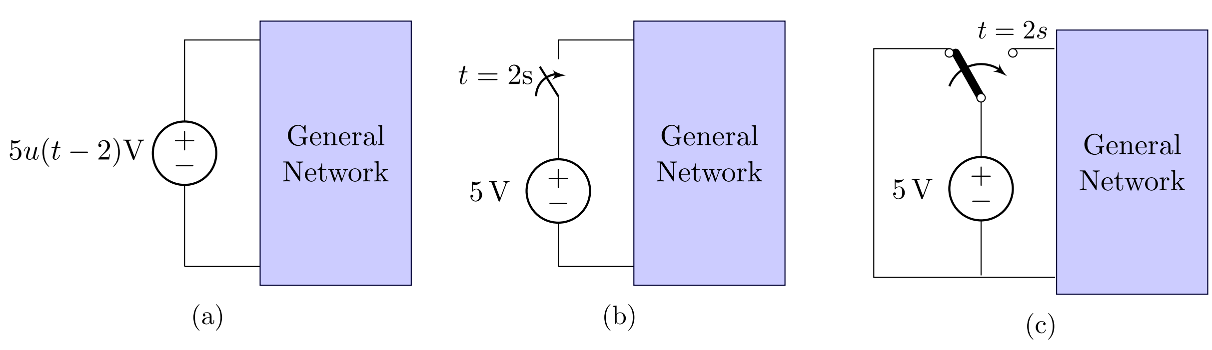


Figure 2.10: (a) The voltage-step forcing function. (b) A possible equivalent. (c) The exact equivalent.

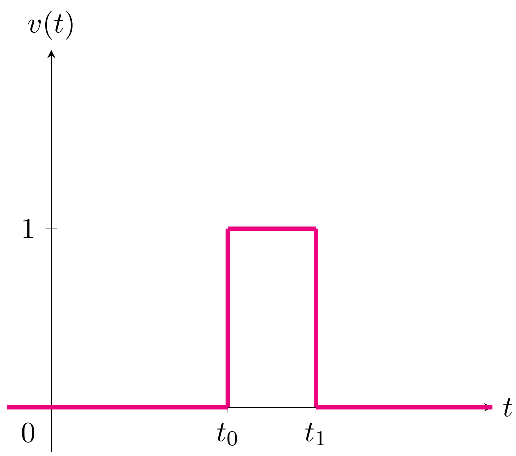
Interestingly, the physical equivalent of the unit step is not actually a simple voltage source in series with a switch. Consider the step voltage source shown in figure 2.10a. Let us try to draw it's physical equivalent. On first thought, one might come up with figure 2.10b. However, these two circuits are not equivalent for $t < 2$s: the voltage across the network in figure 2.10b is completely unspecified in this time interval. The actual equivalent to the step voltage source is the single-pole double throw switch shown in figure 2.10c. Clearly, the voltage is zero for $t < 2$s, and this is consistent with the step voltage function.

2.2.1 The Rectangular Pulse Function

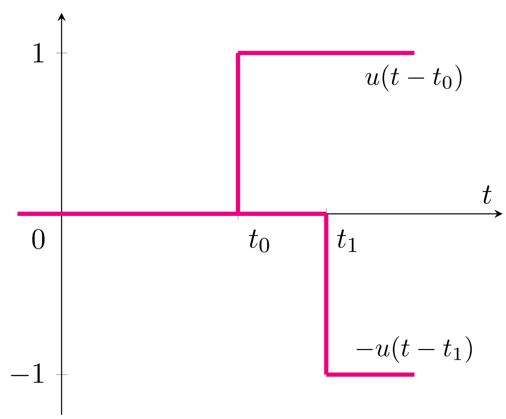
Various rectangular pulses may be obtained by manipulating the unit step function. Let us first consider the pulse given by

$$v(t) = \begin{cases} 0 & t < t_0 \\ V_0 & t_0 < t < t_1 \\ 0 & t > t_1 \end{cases}$$

Let us try to break this down into individual unit step functions. Figure 2.11b shows the superposition of the two step functions $u(t - t_0)$ and $-u(t - t_1)$. Clearly, their superposition gives the rectangular pulse in figure 2.11a.



(a) The Rectangular Pulse



(b) Superposition of two unit step functions

Breaking down rectangular pulses in terms of unit step functions can be quite handy for linear networks, for which we can simply consider individual unit step sources separately and add them to get the total response.

2.3 The Forced Response

Let us now subject a simple network to the sudden application of a DC source. Figure 2.12 shows a circuit for which $i(t)$ is to be determined, consisting of a voltage source in series with a resistance and an inductor, with the switch thrown at $t = 0s$. Note that we also could have removed the switch and represented the voltage source as $V_0 u(t)$ (not *exactly* correct, as said previously).

