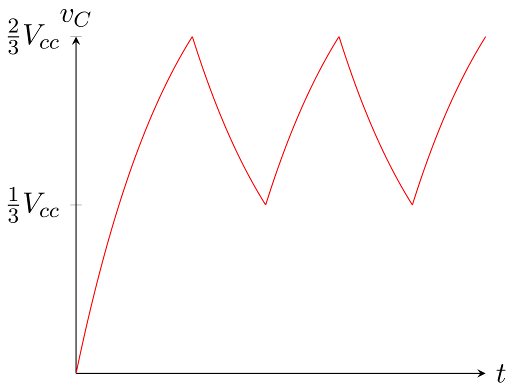
Time Domain Analysis

RL Circuits, RC Circuits, Sinusoidal Forcing Functions

Jay Khandkar



The famous astable 555 waveform

January 18, 2021

Contents

1 Current And Voltage Conventions

The conventions we shall follow, called ***passive sign conventions***, simply state that when current "flows into" the positive terminal of the capacitor/inductor, as indicated by the polarity of v in Fig.1.1, it is taken as positive. We may then write the current-voltage relations:

$$\boxed{\begin{aligned} i &= C\frac{dv}{dt} \\ v &= L\frac{di}{dt} \end{aligned}}$$

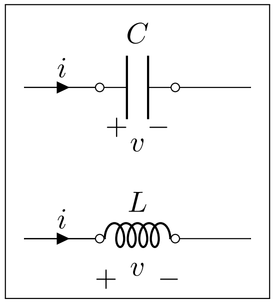


Figure 1.1: The voltage conventions

2 First Order Circuits: RL and RC

A first order circuit is one that is governed by a first order differential equation. Often, it is incorrectly stated that a first order circuit is one that contains only one energy storage element (capacitor/inductor). This is wrong, as there are certain arrangements of $R - L$ and $R - C$ circuits which can be simplified to obtain a first order equation, as we shall see later. So, there being only one energy storage element in a circuit is a *sufficient* but not *necessary* condition for it to be first order.

2.1 The Natural Response

2.1.1 The Source-Free RL Circuit

A ***natural response*** is one that is free of any external voltage/current sources, which are also known as *forcing functions*. It depends on the "general nature" of the circuit (types of elements, sizes and interconnections). It is also known as the ***transient response***, as without any external sources, it must eventually die out. Consider the simple series RL circuit shown in Fig.2.1:

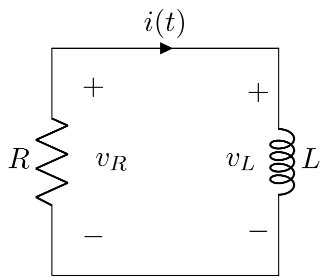


Figure 2.1: Simple R-L circuit for which $i(t)$ is to be determined

Let the time varying current be designated $i(t)$, and the value of i at $t = 0$ be I_0. It may seem counter-intuitive to have a non-zero current initially in a circuit with no sources. In order for a current to be flowing, a source had to be present at some point of time, but we are only bothered with the response of the circuit after $t = 0$, ie. after the source has been removed. Applying Kirchhoff's voltage law, we have

$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0 \tag{2.1}$$

We can separate variables and integrate easily, giving

$$\boxed{i(t) = I_0 e^{-\frac{R}{L}t}} \tag{2.2}$$

where we have incorporated the intitial codition $i(0) = I_0$. The general form of the natural response will be

$$\boxed{i(t) = Ke^{-\frac{R}{L}t}} \tag{2.3}$$

where K is a constant, and is nothing but the value of i at $t = 0$.

Accounting For The Energy

The power dissipated through the resistor at any time t

$$p_R = i^2R = I_0^2 Re^{-\frac{2Rt}{L}}$$

and the total energy turned into heat by the resistor till the response has died out

$$\begin{aligned} w_R &= \int_0^\infty p_R \, dt \\ &= I_0^2 R \int_0^\infty e^{-\frac{2Rt}{L}} \, dt \\ &= I_0^2 R \left(\frac{-L}{2R}\right) e^{-\frac{2Rt}{L}} \Bigg|_0^\infty \\ \boxed{w_R = \frac{1}{2}LI_0^2} \end{aligned}$$

which is the total energy stored initially in the inductor, as expected.

