

# Exercise Walkthrough: Sum of Independent Poisson Random Variables

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## The Exercise

**Statement:** Let  $X \sim \text{Poi}(\lambda_1)$  and  $Y \sim \text{Poi}(\lambda_2)$  be independent random variables. Show that their sum  $S = X + Y$  is also Poisson distributed, specifically  $S \sim \text{Poi}(\lambda_1 + \lambda_2)$ .

**Hint:** Compute the characteristic function of a Poisson RV and use properties of characteristic functions.

## Overview of the Approach

This problem is a classic application of characteristic functions [1]. The strategy is straightforward and follows the hint directly:

1. First, we need a general formula for the characteristic function (CF) of any Poisson-distributed random variable. We will derive this from its definition.
2. Second, we will use the property that for **independent** random variables, the CF of their sum is the product of their individual CFs.
3. Finally, we will examine the resulting CF for the sum  $S = X + Y$ . By the **uniqueness property** of CFs, if we can match this CF to the CF of a known distribution, we have proven that  $S$  follows that distribution.

Let's work through each of these steps in detail.

## 1 Step 1: The Characteristic Function of a Poisson RV

Our first goal is to find the characteristic function  $\phi_X(t)$  for a general random variable  $X \sim \text{Poi}(\lambda)$ .

**Reasoning:** From the script (**Definition 2.46, p. 64**), the characteristic function of a random variable  $X$  is defined as:

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

Since our random variable  $X$  is discrete, the expectation is calculated by summing over all possible values  $k$  that  $X$  can take, weighted by their respective probabilities. The possible outcomes for a Poisson distribution are  $k \in \{0, 1, 2, \dots\}$ .

$$\mathbb{E}[g(X)] = \sum_{k=0}^{\infty} g(k)P(X = k)$$

In our case,  $g(X) = e^{itX}$ . The probability mass function (pmf) for  $X \sim \text{Poi}(\lambda)$  is given in the script (**Example 1.36 (vii), p. 21**) as:

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

**Derivation:** Let's substitute these into the definition of the CF.

$$\begin{aligned} \phi_X(t) &= \mathbb{E}[e^{itX}] \\ &= \sum_{k=0}^{\infty} e^{itk} P(X = k) && \text{(Definition of expectation for discrete RVs)} \\ &= \sum_{k=0}^{\infty} e^{itk} \frac{e^{-\lambda} \lambda^k}{k!} && \text{(Substituting the Poisson pmf [2])} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{it})^k \lambda^k}{k!} && \text{(The term } e^{-\lambda} \text{ is constant w.r.t. } k) \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} && \text{(Combining terms with exponent } k) \end{aligned}$$

Now, we need to recognize the sum. This is the Taylor series expansion for the exponential function [3], which is  $\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$ . In our case,  $z = \lambda e^{it}$ .

$$\begin{aligned} \phi_X(t) &= e^{-\lambda} \exp(\lambda e^{it}) && \text{(Applying the Taylor series formula)} \\ &= \exp(-\lambda + \lambda e^{it}) && \text{(Combining exponents)} \\ &= \exp(\lambda(e^{it} - 1)) && \text{(Factoring out } \lambda) \end{aligned}$$

So, we have our general formula: the characteristic function of a random variable  $X \sim \text{Poi}(\lambda)$  is  $\phi_X(t) = \exp(\lambda(e^{it} - 1))$ .

## 2 Step 2: The Characteristic Function of the Sum

Now we use the result from Step 1 to find the CF of  $S = X + Y$ .

**Reasoning:** The problem states that  $X$  and  $Y$  are **independent**. This is the crucial property we need. As stated in the script (**Remark 2.47 (iv), p. 64**), for independent random variables, the characteristic function of their sum is the product of their individual characteristic functions [4].

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$$

We have  $X \sim \text{Poi}(\lambda_1)$  and  $Y \sim \text{Poi}(\lambda_2)$ . We can use the formula from Step 1 for each of them.

**Derivation:** Let  $S = X + Y$ .

$$\begin{aligned} \phi_S(t) &= \phi_{X+Y}(t) \\ &= \phi_X(t)\phi_Y(t) && \text{(Due to independence)} \\ &= \exp(\lambda_1(e^{it} - 1)) \cdot \exp(\lambda_2(e^{it} - 1)) && \text{(Using the result from Step 1)} \\ &= \exp(\lambda_1(e^{it} - 1) + \lambda_2(e^{it} - 1)) && \text{(Product of exponentials is exp of sum)} \\ &= \exp((\lambda_1 + \lambda_2)(e^{it} - 1)) && \text{(Factoring out } (e^{it} - 1)) \end{aligned}$$

This gives us the characteristic function for the sum  $S$ .