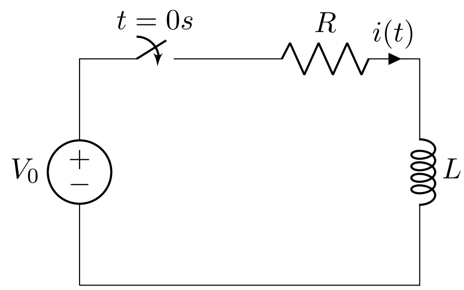


Figure 2.12: The Driven RL Circuit

Clearly, $i(t) = 0$ for $t < 0s$. For $t > 0s$, we may apply Kirchhoff's voltage law and obtain the differential equation

$$Ri + L\frac{di}{dt} = V_0 \tag{2.7}$$

Now, we can separate variables and integrate easily:

$$\begin{aligned} \int_0^i \frac{L}{V_0 - Ri}\, di &= \int_0^t dt \\ \Rightarrow \frac{L}{R} \ln(V_0 - Ri)\Bigg|_0^i &= t \\ \Rightarrow i(t) &= \frac{V_0}{R}(1 - e^{-Rt/L}) \end{aligned}$$

which is the response for $t > 0s$. Note that here $i(0^-) = i(0^+) = 0$. The complete response may be written conveniently in terms of the unit step function as

$$\boxed{i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})u(t)} \tag{2.8}$$

We have arrived at this expression purely by mathematical methods. There is an easier way to arrive at this expression, and it's convenience is highlighted even more when analysing second order driven circuits. For now, let us try to interpret the terms appearing in equation 2.8.

There is one constant, $\frac{V_0}{R}$ and our familiar expression for natural circuits: $\frac{V_0}{R}e^{-t/\tau}$. The constant is nothing but the current at steady state, after transients have died down and is also called the *steady state response, particular solution* or the *particular integral*. It is found simply by analysing the circuit at steady state - shorting all inductors and opening all capacitors.
The other term is the natural response, which approaches zero as the time increases without limit. The amplitude of this natural response will depend on the initial value of the complete response, and hence also on the initial value of the forcing function. Let us now try arriving at equation 2.8 in this new light. The total response will comprise of a natural response, i_n and a forced response, i_f. Hence,

$$i = i_n + i_f$$

where i_n must have the form $Ae^{-Rt/L}$, where A is a constant to be determined using the initial condition of the forcing function. The value of i_f is found by considering the circuit after the natural response has died out, when the inductor will act as a short. Clearly, in this condition, we have a single resistor in series with the voltage source and hence $i_f = \frac{V_0}{R}$. So,

$$i = \frac{V_0}{R} + Ae^{-Rt/L}$$

We may now apply the initial condition $i(0^-) = i(0^+) = 0$:

$$0 = \frac{V_0}{R} + Ae^0$$
$$\Rightarrow A = -\frac{V_0}{R}$$

and so we have

$$i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})u(t)$$

The response is plotted in figure 2.13, and it is clear how i rises from it's intial value of 0 to it's forced value of $\frac{V_0}{R}$. In this case, a tangent drawn to the response curve at $t = 0$ meets the forced response at $t = \tau$.

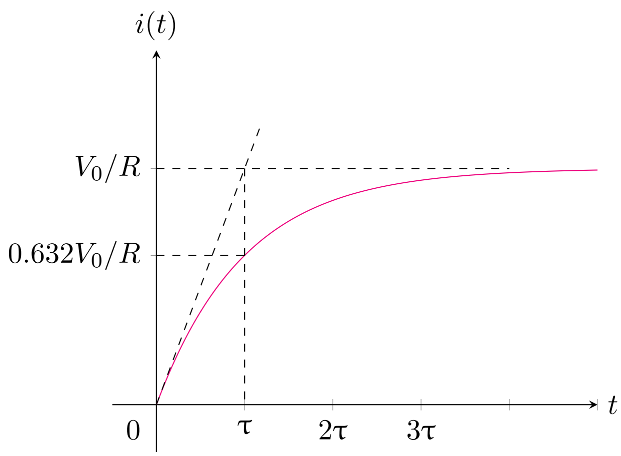


Figure 2.13: The plot of $i(t)$

Why we can break the complete response into two parts

There is a mathematical reason to why we can break the forced response i into a sum of response, i_n and i_f. Consider the differential equation governing the forced response, a first order non-homogenous differential equation:

$$Ri + L\frac{di}{dt} = V_0 \tag{2.9}$$

Let us say we have a solution to the homogenous part of 2.9, i_n, so that

$$Ri_n + L\frac{di_n}{dt} = 0$$

And let us say we have some solution i_f to 2.9 itself, so that

$$Ri_f + L\frac{di_f}{dt} = V_0$$

Adding the above two equations, we have:

$$R(i_n + i_f) + L\frac{d}{dt}(i_n + i_f) = 0$$

meaning that $i_n + i_f$ satisfies 2.9 just as well as i_f alone. So we have no reason to discard the contribution of i_n. From a physical point of view, i_n is the natural response, more commonly known as transient response in mechanics, and i_f is the forced response, known as the steady state response in mechanics. It may be verified that the same holds true for second order non-homogenous differential equations, or second-order forced circuits.