2.1.2 Properties Of The Natural Response

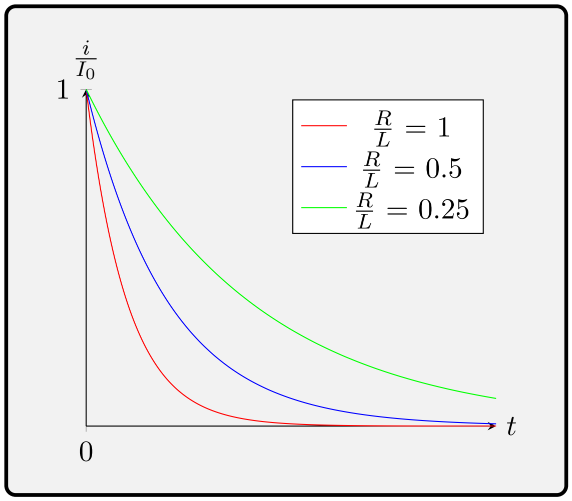


Figure 2.2: Some possible plots for the natural response

From the natural response of the series RL circuit given by Eq.2.2, it is clear that that the current decays exponentially to zero from an initial value of I_0. The graph of $\frac{i}{I_0}$ is plotted in figure 2.2 for three different values of $\frac{R}{L}$, as indicated in the legend.
Consider now the initial rate of decay, which is found by differentiating equation 2.2:

$$\left.\frac{d}{dt}\frac{i}{I_0}\right|_{t=0} = -\frac{R}{L}e^{-Rt/L}\bigg|_{t=0} = -\frac{R}{L}$$

Assuming that the decay continues at this rate, the time τ taken by $\frac{i}{I_0}$ to drop from 1 to 0 is given by

$$\left(\frac{R}{L}\right)\tau = 1$$

or

$$\boxed{\tau = \frac{L}{R}} \tag{2.4}$$

The constant τ has the units of seconds and is called the **time constant** of the circuit. It may be found graphically from the response curve by drawing the tangent to the curve at $t = 0$ and finding where it intersects the x axis, as shown below:

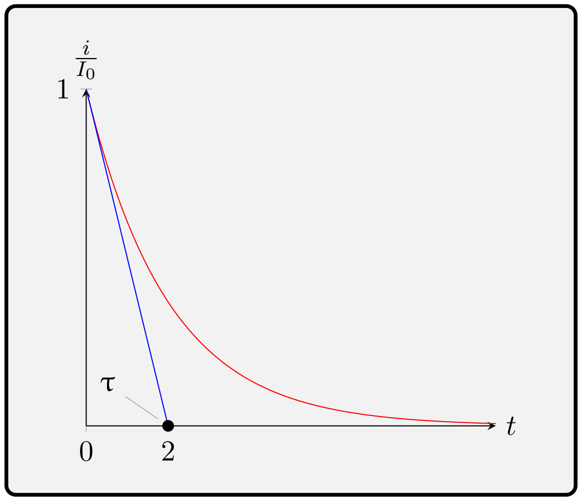
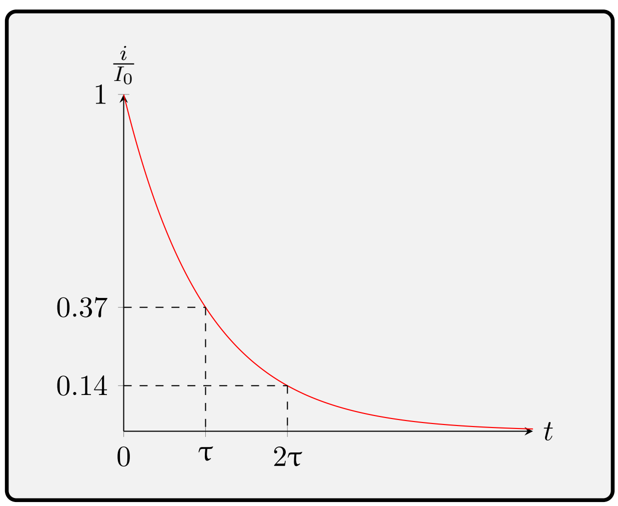


