

Quantum Error Correction and Entanglement Purification

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Outline

- 1 Motivation and Overview
- 2 Quantum Errors
- 3 3- and 9-qubit Codes
- 4 Stabilizer Codes
- 5 Digitization of Quantum Noise
- 6 Entanglement Purification
- 7 One-way Protocols and QECC
- 8 Simulations and Results
 - 3-Bit Code for Bit-Flip Error
 - 3-Bit Code for Phase Error
 - 9-Bit Shor Code for Z Error

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Motivation

- Quantum information is fragile and susceptible to errors which may arise due to faulty gates, environmental decoherence etc.
- Reliable quantum computation and communication require active protection of quantum states which maybe achieved through:
 - **Quantum Error Correction (QEC)**: encode logical qubits into many physical qubits, detect and correct errors.
 - **Entanglement Purification**: distill high-fidelity entangled pairs from many noisy copies.
- In the project, we shall explore some QEC codes and bipartite purification protocols, and implement them in Qiskit. We shall also establish the equivalence between one-way entanglement purification and quantum error correction codes.

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Types of Quantum Errors - Coherent Errors

- These errors arise from imperfect control of quantum gates, leading to small deviations in the intended operations.
- For instance, consider a small rotation $e^{i\epsilon X}$ applied N times to $|0\rangle$.
- We have the following expression for the final state:

$$|\psi\rangle = \prod_{i=1}^N e^{i\epsilon X} |0\rangle = \cos(N\epsilon) |0\rangle + i \sin(N\epsilon) |1\rangle \quad (1)$$

- The probability of measuring $|1\rangle$ is given by:

$$P(|1\rangle) = \sin^2(N\epsilon) \approx N^2\epsilon^2 \quad \text{for small } \epsilon. \quad (2)$$

- Hence, the error probability grows as $\mathcal{O}(N^2\epsilon^2)$.

Types of Quantum Errors - Environmental Decoherence

- Quantum systems can become entangled with their environment, leading to loss of coherence.
- For example, consider an environment with states $|e_0\rangle$ and $|e_1\rangle$ such that the environment state flips when the qubit is in state $|1\rangle$:
- On applying the operation HIH (ideally identity) on the qubit $|0\rangle|e_0\rangle$, we have:

$$|\psi\rangle = HIH|0\rangle|e_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle)|e_0\rangle + \frac{1}{2}(|0\rangle - |1\rangle)|e_1\rangle \quad (3)$$

- Tracing out the environment, the reduced density matrix of the qubit becomes:

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \quad (4)$$

Types of Quantum Errors - Other Imperfections

- **Measurement errors:** Faulty measurement apparatus can yield incorrect outcomes.
 - Consider the operators:

$$F_0 = (1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|$$

$$F_1 = p |0\rangle \langle 0| + (1 - p) |1\rangle \langle 1|$$

- The final state after measurement will be a superposition state weighted by these probabilities.
- **Qubit loss:** Physical qubits may be lost due to decoherence or operational errors, leading to incomplete information about the quantum state.
- **Initialization errors:** Errors during state preparation can lead to incorrect initial states, affecting subsequent computations.
- **Qubit Leakage:** Qubits may leak out of the computational subspace into higher energy levels, which may introduce unwanted dynamics in a quantum circuit.

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3-qubit Bit-flip Code

- Encodes one logical qubit into three physical qubits:

$$|0\rangle_L = |000\rangle, \quad |1\rangle_L = |111\rangle.$$

- **Distance** $d = 3 \Rightarrow$ corrects one bit-flip error ($t = \lfloor (d - 1)/2 \rfloor = 1$).
- Error detection via parity checks using ancilla qubits.
- For coherent X -rotations on each qubit:

$$F_{\text{unencoded}} \approx 1 - \epsilon^2, \quad F_{\text{encoded}} \approx 1 - \epsilon^6$$

(after successful syndrome measurement and correction).

9-qubit Shor Code

- Combines 3-qubit bit-flip and 3-qubit phase-flip codes.

$$|0\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

- Corrects any single-qubit error (arbitrary Pauli X, Y, Z).
- Bit-flip correction: apply 3-qubit repetition code within each block.
- Phase-flip correction: compare phases between the three blocks.

