Quantum Error Correction and Entanglement Purification

Tanay Bhat   Jay Mehta   Rajwardhan Toraskar

Department of Electrical Engineering
Indian Institute of Technology Bombay

Outline

Motivation

- Quantum information is fragile and susceptible to errors which may arise due to faulty gates, environmental decoherence etc.
- Reliable quantum computation and communication require active protection of quantum states which maybe achieved through:
  - **Quantum Error Correction (QEC)**: encode logical qubits into many physical qubits, detect and correct errors.
  - **Entanglement Purification**: distill high-fidelity entangled pairs from many noisy copies.
- In the project, we shall explore some QEC codes and bipartite purification protocols, and implement them in Qiskit. We shall also establish the equivalence between one-way entanglement purification and quantum error correction codes.

Types of Quantum Errors - Coherent Errors

- These errors arise from imperfect control of quantum gates, leading to small deviations in the intended operations.
- For instance, consider a small rotation $e^{i\epsilon X}$ applied N times to $|0\rangle$.
- We have the following expression for the final state:

$$|\psi\rangle = \prod_{i=1}^{N} e^{i\epsilon X} |0\rangle = \cos(N\epsilon)|0\rangle + i\sin(N\epsilon)|1\rangle \tag{1}$$

- The probability of measuring $|1\rangle$ is given by:

$$P(|1\rangle) = \sin^2(N\epsilon) \approx N^2\epsilon^2 \quad \text{for small } \epsilon. \tag{2}$$

- Hence, the error probability grows as $\mathcal{O}(N^2\epsilon^2)$.

Types of Quantum Errors - Environmental Decoherence

- Quantum systems can become entangled with their environment, leading to loss of coherence.
- For example, consider an environment with states $|e_0\rangle$ and $|e_1\rangle$ such that the environment state flips when the qubit is in state $|1\rangle$:
- On applying the operation HIH (ideally identity) on the qubit $|0\rangle|e_0\rangle$, we have:

$$|\psi\rangle = HIH|0\rangle|e_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle)|e_0\rangle + \frac{1}{2}(|0\rangle - |1\rangle)|e_1\rangle \tag{3}$$

- Tracing out the environment, the reduced density matrix of the qubit becomes:

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{4}$$

Types of Quantum Errors - Other Imperfections

- **Measurement errors:** Faulty measurement apparatus can yield incorrect outcomes.
  - Consider the operators:

$$F_0 = (1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|$$
$$F_1 = p\left|0\right\rangle\left\langle 0\right| + (1-p)\left|1\right\rangle\left\langle 1\right|$$

  - The final state after measurement will be a superposition state weighted by these probabilities.
- **Qubit loss:** Physical qubits may be lost due to decoherence or operational errors, leading to incomplete information about the quantum state.
- **Initialization errors:** Errors during state preparation can lead to incorrect initial states, affecting subsequent computations.
- **Qubit Leakage:** Qubits may leak out of the computational subspace into higher energy levels, which may introduce unwanted dynamics in a quantum circuit.

3-qubit Bit-flip Code

- Encodes one logical qubit into three physical qubits:

$$|0\rangle_L = |000\rangle , \quad |1\rangle_L = |111\rangle .$$

- **Distance** $d = 3 \Rightarrow$ corrects one bit-flip error ($t = \lfloor (d-1)/2 \rfloor = 1$).
- Error detection via parity checks using ancilla qubits.
- For coherent X-rotations on each qubit:

$$F_{\text{unencoded}} \approx 1 - \epsilon^2, \qquad F_{\text{encoded}} \approx 1 - \epsilon^6$$

(after successful syndrome measurement and correction).

9-qubit Shor Code

- Combines 3-qubit bit-flip and 3-qubit phase-flip codes.

$$|0\rangle_L = \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)^{\otimes 3}$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)^{\otimes 3}$$

- Corrects any single-qubit error (arbitrary Pauli X, Y, Z).
- Bit-flip correction: apply 3-qubit repetition code within each block.
- Phase-flip correction: compare phases between the three blocks.