Figure 2.3: Finding the time constant graphically

Yet another way to define the time constant is to note that

$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679$$

ie, in one time constant, the response falls to 36.8% of it's initial value. The value of τ may also be determined from this fact:



Example 2.1

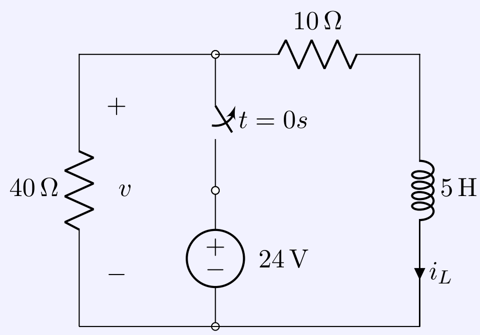
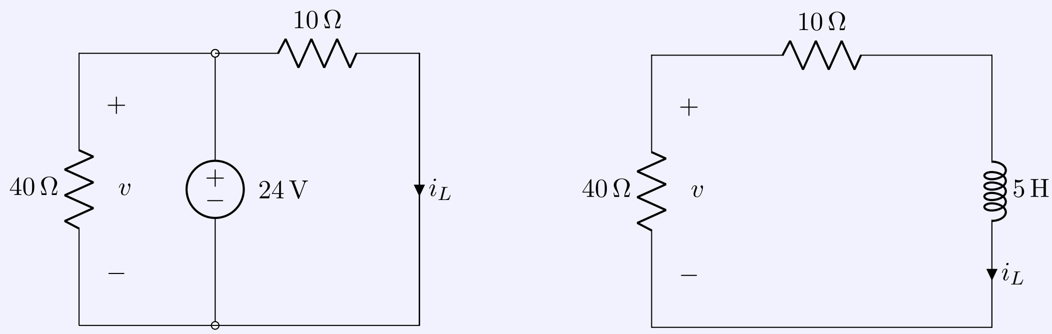


Figure 2.4: A simple RL circuit with a switch thrown at $t = 0$

For the circuit given in figure 2.4, let us try to find the voltage at $t = 200ms$.

Figures 2.5a and 2.5b show the state of the circuit prior to and post the throwing of the switch. In 2.5a, the inductor acts like a short to DC current after all transients have died down.



(a) The circuit prior to $t = 0$ (b) The circuit post $t = 0$

Figure 2.5: The simplified circuit before and after the switch is opened

Applying Kirchhoff's Law in 2.5b, we may write

$$
\begin{aligned}
-v + 10i_L + 5\frac{di_l}{dt} &= 0 \\
\Rightarrow \frac{5}{40}\frac{dv}{dt} + \left(\frac{10}{40} + 1\right)v &= 0 \\
\Rightarrow \frac{dv}{dt} + 10v &= 0
\end{aligned}
$$

where we have taken $i_l = -v/40$ by our conventions. The solution to this differential equation is

$$v(t) = v(0^+)e^{-10t}$$

Note that the initial value of voltage here is denoted by $v(0^+)$, as the voltage across the resistor may change discontinuously when the switch is thrown. We have to use the fact that i_L cannot change discontinuously, and remains the same just before and just after the switch is thrown. From 2.5a, $i_l(0) = 24/10 = 2.4$A. Hence, $v(0^+) = (40) \cdot (-2.4) = -96$V, and so

$$\boxed{v(t) = -96e^{-10t}}$$

which gives $v(0.2) = -12.99$V.

