gibbs

November 3, 2018

Question 1 Algorithm:

We have used a Gibbs sampler to estimate the marginal distribution generated from the conditional distributions:

$p(x|y) \propto ye^{-yx}, 0 < x < B < \infty$

$p(y|x) \propto xe^{-yx}, 0 < y < B < \infty$, where B is a known positive constant.

First, we have used initial values of x and y to be random numbers generated from Uniform[0, B]. Then, we have created a T x 3 matrix called *chain* filled with zeros, where T = number of iterations.

Since the marginal conditional density function of $x|y$, $p(x|y) \propto ye^{-yx}$ and the range of x is from 0 to B, the cumulative density function of $x|y$, $F(x|y) = \int_0^x ye^{-yx}\, dx$.

Note that the conditional density function of $x|y$ follows the exponential distribution. Then, $F(x|y) = \int_0^x ye^{-yx}\, dx = 1 - e^{-yx}$.

Since the ranges of both x and y are from 0 to B, and it is not to infinity, the cumulative functions of both x and y are less than 1. Therefore, we need to rescale this by dividing $\int_0^B xe^{-yx}\, dx$ and $\int_0^B xe^{-yx}\, dy$ respectively. For Inverse Transform Sampling, we have first generated u from Uniform[0, 1] and let $u = \frac{\int_0^x xe^{-yx}\, dx}{\int_o^B xe^{-yx}\, dx} = \frac{1-e^{-yx}}{1-e^{-yB}}$. Solving for x, we have $x = -\frac{1}{y} ln(1 - u(1 - e^{-yB}))$.

Similarly, we have generated v from Uniform[0, 1] and used Inverse Transform Sampling. Let $v = \frac{\int_0^y xe^{-yx}\, dy}{\int_o^B xe^{-yx}\, dy} = \frac{1-e^{-yx}}{1-e^{-xB}}$. Solving for y, we have $y = -\frac{l}{x} ln(1 - v(1 - e^{-xB}))$.

The first column of *chain* will represent the iteration number. From each iteration of the sampler, we have stored value of x into the second column of *chain* and value of y into the third column of *chain*.

Finally, this function returns the matrix *chain* with stored x and y values.

Please note that we have not included thinning process in this function.

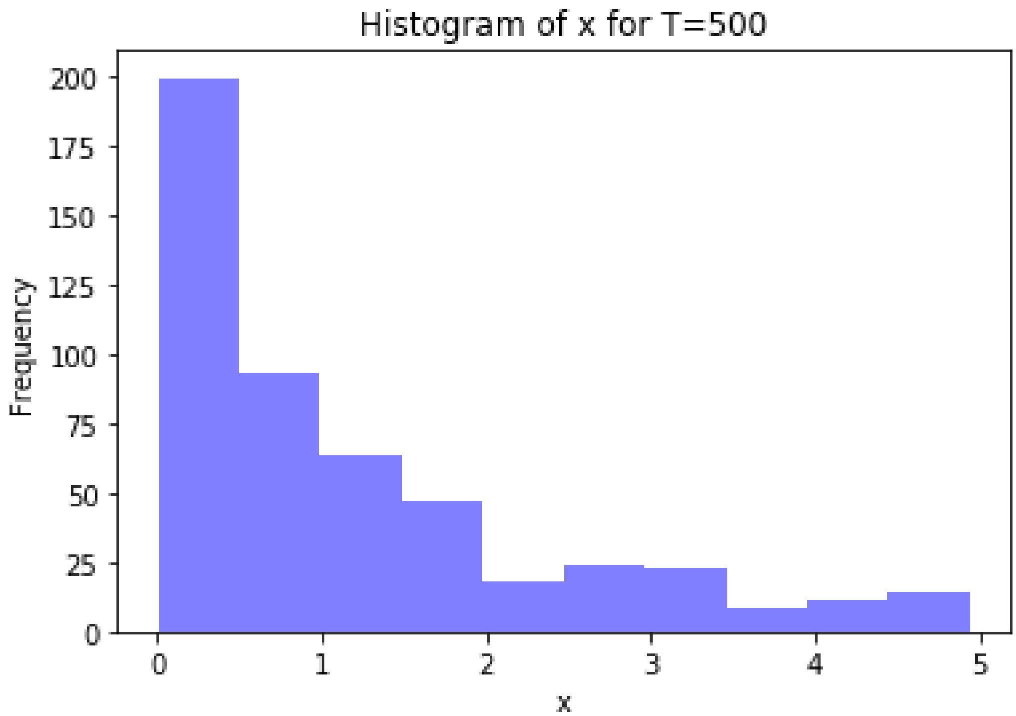
The algorithm is as follows:

