Quantum Error Correction and Entanglement Purification

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Outline

Introduction

Motivation

- Quantum states are quite sensitive and prone to errors arising due to decoherence, imperfect application of gates and measurments.
- To combat these errors, two main strategies are employed:
  - **Entanglement purification** — Use multiple noisy entangled pairs to generate a few high-fidelity pairs.
  - **Quantum error correction (QEC)** — encode logical qubits into many physical qubits to detect and correct errors.
- In this PPT, we shall analyse the 3-bit and 9-bit Shor codes for QEC and also look at the BBPSSW protocol for entanglement purification.

Quantum Errors

Types of Quantum Errors

Common categories:

- **Coherent errors**: Systematic over/under rotations (unitary). Generally arise due to incorrect operation of gates.
- **Environmental decoherence**: Coupling to the environment leads to loss of coherence.
- **Loss / leakage / measurement / initialization errors**.

Coherent errors

Small rotation $U = e^{i\epsilon\sigma_x}$ applied to a qubit N times:

$$|\psi\rangle = \left(e^{i\epsilon\sigma_x}\right)^{\otimes N}|0\rangle = \cos(N\epsilon)|0\rangle + i\sin(N\epsilon)|1\rangle .$$

Measurement probabilities:

$$P(|0\rangle) = \cos^2(N\epsilon) \approx 1 - (N\epsilon)^2, \qquad P(|1\rangle) \approx (N\epsilon)^2.$$

With QEC we can suppress such errors to higher orders ($O(\epsilon^2) \to O(\epsilon^6)$).

Decoherence due to environment

Example: system entangles with environment $|e_0\rangle, |e_1\rangle$. The state of the environment flips if the qubit is $|1\rangle$. Hence on application of the operation HIH to the state $|0\rangle |e_0\rangle$, we get:

$$|\psi\rangle = HIH |0\rangle |e_0\rangle = \frac{1}{2}(|0\rangle + |1\rangle) |e_0\rangle + \frac{1}{2}(|0\rangle - |1\rangle) |e_1\rangle .$$

We can then consider the density matrix of the combined system and trace out the environment to get the reduced density matrix of the system:

$$\rho = \mathrm{tr}_E(|\psi\rangle \langle\psi|) = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|)$$

Hence the qubit is in a completely mixed state, i.e. it has decohered.

Measurement Errors

Two Common Models:

1. **POVM Model:**

$$F_0 = (1 - p_M) |0\rangle \langle 0| + p_M |1\rangle \langle 1| , \quad F_1 = (1 - p_M) |1\rangle \langle 1| + p_M |0\rangle \langle 0|$$

   Outcome probabilities:

$$\mathrm{Tr}(F_0\rho) = (1 - p_M)\mathrm{Tr}(A_0\rho) + p_M \mathrm{Tr}(A_1\rho)$$

   Post-measurement state remains partially coherent:

$$M_0 = \sqrt{1 - p_M} |0\rangle \langle 0| + \sqrt{p_M} |1\rangle \langle 1|$$

2. **Bit-Flip Channel Model:**

$$\rho' = (1 - p_M)\rho + p_M X \rho X$$

   Same measurement statistics, but projects directly to $|0\rangle$ or $|1\rangle$.

For practical systems, both models are equivalent since qubits are reinitialized immediately after measurement.

Qubit Loss

Definition: Disappearance of the physical information carrier.

Model:

$$\rho \to \mathrm{Tr}_i(\rho)$$

reduces the Hilbert space dimension.

Implications:

- Standard QEC assumes all qubits are accessible.
- Requires **non-demolition detection** to identify loss events.
- Lost qubits can be replaced, improving resilience.

Initialization Errors

Two Types:

1. **Incoherent:** Imperfect statistical mixture

$$\rho_i = (1 - p_I)\,|0\rangle\langle 0| + p_I\,|1\rangle\langle 1|$$

2. **Coherent:** Slightly rotated pure state

$$|\psi_i\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad |\beta|^2 \ll 1$$

Effect: Reduces state preparation fidelity and increases measurement errors.

Qubit Leakage

Leakage: Escape from the computational subspace $\{|0\rangle, |1\rangle\}$ into higher levels.

$$U|0\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$$

Consequences:

- Violates two-level assumption.
- Causes extra decoherence if $|2\rangle$ decays quickly.

Mitigation:

- Non-demolition verification of qubit state.
- Pulse refocusing to bring population back to logical subspace.
- Post-fabrication qubit screening.

3-Qubit Bit-Flip Code

3-Qubit Bit-Flip Code — Encoding

Logical codewords:

$$|0\rangle_L = |000\rangle , \qquad |1\rangle_L = |111\rangle .$$

Encode an arbitrary qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as

$$|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle .$$

This code is able to correct for any single bit-flip error.