2.1.3 The Source-Free RC Circuit

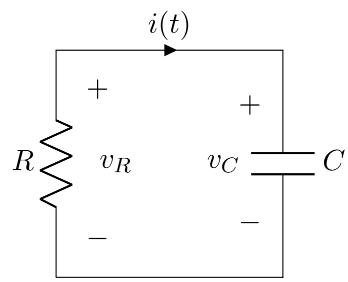


Figure 2.6: Source-Free RC Circuit

The source free response for RC circuits is nearly the same as with LC, but with slightly different expressions for the time constant. In this case, the general form of the response is given by

$$\boxed{v(t) = v(0)e^{-\frac{t}{\tau}}} \tag{2.5}$$

and the expression for the time constant is

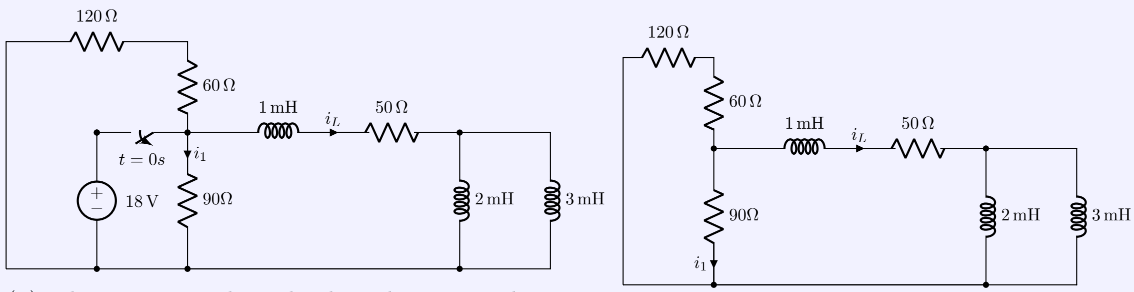
$$\boxed{\tau = R \cdot C} \tag{2.6}$$

It may also be noted that the voltage across a capacitor cannot change discontinuously, as opposed to the current for an inductor.

2.1.4 General RL and RC Circuits

For circuits containing more than one capacitor/inductor and more than one resistor, it is possible to arrive at a single time constant and reduce it to a simple circuit with one resistor and one energy storage element, as long as the Thévenin resistance "seen" by all the elements is the same. In such a case, we can write $\tau = \frac{L_{eq}}{R_{eq}}$ or $\tau = R_{eq} \cdot C_{eq}$. The general form of the natural response would then be $Ae^{-t/\tau}$, where the constant A can be determined using the initial conditions, and would be different for different elements. Let us illustrate this using the following example:

Example 2.2



(a) The circuit with multiple inductors and resistors

(b) The simplified circuit after $t = 0$

For the circuit given in figure 2.7a find i_1 and i_L for $t > 0$.

Firstly, note that the equivalent/Thévenin resistance seen by all three inductances is equal, and equal to $50 + 90||(120 + 60) = 110\Omega$. So, we can reduce it to a simple LR circuit, with

$$L_{eq} = \frac{2 \cdot 3}{2 + 3} + 1 = 2.2mH$$
$$\Rightarrow \tau = \frac{L_{eq}}{R_{eq}} = 20\mu s$$

And so every current and voltage in the network must have the form $Ke^{-t/\tau} = Ke^{-50{,}000t}$. If we consider the circuit prior to opening the switch and after all transients have died down, the inductors act as shorts to DC current and so i_L is easily found to be $i_L = 18/50 = 360mA$. This must also be the value of i_L just after the switch is opened. So,

$$i_L = \begin{cases} 360mA & t < 0 \\ 360e^{-50{,}000t}mA & 0 \leq t \end{cases}$$

Now, i_1 may change discontiuously at $t = 0$ so we have to find it's value at 0^+ using the value of $i_L(0^+)$. Using current division,

$$i_1(0^+) = -i_L(0^+) \cdot \frac{120 + 60}{120 + 60 + 90} = -240mA$$

Hence,

$$i_1 = \begin{cases} 200mA & t < 0 \\ -240e^{-50{,}000t}mA & 0 \leq t \end{cases}$$

2.2 The Unit Step Function

The unit step function, also called the **Heaviside step function** is a step function whose value is zero for negative arguments and one for positive arguments. This function is important as the

switching action of a battery is equivalent to a forcing function which is zero up until the switch is thrown, and equal to the battery voltage thereafter. It is represented by u, θ or H.
If we shift the argument to some arbitrary time t_0, then $u(t - t_0)$ must be zero for all values of t less than t_0 and unity for values of t greater than t_0. The unit step function changes abruptly/discontinuously from 0 to 1, and it's value at $t = t_0$ is not defined.
It can be represented analytically as

