SDG 4 QUALITY EDUCATION

TE 594: Quantum Computing & Information Processing

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BACKGROUND MATH

Probability: How likely a certain event is to occur. Like flipping a coin or rolling a die.

A probability is given as a real number between 0 and 1.

Example: Pr[fair coin lands heads] = 1/2

Adding all the probabilities over all possible events must equal 1

Background Math

A vector is an abstract mathematical object. They are elements of a vector space.

A column vector is a list of numbers.

Examples:

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad u = \begin{pmatrix} 7 \\ 0 \\ 1 \\ 5 \end{pmatrix}$$

Background Math

A matrix is something that maps vectors to vectors.

A matrix is often described by a box of numbers.

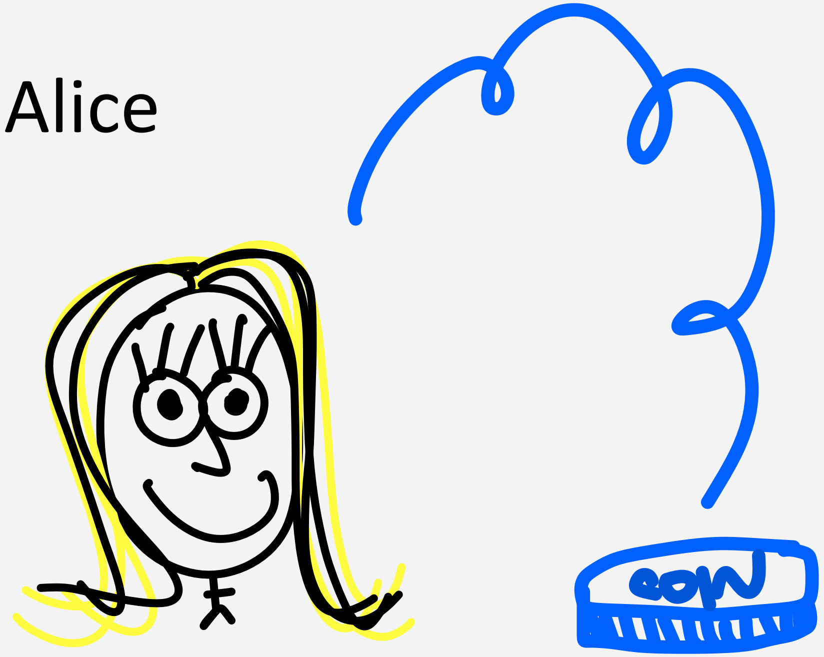
Example: $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Matrix-vector multiplication:

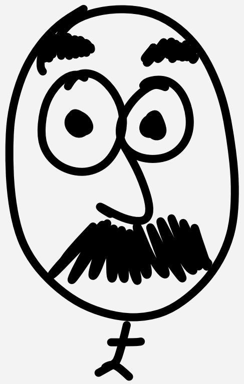
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 \\ 3 \cdot 1 + 4 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Random Bits

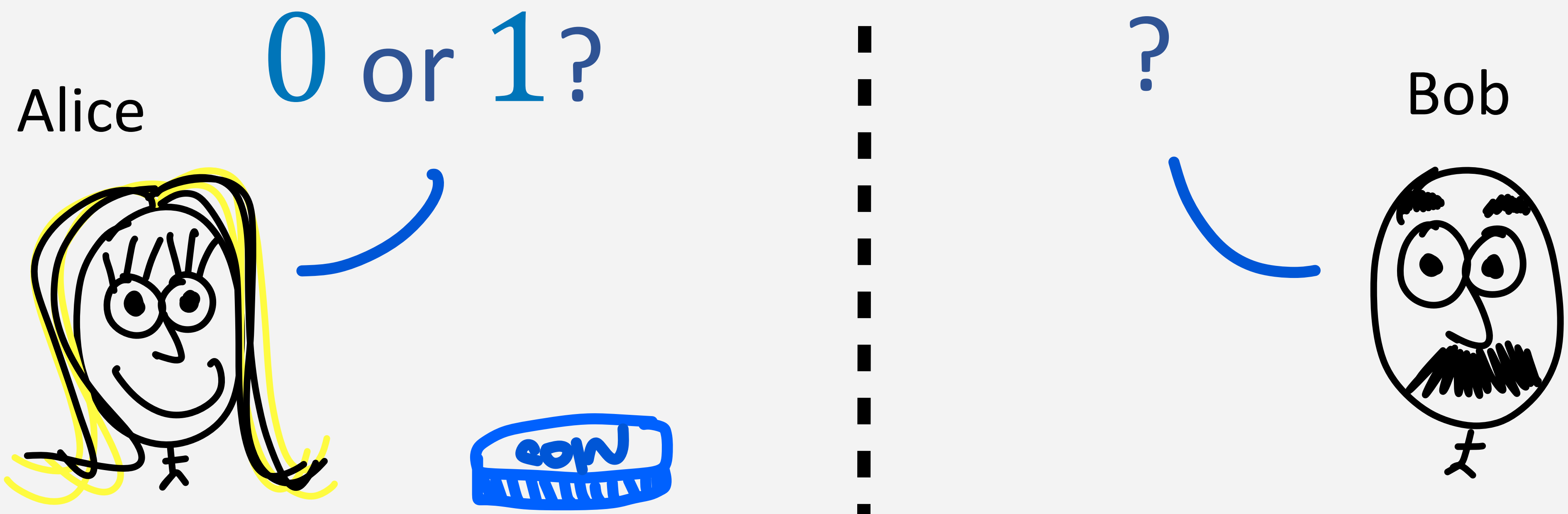
Alice



Bob



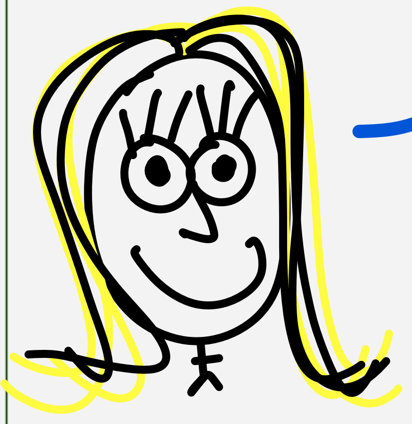
Random Bits



Random Bits

Alice

0 or 1?



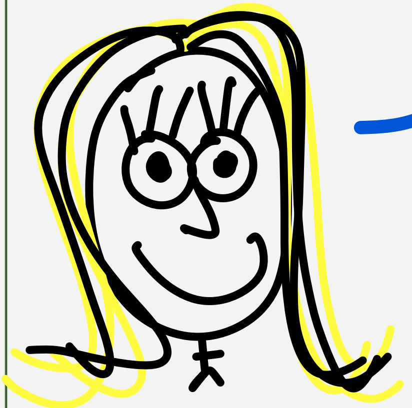
Probability vectors

Fair coin toss: $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Random Bits

Alice

0 or 1?



