Exercise Walkthrough: The Derangement Problem

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Abstract

This document provides a detailed, step-by-step solution to the "derangement problem," often phrased as the hat-check problem or, in this case, the anonymous sheet redistribution problem. We will rigorously define the probabilistic model, apply the Principle of Inclusion-Exclusion as hinted, and analyze the asymptotic behavior of the resulting probability. Each step is explained with reference to the concepts from the "Discrete Probability Theory" script.

1 The Problem Statement

We are given the following exercise:

> *Angelika supervises an exercise group with n sheets being submitted anonymously. After grading, Angelika randomly redistributes the sheets. What is the probability that no one gets their original sheet back? How does this probability behave as $n \to \infty$?*

This is a classic problem in combinatorics. A permutation of elements where no element appears in its original position is called a **derangement**. We are asked to find the probability of a random permutation being a derangement.

2 Step-by-Step Solution

2.1 Step 1: Modeling the Random Process

First, we need to translate the problem into a formal probabilistic framework. This involves defining the sample space Ω, the event space $\mathcal{A}$, and the probability measure P.

Sample Space Ω: The process is the redistribution of n unique sheets to n unique students. We can label the students and their original sheets from 1 to n. A single outcome of this experiment is a complete assignment of sheets to students. We can represent an outcome ω as a tuple $(\omega_1, \omega_2, \dots, \omega_n)$, where ω_i is the original sheet number that student i receives. Since each student receives exactly one sheet and all sheets are distinct, this is a **permutation**[1] of the numbers $\{1, 2, \dots, n\}$.

The problem states this is equivalent to drawing n distinct balls without replacement from an urn, which corresponds to **Lemma 1.33 (ii)** (classical urn models) for ordered draws without replacement. The sample space Ω is the set of all permutations of $[n] := \{1, \dots, n\}$.

$$\Omega = \{\omega = (\omega_1, \dots, \omega_n) \mid \omega_i \in [n] \text{ for all } i, \text{ and } \omega_i \neq \omega_j \text{ for } i \neq j\}$$

The total number of such permutations is $|\Omega| = n!$.

Probability Measure P: The sheets are redistributed "randomly". This is a key word that tells us to assume that every possible permutation is equally likely. This describes a **Laplace probability space**[2]. Following **Example 1.36 (i)** (uniform distribution), the probability of any single outcome $\omega \in \Omega$ is:

$$P(\{\omega\}) = \frac{1}{|\Omega|} = \frac{1}{n!}$$

For any event $E \subseteq \Omega$, its probability is $P(E) = \frac{|E|}{|\Omega|}$.

Event Space $\mathcal{A}$: For a finite sample space like ours, we can consider any subset of Ω to be an event. Therefore, the event space $\mathcal{A}$ is the power set of Ω, i.e., $\mathcal{A} = \mathcal{P}(\Omega)$.

2.2 Step 2: Defining the Events of Interest

The question asks for the probability that *no one* gets their original sheet back. Calculating this directly can be tricky. It's often easier to first calculate the probability of the complementary event: **at least one person gets their original sheet back**.

Let's define A_i as the event that student i gets their own sheet back.

$$A_i = \{\omega \in \Omega \mid \omega_i = i\} \quad \text{for } i = 1, \dots, n$$

The event that "at least one person gets their sheet back" is the union of all these events:

$$A_{total} = A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

The probability we are ultimately looking for is $P(A_{total}^c) = 1 - P(A_{total})$.

2.3 Step 3: Applying the Principle of Inclusion-Exclusion

To find $P(A_{total})$, we use the **Principle of Inclusion-Exclusion**[3] from **Theorem 1.20**. For n events, the formula is:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

This formula looks daunting, but it simplifies nicely due to the symmetry of our problem. Let's break it down by calculating the probability of the intersection terms.

2.4 Step 4: Calculating the Intersection Probabilities

Let's compute $P(A_{i_1} \cap \cdots \cap A_{i_k})$ for some distinct indices $i_1, \dots, i_k$. This intersection represents the event that students $i_1, i_2, \dots, i_k$ all get their own sheets back.

To find the size of this event, $|A_{i_1} \cap \cdots \cap A_{i_k}|$, we count the number of permutations where $\omega_{i_1} = i_1, \omega_{i_2} = i_2, \dots, \omega_{i_k} = i_k$. If these k positions are fixed, the remaining $n-k$ sheets must be distributed among the remaining $n-k$ students. There are $(n-k)!$ ways to arrange these remaining sheets. So, $|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$.

The probability is therefore:

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{|A_{i_1} \cap \cdots \cap A_{i_k}|}{|\Omega|} = \frac{(n-k)!}{n!}$$

Crucially, notice that this probability only depends on the *number* of events in the intersection (k), not on which specific students we chose.