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$$

or graphically as

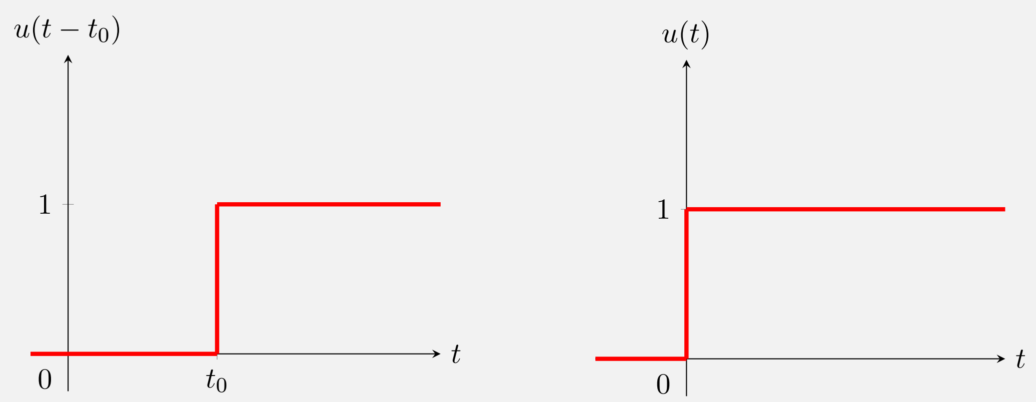


Figure 2.8: The unit step function

Note that the vertical line is not really a part of the unit step's definition but it is usally shown in each drawing as it looks nicer.

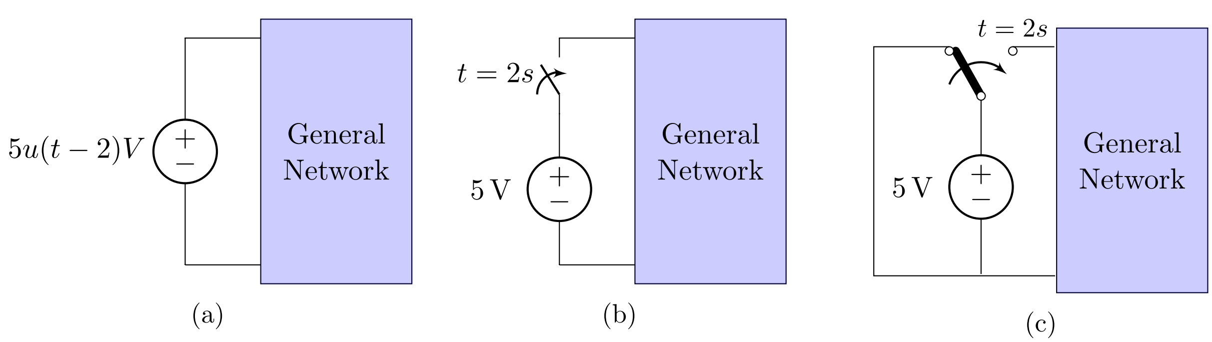


Figure 2.9: (a) The voltage-step forcing function. (b) A possible equivalent. (c) The exact equivalent.