Probability vectors

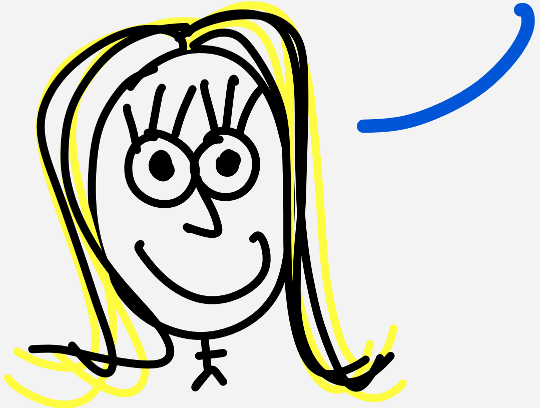
Fair coin toss: $\begin{pmatrix}1/2\\1/2\end{pmatrix}$

Probability the coin (or bit) is 0

Random Bits

Alice

0 or 1?



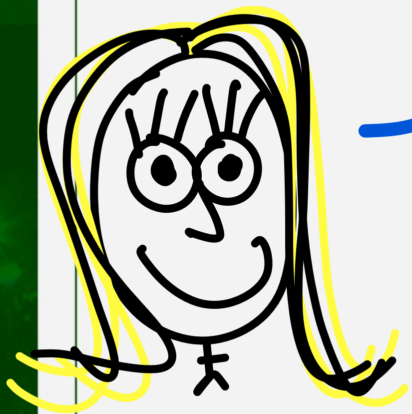
Probability vectors

Fair coin toss: $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Probability the coin (or bit) is 1

Random Bits

Alice



0 or 1?

Probability vectors

Unfair coin toss: $\begin{pmatrix}1\\0\end{pmatrix}$

Random Bits

Alice

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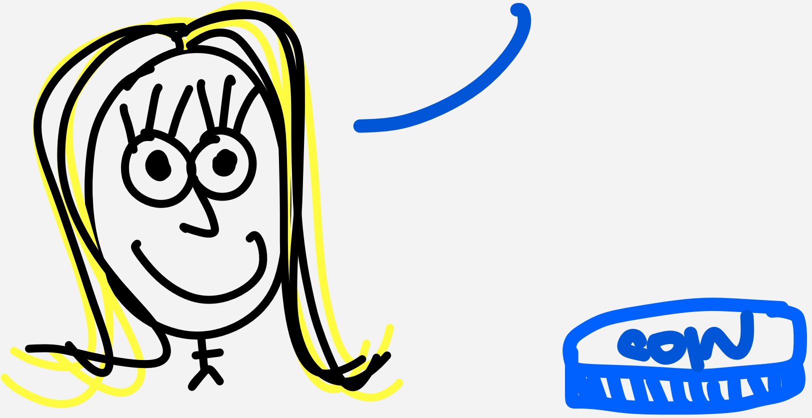

Probability vectors

Unfair coin toss: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Random Bits

Operations

Since random bits are represented by vectors, we need a way to map vectors to vectors. Matrices!

$$M\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Stochastic matrices!

But! We need to preserve $q_0, q_1 \geq 0$ and $q_0 + q_1 = 1$
when $p_0, p_1 \geq 0$ and $p_0 + p_1 = 1$

Random Bits

Operations

Since random bits are represented by vectors, we need a way to map vectors to vectors. Matrices!

$$M \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Stochastic matrices!

Definition: A stochastic matrix has nonnegative entries and columns adding to 1.

Random Bits

Example

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ initializes to } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for all probability vectors } \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

Random Bits

Example

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$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ initializes to } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Random Bits

Example

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ initializes to } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ initializes to } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ does nothing! I.e., } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ for all vectors } \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Random Bits

Example

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ What does this matrix do?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_0 \end{pmatrix}$$ It switched the probabilities!

Random Bits

Example

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ What does this matrix do?

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Random Bits

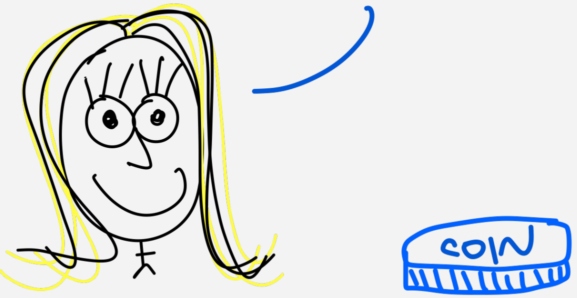
Example

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ NOT operation

Lots of possibilities!

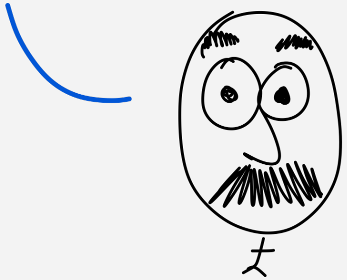
Random Bits

0 or 1?



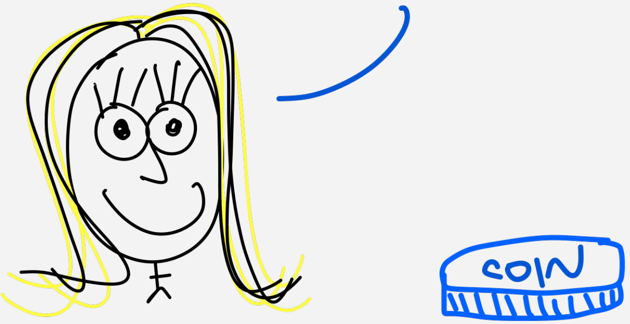
$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

Bob knows the probabilities, but not the outcome!

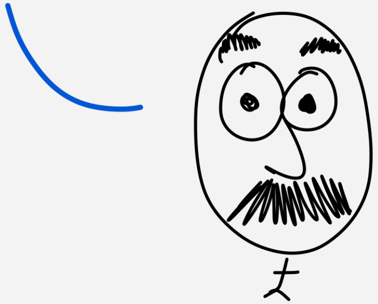


Random Bits

I switched
the outcome!



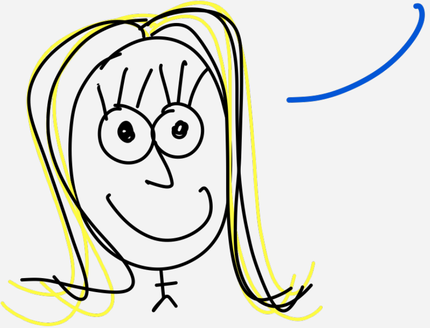
$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \to \begin{pmatrix} p_1 \\ p_0 \end{pmatrix}$$



Note the outcome is either 0 or 1
(it is just that Bob doesn't know)

Random Bits

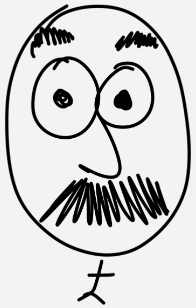
I am rolling
for initiative!