Stabilizer Formalism

- A state $|\psi\rangle$ is *stabilized* by K if $K|\psi\rangle = |\psi\rangle$.
- Stabilizer group G: Abelian subgroup of the n-qubit Pauli group P_n.
- Code space: simultaneous $+1$ eigenspace of all generators.
- For n physical qubits and k independent generators:

$$\dim(\text{code space}) = 2^{n-k}$$

Steane [[7,1,3]] Code

- CSS stabilizer code with 6 generators:
  - 3 X-type stabilizers.
  - 3 Z-type stabilizers.
- Encodes 1 logical qubit, distance $d = 3$ (corrects 1 arbitrary Pauli error).
- Logical operators: e.g. $Z_L = Z^{\otimes 7}$, $X_L = X_1 X_2 X_3$.
- State preparation:
  - Start from $|0\rangle^{\otimes 7}$.
  - Measure X-type stabilizers with ancillas.
  - Apply classically controlled Z corrections to enforce $+1$ eigenvalues.

Digitization via Syndrome Measurement

- Realistic noise is continuous (coherent rotations, Lindblad evolution).
- Any CPTP map can be expanded in the Pauli basis:

$$\rho \to \sum_k A_k \rho A_k^\dagger, \quad A_k = \sum_j \alpha_{k,j} P_j.$$

- Syndrome measurement projects superpositions of errors onto definite Pauli errors:

$$\rho \to P_j \rho P_j \quad \text{with probability } |\alpha_j|^2.$$

- QEC effectively reduces continuous errors to **stochastic Pauli faults** with some error rate p.

Werner States

- Start from general mixed state in Bell basis:

$$\rho'_{AB} = \sum_{k_1,k_2,j_1,j_2} \lambda_{k_1 k_2 j_1 j_2} \left|\phi_{k_1 k_2}\right\rangle \left\langle\phi_{j_1 j_2}\right| .$$

- Random bilateral operations can be used to remove off-diagonal terms without changing fidelity with $|\phi_{00}\rangle$.
- Resulting Werner state:

$$\rho_W(x) = x \left|\phi_{00}\right\rangle \left\langle\phi_{00}\right| + \frac{1-x}{4} I_4 .$$

- Any purification protocol that works for Werner states works for all states with the same fidelity.

BBPSSW Protocol

- Input: two copies of a Werner state with fidelity $F > 1/2$.
- Steps:
  1. Depolarize (twirl) to Werner form.
  2. Apply bilateral CNOT: $A_1 \to A_2$, $B_1 \to B_2$.
  3. Measure target pair; keep source pair only for certain outcomes.
- Updated fidelity (conditional on success):

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + \frac{2F(1-F)}{3} + 5\left(\frac{1-F}{3}\right)^2}.$$

- Fidelity increases with each successful round; yield decreases due to discarding pairs.

DEJMPS Protocol

- Designed for Bell-diagonal states; more efficient than BBPSSW.
- Key ideas:
  - Local basis rotations convert phase errors into bit errors.
  - Bilateral CNOT and post-selection on matching measurement outcomes.
- Coefficients in Bell basis $\{\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}\}$ update as:

$$\lambda'_{00} = \frac{\lambda_{00}^2 + \lambda_{11}^2}{N}, \quad \lambda'_{10} = \frac{2\lambda_{00}\lambda_{11}}{N},$$

$$\lambda'_{01} = \frac{\lambda_{01}^2 + \lambda_{10}^2}{N}, \quad \lambda'_{11} = \frac{2\lambda_{01}\lambda_{10}}{N},$$

$$N = (\lambda_{00} + \lambda_{11})^2 + (\lambda_{01} + \lambda_{10})^2.$$

- Converges to $|\Phi^+\rangle$ for initial fidelity $F = \lambda_{00} > 1/2$.

