Exercise Walkthrough: Modes of Convergence

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The Exercise

Let's carefully work through the following exercise, which explores the properties and relationships between different modes of convergence for random variables.

Exercise 1. *Let X, Y be real-valued random variables (RVRVs) and $(X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}$ be sequences of RVRVs. Verify the following statements.*

1. *If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.*
2. *If $\mathbb{E}[|X_n - X|] \to 0$, then $X_n \xrightarrow{P} X$.*
3. *If $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$ then $P(X = Y) = 1$.*
4. *If $\mathbb{E}[|X_n - X|] \to 0$ and $\mathbb{E}[|X_n - Y|] \to 0$ then $P(X = Y) = 1$.*

Part (i): The Sum Rule for Convergence in Probability

Overview

This part asks us to prove that if two sequences of random variables converge in probability, their sum also converges in probability to the sum of their limits. This is a very intuitive and useful property, often called the Continuous Mapping Theorem for addition. It ensures that convergence behaves nicely with basic arithmetic operations.

Step-by-Step Solution

Proof. Our goal is to show that $X_n + Y_n \xrightarrow{P} X + Y$.

Step 1: State the Goal Formally According to **Definition 2.56 (i)** of the script, we need to show that for any given $\epsilon > 0$, the following holds:

$$\lim_{n\to\infty} P\left(|(X_n + Y_n) - (X + Y)| \geq \epsilon\right) = 0$$

Step 2: Apply the Triangle Inequality The trick is to relate the term we're interested in, $|(X_n + Y_n) - (X + Y)|$, to the terms we know something about, namely $|X_n - X|$ and $|Y_n - Y|$. We can rearrange the term and apply the triangle inequality [2].

$$|(X_n + Y_n) - (X + Y)| = |(X_n - X) + (Y_n - Y)| \leq |X_n - X| + |Y_n - Y|$$

This inequality is the key. It tells us that the error of the sum is at most the sum of the individual errors.

Step 3: Relate the Events Now, let's think about the events. If the sum of the errors $|X_n - X| + |Y_n - Y|$ is less than ϵ, then our target error $|(X_n + Y_n) - (X + Y)|$ must also be less than ϵ. This means that the event $\{|(X_n + Y_n) - (X + Y)| \geq \epsilon\}$ can only happen if the event $\{|X_n - X| + |Y_n - Y| \geq \epsilon\}$ happens. In set notation, this means:

$$\{|(X_n + Y_n) - (X + Y)| \geq \epsilon\} \subseteq \{|X_n - X| + |Y_n - Y| \geq \epsilon\}$$

If $|X_n - X| < \epsilon/2$ and $|Y_n - Y| < \epsilon/2$, their sum is less than ϵ. Therefore, for their sum to be $\geq \epsilon$, at least one of them must be $\geq \epsilon/2$. This allows us to make another subset connection:

$$\{|X_n - X| + |Y_n - Y| \geq \epsilon\} \subseteq \{|X_n - X| \geq \epsilon/2\} \cup \{|Y_n - Y| \geq \epsilon/2\}$$

Step 4: Use the Union Bound By the monotonicity and subadditivity of probability measures (**Proposition 1.17**), we can bound the probability:

$$\begin{aligned} P\left(|(X_n + Y_n) - (X + Y)| \geq \epsilon\right) &\leq P\left(\{|X_n - X| \geq \epsilon/2\} \cup \{|Y_n - Y| \geq \epsilon/2\}\right) \\ &\leq P\left(|X_n - X| \geq \epsilon/2\right) + P\left(|Y_n - Y| \geq \epsilon/2\right) \quad [3] \end{aligned}$$

Step 5: Take the Limit We are given that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. This means that as $n \to \infty$:

$$P(|X_n - X| \geq \epsilon/2) \to 0 \quad \text{and} \quad P(|Y_n - Y| \geq \epsilon/2) \to 0$$

Therefore, the sum of these two probabilities also goes to zero. Since our target probability is non-negative and is less than or equal to a quantity that goes to zero, it must also go to zero.

$$\lim_{n\to\infty} P(|(X_n + Y_n) - (X + Y)| \geq \epsilon) = 0$$

This completes the proof. □

Part (ii): L^1 Convergence implies Convergence in Probability

Overview

Here, we want to show that convergence in mean (specifically, in L^1) [5] is a stronger type of convergence than convergence in probability [1]. This means if you know that the expected absolute difference $\mathbb{E}[|X_n - X|]$ goes to zero, you can be sure that X_n also converges in probability to X. The hint points us to a very powerful tool: Markov's inequality.

Step-by-Step Solution

Proof. We are given that $\lim_{n\to\infty} \mathbb{E}[|X_n - X|] = 0$. We want to show that for any $\epsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$.

