

# Quantum Error Correction and Entanglement Purification

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# Outline

- 1 Motivation and Overview
- 2 Quantum Errors
- 3 3- and 9-qubit Codes
- 4 Stabilizer Codes
- 5 Digitization of Quantum Noise
- 6 Entanglement Purification
- 7 One-way Protocols and QECC
- 8 Simulations and Results
  - 3-Bit Code for Bit-Flip Error
  - 3-Bit Code for Phase Error
  - 9-Bit Shor Code for Z Error

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- Quantum information is fragile and susceptible to errors which may arise due to faulty gates, environmental decoherence etc.
- Reliable quantum computation and communication require active protection of quantum states which maybe achieved through:
  - **Quantum Error Correction (QEC)**: encode logical qubits into many physical qubits, detect and correct errors.
  - **Entanglement Purification**: distill high-fidelity entangled pairs from many noisy copies.
- In the project, we shall explore some QEC codes and bipartite purification protocols, and implement them in Qiskit. We shall also establish the equivalence between one-way entanglement purification and quantum error correction codes.

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# Types of Quantum Errors - Coherent Errors

- These errors arise from imperfect control of quantum gates, leading to small deviations in the intended operations.
- For instance, consider a small rotation  $e^{i\epsilon X}$  applied  $N$  times to  $|0\rangle$ .
- We have the following expression for the final state:

$$|\psi\rangle = \prod_{i=1}^N e^{i\epsilon X} |0\rangle = \cos(N\epsilon) |0\rangle + i \sin(N\epsilon) |1\rangle \quad (1)$$

- The probability of measuring  $|1\rangle$  is given by:

$$P(|1\rangle) = \sin^2(N\epsilon) \approx N^2\epsilon^2 \quad \text{for small } \epsilon. \quad (2)$$

- Hence, the error probability grows as  $\mathcal{O}(N^2\epsilon^2)$ .

# Types of Quantum Errors - Environmental Decoherence

- Quantum systems can become entangled with their environment, leading to loss of coherence.
- For example, consider an environment with states  $|e_0\rangle$  and  $|e_1\rangle$  such that the environment state flips when the qubit is in state  $|1\rangle$ :
- On applying the operation  $HHH$  (ideally identity) on the qubit  $|0\rangle |e_0\rangle$ , we have:

$$|\psi\rangle = HHH |0\rangle |e_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle) |e_0\rangle + \frac{1}{2}(|0\rangle - |1\rangle) |e_1\rangle \quad (3)$$

- Tracing out the environment, the reduced density matrix of the qubit becomes:

$$\rho = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) \quad (4)$$

# Types of Quantum Errors - Other Imperfections

- **Measurement errors:** Faulty measurement apparatus can yield incorrect outcomes.
  - Consider the operators:

$$F_0 = (1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|$$

$$F_1 = p |0\rangle \langle 0| + (1 - p) |1\rangle \langle 1|$$

- The final state after measurement will be a superposition state weighted by these probabilities.
- **Qubit loss:** Physical qubits may be lost due to decoherence or operational errors, leading to incomplete information about the quantum state.
- **Initialization errors:** Errors during state preparation can lead to incorrect initial states, affecting subsequent computations.
- **Qubit Leakage:** Qubits may leak out of the computational subspace into higher energy levels, which may introduce unwanted dynamics in a quantum circuit.



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# 3-qubit Bit-flip Code

- Encodes one logical qubit into three physical qubits:

$$|0\rangle_L = |000\rangle, \quad |1\rangle_L = |111\rangle.$$

- **Distance**  $d = 3 \Rightarrow$  corrects one bit-flip error ( $t = \lfloor (d - 1)/2 \rfloor = 1$ ).
- Error detection via parity checks using ancilla qubits.
- For coherent  $X$ -rotations on each qubit:

$$F_{\text{unencoded}} \approx 1 - \epsilon^2, \quad F_{\text{encoded}} \approx 1 - \epsilon^6$$

(after successful syndrome measurement and correction).

# 9-qubit Shor Code

- Combines 3-qubit bit-flip and 3-qubit phase-flip codes.

$$|0\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

- Corrects any single-qubit error (arbitrary Pauli  $X, Y, Z$ ).
- Bit-flip correction: apply 3-qubit repetition code within each block.
- Phase-flip correction: compare phases between the three blocks.

