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2.3 Matrix character

- positive definite and positive semi-definite

i) positive definite

$x^T A x > 0$ (x is not a zero vector)

↳ "positive definite"

ii) positive semi-definite

$x^T A x \geq 0$

↳ "positive semi-definite"

↳ we could define positive definite or positive semi-definite for a matrix.

↳ ~~However, we define a "~~~~Symmetric~~~~" matrix~~

In general, it applies to the "Symmetric Matrix"

for example)

↳ Identity matrix is "positive definite"

$$x^T I x = [x_1, \cdots, x_N] \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{x_1^2 + x_2^2 + \cdots + x_N^2 > 0}_{\text{"positive definite"}}$$

Practice If $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, is it positive definite or semi-definite?

$x = [x_1 \ x_2 \ x_3]$

$$x^T A x = [x_1 \ x_2 \ x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(2x_1 - x_2) \ (-x_1 + 2x_2 - x_3) (-x_2 + 2x_3)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (2x_1^2 - x_1 x_2) + (-x_1 x_2 + 2x_2^2 - x_2 x_3) + (-x_2 x_3 + 2x_3^2)$$

$$= \underbrace{x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 \geq 0}_{\text{"Positive-definite"}}$$