Exercise Walkthrough: De Morgan's Laws

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1 Overview and Goal

This document provides a step-by-step walkthrough for proving De Morgan's Laws as stated in *Lemma 1.2* of the "Discrete Probability Theory" script (page 9). The exercise asks us to prove the following two identities for any given set Ω and an arbitrary family of its subsets $\{A_\alpha \subseteq \Omega \mid \alpha \in I\}$:

1. $\left(\bigcup_{\alpha\in I} A_\alpha\right)^c = \bigcap_{\alpha\in I} A_\alpha^c$
2. $\left(\bigcap_{\alpha\in I} A_\alpha\right)^c = \bigcup_{\alpha\in I} A_\alpha^c$

These laws are fundamental tools. They describe how the complement operation interacts with unions and intersections. In probability theory, this is crucial for calculating probabilities of complex events, especially since a σ-algebra is defined by its closure under complement and countable union (see *Definition 1.5*).

We will prove the first identity in full detail. The proof for the second identity is very similar, and I will leave it as a short exercise for you to solidify your understanding.

2 The Proof: Step-by-Step

2.1 The Strategy: Proof by Double Inclusion

To prove that two sets, let's call them S and T, are equal ($S = T$), the standard method is to show that each set is a subset of the other. This is called proof by double inclusion [1]. We need to show:

1. $S \subseteq T$: Every element of S is also an element of T.
2. $T \subseteq S$: Every element of T is also an element of S.

For our first identity, this means we will prove:

a) $\left(\bigcup_{\alpha\in I} A_\alpha\right)^c \subseteq \bigcap_{\alpha\in I} A_\alpha^c$

b) $\bigcap_{\alpha\in I} A_\alpha^c \subseteq \left(\bigcup_{\alpha\in I} A_\alpha\right)^c$

2.2 Part 1: Proving the First Inclusion ($\subseteq$)

Goal: Show that $\left(\bigcup_{\alpha\in I} A_\alpha\right)^c \subseteq \bigcap_{\alpha\in I} A_\alpha^c$.

- **Step 1: Start with an arbitrary element.** Let x be an arbitrary element of the set on the left-hand side.

$$x \in \left(\bigcup_{\alpha\in I} A_\alpha\right)^c$$

- **Step 2: Unpack the definition of the complement.** The definition of a set complement [2] states that if an element is in the complement of a set, it is *not* in the set itself.

$$\text{This means that } x \notin \left(\bigcup_{\alpha\in I} A_\alpha\right)$$

- **Step 3: Unpack the definition of the union.** The definition of a union [3] states that an element is in the union if it is in *at least one* of the sets. Since our element x is *not* in the union, it must not be in *any* of the sets A_α.

$$\text{This means that for all } \alpha \in I, \text{ we have } x \notin A_\alpha$$

- **Step 4: Use the definition of the complement again.** If $x \notin A_\alpha$ for every single α, then by the definition of a complement [2], x must be in the complement of every single A_α.

$$\text{This means that for all } \alpha \in I, \text{ we have } x \in A_\alpha^c$$

- **Step 5: Unpack the definition of the intersection.** The definition of an intersection [4] states that if an element is in the intersection of a family of sets, it must be in *every single one* of those sets. Since we've established that $x \in A_\alpha^c$ for all α, it must be in their intersection.

$$\text{This implies that } x \in \bigcap_{\alpha\in I} A_\alpha^c$$

- **Conclusion of Part 1:** We started by picking an arbitrary element x from the set on the left-hand side and, through a series of logical steps based on definitions, we have shown that it must also be an element of the set on the right-hand side. This proves the first inclusion.

2.3 Part 2: Proving the Second Inclusion ($\supseteq$)

Goal: Show that $\bigcap_{\alpha\in I} A_\alpha^c \subseteq \left(\bigcup_{\alpha\in I} A_\alpha\right)^c$.

- **Step 1: Start with an arbitrary element.** Let x be an arbitrary element of the set on the right-hand side (of the original '$\subseteq$' statement).

$$x \in \bigcap_{\alpha\in I} A_\alpha^c$$

- **Step 2: Unpack the definition of the intersection.** According to the definition of intersection [4], if x is in the intersection, it must be an element of every set in the family.

$$\text{This means that for all } \alpha \in I, \text{ we have } x \in A_\alpha^c$$

- **Step 3: Unpack the definition of the complement.** If x is in the complement of every A_α [2], it means it is *not* in any of the original sets A_α.

$$\text{This means that for all } \alpha \in I, \text{ we have } x \notin A_\alpha$$

- **Step 4: Use the definition of the union.** If x is not in *any* of the sets A_α, it cannot possibly be in their union [3] (which contains only elements that are in at least one A_α).

$$\text{This implies that } x \notin \left(\bigcup_{\alpha\in I} A_\alpha\right)$$

- **Step 5: Use the definition of the complement again.** Finally, if an element is not in a set, it must be in that set's complement [2].

$$\text{This means that } x \in \left(\bigcup_{\alpha\in I} A_\alpha\right)^c$$

- **Conclusion of Part 2:** We started with an arbitrary element x from the right-hand side and showed it must also be in the left-hand side. This proves the second inclusion.

