Exercise Walkthrough: Commutativity of σ-Algebra Generation and Restriction

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Abstract

This document provides a detailed, step-by-step walkthrough for an exercise from the Discrete Probability Theory course. The exercise demonstrates a key property of σ-algebras: that generating a σ-algebra from a restricted system of sets is equivalent to restricting the σ-algebra generated from the original system. We will leverage concepts from the lecture script, including measurable maps and inverse images, to construct a clear and formal proof.

1 Overview and Goal

The exercise asks us to prove the following statement:

Theorem. Let $\Omega \neq \emptyset$ be a set, $A \subseteq \Omega$ with $A \neq \emptyset$, and let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ be a collection of subsets of Ω. Then it holds that

$$\sigma(\mathcal{E}|_A) = \sigma(\mathcal{E})|_A.$$

In plain language, this theorem is about whether two different procedures result in the same final collection of sets.

1. **The Left-Hand Side (LHS):** $\sigma(\mathcal{E}|_A)$. This means we first take our collection of "generator" sets $\mathcal{E}$ and restrict every set in it to A (by intersecting each set with A). This gives us a new collection of sets, $\mathcal{E}|_A$, which are all subsets of A. Then, we find the smallest σ-algebra *on the set* A that contains all these restricted sets.
2. **The Right-Hand Side (RHS):** $\sigma(\mathcal{E})|_A$. This means we first take our original collection $\mathcal{E}$ and find the smallest σ-algebra *on the set* Ω that contains it. This gives us $\sigma(\mathcal{E})$. Then, we take this entire σ-algebra and restrict it to A (again, by intersecting every set in it with A).

The theorem states that these two procedures are equivalent. The order of "generating" and "restricting" doesn't matter. The provided solution sketch uses a clever and powerful tool: the **natural injection map** and its inverse. Let's build up the necessary concepts before diving into the proof.

2 Preliminaries and Definitions

To follow the proof, we need to be crystal clear on a few definitions, most of which are from the lecture script.

σ-Algebra (Definition 1.5). A collection of subsets $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra on Ω if it contains the empty set, is closed under complements, and is closed under countable unions.

Generated σ-Algebra (Lemma 1.7). For any system of subsets $\mathcal{E} \subseteq \mathcal{P}(\Omega)$, the σ-algebra generated by $\mathcal{E}$, denoted $\sigma(\mathcal{E})$, is the smallest σ-algebra on Ω that contains $\mathcal{E}$. It is the intersection of all σ-algebras containing $\mathcal{E}$.

Restriction of a System of Sets. The exercise uses the notation $\mathcal{E}|_A$. Let's define it formally. For a system of subsets $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ and a set $A \subseteq \Omega$, the **restriction of $\mathcal{E}$ to A** is a collection of subsets of A defined as:

$$\mathcal{E}|_A := \{E \cap A \mid E \in \mathcal{E}\}.$$

This is analogous to the **induced σ-algebra** from Lemma 1.14 in the script, which uses the notation $\mathcal{A}|_B = \{A \cap B \mid A \in \mathcal{A}\}$ for an existing σ-algebra $\mathcal{A}$. The principle is the same: intersect every set in the collection with the subspace.

The Natural Injection Map. [1] The solution's key insight is to rephrase the "restriction" operation (set intersection) in terms of an inverse map. We define the **natural injection** (or inclusion) map from A into Ω as:

$$\iota_A : A \to \Omega, \quad \text{where} \quad \iota_A(a) = a \quad \text{for all } a \in A.$$

This map simply takes an element from the subset A and views it as an element of the larger set Ω. The magic happens when we look at its inverse. For any set $B \subseteq \Omega$, the inverse image of B under ι_A is:

$$\iota_A^{-1}(B) = \{a \in A \mid \iota_A(a) \in B\} = \{a \in A \mid a \in B\} = A \cap B.$$

This gives us a powerful bridge: restricting a set B to A is identical to taking the inverse image of B under the natural injection map ι_A.

3 The Step-by-Step Proof

Now we have all the tools to follow the solution. We want to show that $\sigma(\mathcal{E}|_A) = \sigma(\mathcal{E})|_A$. We will start with the LHS and transform it step-by-step into the RHS.

$$\begin{aligned}
\sigma(\mathcal{E}|_A) &= \sigma\left(\{E \cap A \mid E \in \mathcal{E}\}\right) && (1)\\
&= \sigma\left(\{\iota_A^{-1}(E) \mid E \in \mathcal{E}\}\right) && (2)\\
&= \sigma\left(\iota_A^{-1}(\mathcal{E})\right) && (3)\\
&= \iota_A^{-1}\left(\sigma(\mathcal{E})\right) && (4)\\
&= \{A \cap B \mid B \in \sigma(\mathcal{E})\} && (5)\\
&= \sigma(\mathcal{E})|_A && (6)
\end{aligned}$$

Let's break down the reasoning for each step.

Step (1): Expanding the definition. Here, we simply apply the definition of the restriction of a system of sets, $\mathcal{E}|_A$, which we established in the preliminaries. This step just makes the notation explicit.

Step (2): Introducing the injection map. This step uses the crucial property of the natural injection map ι_A. As we showed, for any set $E \subseteq \Omega$, its intersection with A is exactly its inverse image under ι_A. So, we replace every instance of $E \cap A$ with $\iota_A^{-1}(E)$. We are just rephrasing the problem.