One-way Hashing Protocol

- Alice and Bob share n noisy Bell pairs (Bell-diagonal state ρ).
- Represent each pair by 2-bit error label (i, j) (phase and amplitude errors).
- Repeatedly:
  - Choose random parity vector s.
  - Use local unitaries + bilateral CNOT to map $s \cdot x$ into one pair.
  - Measure and discard that pair; record the parity.
- After sacrificing $\approx nS(\rho)$ pairs, remaining $m \approx n(1 - S(\rho))$ pairs can be corrected.
- **Asymptotic yield**: $m/n \approx 1 - S(\rho)$.

Equivalence of 1-EPP and QECC

- Let $Q(\chi)$ be the quantum capacity of a channel χ.
- Let $D(M)$ be the one-way distillable entanglement of a mixed state M.
- Using teleportation with noisy entangled pairs:
  - Any QECC of rate R yields a 1-EPP with yield $\geq R$:
$$Q(\chi(M)) \leq D(M).$$
  - Any 1-EPP with yield R yields a QECC of rate $\geq R$:
$$D(M) \leq Q(\chi(M)).$$
- Therefore: **one-way entanglement purification and QECC are equivalent** in terms of achievable rates.

Equivalence Proof: $Q(\chi(M)) \leq D(M)$

Consider a quantum teleportation setup where Alice and Bob share multiple copies of a mixed entangled state M instead of perfect Bell pairs. We construct a purification protocol as follows:

- Alice prepares to send n qubits by preparing m Bell pairs and $n - m$ ancilla qubits in state $|0\rangle$. Encode one half of each Bell pair and the ancillas using a QECC of rate $R = m/n$.
- Alice sends her encoded qubits through the channel $\chi(M)$ to Bob.
- Bob receives the qubits and performs error correction using the QECC and recovers the m halves of the Bell pairs.

Hence n copies of the mixed state M were used to get m high-fidelity Bell pairs, yielding a distillation rate of $R = m/n$. The yield of such a 1-EPP is at least as large as the rate of the QECC used, leading to:

$$Q(\chi(M)) \leq D(M).$$

Equivalence Proof: $Q(\chi(M)) \geq D(M)$

Consider a one-way entanglement purification protocol (1-EPP) that distills m high-fidelity Bell pairs from n copies of a mixed state M. We construct a QECC as follows:

- Alice and Bob use the 1-EPP to distill m high-fidelity Bell pairs from n copies of M.
- Alices prepares to send m qubits of quantum information. She uses the distilled Bell pairs to teleport these qubits to Bob.

Hence we are able to send m qubits of quantum information without error using n uses of the channel $\chi(M)$, yielding a quantum communication rate of $R = m/n$. The rate of such a QECC is at least as large as the yield of the 1-EPP used, leading to:

$$D(M) \leq Q(\chi(M)).$$

Qiskit Implementations

- Implemented in Qiskit:
  - 3-qubit bit-flip code (X error and coherent rotation suppression).
  - 9-qubit Shor code for correcting a Z error.
  - Steane [[7,1,3]] code state preparation for $|0\rangle_L$ and $|1\rangle_L$.
  - BBPSSW and DEJMPS entanglement purification circuits.
- Results:
  - Histograms show deterministic recovery of logical states in 3- and 9-qubit codes.
  - Numerical fidelity matches theoretical predictions ($1 - \epsilon^6$ vs $1 - \epsilon^2$).
  - Purification protocols increase fidelity for initial $F > 0.5$, with non-unit success probability.

Conclusion

- Demonstrated how:
  - Simple repetition and Shor codes protect against bit-flip and phase-flip errors.
  - Stabilizer formalism generalizes QEC and simplifies encoding/decoding.
  - Syndrome measurements digitize continuous noise into discrete Pauli errors.
  - Entanglement purification (BBPSSW, DEJMPS, hashing) enhances entanglement fidelity.
- Established conceptual link and rate-equivalence between 1-EPP and QECC.
- Simulations validate theoretical performance of the studied codes and protocols.