The procedure to find the forced response for driven RC circuits is exactly the same as that for RL circuits. We won't repeat it here, but will instead illustrate it with the following example:

Example 2.3

Determine an expression for $v(t)$ in the circuit of figure 2.14 valid for $t > 0$.

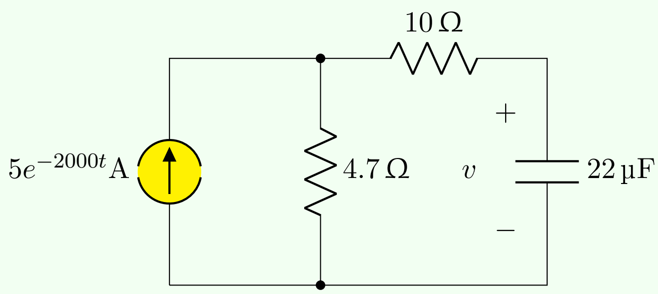


Figure 2.14: An RC circuit forced by an exponentially decaying function

We expect a complete response of the form

$$v(t) = v_f + v_n$$

Where v_n is of the form $Ae^{it/\tau}$ and v_f is reminiscent of the exponential forcing function. To calculate the time constant for this circuit, note that the Thévenin resistance across the capacitor is

$$R_{eq} = 4.7 + 10 = 14.7\Omega$$

and hence our time constant is $\tau = R_{eq} \cdot C = 323.4$µs. At this point, it will be convenient to perform a source transformation resulting in a voltage source $23.5e^{-2000t}$ in series with 14.7Ω and 22µF. Applying Kirchhoff's loop law,

$$23.5e^{-2000t} = (14.7)(22 \cdot 10^{-6})\frac{dv}{dt} + v$$

$$\Rightarrow \frac{dv}{dt} + (3.092 \cdot 10^3)v = 72.67 \cdot 10^3 e^{-2000t}$$

Comparing this with the standard solution to a first order non-homogenous linear equation

$$v(t) = e^{-Pt} \int Qe^{Pt}\, dt + Ae^{-Pt}$$

we find that $P = 1/\tau = 3.092 \cdot 10^3$ and $Q(t) = 72.67 \cdot 10^3 e^{-2000t}$ and after performing the necessary integration, we get

$$v(t) = 66.55e^{-2000t} + Ae^{-3092}$$

and since the voltage across a capacitor cannot change discontinuously with time, our initial conditions are $v(0^-) = v(0^+) = 0$. Incorporating these into our response, finally we get

$$v(t) = 66.55(e^{-2000t} - e^{-3092})\mathrm{V}$$

for $t > 0s$. Is this expected? We have one term of the form e^{-2000t}, our forcing function and one term of the form e^{-3092}, our natural response, and so everything checks out. Interestingly, we can find this response in just 3 or 4 steps using Laplace and Fourier transforms, without even having to write down a single differential equation.

The methods of finding a response for a forced circuit are summarised neatly below:

1. With all independent sources zeroed out, simplify the circuit to determine R_{eq}, C_{eq} and the time constant $\tau = R_{eq} \cdot C_{eq}$
2. Viewing C_{eq} as an open circuit, find $v_c(0^-)$, the capacitor voltage prior to the discontinuity.
3. Viewing C_{eq} as an open circuit, find the forced response $f(\infty)$
4. Write the total response as a sum of forced and natural responses: $f(t) = f(\infty) + Ae^{-t/\tau}$.
5. Find $f(0^+)$ using the fact that the capacitor voltage may not change discontinuously, ie. $v_c(0^+) = v_c(0^-)$.
6. Apply this initial condition $f(0^+)$ to find the value of A, and hence the complete response.

The method to find the complete response for an LR circuit is, of course, the dual of the statements listed above.

2.4 An Application Of First Order Circuits: The Astable 555 Timer IC

Figure 2.15 shows the 555 IC being operated as an astable timer. The components inside the cyan box are all part of the IC, the components outside are under our control. We have chosen to connect an LED across the output of the IC. Let us see how the LED is made to blink periodically, and also find an expression for this time period.