D20 (20-sided die)

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{15} \\ p_{16} \\ p_{17} \\ p_{18} \\ p_{19} \\ p_{20} \end{pmatrix}$$

A probability vector
can have as many
entries as needed



Qubits

Quantum bit = Qubit

Qubits

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$\alpha, \beta \in \mathbb{C}$

$$|\alpha|^2 + |\beta|^2 = 1$$

(unit vectors)

Notation: $|\psi\rangle$ "ket psi"

ψ is the Greek letter psi.

Qubits

My favourite qubits:

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ "Zero state"

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ "One state"

Qubits

My favourite qubits:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{"Zero state"}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{"One state"}$$

Note these are unit vectors

These can be thought of as the 0s and 1s inside a "classical" computer

Qubits

My favourite qubits:

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ "Zero state"

$|+\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ "Plus state"

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ "One state"

$|-\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ "Minus state"

Qubits

More notation: $\langle\psi|$ "bra psi"

$\langle\psi| = |\psi\rangle^*$ (Here * means its adjoint, or conjugate transpose)

Qubits

More notation: $\langle\psi|$ "bra psi"

$\langle\psi| = |\psi\rangle^*$ (Here * means its adjoint, or conjugate transpose)

$= |\psi\rangle^\dagger$ (the dagger "$\dagger$" is alternate notation)

Qubits

More notation: $\langle\psi|$ "bra psi"

$\langle\psi| = |\psi\rangle^*$ (Here * means its adjoint, or conjugate transpose)

If $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ then $\langle\psi| = [\bar{\alpha}, \bar{\beta}]$

Note the complex conjugates!

Qubits

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Note the complex conjugates!

Bracket: $\langle\psi|\phi\rangle = \langle |\psi\rangle, |\phi\rangle \rangle$

"Bra-ket" $\langle\psi|\psi\rangle = 1$ for all quantum states

Qubits

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Bracket: $\langle\psi|\phi\rangle = \langle\, |\psi\rangle, |\phi\rangle \,\rangle$

"Bra-ket"

$\langle\psi|\psi\rangle = 1$ for all quantum states

Why?

Qubits

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$|\psi\rangle$ is in a <u>superposition</u> of $|0\rangle$ & $|1\rangle$

Qubits

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

Amplitudes

$|\psi\rangle$ is in a <u>superposition</u> of $|0\rangle$ & $|1\rangle$

Qubits

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$|\psi\rangle$ is in a <u>superposition</u> of $|0\rangle$ & $|1\rangle$

$|\psi\rangle$ has properties of both states!

Qubits

Question: Is $|0\rangle$ in a superposition of $|+\rangle$ & $|-\rangle$?

Answer: Yes! $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

Qubit operations

Again, we'll use matrices to map vectors to vectors.

$$|\psi\rangle \rightarrow |\phi\rangle$$

What properties do the matrices need to have?

Qubit operations

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$$|\psi\rangle \longrightarrow |\phi\rangle$$

Unit vector   Unit vector

What properties do the matrices need to have?

Qubit operations

Again, we'll use matrices to map vectors to vectors.

$|\psi\rangle \longrightarrow |\phi\rangle$

Unit vector Unit vector

This mapping needs to be <u>unitary</u>.

What properties do the matrices need to have?

Qubit operations

<u>Def'n</u>: A matrix is unitary if it preserves inner products.

Let U be a unitary matrix.

<u>Equivalent definitions:</u>

1. $\langle Ux, Uy \rangle = \langle x, y \rangle$ $\forall$ vectors x, y.
2. $\|Ux\|_2 = \|x\|_2$ $\forall$ vectors x
3. $U^* U = \mathbb{1}$
4. $UU^* = \mathbb{1}$
5. U^* is unitary
6. The columns of U form an orthonormal basis.
7. The rows of U form an orthonormal basis.
8. U is a normal matrix with eigenvalues having modulus 1.

meaning $U^{-1} = U^*$

$\boxed{\lambda = e^{i\theta}}$

Qubit operations

<u>Examples of unitary matrices</u>

$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity

$\mathbb{1}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$

Qubit operations

<u>Examples of unitary matrices</u>

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Identity

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ NOT

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

Qubit operations

Examples of unitary matrices

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$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{NOT}$$

$$X|0\rangle = |1\rangle$$

$$X|1\rangle = |0\rangle$$

$$X|+\rangle = \frac{X|0\rangle + X|1\rangle}{\sqrt{2}}$$

$$= \frac{|1\rangle + |0\rangle}{\sqrt{2}}$$

$$= |+\rangle$$

Qubit operations

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Eigenvector!

Qubit operations

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Eigenvector! Unaffected!

Qubit operations

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$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Local Phase

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$

(Is this OK?)

Qubit operations

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(Is this OK?)

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

$Z|+\rangle = |-\rangle$

Qubit operations

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These two states differ by a local phase

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These two states differ by a local phase

$|\phi\rangle = \alpha|0\rangle + e^{i\theta}\beta|1\rangle$

Qubit operations

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Qubit operations

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These two states differ by a <u>local</u> phase

$|\phi\rangle = e^{i\theta}(\alpha|0\rangle + \beta|1\rangle)$

Qubit operations

<u>Examples of unitary matrices</u>

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Identity

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ NOT

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ Local Phase

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$ Y

Qubit operations

Examples of unitary matrices

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Identity σ_1

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ NOT σ_x

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ Local Phase σ_z

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$ Y σ_y

Pauli Matrices

Qubit operations

Examples of unitary matrices

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Reflections + Rotations are unitary

Qubit operations

Examples of unitary matrices

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Reflections + Rotations are unitary

$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Hadamard

Qubit operations

Examples of unitary matrices

$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity

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Reflections + Rotations are unitary

$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Hadamard

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

$H|+\rangle = |0\rangle$

$H|-\rangle = |1\rangle$

$(H^2 = \mathbb{1})$

Qubit operations

Examples of unitary matrices

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Reflections + Rotations are unitary

$H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Hadamard

And many more!

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$U + V$ are unitary

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = ?$

$\langle\psi| = \langle 0|U^*$ ← still a row vector

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = ?$

$\langle\psi| = \langle 0|U^*$

so $\langle\psi|\phi\rangle = \langle 0|U^* V|0\rangle$

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = \, ?$

$\langle\psi| = \langle 0|U^*$

so $\langle\psi|\phi\rangle = (\langle 0|U^*)(V|0\rangle)$

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = ?$

$\langle\psi| = \langle 0| U^*$

This is like applying 2 unitaries to $|0\rangle$

So $\langle\psi|\phi\rangle = \langle 0|\left(U^* V|0\rangle\right)$

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = ?$

$\langle\psi| = \langle 0| U^*$

So $\langle\psi|\phi\rangle = \left(\langle 0| U^* V\right)|0\rangle$

Worked example

Suppose $|\psi\rangle = U|0\rangle$

$|\phi\rangle = V|0\rangle$

$\langle\psi|\phi\rangle = ?$

$\langle\psi| = \langle 0|U^*$

So $\langle\psi|\phi\rangle = \langle 0|U^*V|0\rangle$

It's just matrix multiplication

What's next?

