

Exercise Walkthrough: Probability of At Least One 'One'

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1 Overview

This document provides a step-by-step solution to the problem of calculating the probability of obtaining at least one 1 when rolling a 5-sided die 4 times. The sample space consists of all possible outcomes, which are sequences of 4 numbers from 1 to 5. The event of interest is the set of outcomes where at least one number is 1. The probability of this event is calculated by finding the complement of the event where no 1s are rolled, and then subtracting this from 1.

2 Step 1: Defining the Probability Space

The probability space is defined by the sample space Ω , the sigma-algebra \mathcal{A} , and the probability measure P . The sample space Ω is the set of all possible outcomes of 4 rolls of a 5-sided die. The sigma-algebra \mathcal{A} is the power set of Ω . The probability measure P is defined by the probability of each outcome, which is $\frac{1}{5^4}$ for each outcome.

The Sample Space Ω consists of all possible outcomes of 4 rolls of a 5-sided die. The sample space is a set of sequences of 4 numbers from 1 to 5. The total number of outcomes is $5^4 = 625$. The event of interest is the set of outcomes where at least one number is 1. The probability of this event is calculated by finding the complement of the event where no 1s are rolled, and then subtracting this from 1.

$$\begin{aligned} & \text{The sample space } \Omega \text{ is the set of all possible outcomes of the experiment.} \\ & \text{In this case, the sample space } \Omega \text{ is the set of all possible outcomes of the experiment.} \end{aligned}$$

$$\Omega' = \Omega^6 = \{(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) \mid \omega_j \in \Omega, j = 1, \dots, 6\}$$

$$\begin{aligned} & \text{The event space } \mathcal{A}' \text{ is the } \sigma\text{-algebra generated by the events } \mathcal{A}. \\ & \text{In this case, the event space } \mathcal{A}' \text{ is the } \sigma\text{-algebra generated by the events } \mathcal{A}. \end{aligned}$$

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$$\mathcal{A}' = \mathcal{P}(\Omega')$$

$$\begin{aligned} & \text{The probability measure } P \text{ is a function that assigns a probability to each event in } \mathcal{A}'. \\ & \text{In this case, the probability measure } P \text{ is a function that assigns a probability to each event in } \mathcal{A}'. \end{aligned}$$

$$P(E) = \frac{|E|}{|\Omega'|} = \frac{|E|}{6^6}$$

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3 Step 2: Defining the Event of Interest

$$\begin{aligned}
& \text{Let } A = \{\omega \in \Omega' \mid \exists j \in \{1, \dots, 6\} \text{ s.t. } \omega_j = 0\} \\
& \text{where } \Omega' = \{\omega \in \Omega \mid \omega_j \in \{0, 1\} \text{ for all } j \in \{1, \dots, 6\}\}
\end{aligned}$$

$$\begin{aligned}
& \text{Let } A^c = \{\omega \in \Omega' \mid \forall j \in \{1, \dots, 6\}, \omega_j \neq 0\} \\
& \text{where } \Omega' = \{\omega \in \Omega \mid \omega_j \in \{0, 1\} \text{ for all } j \in \{1, \dots, 6\}\}
\end{aligned}$$

4 Step 3: Using the Complement Rule

$$\begin{aligned}
& \text{Let } P(A) = 1 - P(A^c) \\
& \text{where } P(A^c) = \sum_{j=1}^6 P(\omega_j = 0)
\end{aligned}$$

$$\begin{aligned}
& \text{Let } P(A^c) = \sum_{j=1}^6 P(\omega_j = 0) \\
& \text{where } P(\omega_j = 0) = \frac{1}{2}
\end{aligned}$$

5 Step 4: Calculating the Probability of the Complement

$$\begin{aligned}
& \text{Let } P(A^c) = \sum_{j=1}^6 P(\omega_j = 0) \\
& \text{where } P(\omega_j = 0) = \frac{1}{2}
\end{aligned}$$

[1] **Power Set (Definition A.3)** $\mathcal{P}(\Omega) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

[2] **Laplace Probability / Uniform Distribution (Example 1.36 (i), 1.37)** $\ast \nabla \triangle \square \circ \times \approx$

$$P(A) = \sum_{\omega \in A} p(\omega) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

[illegible]

[3] **Complement Rule (Derived from Definition 1.1 & 1.18)**

$$\begin{aligned}
 & \int_{\Omega} \mathbb{1}_A = P(A) \\
 & \int_{\Omega} \mathbb{1}_{A^c} = P(A^c) \\
 & \int_{\Omega} \mathbb{1}_{A \cup A^c} = P(A \cup A^c) = P(\Omega) = 1 \\
 & \int_{\Omega} \mathbb{1}_A + \int_{\Omega} \mathbb{1}_{A^c} = P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1
 \end{aligned}$$

- $P(\Omega) = 1$

- $\int_{\Omega} \mathbb{1}_{A \cup A^c} = P(A \cup A^c) = P(A) + P(A^c) = \int_{\Omega} \mathbb{1}_A + \int_{\Omega} \mathbb{1}_{A^c} = 1$

$$\begin{aligned}
 & \int_{\Omega} \mathbb{1}_A + \int_{\Omega} \mathbb{1}_{A^c} = P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1 \\
 & \int_{\Omega} \mathbb{1}_A = P(A) \\
 & \int_{\Omega} \mathbb{1}_{A^c} = P(A^c) \\
 & P(A) = 1 - P(A^c)
 \end{aligned}$$