2.4 Final Conclusion

Since we have successfully shown both inclusions (Part 1 and Part 2), we can conclude by the principle of double inclusion [1] that the two sets are equal.

$$\left(\bigcup_{\alpha\in I} A_\alpha\right)^c = \bigcap_{\alpha\in I} A_\alpha^c$$

The proof for the first of De Morgan's laws is complete.

3 Check Your Understanding

Now it's your turn! The best way to internalize this proof technique is to apply it yourself.

Exercise: Prove the second of De Morgan's laws:

$$\left(\bigcap_{\alpha\in I} A_\alpha\right)^c = \bigcup_{\alpha\in I} A_\alpha^c$$

Hint: Follow the exact same strategy of double inclusion.

1. For the "$\subseteq$" part, start by assuming $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right)^c$ and work your way to showing that $x \in \bigcup_{\alpha\in I} A_\alpha^c$. How does being "not in the intersection" differ from being "not in the union"?
2. For the "$\supseteq$" part, start by assuming $x \in \bigcup_{\alpha\in I} A_\alpha^c$ and show that it implies $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right)^c$.

The logic is perfectly analogous to what we did above.

4 Summary & Key Takeaways

- **De Morgan's Laws** provide a critical identity for simplifying expressions involving complements of unions or intersections. Essentially, the complement "flips" the operation (union to intersection, and vice-versa) and distributes over the individual sets.
- The **Proof by Double Inclusion** ($S \subseteq T$ and $T \subseteq S \implies S = T$) is the standard and most rigorous method for proving that two sets are equal.
- The proof itself is a methodical application of the fundamental **definitions** of set operations: complement, union, and intersection. Being precise with these definitions is key.

These laws are not just abstract rules; they appear frequently when manipulating events in probability theory. For example, if you want to find the probability that "not all of events A_α occur," you are looking for $P\left(\left(\bigcap A_\alpha\right)^c\right)$, which De Morgan's law tells you is equal to $P\left(\bigcup A_\alpha^c\right)$ – the probability that "at least one of the complement events occurs."

In-depth Explanations

[1] Proof of Set Equality (Double Inclusion)

In set theory, two sets S and T are defined to be equal if and only if they contain exactly the same elements. While this sounds simple, proving it requires a formal procedure. The axiom of extensionality states that $S = T$ if and only if $(\forall x : x \in S \iff x \in T)$. This biconditional ($\iff$) is logically equivalent to the conjunction of two conditionals:

1. $(\forall x : x \in S \implies x \in T)$, which is the definition of $S \subseteq T$ (S is a subset of T).
2. $(\forall x : x \in T \implies x \in S)$, which is the definition of $T \subseteq S$ (T is a subset of S).

Therefore, to prove $S = T$, we must prove both $S \subseteq T$ and $T \subseteq S$. This two-part proof is known as proof by double inclusion.

[2] Set Complement (A^c)

Given a universe or sample space Ω, the complement of a set $A \subseteq \Omega$, denoted A^c (or sometimes $\bar{A}$ or $\Omega \setminus A$), is the set of all elements in Ω that are *not* in A. Formally:

$$A^c = \{x \in \Omega \mid x \notin A\}$$

This means that for any element $x \in \Omega$, exactly one of two statements is true: either $x \in A$ or $x \in A^c$.

[3] Set Union ($\bigcup$)

The union of a family of sets $\{A_\alpha \mid \alpha \in I\}$, denoted $\bigcup_{\alpha \in I} A_\alpha$, is the set containing all elements that belong to *at least one* of the sets A_α in the family. Formally:

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in \Omega \mid \exists \alpha \in I \text{ such that } x \in A_\alpha\}$$

The key words are "at least one" which corresponds to the logical "OR" ($\exists$ is the existential quantifier, "there exists"). If an element is *not* in the union, it must not be in *any* of the sets.

[4] Set Intersection ($\bigcap$)

The intersection of a family of sets $\{A_\alpha \mid \alpha \in I\}$, denoted $\bigcap_{\alpha \in I} A_\alpha$, is the set containing only those elements that belong to *every single one* of the sets A_α in the family. Formally:

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in \Omega \mid \forall \alpha \in I, x \in A_\alpha\}$$

The key words are "every single one" which corresponds to the logical "AND" ($\forall$ is the universal quantifier, "for all"). If an element is in the intersection, it must be everywhere.