To do (classical) computations (E.g., what is 5 x 3?) we need to extract information from qubits. In other words, if we have the answer stored in some qubits, how do we read the answer?

Qubit measurement

We will start with the simplest form of a measurement and introduce more as we proceed through the course.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{measure}} \begin{array}{l} \text{See "0" with prob. } |\alpha|^2 \\ \text{See "1" with prob } |\beta|^2 \end{array}$$

On outcome "0", $|\psi\rangle$ collapses to $|0\rangle$

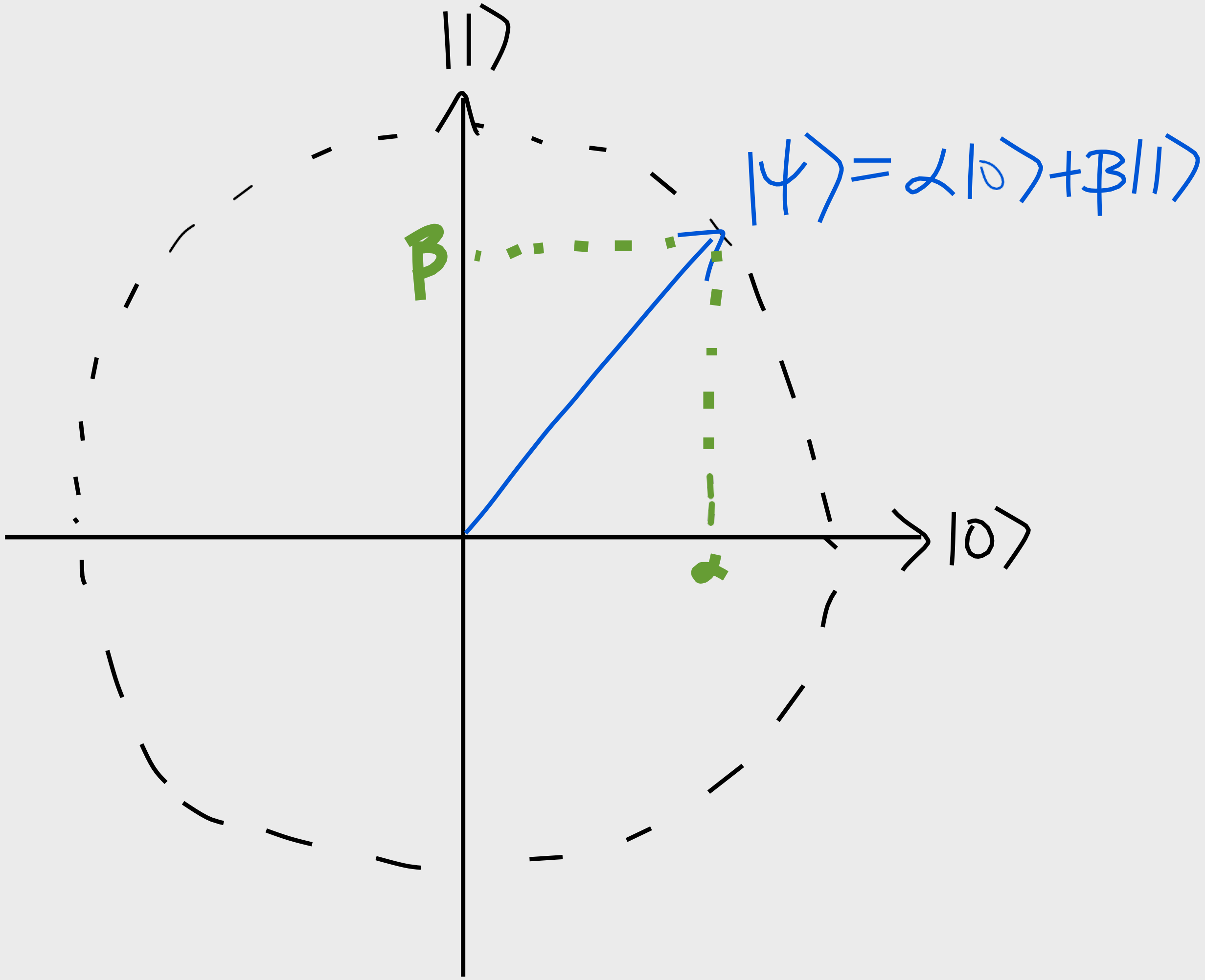
On outcome "1", $|\psi\rangle$ collapses to $|1\rangle$

Qubit measurement

$\text{Qubit} \xrightarrow{\text{measure}} \text{Bit}$

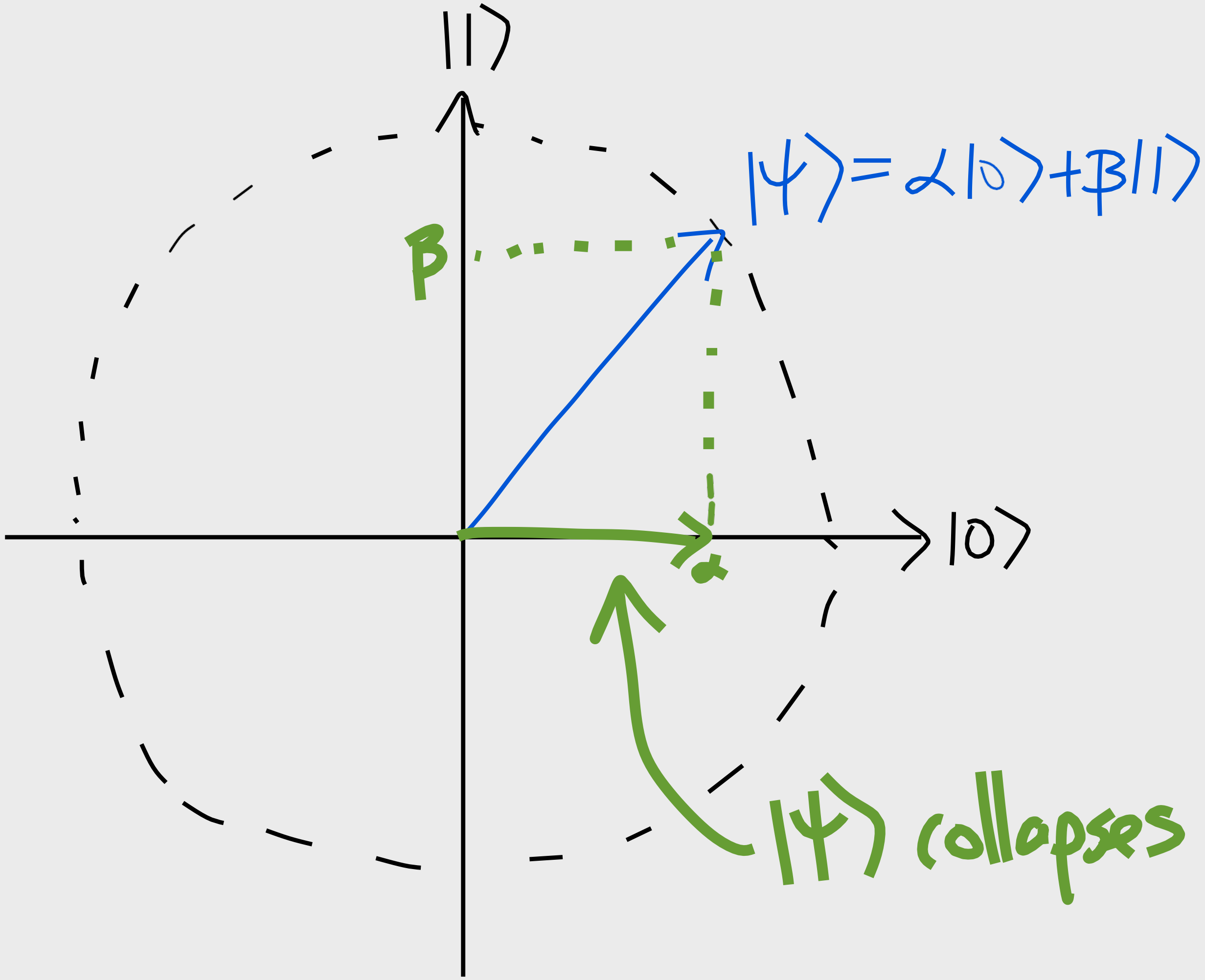
Measuring disturbs the state (make it count!)