Syndrome extraction & correction

Use two ancillas (parity checks). Syndrome table:

Syndrome	Correction
00	No error
01	X_3
10	X_2
11	X_1

Circuit: encode with two CNOTs, measure ancillas, apply conditional X.

Error suppression (coherent small rotations)

For $U = e^{i\epsilon\sigma_x}$ on each physical qubit:

$$E = U^{\otimes 3} = c_0\,\sigma_I\sigma_I\sigma_I + c_1\,(\sigma_x\sigma_I\sigma_I + \dots) + c_2(\dots) + c_3\,\sigma_x\sigma_x\sigma_x,$$

where

$$c_0 = \cos^3\epsilon, \quad c_1 = i\cos^2\epsilon\sin\epsilon, \quad c_2 = -\cos\epsilon\sin^2\epsilon, \quad c_3 = -i\sin^3\epsilon.$$

Unencoded fidelity: $F_{\text{unencoded}} = \cos^2\epsilon \approx 1 - \epsilon^2$.
After post-selecting the no-error syndrome, encoded fidelity:

$$F_{\text{encoded}} \approx 1 - \epsilon^6.$$

(Error reduced from $O(\epsilon^2)$ to $O(\epsilon^6)$.)

9-Qubit Shor Code

Shor 9-Qubit Code — Logical states

Combine phase-flip and bit-flip repetition codes.

$$|0\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}, \quad |1\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}.$$

Each block of 3 corrects an X error; inter-block checks correct Z errors.

How it protects against arbitrary single-qubit errors

An arbitrary single-qubit error can be expanded in Pauli basis. The combined bit-flip and phase-flip checks detect and correct single-qubit X, Z (and hence Y) errors. Coherent rotation generalization:

$$E = \bigotimes_{i=1}^{9} (\cos \epsilon \, \sigma_I + i \sin \epsilon \, \sigma_x),$$

followed by appropriate syndrome circuits increases logical fidelity compared to unencoded qubit.

Entanglement Purification (BBPSSW)

Werner states and depolarization

Any two-qubit state can be depolarized to a Werner form without changing fidelity with $|\phi_{00}\rangle$:

$$\rho_W(x) = x\,|\phi_{00}\rangle\langle\phi_{00}| + \frac{1-x}{4} I_4.$$

Fidelity relative to $|\phi_{00}\rangle$: $F = \langle\phi_{00}|\,\rho_W\,|\phi_{00}\rangle = \frac{3x+1}{4}$.

BBPSSW protocol (recurrence)

Starting with pairs of fidelity $F > 1/2$:

1. Depolarize to Werner form $\rho_W(F)$.
2. Take two copies; apply bilateral CNOT (A1$\rightarrow$A2 and B1$\rightarrow$B2).
3. Measure target qubits (A2,B2) in Z- and X-bases; keep control pair if measurement results match.

Fidelity update and success probability

After one successful purification step the surviving pair (still Werner) has fidelity

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2}{F^2 + 2F\left(\frac{1-F}{3}\right) + 5\left(\frac{1-F}{3}\right)^2},$$

and the success probability is

$$p_{\text{succ}} = F^2 + 2F\left(\frac{1-F}{3}\right) + 5\left(\frac{1-F}{3}\right)^2.$$

Iterating increases fidelity but reduces yield (consume 2 pairs to keep 1).

Implementation & Results

Qiskit simulations

- 3-qubit bit-flip and phase-error simulations: encoding, injecting error, syndrome extraction, correction, decode. Observed fidelity improvement consistent with theory (e.g. $1 - \epsilon^2 \to 1 - \epsilon^6$).
- 9-qubit Shor code simulation: injected single Z error, used 6 ancillas for syndrome extraction; decoded and observed restored logical state.

Code repository (included in report):
`https://github.com/jaymehta132/QuantumErrorCorrection-EE7001`

3-Bit Code for Bit-Flip (X) Error

Process:

1. Encode logical qubit using Hadamard + 2 CNOTs.
2. Inject bit-flip error (X gate) on one qubit.
3. Detect error using 2 ancilla qubits and 4 CNOTs.
4. Apply conditional correction based on syndrome.
5. Decode and measure final qubit.

Result: Measured state “001” with probability 1 — successful correction.

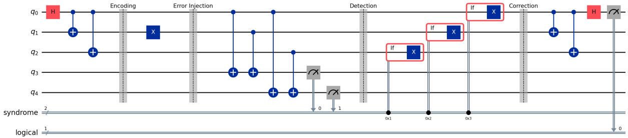


Figure: 3-bit Code Circuit for X Error

3-Bit Code: X Error Results

- Histogram confirms complete recovery of logical qubit.
- Fidelity maintained post-correction.
- Demonstrates robustness of simple repetition code for single X-error.