### 3 Step 3: Concluding the Proof

The final step is to interpret the result from Step 2.

**Reasoning:** The characteristic function we found for  $S$  is:

$$\phi_S(t) = \exp((\lambda_1 + \lambda_2)(e^{it} - 1))$$

Let's compare this to the general form of a Poisson CF we derived in Step 1:  $\phi_Z(t) = \exp(\lambda(e^{it} - 1))$  for a random variable  $Z \sim \text{Poi}(\lambda)$ . Our result for  $\phi_S(t)$  has exactly this form, with the parameter  $\lambda = \lambda_1 + \lambda_2$ .

The uniqueness property of characteristic functions (**Remark 2.47 (v), p. 64**) states that a probability distribution is uniquely determined by its CF. In other words, if two random variables have the same CF, they must follow the same distribution [5].

**Conclusion:** Since the characteristic function of  $S = X + Y$  is that of a Poisson distribution with parameter  $\lambda_1 + \lambda_2$ , we can conclude that  $S$  must be distributed as such.

$$S \sim \text{Poi}(\lambda_1 + \lambda_2)$$

This completes the proof.

### Summary and Takeaways

We have successfully shown that the sum of two independent Poisson random variables is another Poisson random variable whose parameter is the sum of the individual parameters.

The key takeaways from this exercise are:

- **Power of CFs:** Characteristic functions are a powerful tool, especially for problems involving sums of independent random variables. They often transform a complicated convolution of distributions into a simple multiplication.
- **The "Trilogy" of Properties:** The solution relies on a three-step pattern common in such proofs: (1) Find the general CF for the distribution family. (2) Use the independence property ( $\phi_{X+Y} = \phi_X \phi_Y$ ). (3) Use the uniqueness property to identify the resulting distribution.

This method is not limited to Poisson distributions and is frequently used to prove similar closure properties for other distributions, like the Normal or Gamma distributions.

## Further Explanations

[1] **Characteristic Function (CF):** The characteristic function of a random variable  $X$  is defined as  $\phi_X(t) = \mathbb{E}[e^{itX}]$ , where  $i$  is the imaginary unit and  $t \in \mathbb{R}$ . It's a fundamental tool in probability theory because it always exists (unlike the moment-generating function) and, most importantly, it uniquely determines the distribution of the random variable. This means if you know the CF, you know the distribution. See **Definition 2.46, p. 64** for the formal definition.

[2] **Poisson Distribution:** The Poisson distribution, denoted  $\text{Poi}(\lambda)$ , is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event. Its probability mass function (pmf) is  $P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$  for  $k = 0, 1, 2, \dots$ . The parameter  $\lambda$  is the expected number of events in the interval. See **Example 1.36 (vii), p. 21**.

[3] **Taylor Series for  $e^z$ :** The Taylor series (or Maclaurin series, since it's centered at 0) for the exponential function  $e^z$  is one of the most important series in mathematics. It is given by:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

This series converges for all complex numbers  $z$ . Recognizing this pattern is the key to simplifying the sum in our derivation.

[4] **Independence and CFs:** A crucial property of characteristic functions is how they behave with sums of *independent* random variables. If  $X$  and  $Y$  are independent, then the expectation of the product of functions of  $X$  and  $Y$  is the product of their individual expectations, i.e.,  $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$ . Applying this to the CF of the sum  $S = X + Y$ :

$$\phi_S(t) = \mathbb{E}[e^{it(X+Y)}] = \mathbb{E}[e^{itX}e^{itY}] = \mathbb{E}[e^{itX}]\mathbb{E}[e^{itY}] = \phi_X(t)\phi_Y(t)$$

This property is mentioned in **Remark 2.47 (iv), p. 64**.

[5] **Uniqueness Property of CFs:** This property, also known as the Inversion Theorem, establishes a one-to-one correspondence between probability distributions and characteristic functions. If two random variables  $X$  and  $Y$  have the same characteristic function (i.e.,  $\phi_X(t) = \phi_Y(t)$  for all  $t \in \mathbb{R}$ ), then they must have the same probability distribution ( $F_X(z) = F_Y(z)$  for all  $z \in \mathbb{R}$ ). This allows us to identify a distribution by simply identifying its CF. See **Remark 2.47 (v), p. 64**.