Qubit measurement



What happens when you measure $|\psi\rangle$?

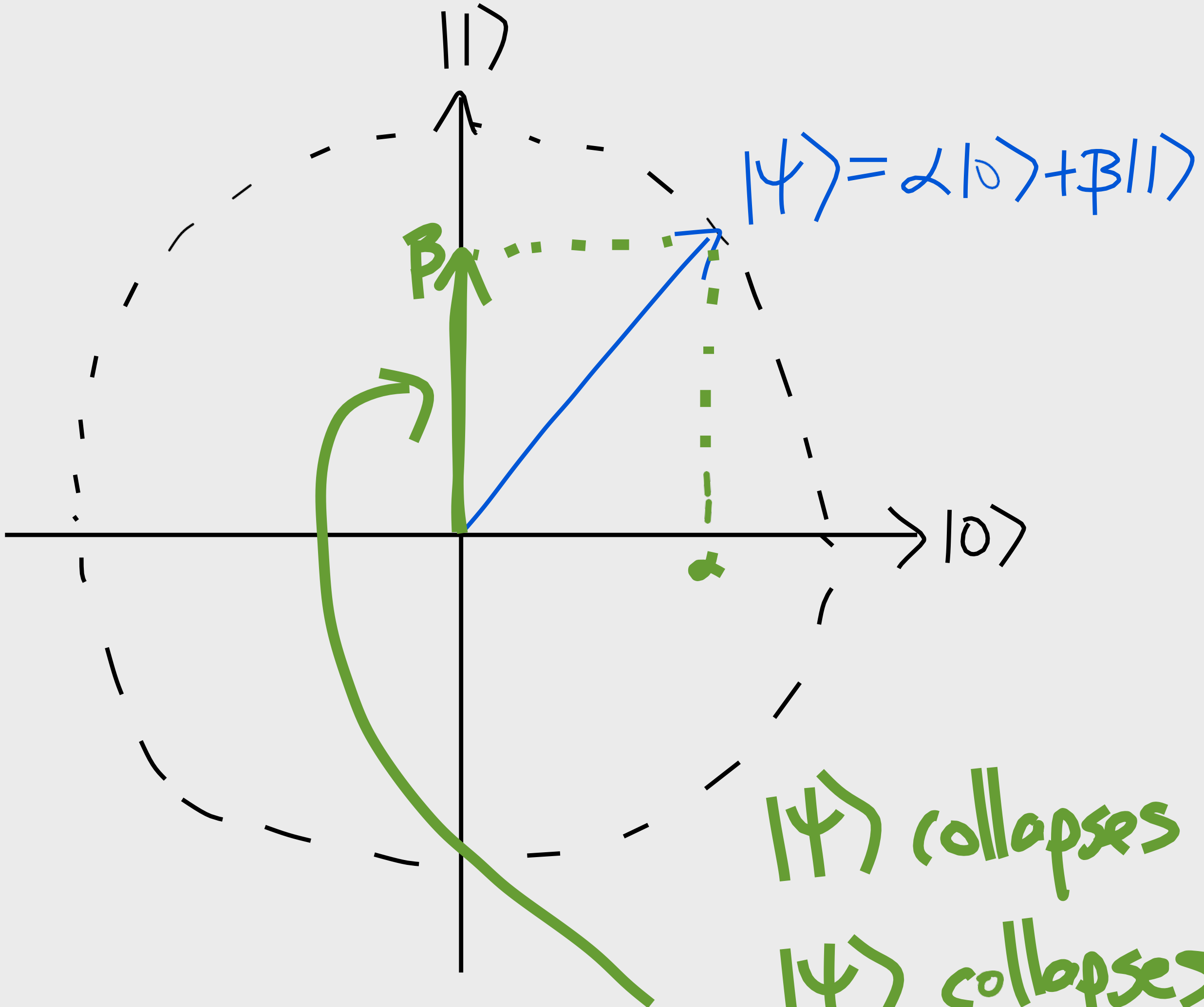
Qubit measurement



what happens when you measure $|\psi\rangle$?

$|\psi\rangle$ collapses to $|0\rangle$ with probability $|\alpha|^2$

Qubit measurement

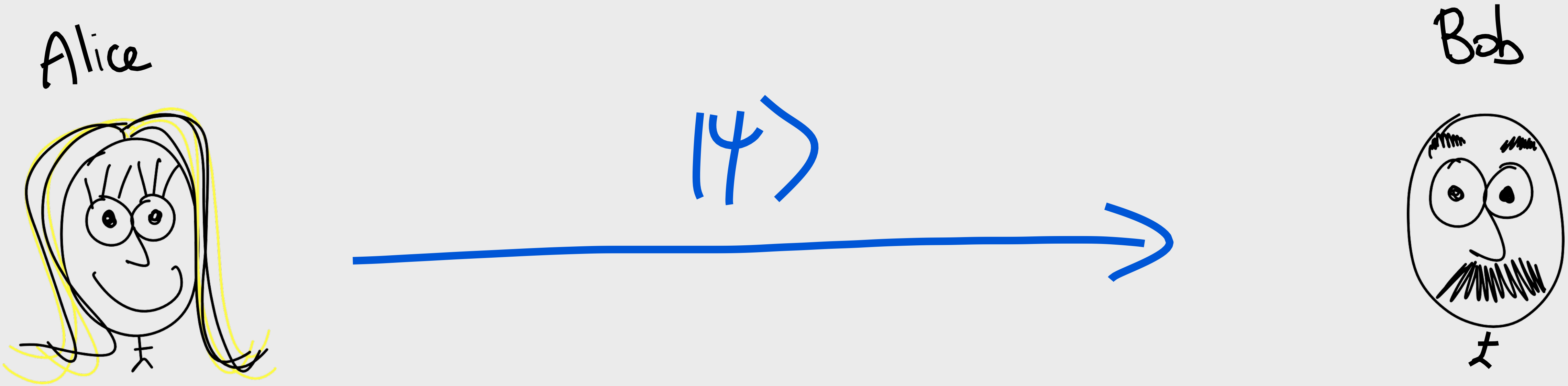


What happens when you measure $|\psi\rangle$?

