Linearity : $f(x)$ , function, operation

1) Superposition $f(x_1+x_2) = f(x_1)+f(x_2)$

2) homogeneity $f(ax_1) = af(x)$

example) · $y = mx + n$ ($\rightarrow n=0,\ m\neq 0$)

· differentiation

$$\frac{d}{dt}\left(x_1(t)+x_2(t)\right) = \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t)$$

· Integration

$$\int x_1(t)+x_2(t)\,dt = \int x_1(t)\,dt + \int x_2(t)\,dt$$

Matrix operation (multiplication)

$$A(X_1+X_2) = AX_1 + AX_2$$

Vector operation

· Basic Notation of Matrices

· Vector $v=(a_1, b_1, c_1)$ , $w=(a_2, b_2, c_2)$

$\hookrightarrow$ Column of a matrix $= \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$

· Linear Combinations $\alpha v + \beta w$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \beta\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

· Central ideas (terms)

- column space : all combinations of the columns
- row space : all combinations of the rows
- rank : # of independent columns (or rows)
- elimination : to find a rank of a matrix

# Chapter 1. Matrix & Gaussian Elimination

## 1.1 Introduction

How to solve linear equations with $n$ unknowns?

(1) Elimination

$$x+2y=3$$
$$4x+5y=6$$

$$(4x+5y=6)-4\times(x+2y=3) \Rightarrow -3y=-6 \Rightarrow y=2,\ x=-1$$

(2) Determinants (Cramer's rule)

$$y=\frac{\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix}}{\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}}=2,\qquad x=\frac{\begin{vmatrix}3 & 2\\ 6 & 5\end{vmatrix}}{\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}}=-1$$

$\Rightarrow$ usually, elimination method is much better than determinants.

## 1.2 Geometry of Linear Equations

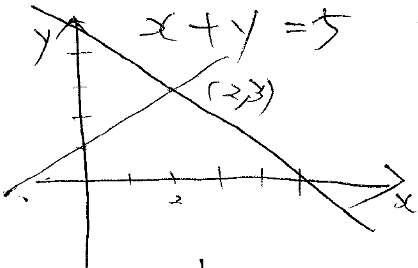
→ 도형의 특성을 이용

→ lines (2 unknowns), planes (3 or more unknowns)

$2x-y=1$

$x+y=5$

(1) row equations

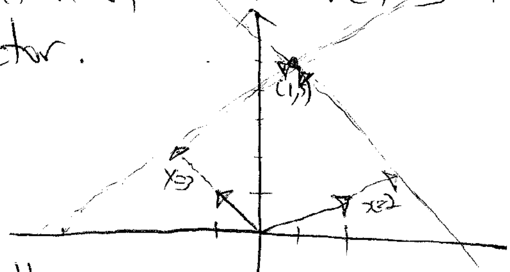



2 lines $\Rightarrow$ solution is the intersection of the two lines.

(2) Column form

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$

→ To find the combination of column vectors that produces the right-side vector.




→ Solution is to make the geometric parallelogram of vectors Page ②

◎ $n=3$ Case

$$\left.\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}\right\} \Rightarrow 3 \text{ planes}$$

(1) row equations

→ Solution is the intersection of 3 planes ⇒ a point

(2) column form

$$u\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

↳ Linear Combinations of 3 column vectors

Row picture (equation) ⇒ intersection of planes
Column picture (equation) ⇒ combination of columns

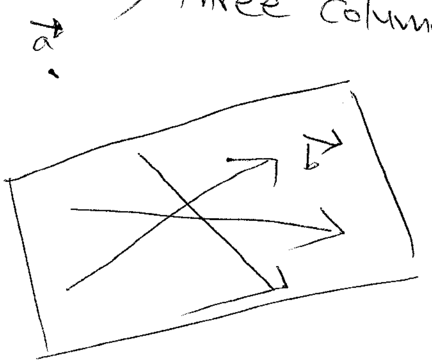
◎ Singular Cases → No solution / Infinite solutions

(1) row form

→ Parallel
→ No intersection of 3 planes ) ⇒ "No Solution"
→ Line of intersection ⇒ "Infinite"

(2) Column picture

⇒ three column vectors are in a plane.



$\vec{a}$ : Not in a plane
↳ No Solution

$\vec{b}$ : in a plane
↳ Infinite Solutions

If $n$ planes have no points in common or infinitely many points
then the $n$ columns lie in the same plane.

## 1.3 Example of Gaussian Elimination

◎ Forward elimination step

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

1st pivot

↓ To eliminate u from the last two equations

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned}$$

2nd pivot

↓ To eliminate v from the last equation.

$$\left.\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}\right\} \Rightarrow \text{Triangular system}$$

$w=2 \Rightarrow v=1 \Rightarrow u=1 \rightarrow$ back-substitution

"By definition, pivots cannot be zero"

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

↳ "Triangular System"

## ⊙ Breakdown of Elimination

When a zero appears in a pivot position, elimination has to stop, and the order of equations has to be changed.