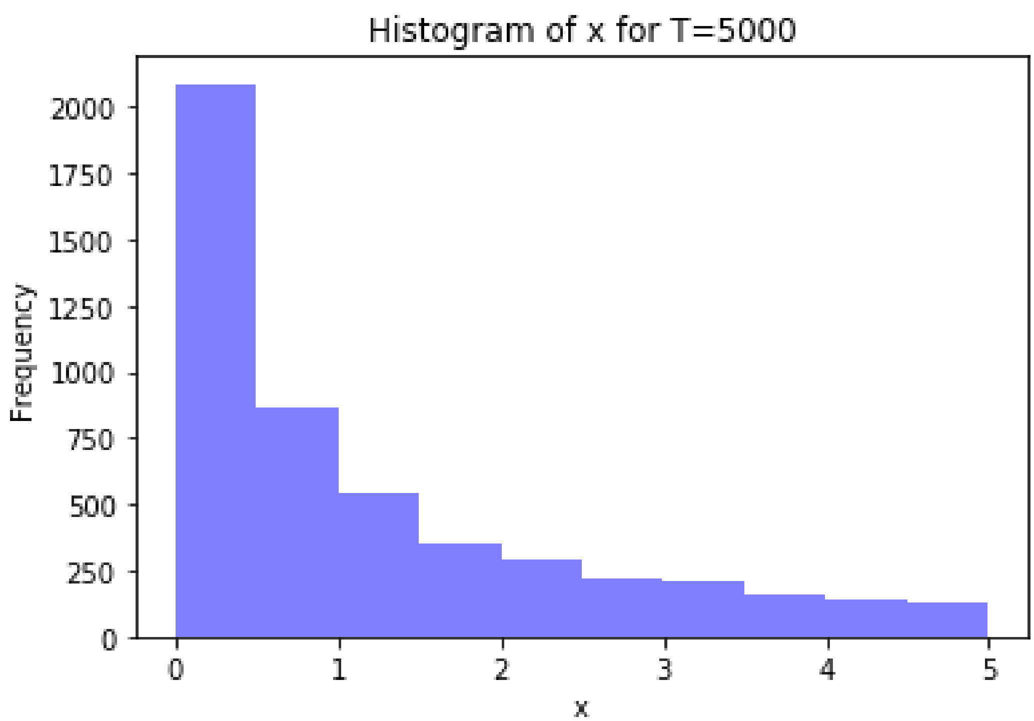
```
In [1]: %matplotlib inline
        import random, math
        from math import *
        from random import *
        import numpy as np
        from numpy import *
        import matplotlib.pyplot as plt
        from matplotlib.pyplot import *

        def gibbs(B, T):
```