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- A state  $|\psi\rangle$  is *stabilized* by  $K$  if  $K|\psi\rangle = |\psi\rangle$ .
- Stabilizer group  $G$ : Abelian subgroup of the  $n$ -qubit Pauli group  $P_n$ .
- Code space: simultaneous  $+1$  eigenspace of all generators.
- For  $n$  physical qubits and  $k$  independent generators:

$$\dim(\text{code space}) = 2^{n-k}$$

- CSS stabilizer code with 6 generators:
  - 3  $X$ -type stabilizers.
  - 3  $Z$ -type stabilizers.
- Encodes 1 logical qubit, distance  $d = 3$  (corrects 1 arbitrary Pauli error).
- Logical operators: e.g.  $Z_L = Z^{\otimes 7}$ ,  $X_L = X_1 X_2 X_3$ .
- State preparation:
  - Start from  $|0\rangle^{\otimes 7}$ .
  - Measure  $X$ -type stabilizers with ancillas.
  - Apply classically controlled  $Z$  corrections to enforce  $+1$  eigenvalues.

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# Digitization via Syndrome Measurement

- Realistic noise is continuous (coherent rotations, Lindblad evolution).
- Any CPTP map can be expanded in the Pauli basis:

$$\rho \rightarrow \sum_k A_k \rho A_k^\dagger, \quad A_k = \sum_j \alpha_{kj} P_j.$$

- Syndrome measurement projects superpositions of errors onto definite Pauli errors:

$$\rho \rightarrow P_j \rho P_j \quad \text{with probability } |\alpha_j|^2.$$

- QEC effectively reduces continuous errors to **stochastic Pauli faults** with some error rate  $p$ .



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- Start from general mixed state in Bell basis:

$$\rho'_{AB} = \sum_{k_1, k_2, j_1, j_2} \lambda_{k_1 k_2 j_1 j_2} |\phi_{k_1 k_2}\rangle \langle \phi_{j_1 j_2}|.$$

- Random bilateral operations can be used to remove off-diagonal terms without changing fidelity with  $|\phi_{00}\rangle$ .
- Resulting Werner state:

$$\rho_W(x) = x |\phi_{00}\rangle \langle \phi_{00}| + \frac{1-x}{4} I_4.$$

- Any purification protocol that works for Werner states works for all states with the same fidelity.

- Input: two copies of a Werner state with fidelity  $F > 1/2$ .
- Steps:
  - ① Depolarize (twirl) to Werner form.
  - ② Apply bilateral CNOT:  $A_1 \rightarrow A_2, B_1 \rightarrow B_2$ .
  - ③ Measure target pair; keep source pair only for certain outcomes.
- Updated fidelity (conditional on success):

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + \frac{2F(1-F)}{3} + 5\left(\frac{1-F}{3}\right)^2}.$$

- Fidelity increases with each successful round; yield decreases due to discarding pairs.

- Designed for Bell-diagonal states; more efficient than BBPSSW.
- Key ideas:
  - Local basis rotations convert phase errors into bit errors.
  - Bilateral CNOT and post-selection on matching measurement outcomes.
- Coefficients in Bell basis  $\{\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}\}$  update as:

$$\lambda'_{00} = \frac{\lambda_{00}^2 + \lambda_{11}^2}{N}, \quad \lambda'_{10} = \frac{2\lambda_{00}\lambda_{11}}{N},$$

$$\lambda'_{01} = \frac{\lambda_{01}^2 + \lambda_{10}^2}{N}, \quad \lambda'_{11} = \frac{2\lambda_{01}\lambda_{10}}{N},$$

$$N = (\lambda_{00} + \lambda_{11})^2 + (\lambda_{01} + \lambda_{10})^2.$$

- Converges to  $|\Phi^+\rangle$  for initial fidelity  $F = \lambda_{00} > 1/2$ .

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# One-way Hashing Protocol

- Alice and Bob share  $n$  noisy Bell pairs (Bell-diagonal state  $\rho$ ).
- Represent each pair by 2-bit error label  $(i, j)$  (phase and amplitude errors).
- Repeatedly:
  - Choose random parity vector  $s$ .
  - Use local unitaries + bilateral CNOT to map  $s \cdot x$  into one pair.
  - Measure and discard that pair; record the parity.
- After sacrificing  $\approx nS(\rho)$  pairs, remaining  $m \approx n(1 - S(\rho))$  pairs can be corrected.
- **Asymptotic yield:**  $m/n \approx 1 - S(\rho)$ .