Step 1: Identify the Right Tool The hint suggests Markov's inequality (**Theorem 2.38**). This inequality relates the probability of a random variable being large to its expectation. For a non-negative random variable Z and a constant $a > 0$, it states:

$$P(Z \geq a) \leq \frac{\mathbb{E}[Z]}{a}$$

Step 2: Apply the Inequality Let's define a new, non-negative random variable $Z_n = |X_n - X|$. We are interested in the probability $P(Z_n \geq \epsilon)$. Applying Markov's inequality with $Z = Z_n$ and $a = \epsilon$, we get:

$$P(|X_n - X| \geq \epsilon) \leq \frac{\mathbb{E}[|X_n - X|]}{\epsilon}$$

Step 3: Take the Limit We can now take the limit as $n \to \infty$ on both sides of the inequality.

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) \leq \lim_{n\to\infty} \frac{\mathbb{E}[|X_n - X|]}{\epsilon}$$

By our initial assumption, the numerator on the right side goes to zero: $\lim_{n\to\infty} \mathbb{E}[|X_n - X|] = 0$. Since ϵ is a fixed positive constant, the entire right side becomes 0.

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) \leq 0$$

Since probability cannot be negative, we must have equality.

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

This is precisely the definition of $X_n \xrightarrow{P} X$. □

Part (iii): Uniqueness of the Limit in Probability

Overview

This part establishes a crucial property: if a sequence converges in probability, its limit is unique. Uniqueness here means that if it converges to two different random variables, X and Y, then X and Y must be the same variable with probability 1. This is also called being equal "almost surely" [6]. Without this property, the concept of a "limit" would be ambiguous.

Step-by-Step Solution

Proof. We are given $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$. Our goal is to show $P(X = Y) = 1$. This is equivalent to showing $P(X \neq Y) = 0$.

Step 1: Use the Triangle Inequality on the Limits As hinted, we look at the distance $|X - Y|$. We can cleverly introduce X_n and use the triangle inequality [2]:

$$|X - Y| = |(X - X_n) + (X_n - Y)| \leq |X - X_n| + |Y - X_n|$$

Step 2: Bound the Probability for a Fixed ϵ Let's fix an arbitrary $\epsilon > 0$. We want to show that $P(|X - Y| \geq \epsilon) = 0$. Using the inequality from Step 1 and the same logic as in Part (i), we have:

$$\{|X - Y| \geq \epsilon\} \subseteq \{|X - X_n| \geq \epsilon/2\} \cup \{|Y - X_n| \geq \epsilon/2\}$$

Using the union bound [3], we get for **any** $n \in \mathbb{N}$:

$$P(|X - Y| \geq \epsilon) \leq P(|X_n - X| \geq \epsilon/2) + P(|X_n - Y| \geq \epsilon/2)$$

Step 3: Take the Limit in n The left-hand side, $P(|X - Y| \geq \epsilon)$, does not depend on n. We can take the limit of the right-hand side as $n \to \infty$. Since $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$, both terms on the right go to 0.

$$\lim_{n\to\infty} \left(P(|X_n - X| \geq \epsilon/2) + P(|X_n - Y| \geq \epsilon/2)\right) = 0 + 0 = 0$$

Since the inequality holds for all n, it must also hold in the limit. Therefore, we have:

$$P(|X - Y| \geq \epsilon) \leq 0$$

As probability is non-negative, this forces $P(|X - Y| \geq \epsilon) = 0$.

Step 4: Conclude for the Event $\{X \neq Y\}$ We have shown that for any $\epsilon > 0$, the probability of X and Y differing by at least ϵ is zero. The event that X and Y are not equal, $\{X \neq Y\}$, can be written as a countable union of such events:

$$\{X \neq Y\} = \bigcup_{k=1}^{\infty} \left\{|X - Y| \geq \frac{1}{k}\right\}$$

Using the union bound (σ-subadditivity) for countable unions:

$$P(X \neq Y) = P\left(\bigcup_{k=1}^{\infty} \left\{|X - Y| \geq \frac{1}{k}\right\}\right) \leq \sum_{k=1}^{\infty} P\left(|X - Y| \geq \frac{1}{k}\right)$$

From Step 3, we know every term in the sum is 0. So, $P(X \neq Y) \leq \sum_{k=1}^{\infty} 0 = 0$. Again, since probability is non-negative, $P(X \neq Y) = 0$, which means $P(X = Y) = 1$. □

Part (iv): Uniqueness of the Limit in L^1

Overview

Finally, we show that the limit in L^1 (in mean) is also unique almost surely. We can solve this elegantly by combining our previous results, or by a more direct proof that mirrors the logic of Part (iii) but uses expectations.

Step-by-Step Solution (Method 1: Combining Previous Results)

Proof. This is the most straightforward path.

1. We are given $\mathbb{E}[|X_n - X|] \to 0$ and $\mathbb{E}[|X_n - Y|] \to 0$.
2. From Part (ii), we know that L^1 convergence implies convergence in probability. Therefore, we have $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$.
3. From Part (iii), we know that if a sequence converges in probability to two limits, those limits must be almost surely equal.
4. Therefore, we can immediately conclude that $P(X = Y) = 1$.

