# Exercise Walkthrough: Independence vs. Uncorrelatedness

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**Abstract**

This document provides a detailed, step-by-step walkthrough for an exercise on the relationship between independence and uncorrelatedness of random variables. We will first prove that independence implies uncorrelatedness. Then, we will construct a counterexample to show that the reverse is not true: two random variables can be uncorrelated without being independent. Each step is justified with reference to definitions and theorems from the "Discrete Probability Theory" script by Niki Kilbertus.

## 1 Overview of the Problem

The exercise asks us to explore the link between two key properties of random variables (RVs), $X$ and $Y$:

1. **Independence**: A strong condition meaning that the value of one variable gives no information about the value of the other. [1]
2. **Uncorrelatedness**: A weaker condition meaning that there is no *linear* relationship between the variables. [2]

We will tackle this in two parts:

- **Part (i):** Show that independence is the stronger property by proving that if $X$ and $Y$ are independent, they must also be uncorrelated.
- **Part (ii):** Show that the relationship doesn't go the other way by providing an example of two variables that are uncorrelated but are clearly dependent.

## 2 Part (i): Independence implies Uncorrelatedness

### 2.1 Goal and Strategy

Our goal is to show that if two random variables $X$ and $Y$ are independent, then their covariance is zero, which by definition means they are uncorrelated.

**Our strategy is as follows:**

1. Start with the definition of covariance.
2. Calculate the term $\mathbb{E}[XY]$ by using its integral definition.
3. Apply the property of independence, which allows us to factorize the joint probability density function (pdf).
4. Show that this factorization leads to $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
5. Substitute this result back into the covariance formula to show it equals zero.

### 2.2 Step-by-Step Proof

**Step 1: Recall the definition of covariance.** From **Remark 2.10** (computational formula for covariance), we know that the covariance between $X$ and $Y$ is given by:

$$\mathrm{cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{1}$$

To show that $X$ and $Y$ are uncorrelated, we need to prove that $\mathrm{cov}[X,Y] = 0$. This is equivalent to showing that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

**Step 2: Express $\mathbb{E}[XY]$ as an integral.** Using the Law of the Unconscious Statistician (**Lemma 2.2**) for a function $g(X,Y) = XY$, the expected value $\mathbb{E}[XY]$ is computed by integrating over the joint pdf $p_{X,Y}(x,y)$:

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p_{X,Y}(x,y)\,dx\,dy \tag{2}$$

**Step 3: Apply the independence assumption.** Here comes the crucial step. We are given that $X$ and $Y$ are independent. According to **Definition 1.72 (iii)**, for independent continuous random variables, their joint pdf factorizes into the product of their marginal pdfs:

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

We can substitute this into our integral for $\mathbb{E}[XY]$ from Equation 2:

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot p_X(x)p_Y(y)\,dx\,dy$$

**Step 4: Separate the integral.** Since the integrand is now a product of a function of $x$ and a function of $y$, and the integration limits are constant, we can separate the double integral into a product of two single integrals (an application of Fubini's Theorem):

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty} y \cdot p_Y(y)\left(\int_{-\infty}^{\infty} x \cdot p_X(x)\,dx\right)\,dy \\
&= \left(\int_{-\infty}^{\infty} x \cdot p_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} y \cdot p_Y(y)\,dy\right)
\end{aligned}$$

**Step 5: Relate back to expectation.** We recognize these single integrals. By the definition of expectation (**Definition 2.1**), they are precisely $\mathbb{E}[X]$ and $\mathbb{E}[Y]$:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x \cdot p_X(x)\,dx \\
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} y \cdot p_Y(y)\,dy
\end{aligned}$$

Therefore, we have shown:

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \tag{3}$$

**Step 6: Conclude the proof.** Substituting Equation 3 back into the covariance formula (Equation 1):

$$\mathrm{cov}[X,Y] = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

Since $\mathrm{cov}[X,Y] = 0$, the random variables $X$ and $Y$ are, by definition, uncorrelated.