2.5 Step 5: Simplifying the Inclusion-Exclusion Sum

Now we can simplify the inner sum in the inclusion-exclusion formula for a fixed k:

$$S_k = \sum_{1 \le i_1 < \cdots < i_k \le n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

This is a sum over all possible combinations of k students. The number of terms in this sum is the number of ways to choose k indices from n, which is given by the **binomial coefficient**[4] $\binom{n}{k}$. Since every term in the sum is equal to $\frac{(n-k)!}{n!}$, we have:

$$S_k = \binom{n}{k} \cdot \frac{(n-k)!}{n!} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{n!} = \frac{1}{k!}$$

This is a wonderful simplification! Now we can write the full probability for $P(A_{total})$:

$$P(A_{total}) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} S_k = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

2.6 Step 6: Finding the Probability of No Matches

We are looking for the probability that *no one* gets their sheet back, which is $P(A^c_{total})$.

$$P(A^c_{total}) = 1 - P(A_{total}) = 1 - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

Let's expand the sum to see what this looks like:

$$\begin{aligned} P(A^c_{total}) &= 1 - \left(\frac{(-1)^2}{1!} + \frac{(-1)^3}{2!} + \frac{(-1)^4}{3!} + \cdots + \frac{(-1)^{n+1}}{n!} \right) \\ &= 1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!} \right) \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{(-1)^{n+1}}{n!} \end{aligned}$$

We can write 1 as $\frac{1}{0!}$. And the last term is $(-1) \cdot \frac{(-1)^{n+1}}{n!} = \frac{(-1)^{n+2}}{n!} = \frac{(-1)^n}{n!}$ if we are careful with the signs. A cleaner way is:

$$P(A^c_{total}) = 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} = \frac{(-1)^0}{0!} + \sum_{k=1}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

So, the probability that no one gets their original sheet back is:

$$\mathbf{P}(\text{no one gets their sheet back}) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}$$

2.7 Step 7: The Asymptotic Behavior (n $\to \infty$)

The final part of the question asks what happens to this probability as n becomes very large. We need to evaluate:

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

This sum is the partial sum of the infinite series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$. As the hint suggests, this is directly related to the **Taylor series expansion of the exponential function**[5], $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. By setting $x = -1$, we get:

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

Therefore, the limit of our probability is:

$$\lim_{n\to\infty} P(\text{no one gets their sheet back}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e} \approx 0.36788$$

3 Summary and Takeaways

- The probability that no student gets their own sheet back in a random redistribution among n students is $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$.
- This problem is a classic example of derangements.
- The solution beautifully showcases the power of the Principle of Inclusion-Exclusion, where a complex counting problem is solved by systematically adding and subtracting probabilities of simpler events.
- As the number of students n grows, this probability surprisingly converges to a constant value, $1/e$. The convergence is very fast; for $n = 8$, the probability is already ≈ 0.367881, which is very close to $1/e$. This means that in a large class, there's about a 36.8% chance that nobody gets their own graded sheet back.

4 Further Explanations

Here are more in-depth explanations of the key concepts used in the solution.

4.1 Permutations

A permutation of a set of objects is an arrangement of those objects into a particular sequence or order. For a set with n distinct objects, the number of different permutations is given by $n!$ (n-factorial), which is the product of all positive integers up to n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Reasoning: For the first position in the sequence, we have n choices. For the second, we have $n-1$ remaining choices. This continues until the last position, where we only have 1 choice left. The total number of arrangements is the product of these choices. In our exercise, a redistribution of sheets is a permutation of the original sheet numbers, which is why $|\Omega| = n!$. This corresponds to the urn model of ordered draws without replacement in **Lemma 1.33**.

4.2 Laplace Probability Space

A Laplace probability space is a model used when all outcomes of an experiment are equally likely. It is named after Pierre-Simon Laplace. If the sample space Ω is finite and all elementary outcomes $\{\omega\}$ have the same probability, then for any event $E \subseteq \Omega$, its probability is defined as:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}} = \frac{|E|}{|\Omega|}$$

This is the most fundamental model for problems involving things like fair dice, shuffled cards, or "random" choices from a set, as described in **Example 1.36 (i)**.

4.3 Principle of Inclusion-Exclusion (PIE)

The Principle of Inclusion-Exclusion (**Theorem 1.20**) is a counting technique to find the size (or probability) of the union of multiple sets. The main idea is to avoid double-counting. For three events A, B, C, it states:

$$\begin{aligned} P(A \cup B \cup C) = &P(A) + P(B) + P(C) \\ &- [P(A \cap B) + P(A \cap C) + P(B \cap C)] \\ &+ P(A \cap B \cap C) \end{aligned}$$

We *include* the probabilities of the individual events, *exclude* the probabilities of pairwise intersections (which were counted twice), and then *include* back the probability of the three-way intersection (which was added three times and removed three times). This generalizes to any number of sets, leading to the alternating sum formula we used.

4.4 Binomial Coefficient

The binomial coefficient, written as $\binom{n}{k}$ and read "n choose k," counts the number of ways to choose a subset of k elements from a larger set of n elements, where the order of selection does not matter. The formula is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In our solution, we needed to know how many intersection terms of size k there were (i.e., how many ways to choose k students out of n who get their own sheets back). This is precisely what $\binom{n}{k}$ calculates.

4.5 Taylor Series for the Exponential Function

A Taylor series is a representation of a function as an infinite sum of terms, calculated from the values of the function's derivatives at a single point. For many well-behaved functions (like e^x), this series converges to the function itself. The Taylor series for e^x expanded around $x = 0$ (also called a Maclaurin series) is:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

This series is valid for all real numbers x. By substituting $x = -1$, we get the specific series used in our limit calculation, which is a fundamental result from analysis.