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Stabilizer Formalism

- A state $|\psi\rangle$ is *stabilized* by K if $K|\psi\rangle = |\psi\rangle$.
- Stabilizer group G : Abelian subgroup of the n -qubit Pauli group P_n .
- Code space: simultaneous +1 eigenspace of all generators.
- For n physical qubits and k independent generators:

$$\dim(\text{code space}) = 2^{n-k}$$

Steane [[7,1,3]] Code

- CSS stabilizer code with 6 generators:
 - 3 X -type stabilizers.
 - 3 Z -type stabilizers.
- Encodes 1 logical qubit, distance $d = 3$ (corrects 1 arbitrary Pauli error).
- Logical operators: e.g. $Z_L = Z^{\otimes 7}$, $X_L = X_1X_2X_3$.
- State preparation:
 - Start from $|0\rangle^{\otimes 7}$.
 - Measure X -type stabilizers with ancillas.
 - Apply classically controlled Z corrections to enforce +1 eigenvalues.

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Digitization via Syndrome Measurement

- Realistic noise is continuous (coherent rotations, Lindblad evolution).
- Any CPTP map can be expanded in the Pauli basis:

$$\rho \rightarrow \sum_k A_k \rho A_k^\dagger, \quad A_k = \sum_j \alpha_{k,j} P_j.$$

- Syndrome measurement projects superpositions of errors onto definite Pauli errors:

$$\rho \rightarrow P_j \rho P_j \quad \text{with probability } |\alpha_j|^2.$$

- QEC effectively reduces continuous errors to **stochastic Pauli faults** with some error rate p .

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Werner States

- Start from general mixed state in Bell basis:

$$\rho'_{AB} = \sum_{k_1, k_2, j_1, j_2} \lambda_{k_1 k_2 j_1 j_2} |\phi_{k_1 k_2}\rangle \langle \phi_{j_1 j_2}|.$$

- Random bilateral operations can be used to remove off-diagonal terms without changing fidelity with $|\phi_{00}\rangle$.
- Resulting Werner state:

$$\rho_W(x) = x |\phi_{00}\rangle \langle \phi_{00}| + \frac{1-x}{4} I_4.$$

- Any purification protocol that works for Werner states works for all states with the same fidelity.

BPSSW Protocol

- Input: two copies of a Werner state with fidelity $F > 1/2$.
- Steps:
 - ① Depolarize (twirl) to Werner form.
 - ② Apply bilateral CNOT: $A_1 \rightarrow A_2, B_1 \rightarrow B_2$.
 - ③ Measure target pair; keep source pair only for certain outcomes.
- Updated fidelity (conditional on success):

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + \frac{2F(1-F)}{3} + 5\left(\frac{1-F}{3}\right)^2}.$$

- Fidelity increases with each successful round; yield decreases due to discarding pairs.

DEJMPS Protocol

- Designed for Bell-diagonal states; more efficient than BBPSSW.
- Key ideas:
 - Local basis rotations convert phase errors into bit errors.
 - Bilateral CNOT and post-selection on matching measurement outcomes.
- Coefficients in Bell basis $\{\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}\}$ update as:

$$\lambda'_{00} = \frac{\lambda_{00}^2 + \lambda_{11}^2}{N}, \quad \lambda'_{10} = \frac{2\lambda_{00}\lambda_{11}}{N},$$

$$\lambda'_{01} = \frac{\lambda_{01}^2 + \lambda_{10}^2}{N}, \quad \lambda'_{11} = \frac{2\lambda_{01}\lambda_{10}}{N},$$

$$N = (\lambda_{00} + \lambda_{11})^2 + (\lambda_{01} + \lambda_{10})^2.$$

- Converges to $|\Phi^+\rangle$ for initial fidelity $F = \lambda_{00} > 1/2$.

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One-way Hashing Protocol

- Alice and Bob share n noisy Bell pairs (Bell-diagonal state ρ).
- Represent each pair by 2-bit error label (i, j) (phase and amplitude errors).
- Repeatedly:
 - Choose random parity vector s .
 - Use local unitaries + bilateral CNOT to map $s \cdot x$ into one pair.
 - Measure and discard that pair; record the parity.
- After sacrificing $\approx nS(\rho)$ pairs, remaining $m \approx n(1 - S(\rho))$ pairs can be corrected.
- **Asymptotic yield:** $m/n \approx 1 - S(\rho)$.