### 2.3 Summary of Part (i)

The logic is a direct chain: independence allows the joint pdf to be factorized, which in turn allows the integral for $\mathbb{E}[XY]$ to be separated into $\mathbb{E}[X]\mathbb{E}[Y]$. This directly leads to the covariance being zero.

## 3 Part (ii): Uncorrelated but not Independent

### 3.1 Goal and Strategy

Now we need a counterexample: a pair of RVs $(X, Y)$ that are uncorrelated (cov$[X, Y] = 0$) but are *not* independent. The hint suggests letting $Y$ be a deterministic function of $X$, specifically $Y = X^2$. This is a great choice because if $Y$ is a function of $X$, it is clearly dependent on $X$. If we can choose the distribution of $X$ cleverly so that cov$[X, X^2] = 0$, we have our counterexample.

**Our strategy is as follows:**

1. Define $X$ and $Y = X^2$. A uniform distribution symmetric around zero, like $X \sim$ Unif$(-1, 1)$, is a good candidate.
2. Show that $X$ and $Y$ are uncorrelated by computing cov$[X, Y]$ and showing it is zero.
3. Show that $X$ and $Y$ are not independent by showing that their joint CDF, $F_{X,Y}(x, y)$, does not equal the product of their marginal CDFs, $F_X(x)F_Y(y)$.

### 3.2 Step-by-Step Construction

**Step 1: Define the random variables.** Let $X \sim$ Unif$(-1, 1)$. The pdf of $X$ is:

$$p_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$.

**Step 2: Show $X$ and $Y$ are uncorrelated.** We again use the formula cov$[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. First, let's compute $\mathbb{E}[X]$. From **Lemma 2.27 (i)**, for a uniform distribution Unif(a,b), the mean is $(a + b)/2$.

$$\mathbb{E}[X] = \frac{-1 + 1}{2} = 0$$

Because $\mathbb{E}[X] = 0$, the covariance formula simplifies to:

$$\text{cov}[X, Y] = \mathbb{E}[XY] - 0 \cdot \mathbb{E}[Y] = \mathbb{E}[XY]$$

Now we substitute $Y = X^2$:

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot X^2] = \mathbb{E}[X^3]$$

We compute $\mathbb{E}[X^3]$ using LOTUS (**Lemma 2.2**):

$$\mathbb{E}[X^3] = \int_{-\infty}^{\infty} x^3 p_X(x)\, dx = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\, dx$$

This is an integral of an odd function ($f(x) = x^3$) over a symmetric interval $[-1, 1]$. Such integrals are always zero. [4] For completeness, the calculation is:

$$\frac{1}{2}\int_{-1}^{1} x^3\, dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right) = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0$$

So, cov$[X, Y] = 0$. $X$ and $Y$ are uncorrelated.

**Step 3: Show $X$ and $Y$ are not independent.** To show they are not independent, we will show that $F_{X,Y}(x, y) \neq F_X(x)F_Y(y)$ for some $(x, y)$. According to **Definition 1.72 (i)**, this is sufficient to prove non-independence.

First, let's find the marginal CDFs, $F_X(x)$ and $F_Y(y)$, for $x \in [-1, 1]$ and $y \in [0, 1]$ (the supports of $X$ and $Y$).

$$
\begin{aligned}
F_X(x) &= P(X \leq x) = \int_{-1}^{x} \frac{1}{2}\,dt = \frac{1}{2}[t]_{-1}^{x} = \frac{x+1}{2} \\
F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dt = \frac{1}{2}[t]_{-\sqrt{y}}^{\sqrt{y}} = \frac{1}{2}(\sqrt{y} - (-\sqrt{y})) = \sqrt{y}
\end{aligned}
$$

So, the product is $F_X(x)F_Y(y) = \frac{(x+1)\sqrt{y}}{2}$.

