Exercise Walkthrough: Properties of Inverse Images

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25. June 2025

Overview

This document provides a step-by-step walkthrough of a fundamental exercise on the properties of inverse images. These properties are crucial in measure theory and probability. They form the logical foundation for the definition of a measurable function and, consequently, a random variable (as seen in **Definition 1.45** of the script). We will prove that the inverse image operation, f^{-1}, interacts predictably with basic set operations like subsets, unions, and complements. This ensures that structure is preserved when mapping from one space to another.

Exercise 1. *Let Ω_1, Ω_2 be non-empty sets, $f : \Omega_1 \to \Omega_2$ an arbitrary mapping, and $\mathcal{C} \subseteq \mathcal{P}(\Omega_2)$ an arbitrary collection of subsets of Ω_2. Then the following statements hold.*

(i) If $A, B \subseteq \Omega_2$ and $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.

(ii) The inverse image of the union is equal to the union of the inverse images, meaning

$$f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} f^{-1}(A).$$

(iii) Given a subset $A \subseteq \Omega_2$, the inverse image of the complement is equal to the complement of the inverse image, meaning

$$f^{-1}(A^c) = (f^{-1}(A))^c.$$

Step-by-Step Solution

We will tackle each part of the exercise individually, explaining the reasoning for each step.

(i) Monotonicity of the Inverse Image

Claim 1. *If $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Solution.

1. **Goal:** We want to prove the set inclusion $f^{-1}(A) \subseteq f^{-1}(B)$.
2. **Strategy:** To prove this, we will use the standard method for proving set inclusion [2]. We must show that any arbitrary element of the set on the left-hand side is also an element of the set on the right-hand side.
3. Let ω be an arbitrary element in $f^{-1}(A)$. So, $\omega \in f^{-1}(A)$.

4. By the definition of the inverse image [1], if $\omega \in f^{-1}(A)$, then its image $f(\omega)$ must be an element of A. So, $f(\omega) \in A$.
5. We are given the condition that $A \subseteq B$. By the definition of a subset, since $f(\omega) \in A$, it must also be true that $f(\omega) \in B$.
6. Now, since $f(\omega) \in B$, we can again use the definition of the inverse image [1] to conclude that ω must be in the inverse image of B. So, $\omega \in f^{-1}(B)$.
7. **Conclusion:** We started with an arbitrary element $\omega \in f^{-1}(A)$ and showed that it must also be in $f^{-1}(B)$. Therefore, we have proven that $f^{-1}(A) \subseteq f^{-1}(B)$.

□

(ii) Preservation of Unions

Claim 2. $f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} f^{-1}(A)$.

Solution. 1. **Goal:** We want to prove the set equality $f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} f^{-1}(A)$.

2. **Strategy:** To prove this, we use the standard method of double inclusion [3]. We will prove the inclusion in both directions.
3. **Part 1: Show $f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right) \subseteq \bigcup_{A\in\mathcal{C}} f^{-1}(A)$.**
   - Let $\omega \in f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right)$.
   - By definition of the inverse image [1], this means $f(\omega) \in \bigcup_{A\in\mathcal{C}} A$.
   - By definition of a union of a collection of sets [5], there must exist at least one set, let's call it A', in the collection $\mathcal{C}$ such that $f(\omega) \in A'$.
   - Since $f(\omega) \in A'$, the definition of the inverse image [1] tells us that $\omega \in f^{-1}(A')$.
   - Since ω is in one of the sets of the collection $\{f^{-1}(A) \mid A \in \mathcal{C}\}$, it must also be in the union of this collection. Therefore, $\omega \in \bigcup_{A\in\mathcal{C}} f^{-1}(A)$.
4. **Part 2: Show $\bigcup_{A\in\mathcal{C}} f^{-1}(A) \subseteq f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right)$.**
   - Let $\omega \in \bigcup_{A\in\mathcal{C}} f^{-1}(A)$.
   - By definition of union [5], this means there exists at least one set, let's call it A'', in the collection $\mathcal{C}$ such that $\omega \in f^{-1}(A'')$.
   - By definition of the inverse image [1], this implies that $f(\omega) \in A''$.
   - Since $f(\omega)$ is in one of the sets of the collection $\mathcal{C}$, it must also be in the union of all sets in that collection. Therefore, $f(\omega) \in \bigcup_{A\in\mathcal{C}} A$.
   - Finally, by definition of the inverse image [1], this means that $\omega \in f^{-1}\left(\bigcup_{A\in\mathcal{C}} A\right)$.
5. **Conclusion:** Since we have shown inclusion in both directions, the two sets must be equal.