Equivalence of 1-EPP and QECC

- Let $Q(\chi)$ be the quantum capacity of a channel χ .
- Let $D(M)$ be the one-way distillable entanglement of a mixed state M .
- Using teleportation with noisy entangled pairs:
 - Any QECC of rate R yields a 1-EPP with yield $\geq R$:

$$Q(\chi(M)) \leq D(M).$$

- Any 1-EPP with yield R yields a QECC of rate $\geq R$:

$$D(M) \leq Q(\chi(M)).$$

- Therefore: **one-way entanglement purification and QECC are equivalent** in terms of achievable rates.

Equivalence Proof: $Q(\chi(M)) \leq D(M)$

Consider a quantum teleportation setup where Alice and Bob share multiple copies of a mixed entangled state M instead of perfect Bell pairs. We construct a purification protocol as follows:

- Alice prepares to send n qubits by preparing m Bell pairs and $n - m$ ancilla qubits in state $|0\rangle$. Encode one half of each Bell pair and the ancillas using a QECC of rate $R = m/n$.
- Alice sends her encoded qubits through the channel $\chi(M)$ to Bob.
- Bob receives the qubits and performs error correction using the QECC and recovers the m halves of the Bell pairs.

Hence n copies of the mixed state M were used to get m high-fidelity Bell pairs, yielding a distillation rate of $R = m/n$. The yield of such a 1-EPP is at least as large as the rate of the QECC used, leading to:

$$Q(\chi(M)) \leq D(M).$$

Equivalence Proof: $Q(\chi(M)) \geq D(M)$

Consider a one-way entanglement purification protocol (1-EPP) that distills m high-fidelity Bell pairs from n copies of a mixed state M . We construct a QECC as follows:

- Alice and Bob use the 1-EPP to distill m high-fidelity Bell pairs from n copies of M .
- Alice prepares to send m qubits of quantum information. She uses the distilled Bell pairs to teleport these qubits to Bob.

Hence we are able to send m qubits of quantum information without error using n uses of the channel $\chi(M)$, yielding a quantum communication rate of $R = m/n$. The rate of such a QECC is at least as large as the yield of the 1-EPP used, leading to:

$$D(M) \leq Q(\chi(M)).$$

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Qiskit Implementations

- Implemented in Qiskit:
 - 3-qubit bit-flip code (X error and coherent rotation suppression).
 - 9-qubit Shor code for correcting a Z error.
 - Steane [[7,1,3]] code state preparation for $|0\rangle_L$ and $|1\rangle_L$.
 - BBPSSW and DEJMPS entanglement purification circuits.
- Results:
 - Histograms show deterministic recovery of logical states in 3- and 9-qubit codes.
 - Numerical fidelity matches theoretical predictions ($1 - \epsilon^6$ vs $1 - \epsilon^2$).
 - Purification protocols increase fidelity for initial $F > 0.5$, with non-unit success probability.

3-Bit Code for Bit-Flip (X) Error

Process:

- ① Encode logical qubit using Hadamard + 2 CNOTs.
- ② Inject bit-flip error (X gate) on one qubit.
- ③ Detect error using 2 ancilla qubits and 4 CNOTs.
- ④ Apply conditional correction based on syndrome.
- ⑤ Decode and measure final qubit.

Result: Measured state “001” with probability 1 — successful correction.

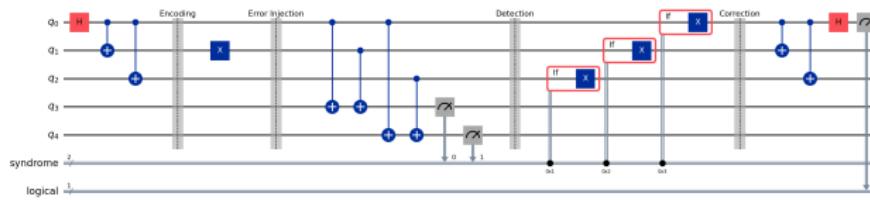


Figure: 3-bit Code Circuit for X Error

3-Bit Code: X Error Results

- Histogram confirms complete recovery of logical qubit.
- Fidelity maintained post-correction.
- Demonstrates robustness of simple repetition code for single X-error.