$|\psi\rangle$ collapses to $|0\rangle$ with probability $|\alpha|^2$

$|\psi\rangle$ collapses to $|1\rangle$ with probability $|\beta|^2$

Quantum state discrimination



Alice prepares

$|\psi\rangle = |0\rangle$

or//

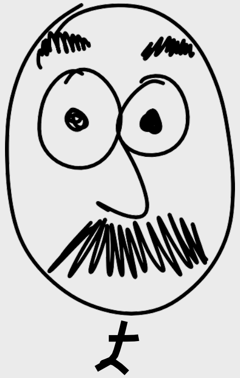
$|\psi\rangle = |1\rangle$

Quantum state discrimination

Alice

$|\psi\rangle$ $\longrightarrow$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or//

$|\psi\rangle = |1\rangle$

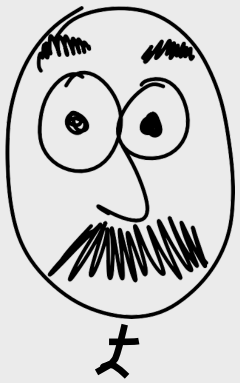
Reality check: Alice is sending a particle encoded in state ket psi, she is not sending a vector!

Quantum state discrimination

Alice

$$\xrightarrow{\quad |\psi\rangle \quad}$$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or//

$|\psi\rangle = |1\rangle$

Can Bob learn what state Alice sent?

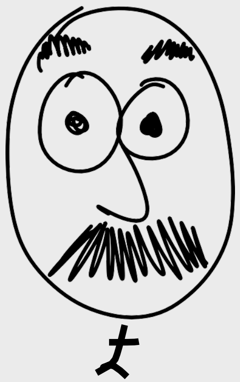
(He knows the two choices)

Quantum state discrimination

Alice

$|\psi\rangle \longrightarrow$

Bob



Alice prepares $|\psi\rangle = |0\rangle$

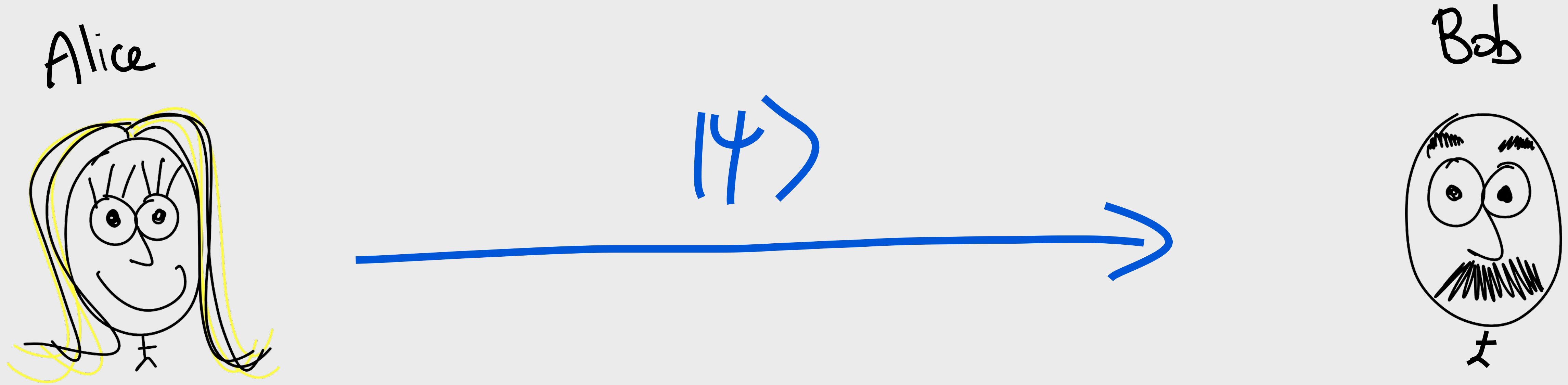
or

$|\psi\rangle = |1\rangle$

Yes! He can measure it.

- If he gets "0" then he knows $|\psi\rangle$ was $|0\rangle$
- If he gets "1" then he knows $|\psi\rangle$ was $|1\rangle$

Quantum state discrimination



Alice prepares

$|\psi\rangle = |+\rangle$

or

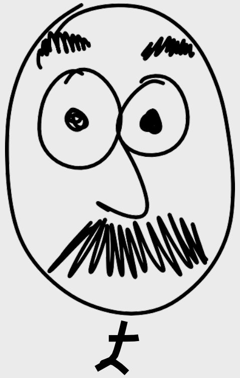
$|\psi\rangle = |-\rangle$

Now, we've changed the states!

Quantum state discrimination

Alice

Bob



$|\psi\rangle$

Alice prepares
$|\psi\rangle = |+\rangle$
or
$|\psi\rangle = |-\rangle$

Now, we've changed the states!

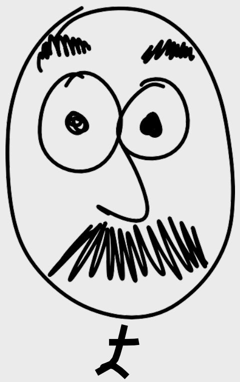
What can Bob do now?