□

(iii) Preservation of Complements

Claim 3. $f^{-1}(A^c) = (f^{-1}(A))^c$.

Solution. 1. **Goal:** We want to prove the set equality $f^{-1}(A^c) = (f^{-1}(A))^c$.

2. **Strategy:** For this proof, we can use a more direct chain of logical equivalences ("if and only if", denoted by $\iff$). This is often more elegant than double inclusion when it's possible. An element ω is in the left set if and only if it is in the right set.

3. Let ω be an arbitrary element in Ω_1. Then:

$$\begin{aligned}
\omega \in f^{-1}(A^c) &\iff f(\omega) \in A^c && \text{(by definition of inverse image [1])}\\
&\iff f(\omega) \notin A && \text{(by definition of complement [4])}\\
&\iff \omega \notin f^{-1}(A) && \text{(by definition of inverse image [1])}\\
&\iff \omega \in (f^{-1}(A))^c && \text{(by definition of complement [4])}
\end{aligned}$$

4. **Conclusion:** Since we have established a chain of equivalences from an element being in $f^{-1}(A^c)$ to it being in $(f^{-1}(A))^c$, the two sets must contain exactly the same elements and are therefore equal.

$\square$

Summary and Further Explanations

Summary

We have formally proven three key properties of the inverse image operation:

- It preserves subset relations: $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$.
- It distributes over arbitrary unions: $f^{-1}(\cup A_i) = \cup f^{-1}(A_i)$.
- It commutes with the complement operation: $f^{-1}(A^c) = (f^{-1}(A))^c$.

These results are essential for probability theory. When we define a random variable X as a measurable function from a probability space $(\Omega, \mathcal{A}, P)$ to a measurable space $(\Omega', \mathcal{A}')$, we require that for any measurable event $A' \in \mathcal{A}'$, its inverse image $X^{-1}(A')$ is also a measurable event in $\mathcal{A}$. The properties we just proved are exactly what you need to show that the collection of all such inverse images, $\{X^{-1}(A') \mid A' \in \mathcal{A}'\}$, itself forms a σ-algebra. This allows us to "pull back" the event structure from the output space to the original sample space, which is how we assign probabilities to outcomes of random variables.

Explanations of Key Concepts

Here are more detailed explanations of the concepts referenced in the proofs.

[1] Inverse Image (Preimage): For a function $f : \Omega_1 \to \Omega_2$ and a subset $S \subseteq \Omega_2$, the inverse image (or preimage) of S under f is the set of all elements in the domain Ω_1 that map into S. It is defined as:

$$f^{-1}(S) := \{\omega \in \Omega_1 \mid f(\omega) \in S\}$$

Note that f^{-1} here does not imply that f has an inverse function; it is notation for an operation on sets. This is central to **Definition 1.45 (random variable)**.

[2] Proving Set Inclusion ($\subseteq$): To prove that a set X is a subset of a set Y, denoted $X \subseteq Y$, you must show that every element of X is also an element of Y. The standard proof structure is:

1. "Let x be an arbitrary element of X."
2. Use definitions and given properties to show that x must also be an element of Y.
3. Conclude that since x was arbitrary, the inclusion $X \subseteq Y$ holds.

[3] Proving Set Equality ($=$): To prove that two sets, X and Y, are equal, you must show they contain exactly the same elements. The most common method is **double inclusion**:

1. Prove $X \subseteq Y$.
2. Prove $Y \subseteq X$.

If both inclusions hold, it must be that $X = Y$.

[4] Set Complement (A^c): Given a universe set Ω and a subset $A \subseteq \Omega$, the complement of A, denoted A^c, is the set of all elements in Ω that are not in A.

$$A^c := \Omega \setminus A = \{\omega \in \Omega \mid \omega \notin A\}$$

In our exercise, for $A \subseteq \Omega_2$, $A^c = \Omega_2 \setminus A$, and for $f^{-1}(A) \subseteq \Omega_1$, $(f^{-1}(A))^c = \Omega_1 \setminus f^{-1}(A)$.

[5] Arbitrary Union of Sets ($\cup$): For a collection of sets $\mathcal{C} = \{A_i \mid i \in I\}$, where I is an index set, their union contains all elements that are in at least one of the sets in the collection.

$$\omega \in \bigcup_{i \in I} A_i \iff \exists i \in I \text{ such that } \omega \in A_i$$