# Equivalence of 1-EPP and QECC

- Let  $Q(\chi)$  be the quantum capacity of a channel  $\chi$ .
- Let  $D(M)$  be the one-way distillable entanglement of a mixed state  $M$ .
- Using teleportation with noisy entangled pairs:
  - Any QECC of rate  $R$  yields a 1-EPP with yield  $\geq R$ :

$$Q(\chi(M)) \leq D(M).$$

- Any 1-EPP with yield  $R$  yields a QECC of rate  $\geq R$ :

$$D(M) \leq Q(\chi(M)).$$

- Therefore: **one-way entanglement purification and QECC are equivalent** in terms of achievable rates.

## Equivalence Proof: $Q(\chi(M)) \leq D(M)$

Consider a quantum teleportation setup where Alice and Bob share multiple copies of a mixed entangled state  $M$  instead of perfect Bell pairs. We construct a purification protocol as follows:

- Alice prepares to send  $n$  qubits by preparing  $m$  Bell pairs and  $n - m$  ancilla qubits in state  $|0\rangle$ . Encode one half of each Bell pair and the ancillas using a QECC of rate  $R = m/n$ .
- Alice sends her encoded qubits through the channel  $\chi(M)$  to Bob.
- Bob receives the qubits and performs error correction using the QECC and recovers the  $m$  halves of the Bell pairs.

Hence  $n$  copies of the mixed state  $M$  were used to get  $m$  high-fidelity Bell pairs, yielding a distillation rate of  $R = m/n$ . The yield of such a 1-EPP is at least as large as the rate of the QECC used, leading to:

$$Q(\chi(M)) \leq D(M).$$



## Equivalence Proof: $Q(\chi(M)) \geq D(M)$

Consider a one-way entanglement purification protocol (1-EPP) that distills  $m$  high-fidelity Bell pairs from  $n$  copies of a mixed state  $M$ . We construct a QECC as follows:

- Alice and Bob use the 1-EPP to distill  $m$  high-fidelity Bell pairs from  $n$  copies of  $M$ .
- Alice prepares to send  $m$  qubits of quantum information. She uses the distilled Bell pairs to teleport these qubits to Bob.

Hence we are able to send  $m$  qubits of quantum information without error using  $n$  uses of the channel  $\chi(M)$ , yielding a quantum communication rate of  $R = m/n$ . The rate of such a QECC is at least as large as the yield of the 1-EPP used, leading to:

$$D(M) \leq Q(\chi(M)).$$

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# Qiskit Implementations

- Implemented in Qiskit:

- 3-qubit bit-flip code (X error and coherent rotation suppression).
- 9-qubit Shor code for correcting a  $Z$  error.
- Steane  $[[7,1,3]]$  code state preparation for  $|0\rangle_L$  and  $|1\rangle_L$ .
- BBPSSW and DEJMPS entanglement purification circuits.

- Results:

- Histograms show deterministic recovery of logical states in 3- and 9-qubit codes.
- Numerical fidelity matches theoretical predictions ( $1 - \epsilon^6$  vs  $1 - \epsilon^2$ ).
- Purification protocols increase fidelity for initial  $F > 0.5$ , with non-unit success probability.

# 3-Bit Code for Bit-Flip (X) Error

## Process:

- 1 Encode logical qubit using Hadamard + 2 CNOTs.
- 2 Inject bit-flip error (X gate) on one qubit.
- 3 Detect error using 2 ancilla qubits and 4 CNOTs.
- 4 Apply conditional correction based on syndrome.
- 5 Decode and measure final qubit.

**Result:** Measured state “001” with probability 1 — successful correction.

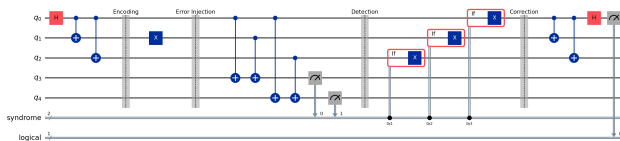


Figure: 3-bit Code Circuit for X Error

## 3-Bit Code: X Error Results

- Histogram confirms complete recovery of logical qubit.
- Fidelity maintained post-correction.
- Demonstrates robustness of simple repetition code for single X-error.