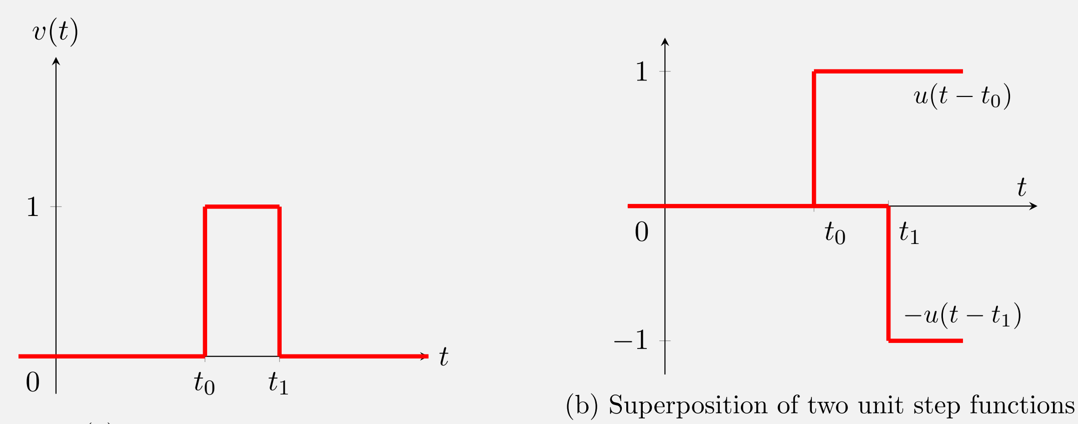
Interestingly, the physical equivalent of the unit step is not actually a simple voltage source in series with a switch. Consider the step voltage source shown in figure 2.9a. Let us try to draw it's physical equivalent. On first thought, one might come up with figure 2.9b. However, these two circuits are not equivalent for $t < 2s$: the voltage across the network in figure 2.9b is completely unspecified in this time interval. The actual equivalent to the step voltage source is the single-pole double throw switch shown in figure 2.9c. Clearly, the voltage is zero for $t < 2s$, and this is consistent with the step voltage function.

2.2.1 The Rectangular Pulse Function

Various rectangular pulses may be obtained by manipulating the unit step function. Let us first consider the pulse given by

$$v(t) = \begin{cases} 0 & t < t_0 \\ V_0 & t_0 < t < t_1 \\ 0 & t > t_1 \end{cases}$$

Let us try to break this down into individual unit step functions. Figure 2.10b shows the superposition of the two step functions $u(t - t_0)$ and $-u(t - t_1)$. Clearly, their superposition gives the rectangular pulse in figure 2.10a.



(a) The Rectangular Pulse

(b) Superposition of two unit step functions

Breaking down rectangular pulses in terms of unit step functions can be quite handy for linear networks, for which we can simply consider individual unit step sources separately and add them to get the total response.

2.3 The Forced Response

Let us now subject a simple network to the sudden application of a DC source. Figure 2.11 shows a circuit for which $i(t)$ is to be determined, consisting of a voltage source in series with a resistance and an inductor, with the switch thrown at $t = 0s$. Note that we also could have removed the switch and represented the voltage source as $V_0 u(t)$ (not *exactly* correct, as said previously).

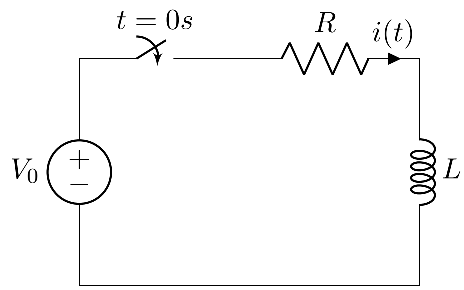


Figure 2.11: The Driven RL Circuit

Clearly, $i(t) = 0$ for $t < 0s$. For $t > 0s$, we may apply Kirchhoff's voltage law and obtain the differential equation

$$Ri + L\frac{di}{dt} = V_0 \tag{2.7}$$

Now, we can separate variables and integrate easily:

$$\int_0^i \frac{L}{V_0 - Ri}\, di = \int_0^t dt$$
$$\Rightarrow \frac{L}{R}\ln(V_0 - Ri)\Bigg|_0^i = t$$
$$\Rightarrow i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})$$

which is the response for $t > 0s$. Note that here $i(0^-) = i(0^+) = 0$. The complete response may be written conveniently in terms of the unit step function as

$$\boxed{i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})u(t)} \tag{2.8}$$

We have arrived at this expression purely by mathematical methods. There is an easier way to arrive at this expression, and it's convenience is highlighted even more when analysing second order driven circuits. For now, let us try to interpret the terms appearing in equation 2.8.