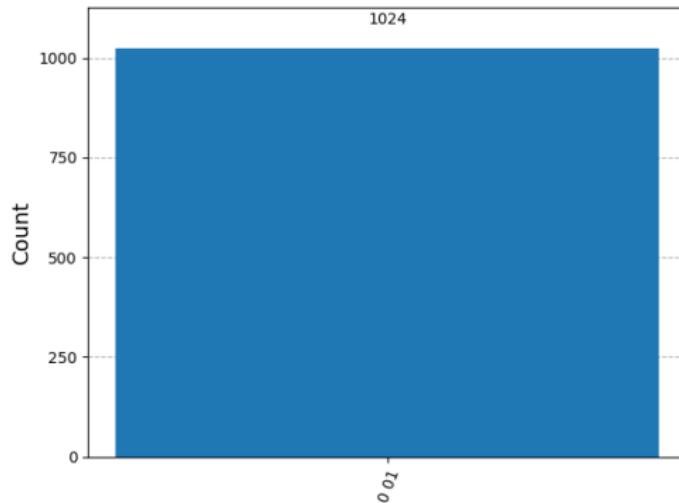


Figure: 3-bit Code Results for X Error

3-Bit Code for Phase Error

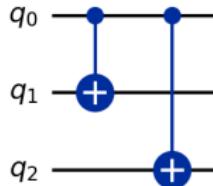
Goal: Verify suppression of coherent rotation errors $U = e^{i\epsilon\sigma_x}$.

Procedure:

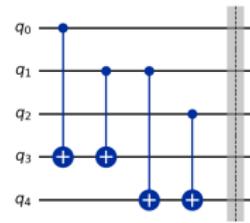
- Encode qubit via CNOTs.
- Apply small X-rotation (ϵ) to all three data qubits.
- Measure syndrome using two ancillas.
- Decode and compute fidelity.

Observation:

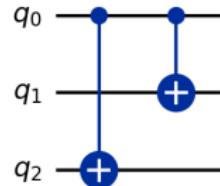
$$F_{\text{unencoded}} = 1 - \epsilon^2 \quad \text{vs.} \quad F_{\text{encoded}} = 1 - \epsilon^6$$



(a) Encoding Circuit



(b) Syndrome
Measurement Circuit



(c) Decoding Circuit

9-Bit Shor Code for Z Error

Overview:

- Protects logical qubit against arbitrary single-qubit errors.
- Encodes $|+\rangle$ using layered 3x3 block structure:
 - Inner: Bit-flip code
 - Outer: Phase-flip code
- Inject phase-flip (Z) error on one qubit.
- Use 6 ancillas for syndrome extraction.

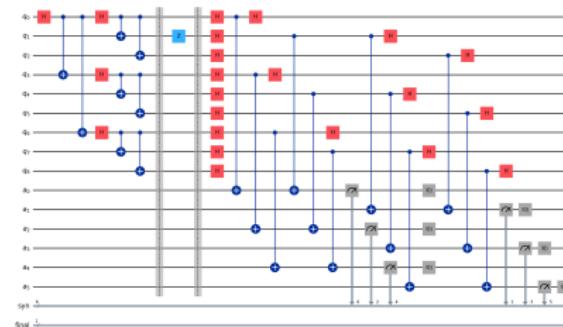


Figure: 9-bit Shor Code — Syndrome Measurement

9-Bit Shor Code Results

- Syndrome bits identify error location.
- Corrective Z operation restores logical state.
- Decoded qubit measured in X-basis gives deterministic “0”.

Result: Successful recovery of logical qubit — validates Shor code's ability to correct any single-qubit error.