Quantum state discrimination

Alice

$|\psi\rangle$

Bob



Alice prepares

$|\psi\rangle = |+\rangle$

or

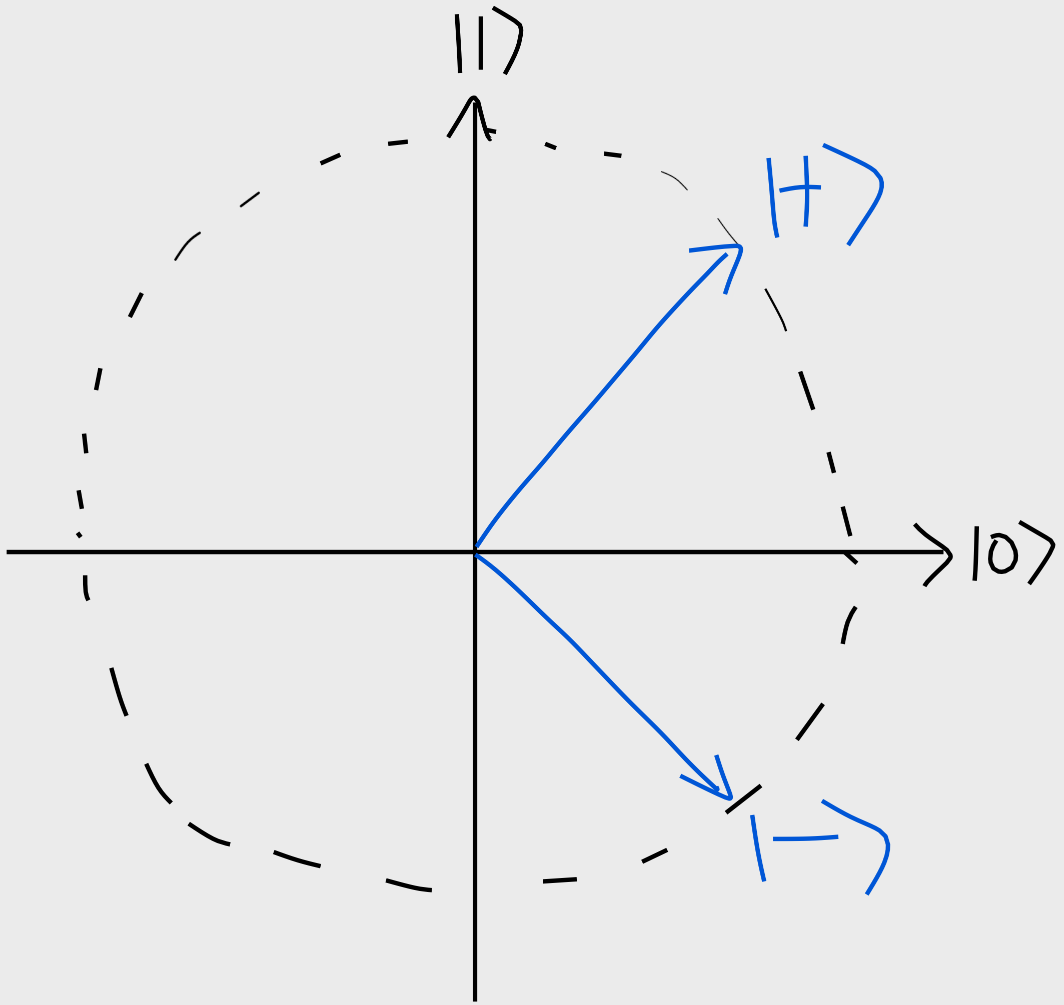
$|\psi\rangle = |-\rangle$

Bob can apply H and then measure.

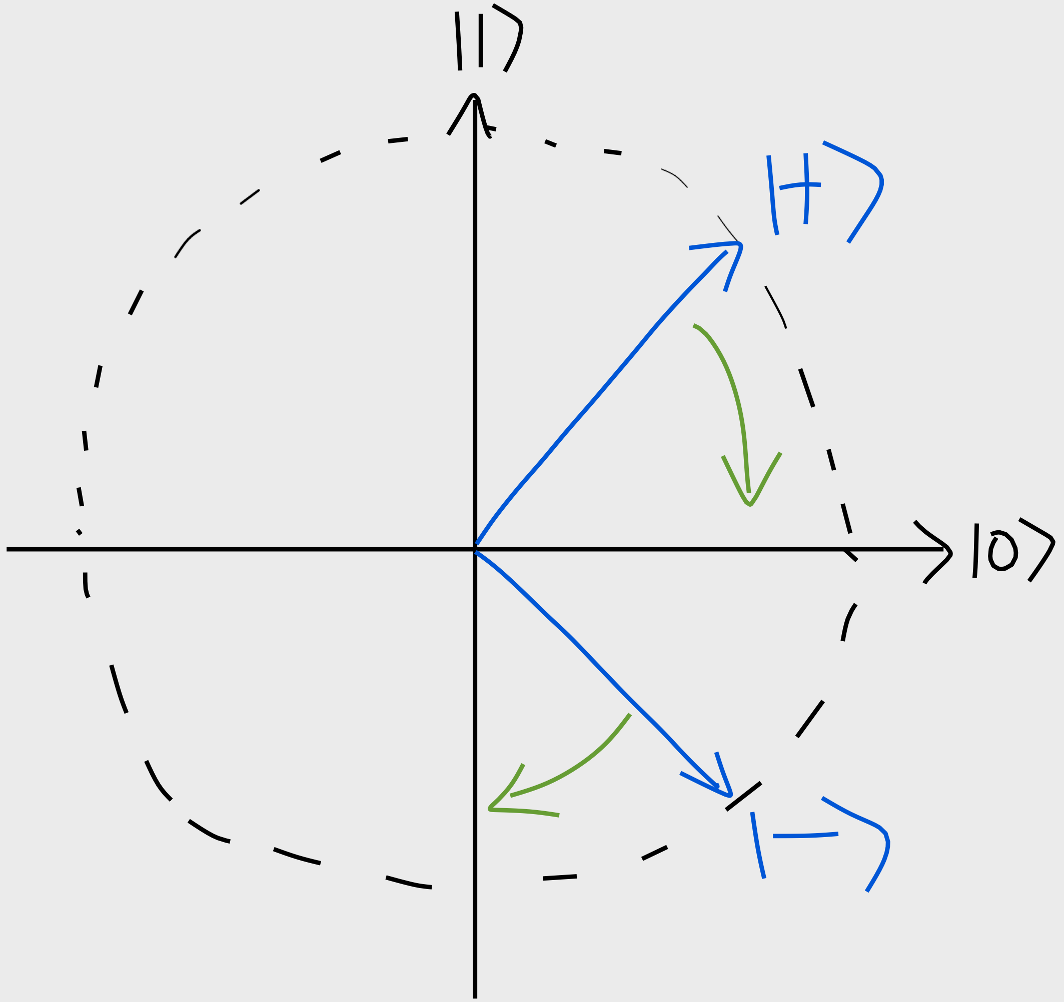
If $|\psi\rangle = |+\rangle$, then $H|\psi\rangle = |0\rangle$