There is one constant, $\frac{V_0}{R}$ and our familiar expression for natural circuits: $\frac{V_0}{R}e^{-t/\tau}$. The constant is nothing but the current at steady state, after transients have died down and is also called the *steady state response, particular solution* or the *particular integral*. It is found simply by analysing the circuit at steady state - shorting all inductors and opening all capacitors.
The other term is the natural response, which approaches zero as the time increases without limit. The amplitude of this natural response will depend on the initial value of the complete response, and hence also on the initial value of the forcing function. Let us now try arriving at equation 2.8 in this new light. The total response will comprise of a natural response, i_n and a forced response, i_f. Hence,

$$i = i_n + i_f$$

where i_n must have the form $Ae^{-Rt/L}$, where A is a constant to be determined using the initial condition of the forcing function. The value of i_f is found by considering the circuit after the natural response has died out, when the inductor will act as a short. Clearly, in this condition, we have a single resistor in series with the voltage source and hence $i_f = \frac{V_0}{R}$. So,

$$i = \frac{V_0}{R} + Ae^{-Rt/L}$$

We may now apply the initial condition $i(0^-) = i(0^+) = 0$:

$$0 = \frac{V_0}{R} + Ae^0$$
$$\Rightarrow A = -\frac{V_0}{R}$$

and so we have

$$i(t) = \frac{V_0}{R}(1 - e^{-Rt/L})u(t)$$

The response is plotted in figure 2.12, and it is clear how i rises from it's intial value of 0 to it's forced value of $\frac{V_0}{R}$. In this case, a tangent drawn to the response curve at $t = 0$ meets the forced response at $t = \tau$.

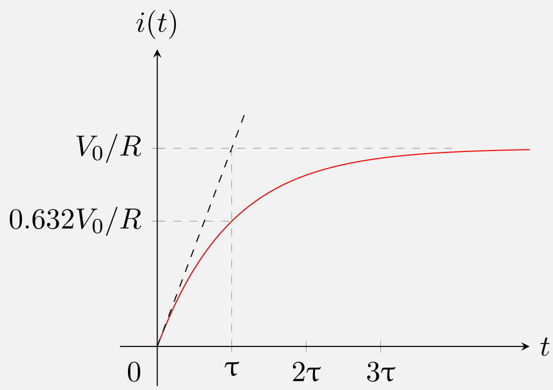


Figure 2.12: The plot of $i(t)$

The procedure to find the forced response for driven RC circuits is exactly the same as that for RL circuits. We won't repeat it here, but will instead illustrate it with the following example:

Example 2.3

Determine an expression for $v(t)$ in the circuit of figure 2.13 valid for $t > 0$.

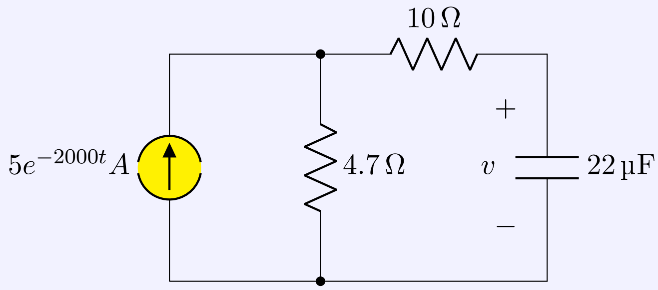


Figure 2.13: An RC circuit forced by an exponentially decaying function

We expect a complete response of the form

$$v(t) = v_f + v_n$$

Where v_n is of the form $Ae^{it/\tau}$ and v_f is reminiscent of the exponential forcing function. To calculate the time constant for this circuit, note that the Thévenin resistance across the capacitor is

$$R_{eq} = 4.7 + 10 = 14.7\Omega$$

and hence our time constant is $\tau = R_{eq} \cdot C = 323.4\mu s$. At this point, it will be convenient to perform a source transformation resulting in a voltage source $23.5e^{-2000t}$ in series with 14.7Ω and $22\mu F$. Applying Kirchhoff's loop law,

$$23.5e^{-2000t} = (14.7)(22 \cdot 10^{-6})\frac{dv}{dt} + v$$
$$\Rightarrow \frac{dv}{dt} + (3.092 \cdot 10^3)v = 72.67 \cdot 10^3 e^{-2000t}$$