This shows how these concepts build upon each other. □

Step-by-Step Solution (Method 2: Direct Proof)

Proof. This method provides more insight into the properties of expectation.

Step 1: Use the Triangle Inequality Just as in Part (iii), we start with the inequality:

$$|X - Y| \leq |X - X_n| + |Y - X_n|$$

Step 2: Take the Expectation We take the expectation of both sides. By the linearity and monotonicity of expectation (**Proposition 2.4**), we get:

$$\begin{aligned}\mathbb{E}[|X - Y|] &\leq \mathbb{E}[|X - X_n| + |Y - X_n|] \\ &= \mathbb{E}[|X - X_n|] + \mathbb{E}[|Y - X_n|]\end{aligned}$$

Step 3: Take the Limit in n This inequality holds for all n. The left side does not depend on n. We can take the limit of the right side as $n \to \infty$. From our initial assumptions:

$$\lim_{n\to\infty} (\mathbb{E}[|X - X_n|] + \mathbb{E}[|Y - X_n|]) = 0 + 0 = 0$$

This implies $\mathbb{E}[|X - Y|] \leq 0$. Since $|X - Y|$ is a non-negative random variable, its expectation must be non-negative, i.e., $\mathbb{E}[|X - Y|] \geq 0$. The only way for both to be true is if $\mathbb{E}[|X - Y|] = 0$.

Step 4: Show that Zero Expectation Implies Almost Sure Equality If the expected value of a non-negative random variable (like $|X - Y|$) is zero, the random variable itself must be zero with probability 1. We can show this using Markov's inequality. For any $k \in \mathbb{N}_{>0}$:

$$P\left(|X - Y| \geq \frac{1}{k}\right) \leq \frac{\mathbb{E}[|X - Y|]}{1/k} = \frac{0}{1/k} = 0$$

This means $P(|X - Y| \geq 1/k) = 0$ for all k. As we argued in Part (iii), the event $\{X \neq Y\}$ is the countable union $\bigcup_{k=1}^{\infty}\{|X - Y| \geq 1/k\}$, and its probability is therefore 0. Thus, $P(X = Y) = 1$. $\square$

In-depth Explanations

[1] Convergence in Probability ($X_n \xrightarrow{P} X$)

As defined in **Definition 2.56 (i)**, a sequence of random variables (X_n) converges in probability to a random variable X if, for any arbitrarily small positive number ϵ, the probability that X_n and X differ by more than ϵ becomes vanishingly small as n gets large. **Formally:** $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$ for all $\epsilon > 0$. **Intuition:** For large n, it's extremely unlikely that X_n will be far from X.

[2] The Triangle Inequality

A fundamental property of absolute values (and more generally, norms). For any real numbers a and b, it states:

$$|a + b| \leq |a| + |b|$$

Intuition: The length of one side of a triangle is never greater than the sum of the lengths of the other two sides. In our proofs, it's a crucial tool for splitting an error term into more manageable parts.

[3] The Union Bound (Boole's Inequality)

This is a direct consequence of the axioms of probability, specifically σ-additivity. For any finite or countable sequence of events $A_1, A_2, \ldots$, it states:

$$P(A_1 \cup A_2 \cup \ldots) \leq P(A_1) + P(A_2) + \ldots$$

It's an incredibly useful tool for bounding the probability of a complex event by breaking it down into simpler events. We used the finite version $P(A \cup B) \leq P(A) + P(B)$.

[4] Markov's Inequality

As seen in **Theorem 2.38**, this inequality provides a (often loose) upper bound on the probability that a non-negative random variable Z is greater than or equal to some positive constant a. **Formally:** $P(Z \geq a) \leq \frac{\mathbb{E}[Z]}{a}$ for $a > 0$. **Intuition:** A random variable can't be "very large, very often" if its average value is small. This inequality is foundational for proving other important results, like Chebyshev's inequality.

[5] Convergence in Mean (L^1-convergence)

This is a stronger mode of convergence. A sequence (X_n) converges in mean (or in L^1) to X if the expected value of the absolute difference between X_n and X approaches zero. **Formally:** $\lim_{n\to\infty} \mathbb{E}[|X_n - X|] = 0$. **Intuition:** Not only do large deviations become improbable (as in convergence in probability), but the "average size" of the deviation, considering all possible outcomes, goes to zero.

[6] Almost Sure Equality ($P(X = Y) = 1$)

Two random variables X and Y are said to be equal almost surely (a.s.) if the set of outcomes ω in the sample space Ω for which $X(\omega) \neq Y(\omega)$ has a total probability of zero. **Formally:** $P(\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}) = 1$. **Intuition:** For all practical purposes in probability theory, X and Y are interchangeable. They might differ on a set of outcomes, but the probability of one of those outcomes occurring is zero. This is the standard notion of "uniqueness" for limits of random variables.