If $|\psi\rangle = |-\rangle$, then $H|\psi\rangle = |1\rangle$

Quantum state discrimination



Quantum state discrimination



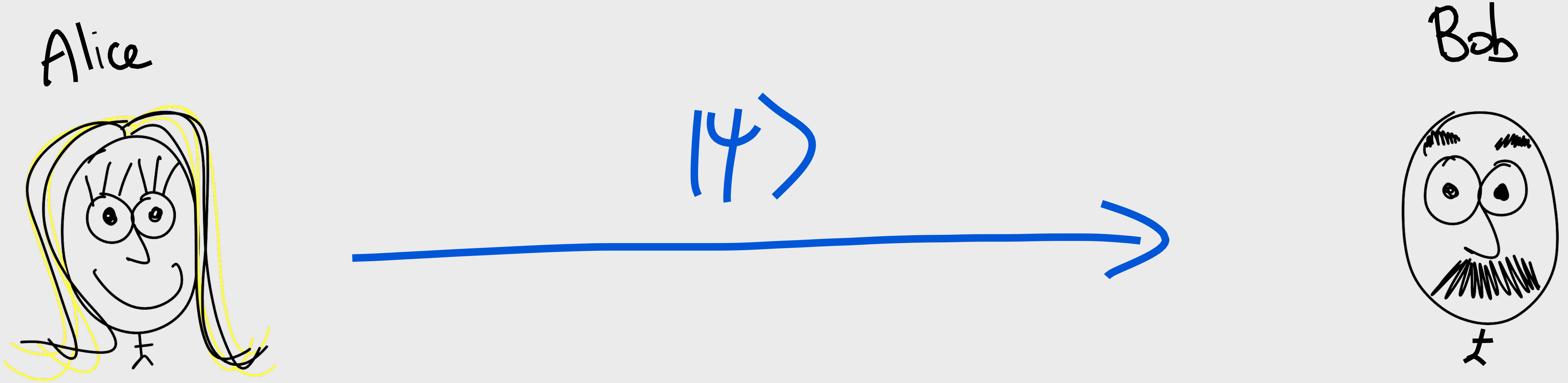
Apply a rotation R.

$R|+\rangle = |0\rangle$

$R|-\rangle = -|1\rangle$

Why does this work?

Quantum state discrimination



Alice prepares

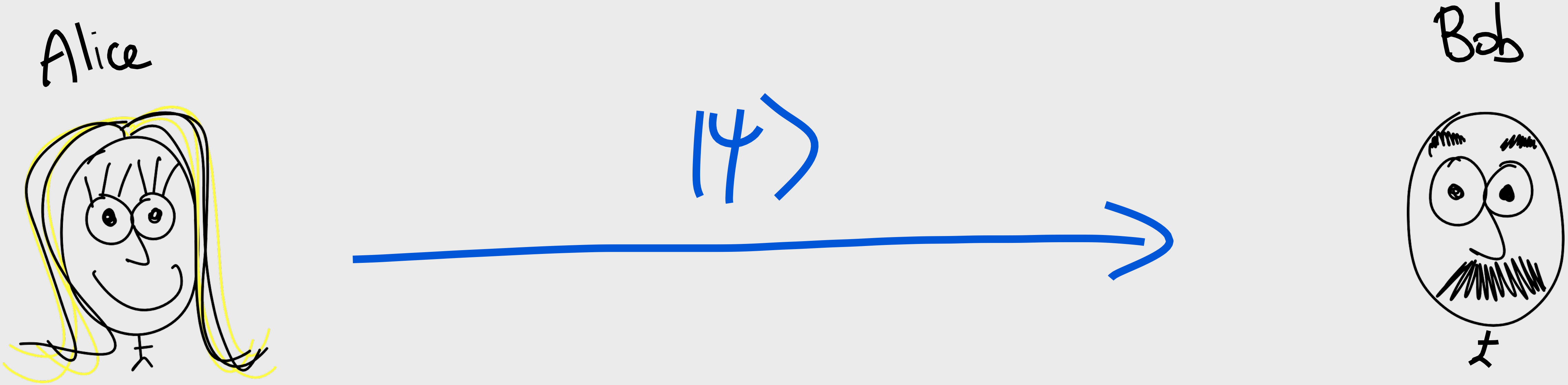
$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Another change!

Quantum state discrimination



Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Another change!

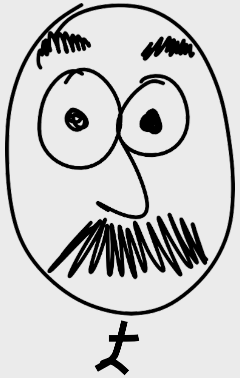
Does Bob have any hope?

Quantum state discrimination

Alice

$|\psi\rangle$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Claim: There does not exist a unitary U such that $U|0\rangle = |0\rangle$ and $U|+\rangle = |1\rangle$

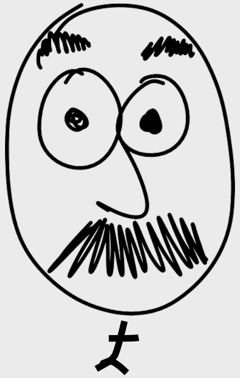
Why?

Quantum state discrimination

Alice

$$\xrightarrow{|\psi\rangle}$$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Claim: There does not exist a unitary $\mathcal{U}$ such that $\mathcal{U}|0\rangle = |0\rangle$ and
$\mathcal{U}|+\rangle = |1\rangle$

Because $\langle 0|+\rangle \neq \langle 0|1\rangle$

Quantum state discrimination

Alice

Alice prepares

$|\psi\rangle = |0\rangle$

or//

$|\psi\rangle = |+\rangle$

Strategy #1: Measure $|\psi\rangle$ anyways

If you get "0", guess $|\psi\rangle = |0\rangle$

If you get "1", guess $|\psi\rangle = |+\rangle$

Quantum state discrimination

Alice

Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Strategy #1: Measure $|\psi\rangle$ anyways

If you get "0", guess $|\psi\rangle = |0\rangle$

If you get "1", guess $|\psi\rangle = |+\rangle$

- If $|\psi\rangle = |0\rangle$, always get outcome "0"
- If $|\psi\rangle = |+\rangle$, get outcome "0" with prob. $\frac{1}{2}$
  get outcome "1" with prob. $\frac{1}{2}$

Quantum state discrimination

Alice

Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Strategy #1: Measure $|\psi\rangle$ anyways

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  get outcome "1" with prob. $\frac{1}{2}$

Worst case error $= 1/2$

Quantum state discrimination

Alice

Alice prepares $|\psi\rangle = |0\rangle$ or $|\psi\rangle = |+\rangle$

Strategy #1: Measure $|\psi\rangle$ anyways

If you get "0", guess $|\psi\rangle = |0\rangle$
If you get "1", guess $|\psi\rangle = |+\rangle$

- If $|\psi\rangle = |0\rangle$, always get outcome "0"
- If $|\psi\rangle = |+\rangle$, get outcome "0" with prob. $\frac{1}{2}$
  get outcome "1" with prob. $\frac{1}{2}$

Worst case error = $1/2$

Better strategy?

Quantum state discrimination

Alice

Fun time question: Is this game fun in the classical world?

General quantum states

$|\psi\rangle \in \mathbb{C}^n$ is a quantum state if $\| |\psi\rangle \|_2 = 1$

$|\psi\rangle \in \mathbb{C}^2$ is a qubit, $|\psi\rangle \in \mathbb{C}^3$ is a qutrit, etc.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

General quantum states

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Measurements and Unitaries work in an analogous way!