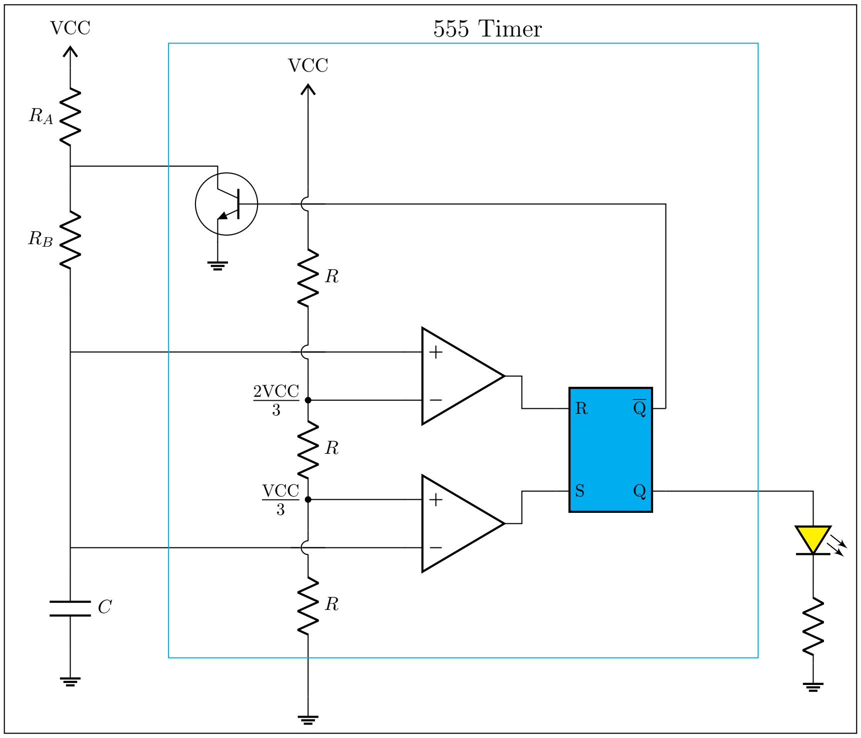


Figure 2.15: The 555 IC being operated as an astable timer, with a simplified version of it's internal circuitry

The three resistances between ground and V_{CC} in the IC are equal, and hence we can assign the node voltages $\frac{\text{VCC}}{3}$ and $\frac{2\text{VCC}}{3}$ at the positive of the lower comparator and the negative of the upper comparator, respectively. Let us say we have connected the capacitor at $t = 0$. At this instant, the voltage at the positive of the upper comparator and the negative of the lower comparator will be $0V$, and hence the upper comparator is off and the lower comparator is on. So, the SR latch is *set* and the output Q is high, the LED is glowing. When the voltage across the capacitor reaches $\frac{\text{VCC}}{3}V$, the lower comparator is no longer on. However, this does not change anything, since the reset pin of the latch is also low, and the output Q remains high. When the voltage across the capacitor reaches $\frac{2\text{VCC}}{3}V$, the upper comparator is turned on and the latch is *reset*. The output goes low and the LED stops glowing. Now, $\overline{Q}$ turns on and hence the BJT also turns on, and current can flow from the capacitor to ground. The capacitor starts discharging.

When the voltage across the capacitor drops below $\frac{\text{VCC}}{3}V$, the set pin goes high again and the output Q goes high. The LED starts glowing, the capacitor starts charging till it reaches $\frac{\text{VCC}}{3}V$ and the entire process is repeated again. We now have an idea of how this timer works, and we can try and calculate the exact time period for charging and discharging.

Firstly, note that the capacitor charges through both the resistors R_A and R_B and discharges only through R_B. The output waveform obtained at Q will therefore be rectangular. For the charging and discharging of the capacitor, the general equations are

$$v(t) = v_0 \cdot (1 - e^{-t/(R_A+R_B)C})$$
$$v(t) = v_0 \cdot e^{-t/(R_B)C}$$

Where, of course, v_0 can be any one of $V_{CC}/3$, $2V_{CC}/3$ or V_{CC}. Consider first the time taken to charge the capacitor. Incorporating the initial condition $v(0) = V_{CC}/3$, the equation for the charging half is

