# The Coupled Pendulum

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## 1 Two Coupled Pendulums

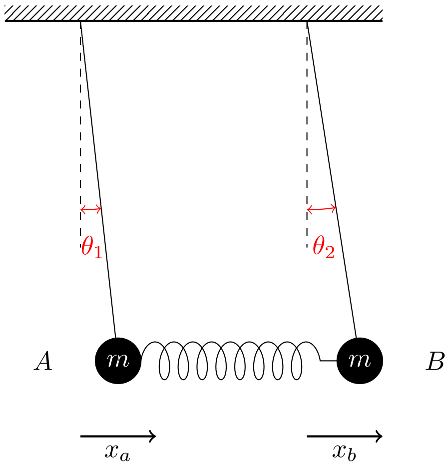


Figure 1: Two pendulums coupled by an ideal spring

Consider two pendulums of equal mass $m$, $A$ and $B$, coupled by an ideal massless spring of spring constant $k$, as shown in Figure 1. The string may be considered sufficiently light so that it's mass may be neglected compared to the bobs. The equations of motion, considering small angle approximations ($\sin\theta \approx \theta, \ddot{y} \approx 0$), for the two pendulums are

$$m\frac{d^2x_a}{dt^2} = -mg\frac{x_a}{l} + k(x_b - x_a)$$
$$m\frac{d^2x_b}{dt^2} = -mg\frac{x_b}{l} - k(x_b - x_a)$$

or

$$\ddot{x}_a + (\omega_0^2 + \omega_c^2)x_a - \omega_c^2 x_b = 0 \tag{1}$$
$$\ddot{x}_b + (\omega_0^2 + \omega_c^2)x_b - \omega_c^2 x_a = 0 \tag{2}$$

where we have let $\omega_0^2 = \frac{g}{l}$ and $\omega_c^2 = \frac{k}{m}$

## 2 Normal Modes

Before we try to solve equations 1 and 2 for the most general motion of the system, let us consider what happens when we draw both $A$ and $B$ aside by equal amounts and release them. The spring remains relaxed and exerts no force on either masses. Both of them then oscillate with the same natural frequency $\omega_0$:

$$x_a = C\cos\omega_0 t$$
$$x_b = C\cos\omega_0 t$$

This is known as a *normal mode of oscillation*, where all masses oscillate with the same frequency. For this system, there is yet another normal mode: pull $A$ and $B$ aside by equal amounts but in opposite direction. The equation of motion for $A$ is

$$\ddot{x}_a + (\omega_0^2 + 2\omega_c^2)x_a = 0$$

which is readily identified as a simple harmonic motion of frequency $\omega\prime = (\omega_0^2 + 2\omega_c^2)^{1/2}$, so that

$$x_a = C\cos\omega^{'} t$$
$$x_b = -C\cos\omega^{'} t$$

The two normal modes are illustrated in figure 2.

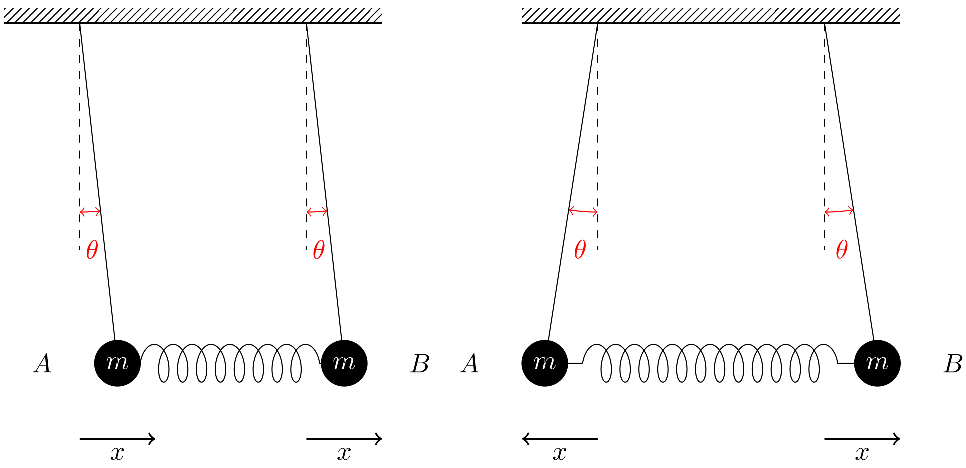


Figure 2: The two normal modes

# 3 Superposition of the normal modes

Let us now try to solve equations 1 and 2 for general initial conditions. The symmetry of the two equations suggests that adding and subtracting them might give some joy:

$$\ddot{x_a} + \ddot{x_b} + \omega_0^2(x_a + x_b) = 0$$
$$\ddot{x_a} - \ddot{x_b} + (\omega_0^2 + 2\omega_c^2)(x_a - x_b) = 0$$

Indeed, these are two simple harmonic motions. Introducing the *normal co-ordinates* $q_1 = x_a + x_b$ and $q_2 = x_a - x_b$,

$$\ddot{q_1} + \omega_0^2 q_1 = 0$$
$$\ddot{q_2} + \omega\prime^2 q_2 = 0$$

whose solutions are

$$q_1 = C\cos\omega_0 t$$
$$q_2 = D\cos\omega\prime t$$

where we have already assumed two initial conditions (initial phases are zero) for simplicity. In terms of our original co-ordinates,

$$x_a = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}C\cos\omega_0 t + \frac{1}{2}D\cos\omega\prime t$$
$$x_b = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}C\cos\omega_0 t - \frac{1}{2}D\cos\omega\prime t$$

We see that the general motion of the oscillator is a superposition of it's normal modes. This is a general result for any number of coupled oscillators. Let us now consider what happens when we pull aside pendulum $A$ by a small amount while keeping $B$ fixed, ie. the above equations subject to the initial conditions at $t = 0$

$$x_a = A_0$$
$$\dot{x_a} = 0$$
$$x_b = 0$$
$$\dot{x_b} = 0$$

We obtain

$$x_a = A_0\cos\frac{\omega\prime - \omega_0}{2}t\cos\frac{\omega\prime + \omega_0}{2}t$$
$$x_b = A_0\sin\frac{\omega\prime - \omega_0}{2}t\sin\frac{\omega\prime + \omega_0}{2}t$$

These motions are plotted in figure . We see that $A$ starts swinging initially, but it's amplitude continuously decreases. Pendulum $B$, initially at rest, starts oscillating and soon the amplitudes of $A$ and $B$ become equal. The amplitude of $A$ then diminishes towards zero and the amplitude of $B$ becomes that of $A$ originally. The spring is therefore acting as a kind of energy transfer agent between the two pendulums.