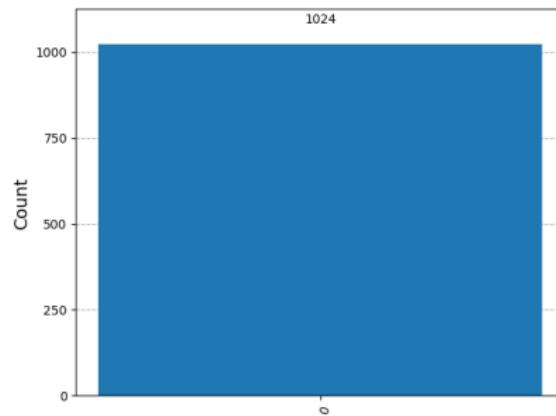


Figure: 9-bit Code Results for Z Error

Steane [[7,1,3]] Code State Preparation

Objective: Prepare logical states $|0\rangle_L$ and $|1\rangle_L$ using Steane code.

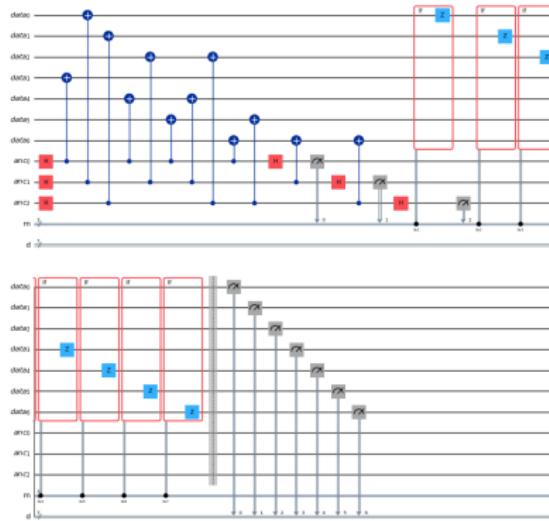


Figure: Steane Code Encoding Circuit

If we start with data qubits in state $|0\rangle^{\otimes 7}$, we encode $|0\rangle_L$. To encode $|1\rangle_L$, we flip all the data qubits before encoding i.e. start with $|1\rangle^{\otimes 7}$.

Steane Code Results

Expected outcomes for logical states:

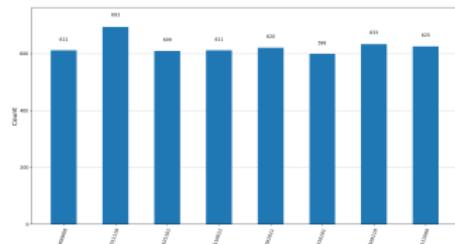
$$\begin{aligned} |0\rangle_L = \frac{1}{\sqrt{8}}(&|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle) \end{aligned} \quad (5)$$

$$\begin{aligned} |1\rangle_L = \frac{1}{\sqrt{8}}(&|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\ &+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle) \end{aligned}$$

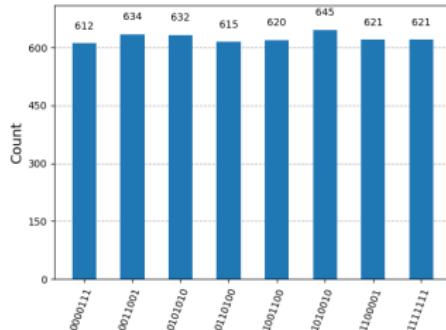
Steane Code Results (Contd.)

- Measured histograms for both $|0\rangle_L$ and $|1\rangle_L$ shows expected outcomes as shown in Fig. 7.
- Observed probabilities closely match theoretical values of $1/8 = 0.125$ for each expected outcome.
- Minor deviations attributed to statistical fluctuations and simulator noise.
- Confirms successful preparation of logical states using Steane code.

Note: The bit order is flipped in the histogram results.



(a) Logical $|0\rangle_L$



(b) Logical $|1\rangle_L$

BBPSSW Entanglement Purification

Protocol Steps:

- ① Start with two Werner states $\rho_W(F)$ shared between Alice and Bob.
- ② Apply bilateral CNOTs: $A_1 \rightarrow A_2, B_1 \rightarrow B_2$.
- ③ Measure target pair (A_2, B_2) in computational basis.
- ④ Keep source pair (A_1, B_1) only if measurement outcomes match.