$$v(t) = \frac{2V_{CC}}{3}(1 - e^{-t/(R_A+R_B)C}) + \frac{V_{CC}}{3}$$

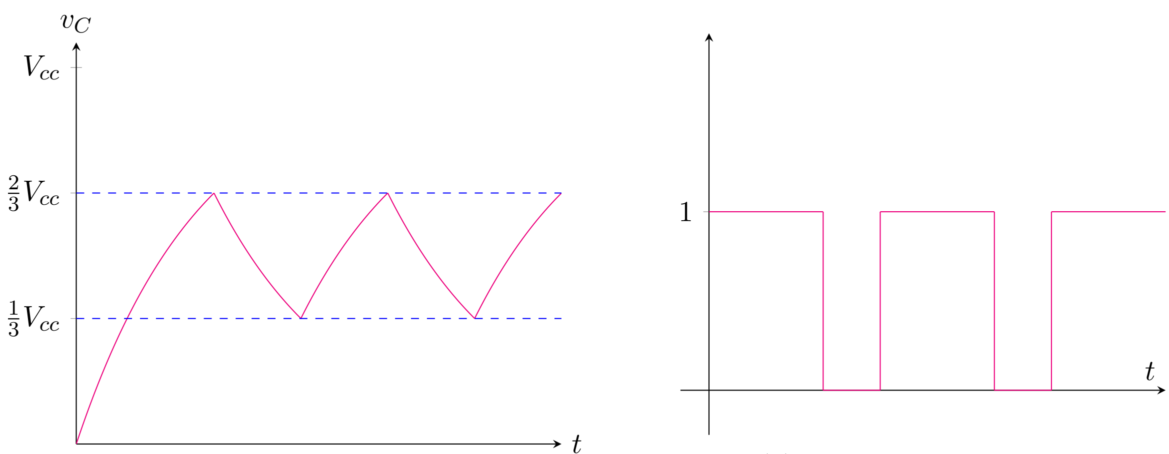
Substituting $v(t) = 2V_{CC}/3$, we find that the time taken t_1 by the capacitor to charge is

$$t_1 = \ln(2) \cdot (R_A + R_B) \cdot C \tag{2.10}$$

Similarly, it may be shown that the time t_2 taken by the capacitor to discharge is

$$t_2 = \ln(2) \cdot (R_B) \cdot C \tag{2.11}$$

These are the standard expressions one may find in any datasheet. The output waveform across the capacitor and Q is plotted in figure 2.16.



(a) The waveform across C

(b) The waveform across Q

Figure 2.16: The output waveforms of the 555 astable timer

3 Second Order Circuits

Let us now consider circuits governed by a second order differential equation, those containing more than one capacitor and inductor. The response shall vary according to the value of the components R, L and C. We will first consider parallel RLC circuits.

3.1 The Parallel RLC

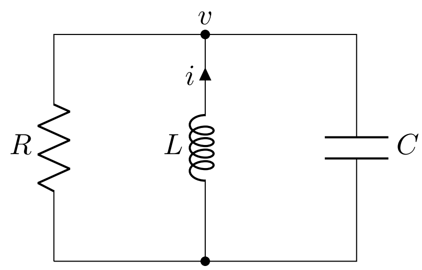


Figure 3.1: The source-free parallel RLC circuit

Let us assume that there is some energy stored initially in the capacitor and the inductor. With reference to figure 3.1, we may then write the nodal equation, using Kirchhoff's current law:

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v\,dt' - i(t_0) + C\frac{dv}{dt} = 0$$

subject to the initial conditions $i(0^+) = I_0$, $v(0^+) = V_0$. Differentiating the above equation, we obtain the differential equation to the parallel RLC circuit:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{3.1}$$

whose solution $v(t)$ is the desired natural response. The auxilliary equation to 3.1 is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Using the quadratic formula, we obtain the roots:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \tag{3.2}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \tag{3.3}$$

At this stage, it is convenient to define

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{3.4}$$

and

$$\alpha = \frac{1}{2RC} \tag{3.5}$$

The term ω_0 is called the ***resonant frequency***, or the ***neper frequency*** and the term α is called the ***exponential damping coefficient***. The two roots are then conveniently given by

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \tag{3.6}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \tag{3.7}$$

and the natural response of the parallel RLC is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{3.8}$$

where A_1 and A_2 may be complex numbers. The three conditions imposed on the discriminant of the quadratic give rise to three different types of responses, known as

- $\alpha > \omega_0$, both roots are real: **overdamped**
- $\alpha = \omega_0$, both roots are equal: **critically damped**
- $\alpha < \omega_0$, both roots are imaginary: **underdamped**

We will consider these cases one by one.

3.1.1 Overdamped Response

For the overdamped case, it may be easily shown that both roots are *negative* real numbers. The response,

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

is then the sum of two exponentially decreasing terms, both of which approach zero as the time increases. What remains is to determine the constants A_1 and A_2 in conformance with the initial conditions. We shall illustrate this using an example.

Example 3.1

For the circuit given in Fig.3.2, determine an expression for the resistor current i_R valid for all time.

