Exercise Walkthrough: Intersection of σ-algebras

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Abstract

This document provides a detailed, step-by-step walkthrough of the proof that the intersection of any family of σ-algebras is also a σ-algebra. We will base our reasoning on the definitions and concepts introduced in the "Discrete Probability Theory" script by Niki Kilbertus. Each step is explained in plain language, aiming to build a solid understanding of this fundamental result.

1 Introduction and Goal

Hello! Today, we're going to tackle a foundational proof in probability theory. The exercise might seem a bit abstract at first, but the result is incredibly important. It's the theoretical bedrock that allows us to construct the specific σ-algebras we need, like the ones generated by a collection of sets.

Our goal is to prove that if you take any collection of σ-algebras on a given sample space Ω, their intersection is also a σ-algebra. To do this, we first need to remember what a σ-algebra is.

Recall: What is a σ-algebra? From the script (Definition 1.5, page 10 [1]), a σ-algebra is a collection of subsets of Ω (called events) that satisfies three specific rules. It must contain the empty set, be closed under complementation, and be closed under countable unions. We'll use these three rules as a checklist for our proof.

2 The Exercise Statement

Let's formally state the problem we are going to solve.

Exercise 1. *Let $\Omega \neq \emptyset$ be a set, I be a nonempty index set [2] (finite, countable, or even uncountable), and let $\mathcal{A}_\alpha$ be a σ-algebra over Ω for each $\alpha \in I$. Then the set $\mathcal{A}$ defined as the intersection [3] of all these σ-algebras,*

$$\mathcal{A} := \bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

is also a σ-algebra over Ω.

3 The Step-by-Step Proof

Our Strategy: To prove that $\mathcal{A}$ is a σ-algebra, we must show that it satisfies the three properties listed in Definition 1.5 [1]. We will check them one by one.

The core idea is simple: if something is in the intersection $\mathcal{A}$, it must be in *every single* $\mathcal{A}_\alpha$. Since every $\mathcal{A}_\alpha$ is a well-behaved σ-algebra, we can use their properties to show that $\mathcal{A}$ must also be well-behaved.

Step 1: Checking for the Empty Set (Property i)

The first rule for a σ-algebra is that it must contain the empty set, $\emptyset$.

- **What we know:** By definition, each $\mathcal{A}_\alpha$ is a σ-algebra. Therefore, according to Property (i) of the definition, we know that $\emptyset \in \mathcal{A}_\alpha$ for every single $\alpha \in I$.
- **The logical step:** The intersection $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ contains only the elements that are present in *all* of the sets $\mathcal{A}_\alpha$. Since we've just established that $\emptyset$ is in every $\mathcal{A}_\alpha$, it must also be in their intersection.
- **Conclusion:** Therefore, $\emptyset \in \mathcal{A}$. The first property is satisfied.

(Note: The provided solution in the prompt mentions Ω. This is also correct. If $\emptyset \in \mathcal{A}_\alpha$, then its complement $\Omega = \emptyset^c$ must also be in $\mathcal{A}_\alpha$ by property (ii). So Ω is in every $\mathcal{A}_\alpha$ and thus in their intersection. Starting with either $\emptyset$ or Ω works fine.)

Step 2: Checking for Closure Under Complementation (Property ii)

The second rule is: if a set A is in the collection, its complement A^c must also be in the collection.

- **What we need to show:** For any arbitrary set $S \in \mathcal{A}$, we must prove that its complement S^c is also in $\mathcal{A}$.
- **The logical step:**
  1. Let's pick an arbitrary set S such that $S \in \mathcal{A}$.
  2. By the definition of an intersection, if $S \in \mathcal{A}$, then S must be an element of every single σ-algebra in the family: $S \in \mathcal{A}_\alpha$ for all $\alpha \in I$.
  3. Now, we know that each $\mathcal{A}_\alpha$ is a σ-algebra. This means each $\mathcal{A}_\alpha$ is closed under complementation (Property ii). So, if $S \in \mathcal{A}_\alpha$, then its complement S^c must also be in $\mathcal{A}_\alpha$. This is true for all $\alpha \in I$.
  4. We have now shown that S^c is an element of every $\mathcal{A}_\alpha$.
  5. Again, by the definition of intersection, if S^c is in all of the sets $\mathcal{A}_\alpha$, it must be in their intersection, $\mathcal{A}$.
- **Conclusion:** We started with an arbitrary $S \in \mathcal{A}$ and showed that $S^c \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under complementation. The second property is satisfied.

Step 3: Checking for Closure Under Countable Unions (Property iii)

The third and final rule is: if you have a countable sequence of sets that are all in the collection, their union must also be in the collection.