Now, let's find the joint CDF, $F_{X,Y}(x, y)$:

$$
\begin{aligned}
F_{X,Y}(x, y) &= P(X \leq x, Y \leq y) = P(X \leq x, X^2 \leq y) \\
&= P(X \leq x, -\sqrt{y} \leq X \leq \sqrt{y}) \\
&= P(X \in [-\sqrt{y}, \sqrt{y}] \cap (-\infty, x]) \\
&= P(-\sqrt{y} \leq X \leq \min(x, \sqrt{y})) \\
&= \int_{-\sqrt{y}}^{\min(x,\sqrt{y})} \frac{1}{2}\,dt = \frac{1}{2}[t]_{-\sqrt{y}}^{\min(x,\sqrt{y})} = \frac{\min(x, \sqrt{y}) + \sqrt{y}}{2}
\end{aligned}
$$

Now we must check if $F_{X,Y}(x, y) = F_X(x)F_Y(y)$:

$$
\frac{\min(x, \sqrt{y}) + \sqrt{y}}{2} \stackrel{?}{=} \frac{(x+1)\sqrt{y}}{2}
$$

Let's pick a test point, for instance $x = 0.5$ and $y = 0.09$. Then $\sqrt{y} = 0.3$.

- LHS: $\frac{\min(0.5,0.3)+0.3}{2} = \frac{0.3+0.3}{2} = 0.3$
- RHS: $\frac{(0.5+1)\sqrt{0.09}}{2} = \frac{1.5 \cdot 0.3}{2} = \frac{0.45}{2} = 0.225$

Since $0.3 \neq 0.225$, the equality does not hold. Therefore, $X$ and $Y$ are not independent.

### 3.3 Visual Intuition for Part (ii)

The relationship $Y = X^2$ means that all the probability mass of the joint distribution $(X, Y)$ lies on the parabola $y = x^2$ for $x \in [-1, 1]$. If you were to create a scatter plot of samples from this joint distribution, they would form a perfect parabolic arc.

- **Dependence:** This is the picture of perfect dependence. If you know the value of $X$, you know the value of $Y$ with certainty. For example, if $X = 0.5$, then $Y$ must be $0.25$. This is the opposite of independence.
- **Uncorrelatedness:** Correlation measures the *linear* trend. For $x \in [0, 1]$, as $x$ increases, $y$ increases. This suggests a positive correlation. But for $x \in [-1, 0]$, as $x$ increases, $y$ decreases. This suggests a negative correlation. Because the distribution of $X$ is symmetric around 0, these two opposing linear trends perfectly cancel each other out, resulting in a net linear relationship of zero.

This example beautifully illustrates that "uncorrelated" is not the same as "unrelated".

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## More In-Depth Explanations

**[1] Independence of Random Variables** Independence is a core concept stating that the outcome of one RV provides no probabilistic information about the outcome of another. For continuous RVs $X$ and $Y$, this is formally defined by the factorization of their joint probability density function (pdf) into the product of their marginal pdfs: $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all $x, y$. This must hold for their CDFs as well: $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.

**[2] Covariance and Correlation** Covariance measures the joint variability of two RVs. A positive covariance indicates they tend to move in the same direction, while a negative covariance indicates they move in opposite directions. However, its value depends on the units of the RVs.

$$\text{cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

Correlation, $\rho[X, Y]$, is the normalized version of covariance, which is dimensionless and always lies in $[-1, 1]$. It measures the strength and direction of the *linear* relationship between two RVs. A correlation of 0 means there is no linear relationship, and the variables are called *uncorrelated.*

$$\rho[X, Y] = \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

**[3] Law of the Unconscious Statistician (LOTUS)** This is a very useful theorem (**Lemma 2.2**) that allows us to calculate the expectation of a function of a random variable, say $g(X)$, without first finding the probability distribution of the new random variable $Z = g(X)$. We can compute it directly by integrating $g(x)$ against the pdf of the original variable $X$:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx$$

**[4] Integral of an Odd Function over a Symmetric Interval** A function $f(x)$ is called *odd* if $f(-x) = -f(x)$ for all $x$. Examples include $x, x^3, \sin(x)$. When an odd function is integrated over an interval that is symmetric about the origin, like $[-a, a]$, the result is always zero. This is because the area below the x-axis for $x < 0$ perfectly cancels out the area above the x-axis for $x > 0$.

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f \text{ is an odd function.}$$