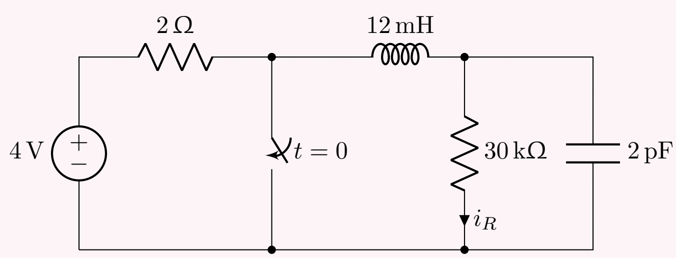


Figure 3.2: Circuit for which i_R is required

After the switch is thrown, we have $R = 30\text{k}\Omega$, $L = 12\text{mH}$ and $C = 2\text{pF}$. Thus, $\alpha = 8.333 \times 10^6 \text{s}^{-1}$ and $\omega_0 = 6.455 \times 10^6 \text{rad s}^{-1}$. This is an overdamped response, with $s_1 = -3.063 \times 10^6 \text{s}^{-1}$ and $s_2 = -13.60 \times 10^6 \text{s}^{-1}$, so that

$$i_R(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

for $t > 0$. To determine the initial conditions, note that $i_L(0^-) = i_R(0^-) = 4/32 \times 10^3 = 125\mu\text{A}$ and $v_C(0^-) = 4 \times 30/32 = 3.75\text{V}$.
For $t > 0$, we only know that $i_L(0^+) = 125\mu\text{A}$ and $v_C(0^-) = 3.75\text{V}$. By Ohm's law, we find $i_R(0^+) = 3.75/30 \times 10^3 = 125\mu\text{A}$, our first initial condition. So,

$$i_R(0) = A_1 + A_2 = 125\mu\text{A} \tag{3.9}$$

Also,

$$i_C(0^+) = i_L(0^+) - i_R(0^+) = 0$$

So,

$$C\frac{dv_C}{dt}\bigg|_{t=0} = 0$$

$$\Rightarrow (2 \times 10^-12) \cdot (30 \times 10^3) \cdot (3.063 \times 10^6 A_1 + 13.60 \times 10^6 A_2) = 0 \tag{3.10}$$

Solving equations 3.9 and 3.10, we get

$$i_R = \begin{cases} 125\mu\text{A} & t < 0 \\ 161.3e^{-3.063 \times 10^6 t} - 36.34e^{-13.60 \times 10^6 t} & t > 0 \end{cases}$$

Graphical Form of the Overdamped Response

Let us now try to graphically represent the overdamped response. For this purpose, we shall consider a parallel RLC circuit with "nicer" values of the components: $R = 6\Omega$, $L = 7\text{H}$ and $C = \frac{1}{42}\text{F}$. With the initial conditions $v(0) = 0$ and $i_L(0) = 10$, the response is

$$v(t) = 84(e^{-t} - e^{-6t})$$

This response is plotted in Fig.3.3 .

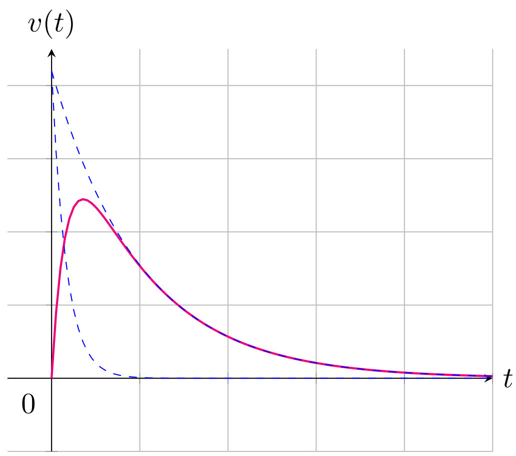


Figure 3.3: The overdamped response

If we consider this response as the superposition of $84e^{-t}$ and $84e^{-6t}$, it is clear from the figure that as time goes on, the $84e^{-t}$ factor is more dominant and $84e^{-6t}$ becomes negligible.

3.1.2 Critical Damping

Critical damping is achieved when

$$critical\ damping \begin{cases} \alpha = \omega_0 \\ LC = 4R^2C^2 \\ L = 4R^2C \end{cases}$$

Here the common root s comes out to be equal to α.
The response in this case is *not* given by Ae^{st}. Even intuitively, we can observe that this soution contains only one arbitrary constant, whereas our differential equation is second order. From any standard text on differential equations, we find that the actual response is

$$v(t) = e^{-\alpha t}(A_1t + A_2) \tag{3.11}$$

Note that this equation contains two arbitrary constants A_1 and A_2 and therefore is the correct solution to our differential equation.