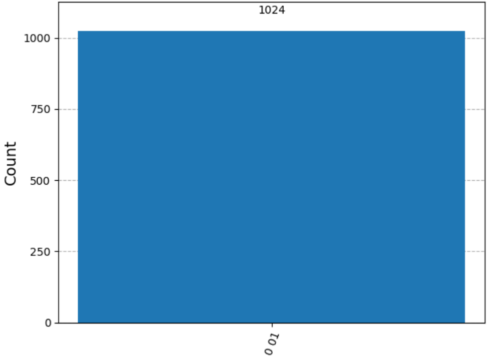


Figure: 3-bit Code Results for X Error

3-Bit Code for Phase Error

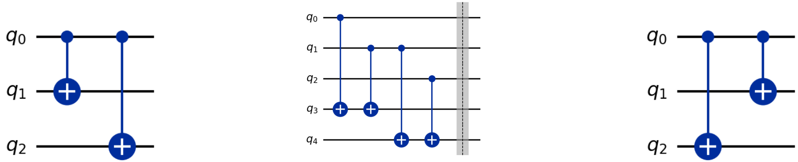
Goal: Verify suppression of coherent rotation errors $U = e^{i\epsilon\sigma_x}$.

Procedure:

- Encode qubit via CNOTs.
- Apply small X-rotation (ϵ) to all three data qubits.
- Measure syndrome using two ancillas.
- Decode and compute fidelity.

Observation:

$$F_{\text{unencoded}} = 1 - \epsilon^2 \quad \text{vs.} \quad F_{\text{encoded}} = 1 - \epsilon^6$$



(a) Encoding Circuit

(b) Syndrome Measurement Circuit

(c) Decoding Circuit

Figure: 3-bit Code Circuits for Phase error

9-Bit Shor Code for Z Error

Overview:

- Protects logical qubit against arbitrary single-qubit errors.
- Encodes $|+\rangle$ using layered 3x3 block structure:
  - Inner: Bit-flip code
  - Outer: Phase-flip code
- Inject phase-flip (Z) error on one qubit.
- Use 6 ancillas for syndrome extraction.

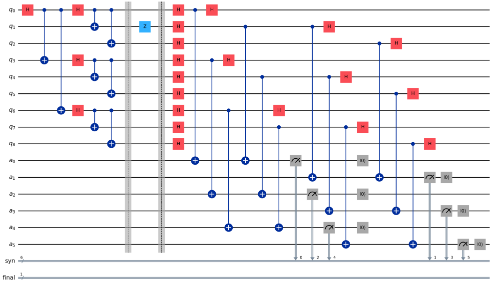


Figure: 9-bit Shor Code — Syndrome Measurement

9-Bit Shor Code Results

- Syndrome bits identify error location.
- Corrective Z operation restores logical state.
- Decoded qubit measured in X-basis gives deterministic "0".

Result: Successful recovery of logical qubit — validates Shor code's ability to correct any single-qubit error.

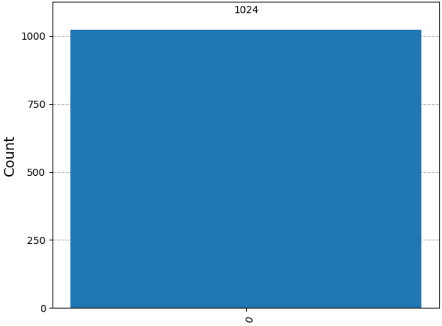


Figure: 9-bit Code Results for Z Error

Summary of Simulations

- **3-bit Code (X error):** Corrects single bit-flip errors with fidelity ≈ 1.
- **3-bit Code (Phase error):** Reduces error probability from $O(\epsilon^2)$ to $O(\epsilon^6)$.
- **9-bit Shor Code:** Corrects arbitrary single-qubit errors using nested redundancy.

Takeaway:

- Qiskit simulations validate theoretical predictions.
- Demonstrate the feasibility of small-scale QEC on near-term quantum hardware.

Conclusion

Conclusions & Outlook

- QEC and entanglement purification are foundational for fault-tolerant quantum computing and long-distance quantum communication.
- Simple codes (3- and 9-qubit) illustrate suppression of coherent errors and correction of arbitrary single-qubit errors.
- Real devices: gate errors, measurement noise, and leakage limit achievable fidelities; fault-tolerance requires concatenation or more advanced topological/subsystem codes and careful experimental engineering.

References I

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