Ex1) $\begin{aligned} u+v+w&=* \\ 2u+2v+5w&=* \\ 4u+6v+8w&=* \end{aligned} \Rightarrow \begin{aligned} u+v+w&=* \\ 3w&=* \\ 2v+4w&=* \end{aligned} \Rightarrow \begin{aligned} u+v+w&=* \\ 2v+4w&=* \\ 3w&=* \end{aligned}$

$\Rightarrow$ Triangular matrix $\Rightarrow$ Non-Singular
with all pivots (Non-Zero)

Ex2) $\left.\begin{aligned} u+v+w&=a \\ 2u+2v+5w&=b \\ 4u+4v+8w&=c \end{aligned}\right) \Rightarrow \left.\begin{aligned} u+v+w&=a \\ 3w&=b-2a \\ 4w&=c-4a \end{aligned}\right) \Rightarrow$ Singular

↳ No exchange for non-zero pivots

If $\dfrac{b-2a}{3} = \dfrac{c-4a}{4} \Rightarrow$ Infinite

else $\Rightarrow$ No solution

$\Rightarrow$ By exchanging the order of equations for non-zero full pivots, we can find a solution by elimination.

# 1.6 Inverse and Transpose

- $Ax = b \rightarrow A^{-1}b = x \Leftrightarrow AA^{-1} = A^{-1}A = I$
- Not all A has its inverse matrix.

Note 1. The inverse exists if elimination produces n pivots.
(row exchanges are allowed)

Note 2. The inverse matrix is unique.

when $A^{-1} = B$ or $C$

$$(BA)C = B(AC) \Leftrightarrow C = B$$

Note 3. If A is invertible,

$Ax = b \qquad A^{-1}Ax = A^{-1}b \Leftrightarrow x = A^{-1}b$

Note 4. Assume there is a non zero vector $x$ such that $Ax = 0$
Then, A can't have an Inverse.

$Ax = 0 \rightarrow x = 0$ for invertible A

Note 5. for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\mathrm{Det}(A) = ad - bc \neq 0 \rightarrow$ invertible

In matlab, finding n non-zero pivots for invertibility.

Note 6. Diagonal Matrix

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & \ddots & \\ & & & 1/d_n \end{bmatrix}$$

ⓔ The inverses come in reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

ⓔ Calculation of $A^{-1}$ : Gauss-Jordan Method.

$$AA^{-1} = I \quad , \quad A^{-1} = [x_1 \; x_2 \; \cdots \; x_n]$$

ex) $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} [x_1 \; x_2 \; x_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ 3 system equations for $x_1, x_2, x_3$ with the same $A$.

· elimination process at the same time for 3 system equations.

$\Rightarrow$ Gauss-Jordan Method : $A = LU \Leftrightarrow A^{-1} = U^{-1}L^{-1}$

$$[A \; e_1 \; e_2 \; e_3] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \text{pivot } 2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$\cdot \text{pivot } -8 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [U \; L^{-1}]$ ← 1st half elimination

$E_{32}E_{31}E_{21} = L^{-1}$   GE의 $L^{-1}$

by multiplying $U^{-1} \Rightarrow U^{-1}[U \; L^{-1}] = [I \; U^{-1}L^{-1}] = [I \; A^{-1}]$

$\Rightarrow$ 좌측 3×3 부분을 $I$로 변환 $\Rightarrow$ 이 과정이 $U^{-1}$ 곱하기

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & -\frac{5}{8} & -\frac{3}{4} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [I \; U^{-1}L^{-1}] = [I \; A^{-1}]$$

※ $\det A$ = the product of the pivots $= (2)\cdot(-8)\cdot(1) = \underline{-16}$

Remark 1. $Lc = b$ and $Ux = c$ are better than $A^{-1}b = x$

◎ Invertible = non-singular (n pivots)

◎ Transpose Matrix

$(A^T)_{ij} = A_{ji}$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$(A^{-1})^T = (A^T)^{-1}$ $\quad A^{-1}A = I \Leftrightarrow (A^{-1}A)^T = I^T = I$

$\Leftrightarrow A^T \underbrace{(A^{-1})^T}_{\Downarrow} = I$

$\Leftrightarrow \quad (A^T)^{-1}$

◎ Symmetric Matrix

$A^T = A$ ⇒ Square matrix

$a_{ij} = a_{ji}$

· If $A$ is symmetric and invertible, $A^{-1}$ is too.

pf) $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I$

$(A^{-1})^T A^T = \underbrace{(A^{-1})^T}_{\Downarrow} A = I$

$A^{-1} = (A^{-1})^T$ ⇒ Symmetric

∘ Symmetric Products $R^T R$, $RR^T$, and $LDL^T$

⇒ $R^T R \neq RR^T$

∘ Suppose $A = A^T$, $A = LDU \Rightarrow A^T = A = U^T D^T L^T = U^T D L^T = LDL^T$

$(L^T = U)$

by symmetricity, we reduce elimination processes.