Graphical Form of the Critically Damped Response

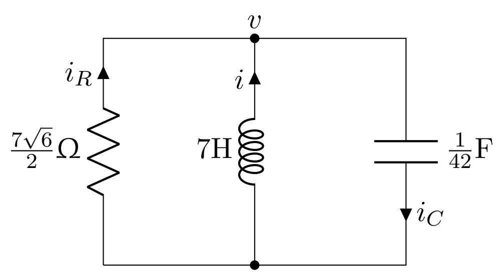


Figure 3.4

Let us take up a critically damped parallel LCR circuit with $R = 7\sqrt{6}/2\Omega$, $L = 7$H and $C = 1/42$F with the initial conditions $v(0) = 0$ and $i_L(0) = 10$. The circuit is shown in figure 3.4.We expect a response of the form

$$v(t) = e^{-\sqrt{6}t}(A_1t + A_2)$$

The first condition gives $A_2 = 0$. The second initial condition gives

$$\left.\frac{dv}{dt}\right|_{t=0} = 0 = \frac{i_C(0)}{C} = \frac{i_R(0)}{C} + \frac{i(0)}{C} = A_1$$

so that $A_1 = 420$V. Therefore, the response is

$$v(t) = 420te^{-2.45t}$$

Let us try to represent it graphically. It's value at $t = 0$ is 0 and for large values of t,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{420t}{e^{2.45t}} = \frac{420}{2.45} \lim_{t\to\infty} \frac{1}{e^{2.45t}}$$

using L'Hôspital's rule. By differentiating, we find that a maximum value occurs at time $t_m = 0.408s$ when $v = 63.1$V. The response is plotted in figure 3.5.

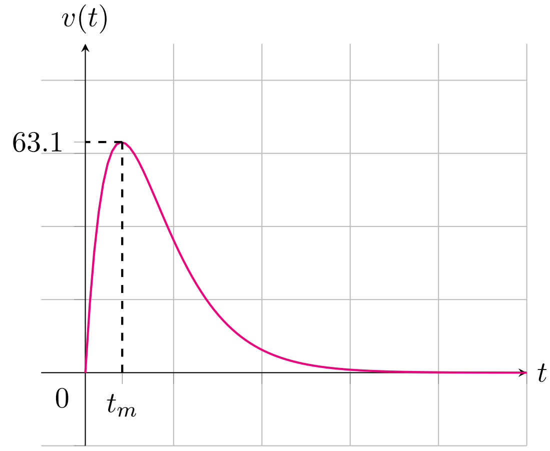


Figure 3.5: The critically damped response

Example 3.2

Find a value of R_1 such that the circuit in figure 3.6 will be characterized by a critically damped response for $t > 0s$, and a value of R_2 such that $v(0) = 2\text{V}$

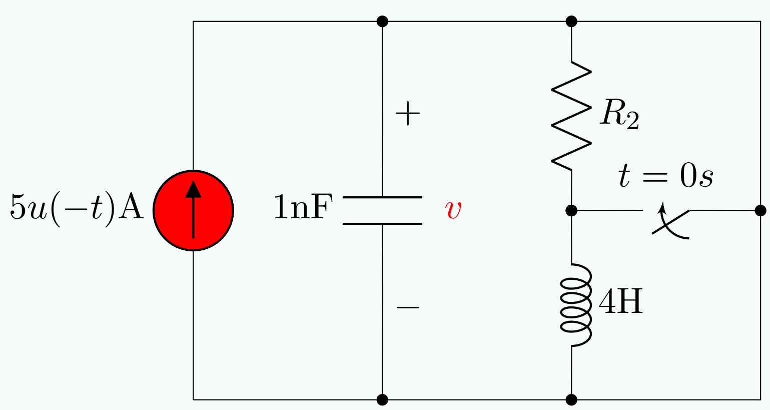


Figure 3.6: Circuit reduces to a parallel RLC after the switch is thrown

At $t = 0^-$, the current source is on and the inductor may be treated as a short. Thus,

$$v(0^-) = 5R_2$$

and so $R_2 = 500\text{m}\Omega$ for $v(0) = 2\text{V}$. For $t > 0s$, the current source is off and R_2 is shorted. Thus it is a parallel RLC circuit and

$$\begin{aligned}\alpha &= \frac{1}{2RC} \\ &= \frac{1}{2 \cdot 10^{-9}C}\end{aligned}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= 15,810\mathrm{rad\,s^{-1}}$$

For a critically damped response, these two values must be equal, hence $R_1 = 31.63\mathrm{k}\Omega$.