Comparing this with the standard solution to a first order non-homogenous linear equation

$$v(t) = e^{-Pt}\int Qe^{Pt}\,dt + Ae^{-Pt}$$

we find that $P = 1/\tau = 3.092 \cdot 10^3$ and $Q(t) = 72.67 \cdot 10^3 e^{-2000t}$ and after performing the necessary integration, we get

$$v(t) = 66.55e^{-2000t} + Ae^{-3092}$$

and since the voltage across a capacitor cannot change discontinuously with time, our initial conditions are $v(0^-) = v(0^+) = 0$. Incorporating these into our response, finally we get

$$v(t) = 66.55(e^{-2000t} - e^{-3092})V$$

for $t > 0s$. Is this expected? We have one term of the form e^{-2000t}, our forcing function and one term of the form e^{-3092}, our natural response, and so everything checks out. Interestingly, we can find this response in just 3 or 4 steps using Laplace and Fourier transforms, without even having to write down a single differential equation.

The methods of finding a response for a forced circuit are summarised neatly below:

1. With all independent sources zeroed out, simplify the circuit to determine R_{eq}, C_{eq} and the time constant $\tau = R_{eq} \cdot C_{eq}$
2. Viewing C_{eq} as an open circuit, find $v_c(0^-)$, the capacitor voltage prior to the discontinuity.
3. Viewing C_{eq} as an open circuit, find the forced response $f(\infty)$
4. Write the total response as a sum of forced and natural responses: $f(t) = f(\infty) + Ae^{-t/\tau}$.
5. Find $f(0^+)$ using the fact that the capacitor voltage may not change discontinuously, ie. $v_c(0^+) = v_c(0^-)$.
6. Apply this initial condition $f(0^+)$ to find the value of A, and hence the complete response.

The method to find the complete response for an LR circuit is, of course, the dual of the statements listed above.

2.4 An Application Of First Order Circuits: The Astable 555 Timer IC

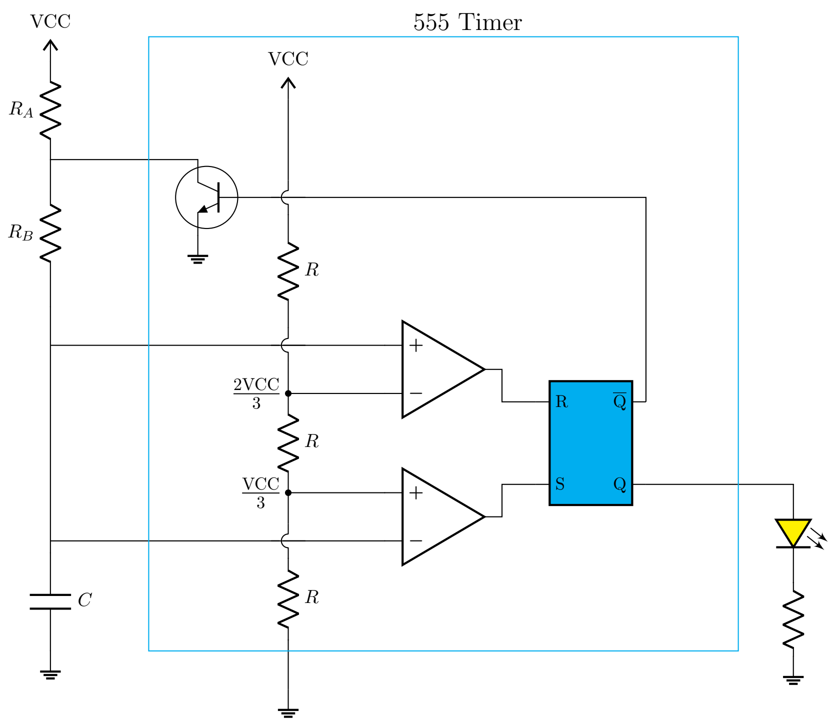


Figure 2.14: The 555 IC being operated as an astable timer, with a simplified version of it's internal circuitry