- **What we need to show:** For any countable sequence of sets $S_1, S_2, S_3, \dots$ where each $S_i \in \mathcal{A}$, we must prove that their union $\bigcup_{i=1}^{\infty} S_i$ is also in $\mathcal{A}$.
- **The logical step:**
  1. Let's take a countable sequence of sets $S_1, S_2, S_3, \dots$ where every set S_i is in our intersection $\mathcal{A}$.
  2. By the definition of intersection, this means that the entire sequence of sets $\{S_i\}_{i \in \mathbb{N}}$ is present in *each* $\mathcal{A}_\alpha$ for all $\alpha \in I$.
  3. We know that each $\mathcal{A}_\alpha$ is a σ-algebra, so it must be closed under countable unions (Property iii). Therefore, the union of our sequence, $\bigcup_{i=1}^{\infty} S_i$, must be an element of $\mathcal{A}_\alpha$ for every $\alpha \in I$.
  4. Since the union $\bigcup_{i=1}^{\infty} S_i$ is in every single $\mathcal{A}_\alpha$, it must, by definition, be in their intersection $\mathcal{A}$.
- **Conclusion:** We have shown that $\mathcal{A}$ is closed under countable unions. The third property is satisfied.

4 Summary and a Glimpse Forward

We have successfully checked all three properties from the definition of a σ-algebra.

✓ $\emptyset \in \mathcal{A}$

✓ For any $S \in \mathcal{A}$, we have $S^c \in \mathcal{A}$

✓ For any countable sequence $S_1, S_2, \dots$ in $\mathcal{A}$, we have $\bigcup S_i \in \mathcal{A}$

Since $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ fulfills all the necessary conditions, we can confidently conclude that it is a σ-algebra.

Why does this matter? This result is not just a theoretical curiosity. It's the key that lets us define the concept of a **generated σ-algebra** (Lemma 1.7, page 10 [4]). Often, we start with a simple collection of subsets $\mathcal{E}$ (e.g., all open intervals on the real line) and we want to find the "smallest" σ-algebra that contains all of them. How do we know such a "smallest" one exists?

Our proof provides the answer! We can consider the (very large) family of *all possible* σ-algebras that contain $\mathcal{E}$. We know this family is not empty because the power set $\mathcal{P}(\Omega)$ is always one such σ-algebra. By the theorem we just proved, the intersection of this entire family is also a σ-algebra. By its very construction, it is the smallest one containing $\mathcal{E}$. This is precisely the definition of the generated σ-algebra, $\sigma(\mathcal{E})$.

Deeper Dive: Explanations

Here are more detailed explanations of the concepts marked with clickable numbers.

[1] **Definition 1.5: σ-algebra.** A σ-algebra (or sigma-field) on a set Ω is a collection $\mathcal{A}$ of subsets of Ω that satisfies the following three axioms:

(i) **Contains the empty set:** $\emptyset \in \mathcal{A}$. This represents the "impossible event".

(ii) **Closed under complementation:** If a set A is in $\mathcal{A}$, then its complement $A^c = \Omega \setminus A$ must also be in $\mathcal{A}$. (If an event can happen, the event "it doesn't happen" is also an event).

(iii) **Closed under countable unions:** If you have a sequence of sets $A_1, A_2, A_3, \ldots$ and all of them are in $\mathcal{A}$, then their union $\bigcup_{i=1}^{\infty} A_i$ must also be in $\mathcal{A}$. (If we can measure the probability of individual events, we can also measure the probability of "at least one of them happening").

This structure is essential because it defines the set of "measurable events"—all the questions to which we can assign a probability.

[2] **Index Set.** An index set I is simply a set used to label the elements of another collection. In this exercise, we have a collection of σ-algebras, and we use the elements α from the set I to give each one a unique label, $\mathcal{A}_\alpha$. The exercise states that I can be finite (e.g., $I = \{1, 2, 3\}$), countably infinite (e.g., $I = \mathbb{N}$), or even uncountably infinite (e.g., $I = \mathbb{R}$). Our proof works regardless of the size of I, which makes the result very powerful.

[3] **Intersection.** The intersection of a family of sets, denoted $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$, is the set of all elements that are members of *every* set $\mathcal{A}_\alpha$ in the family. For an element x to be in the intersection, it must satisfy $x \in \mathcal{A}_\alpha$ for all $\alpha \in I$. If it is missing from even one of them, it is not in the intersection.

[4] **Lemma 1.7: Generated σ-algebra.** For any collection of subsets $\mathcal{E} \subseteq \mathcal{P}(\Omega)$, the σ-algebra generated by $\mathcal{E}$, denoted $\sigma(\mathcal{E})$, is defined as the intersection of all σ-algebras on Ω that contain $\mathcal{E}$. The proof we just completed guarantees that this intersection is indeed a σ-algebra. It is also, by construction, the *smallest* σ-algebra containing $\mathcal{E}$. This is the standard way to build complex event spaces, like the Borel σ-algebra on $\mathbb{R}$, which is generated by the set of all open intervals.