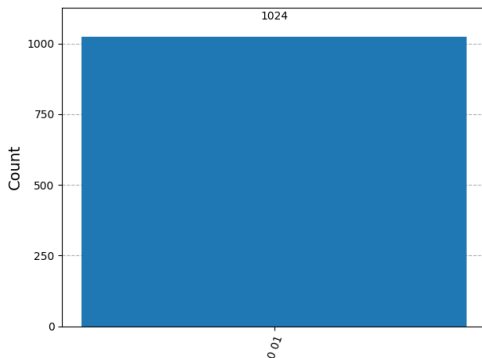


Figure: 3-bit Code Results for X Error

# 3-Bit Code for Phase Error

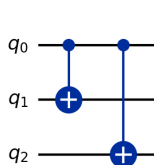
**Goal:** Verify suppression of coherent rotation errors  $U = e^{i\epsilon\sigma_x}$ .

**Procedure:**

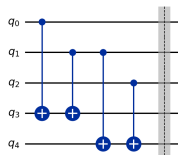
- Encode qubit via CNOTs.
- Apply small X-rotation ( $\epsilon$ ) to all three data qubits.
- Measure syndrome using two ancillas.
- Decode and compute fidelity.

**Observation:**

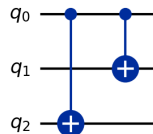
$$F_{\text{unencoded}} = 1 - \epsilon^2 \quad \text{vs.} \quad F_{\text{encoded}} = 1 - \epsilon^6$$



(a) Encoding Circuit



(b) Syndrome  
Measurement Circuit



(c) Decoding Circuit

# 9-Bit Shor Code for Z Error

## Overview:

- Protects logical qubit against arbitrary single-qubit errors.
- Encodes  $|+\rangle$  using layered 3x3 block structure:
  - Inner: Bit-flip code
  - Outer: Phase-flip code
- Inject phase-flip (Z) error on one qubit.
- Use 6 ancillas for syndrome extraction.

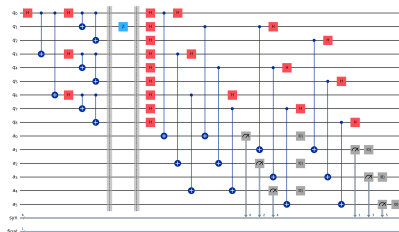


Figure: 9-bit Shor Code — Syndrome Measurement

# 9-Bit Shor Code Results

- Syndrome bits identify error location.
- Corrective Z operation restores logical state.
- Decoded qubit measured in X-basis gives deterministic “0”.

**Result:** Successful recovery of logical qubit — validates Shor code's ability to correct any single-qubit error.

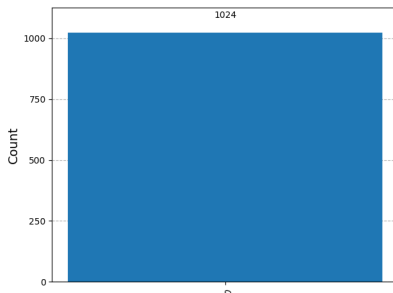


Figure: 9-bit Code Results for Z Error



# Steane $[[7,1,3]]$ Code State Preparation

**Objective:** Prepare logical states  $|0\rangle_L$  and  $|1\rangle_L$  using Steane code.

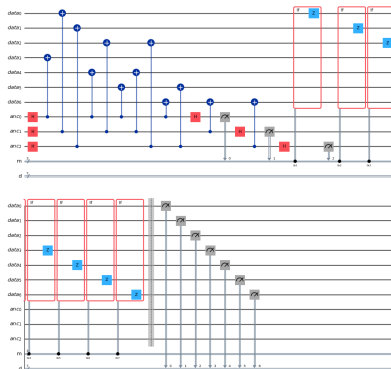


Figure: Steane Code Encoding Circuit

If we start with data qubits in state  $|0\rangle^{\otimes 7}$ , we encode  $|0\rangle_L$ . To encode  $|1\rangle_L$ , we flip all the data qubits before encoding i.e. start with  $|1\rangle^{\otimes 7}$ .

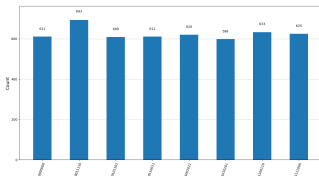
**Expected outcomes for logical states:**

$$\begin{aligned} |0\rangle_L &= \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &\quad + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle) \\ |1\rangle_L &= \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\ &\quad + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle) \end{aligned} \tag{5}$$

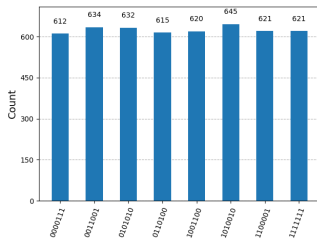
# Steane Code Results (Contd.)