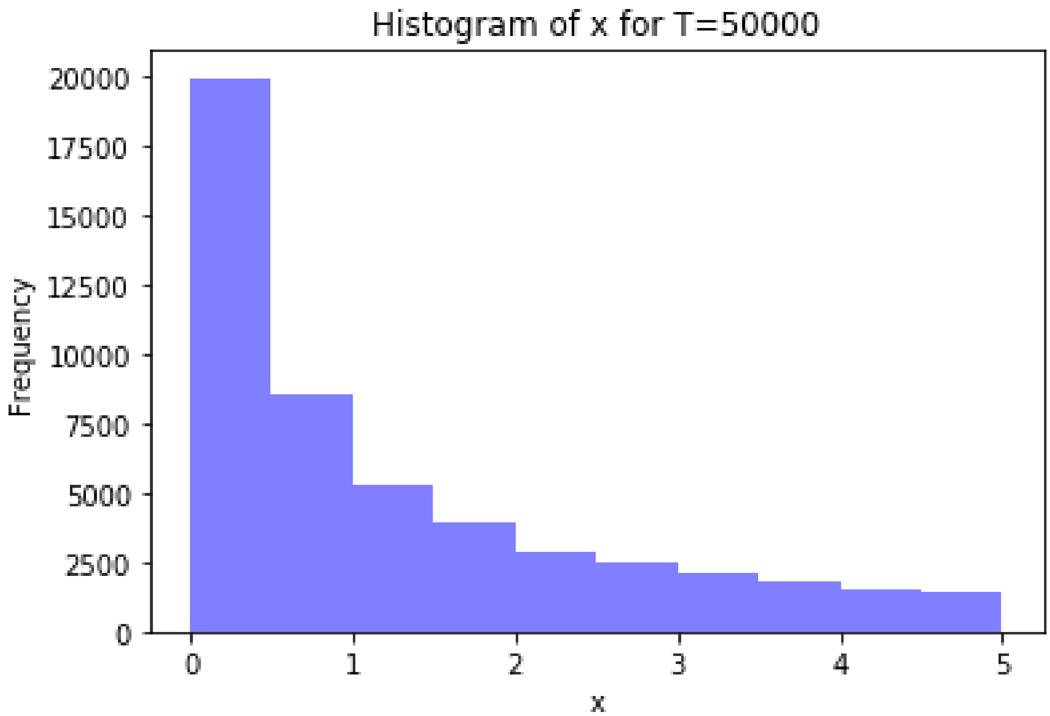
```
    x = float(np.random.uniform(0, B, 1))
    y = float(np.random.uniform(0, B, 1))
    chain = np.zeros((T, 3))
    chain[:, 0] = range(1, T + 1)
    for i in range(T):
        u = float(np.random.uniform(0, 1, 1))
        v = float(np.random.uniform(0, 1, 1))
        x = -np.log(1 - u * (1 - np.exp(-y * B))) / y
        y = -np.log(1 - v * (1 - np.exp(-x * B))) / x
        chain[i, 1] = x
        chain[i, 2] = y
    return(chain)
```

Question 2 Histograms of values for x, for B = 5 and for sample sizes T = 500, 5000, 50000 are as follows:

```
In [2]: G1 = gibbs(B = 5, T = 500)
        G2 = gibbs(B = 5, T = 5000)
        G3 = gibbs(B = 5, T = 50000)

        xx = G1[:,1]
        x2 = G2[:,1]
        x3 = G3[:,1]

        n,bins,patches = plt.hist(xx, facecolor='blue', alpha=0.5)
        plt.xlabel('x')
        plt.ylabel('Frequency')
        plt.title(r'Histogram of x for T=500')
        plt.show()

        n,bins,patches = plt.hist(x2, facecolor='blue', alpha=0.5)
        plt.xlabel('x')
        plt.ylabel('Frequency')
        plt.title(r'Histogram of x for T=5000')
        plt.show()

        n,bins,patches = plt.hist(x3, facecolor='blue', alpha=0.5)
        plt.xlabel('x')
        plt.ylabel('Frequency')
        plt.title(r'Histogram of x for T=50000')
        plt.show()
```


Histogram of x for T=500
Frequency
200
175
150
125
100
75
50
25
0
0
1
2
3
4
5
x


Histogram of x for T=5000
Frequency
2000
1750
1500
1250
1000
750
500
250
0
0
1
2
3
4
5
x



Note that the histograms of x follows exponential distribution. As we increase the number of iteration (T), we get thinner tails.

Question 3 An estimate of the expectation of X, $\mathbb{E}_{p(X)}[X]$, by using the 500, 5000, and 50000 samples from the sampler is as follows:

```
In [4]: print("Estimate of the expectation for T = 500:", round(mean(xx), 4))
        print("Estimate of the expectation for T = 5000:", round(mean(x2), 4))
        print("Estimate of the expectation for T = 50000:", round(mean(x3), 4))
```

```
Estimate of the expectation for T = 500: 1.1659
Estimate of the expectation for T = 5000: 1.2091
Estimate of the expectation for T = 50000: 1.2772
```