Step (3): Simplifying notation. This is a purely notational change. The set $\{\iota_A^{-1}(E) \mid E \in \mathcal{E}\}$ is more compactly written as $\iota_A^{-1}(\mathcal{E})$, which means "the collection of all inverse images of sets in $\mathcal{E}$".

Step (4): The core argument. [2] This is the most important step of the proof. It states that generating a σ-algebra from a collection of inverse images is the same as taking the inverse image of the σ-algebra generated from the original collection. Formally, for a map $f : X \to Y$ and a generator system $\mathcal{E} \subseteq \mathcal{P}(Y)$, it holds that $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$. The exercise hint says to "use the result from Exercise 1", which is precisely this property. This result is a standard theorem in measure theory because the inverse image operation preserves all the necessary set operations (unions, intersections, complements) required to build a σ-algebra.

Step (5): Returning to intersections. Here, we reverse what we did in Step (2). We expand the compact notation $\iota_A^{-1}(\sigma(\mathcal{E}))$ back into its definition: it is the collection of all sets $A \cap B$ where B is a set from the σ-algebra $\sigma(\mathcal{E})$.

Step (6): Applying the definition of restriction. Finally, we recognize that the expression $\{A \cap B \mid B \in \sigma(\mathcal{E})\}$ is exactly the definition of the restriction of the σ-algebra $\sigma(\mathcal{E})$ to the subset A. This is the RHS we wanted to reach.

This completes the proof. We have shown that $\sigma(\mathcal{E}|_A) = \sigma(\mathcal{E})|_A$.

4 Check Your Understanding

To solidify this concept, try to work through a concrete example.

Exercise: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ (a standard die roll). Let $A = \{1, 3, 5\}$ be the event of rolling an odd number. Let the generator system be $\mathcal{E} = \{\{1, 2\}, \{4, 5\}\}$.

1. Explicitly compute the left-hand side:
   - First find $\mathcal{E}|_A$.
   - Then find $\sigma(\mathcal{E}|_A)$, which will be a σ-algebra on A.
2. Explicitly compute the right-hand side:
   - First find $\sigma(\mathcal{E})$, which will be a σ-algebra on Ω.
   - Then find $\sigma(\mathcal{E})|_A$.
3. Verify that your results from both parts are identical.

5 Summary and Takeaways

- **Main Result:** The operations of generating a σ-algebra and restricting to a subspace are commutative. This is a useful and elegant property.
- **Proof Technique:** The key to the formal proof was translating the geometric idea of "restriction" or "intersection" into the language of functions and their inverse images using the natural injection map ι_A.
- **The Power of Inverse Images:** The property $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$ is fundamental for working with measurable functions and induced structures. It essentially states that the inverse image is "well-behaved" with respect to the operations that define a σ-algebra.

This result is a building block for more advanced topics, like understanding the distributions of random variables, which are themselves defined as measurable maps.

In-Depth Explanations

[1] The Natural Injection Map

The natural injection map is a simple but powerful concept. When we have a subset $A \subseteq \Omega$, the map $\iota_A : A \to \Omega$ defined by $\iota_A(a) = a$ formalizes the idea that A is "sitting inside" Ω.

The real utility comes from its inverse image, ι_A^{-1}. For any subset $B \subseteq \Omega$, the inverse image $\iota_A^{-1}(B)$ asks: "Which elements *in the domain* A are mapped into the set B?" Since the mapping rule is just identity ($\iota_A(a) = a$), an element $a \in A$ is mapped into B if and only if a is itself an element of B. Therefore, the elements we are looking for are those that are in A AND in B. This is precisely the definition of the set intersection $A \cap B$.

This turns a set operation $(A \cap \cdot)$ into a map operation $(\iota_A^{-1}(\cdot))$, allowing us to use powerful theorems about maps.

[2] Commutativity of Inverse Image and σ-Algebra Generation

The property $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$ is the cornerstone of the proof. Let's briefly sketch why it's true for a general map $f : X \to Y$ and a generator $\mathcal{E} \subseteq \mathcal{P}(Y)$.

The proof is a classic "show two sets are equal by showing each is a subset of the other" argument.

1. **Show** $\sigma(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\sigma(\mathcal{E}))$: We know that $\mathcal{E} \subseteq \sigma(\mathcal{E})$. Applying the inverse map to both sides gives $f^{-1}(\mathcal{E}) \subseteq f^{-1}(\sigma(\mathcal{E}))$. Now, we can show that $f^{-1}(\sigma(\mathcal{E}))$ is itself a σ-algebra on X. This is because the inverse image operation commutes with set operations:

   - $f^{-1}(Y) = X$ (and $f^{-1}(\emptyset) = \emptyset$).
   - $f^{-1}(B^c) = (f^{-1}(B))^c$.
   - $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$.

   Since $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra that contains the generator system $f^{-1}(\mathcal{E})$, it must be larger than or equal to the *smallest* σ-algebra containing $f^{-1}(\mathcal{E})$, which is $\sigma(f^{-1}(\mathcal{E}))$. Thus, the inclusion holds.

2. **Show** $f^{-1}(\sigma(\mathcal{E})) \subseteq \sigma(f^{-1}(\mathcal{E}))$: This direction is more involved. It requires showing that the collection of sets $\{B \subseteq Y \mid f^{-1}(B) \in \sigma(f^{-1}(\mathcal{E}))\}$ is a σ-algebra on Y that contains $\mathcal{E}$. By the minimality of $\sigma(\mathcal{E})$, this implies $\sigma(\mathcal{E})$ is a subset of this collection, which proves the inclusion.

For our purposes, accepting this as a given theorem (as instructed by the exercise) is sufficient.