- Measured histograms for both  $|0\rangle_L$  and  $|1\rangle_L$  shows expected outcomes as shown in Fig. 7.
- Observed probabilities closely match theoretical values of  $1/8 = 0.125$  for each expected outcome.
- Minor deviations attributed to statistical fluctuations and simulator noise.
- Confirms successful preparation of logical states using Steane code.

**Note:** The bit order is flipped in the histogram results.



(a) Logical  $|0\rangle_L$



(b) Logical  $|1\rangle_L$

# BBPSSW Entanglement Purification

## Protocol Steps:

- 1 Start with two Werner states  $\rho_W(F)$  shared between Alice and Bob.
- 2 Apply bilateral CNOTs:  $A_1 \rightarrow A_2$ ,  $B_1 \rightarrow B_2$ .
- 3 Measure target pair  $(A_2, B_2)$  in computational basis.
- 4 Keep source pair  $(A_1, B_1)$  only if measurement outcomes match.

**Fidelity Update:** After successful purification, the fidelity updates as:

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + \frac{2F(1-F)}{3} + 5\left(\frac{1-F}{3}\right)^2}.$$

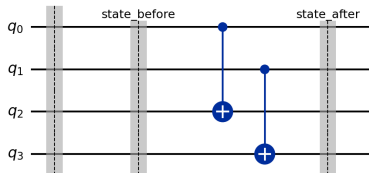


Figure: BBPSSW Entanglement Purification Circuit

# BBPSSW Results

- Simulated BBPSSW protocol for various initial fidelities  $F$ .
- Observed increase in fidelity  $F'$  after purification, matching theoretical predictions.
- Success probability decreases with lower initial fidelity.
- Confirms effectiveness of BBPSSW in enhancing entanglement quality.

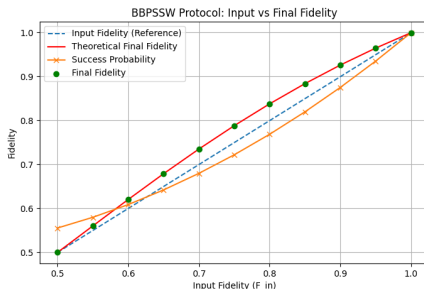


Figure: BBPSSW Purification Results

# DEJMPS Entanglement Purification

## Protocol Steps:

- 1 Start with two Bell-diagonal states shared between Alice and Bob.
- 2 Apply local basis rotations to convert phase errors into bit errors.
- 3 Apply bilateral CNOTs:  $A_1 \rightarrow A_2$ ,  $B_1 \rightarrow B_2$ .
- 4 Measure target pair  $(A_2, B_2)$  in computational basis.
- 5 Keep source pair  $(A_1, B_1)$  only if measurement outcomes match.

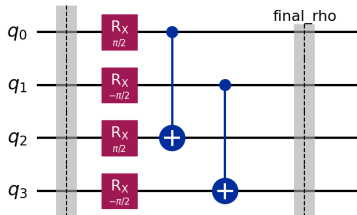


Figure: DEJMPS Entanglement Purification Circuit

# DEJMPS Results

- Simulated DEJMPS protocol for various initial fidelities  $F$ .
- Observed increase in fidelity  $F'$  after purification, consistent with theoretical expectations.
- Higher efficiency compared to BBPSSW for the same initial fidelity.
- Validates DEJMPS as a superior entanglement purification method.

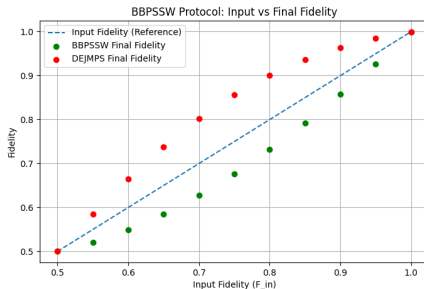


Figure: DEJMPS Purification Results

- Demonstrated how:
  - Simple repetition and Shor codes protect against bit-flip and phase-flip errors.
  - Stabilizer formalism generalizes QEC and simplifies encoding/decoding.
  - Syndrome measurements digitize continuous noise into discrete Pauli errors.
  - Entanglement purification (BBPSSW, DEJMPS, hashing) enhances entanglement fidelity.
- Established conceptual link and rate-equivalence between 1-EPP and QECC.
- Simulations validate theoretical performance of the studied codes and protocols.