3.1.3 Underdamped Response

Form of the Underdamped Response

The exponential form of the response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

For the overdamped case, we have $\alpha < \omega_0$, and so we let

$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\sqrt{\omega_0^2 - \alpha^2}$$

where $\sqrt{\omega_0^2 - \alpha^2}$ is a real quantity and is denoted by ω_d, called the ***natural resonant frequency***:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The response may now be written in terms of ω_d, converting complex exponentials into sines and cosines:

$$\begin{aligned} v(t) &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{j\omega_d t}) \\ &= e^{-\alpha t}((A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t) \end{aligned}$$

Introducing new constants:

$$v(t) = e^{-\alpha t}(B\cos\omega_d t + C\sin\omega_d t) \tag{3.12}$$

At first it may seem a little strange that the complex part of our response has simply vanished. However, we had originally allowed for A_1 and A_2 to be complex. In any case it is easily verified that equation 3.12 is indeed a solution of 3.1 by simple substitution.
Let us now return to our previous example of figure 3.2, but with $R = 10.5\Omega$, $C = 1/42\mathrm{F}$ and $L = 7\mathrm{H}$. Thus,

$$\begin{aligned} \alpha &= \frac{1}{2RC} = 2s^{-2} \\ \omega_0 &= \frac{1}{\sqrt{LC}} = \sqrt{6}\mathrm{rad\,s^{-1}} \\ \omega_d &= \sqrt{\omega_0^2 - \alpha^2} = \sqrt{6-4} = \sqrt{2}\mathrm{rad\,s^{-1}} \end{aligned}$$

and so our response is

$$v(t) = e^{-2t}(B_1 \cos\sqrt{2}t + B_2 \sin\sqrt{2}t)$$

Fitting in our initial conditions $i(0) = 0$ and $v(0) = 0$, we find that the exact response is

$$v(t) = 210\sqrt{2}e^{-2t} \sin\sqrt{2}t$$

Graphical Form of the Underdamped Response

The underdamped response of equation 3.12 can be thought of as a sinusoid with exponentially decreasing amplitude. For large values of α, the response quickly dies out and the oscillatory nature is not very clear. The response of our previous example is plotted in figure 3.7.

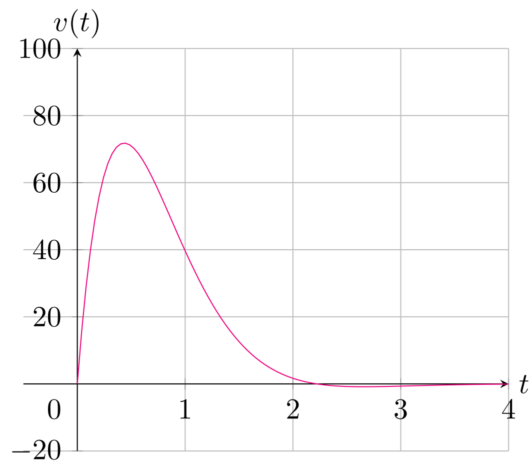


Figure 3.7: The response for our example

Let us now plot a response characterized by

$$v(t) = e^{-0.5x} \cos 10x$$

which is plotted in figure 3.8 and shows much sharper oscillatory behaviour.
If α is 0, corresponding to an infinitely large resistance, the response is an undamped sinusoid of constant amplitude. In such a case, we provide the circuit with some initial energy and give it no means to dissipate that energy: it simply keeps oscillating back and forth between the capacitor and inductor.
The effect of a finite resistance in the RLC circuit can be thought of as an electrical transfer agent. Everytime energy is transferred from L to C, some amount is dissipated through the resistor. Eventually, the resistor has dissipated all of the energy and the capacitor and inductor are left with none.

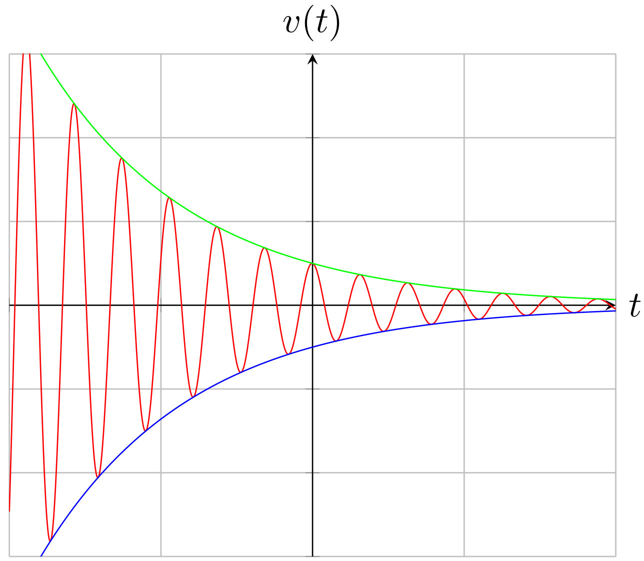


Figure 3.8: An underdamped response with a small value of α