Fidelity Update: After successful purification, the fidelity updates as:

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + \frac{2F(1-F)}{3} + 5\left(\frac{1-F}{3}\right)^2}.$$

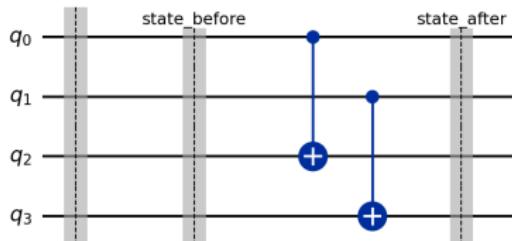


Figure: BBPSSW Entanglement Purification Circuit

BBPSSW Results

- Simulated BBPSSW protocol for various initial fidelities F .
- Observed increase in fidelity F' after purification, matching theoretical predictions.
- Success probability decreases with lower initial fidelity.
- Confirms effectiveness of BBPSSW in enhancing entanglement quality.

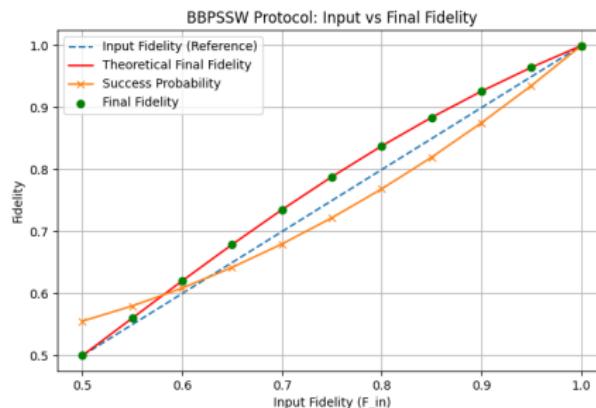


Figure: BBPSSW Purification Results

DEJMPS Entanglement Purification

Protocol Steps:

- ① Start with two Bell-diagonal states shared between Alice and Bob.
- ② Apply local basis rotations to convert phase errors into bit errors.
- ③ Apply bilateral CNOTs: $A_1 \rightarrow A_2, B_1 \rightarrow B_2$.
- ④ Measure target pair (A_2, B_2) in computational basis.
- ⑤ Keep source pair (A_1, B_1) only if measurement outcomes match.

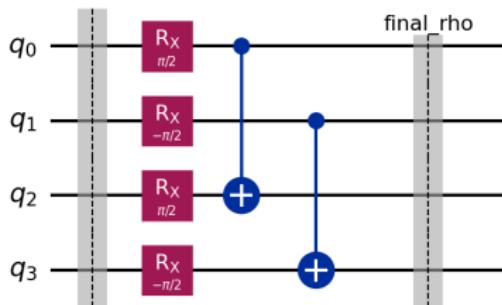


Figure: DEJMPS Entanglement Purification Circuit

DEJMPS Results

- Simulated DEJMPS protocol for various initial fidelities F .
- Observed increase in fidelity F' after purification, consistent with theoretical expectations.
- Higher efficiency compared to BBPSSW for the same initial fidelity.
- Validates DEJMPS as a superior entanglement purification method.

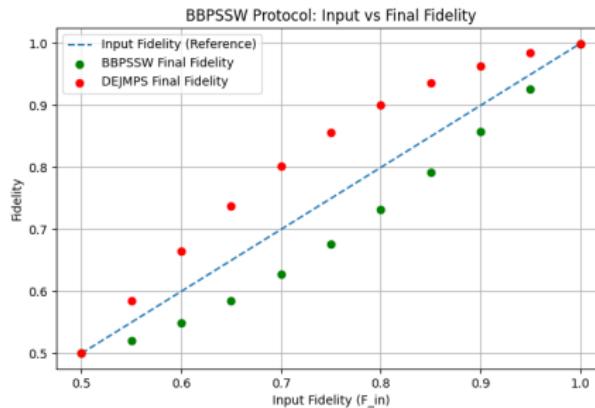


Figure: DEJMPS Purification Results

Conclusion

- Demonstrated how:
 - Simple repetition and Shor codes protect against bit-flip and phase-flip errors.
 - Stabilizer formalism generalizes QEC and simplifies encoding/decoding.
 - Syndrome measurements digitize continuous noise into discrete Pauli errors.
 - Entanglement purification (BBPSSW, DEJMPS, hashing) enhances entanglement fidelity.
- Established conceptual link and rate-equivalence between 1-EPP and QECC.
- Simulations validate theoretical performance of the studied codes and protocols.