

Metropolis-Hastings

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Our target distribution is $\text{Beta}(6,4)$ and our proposal distribution is $\phi_{prop}|\phi_{old} \sim \text{Beta}(c\phi_{old}, c(1-\phi_{old}))$. First, we implement several functions which will be used in calculating posterior distribution. Here, we define likelihood function which gives the log-likelihood value of the parameter, and also prior function that gives the log density value of the parameter. The density used in the likelihood function is $\text{Beta}(6,4)$ since it is our target distribution, and the density used in the prior function is $\text{Uniform}(0,1)$ since our start value is from $\text{Uniform}(0,1)$. In addition, since the parameters in Beta distribution have to be larger than 0, for $c>0$, $\phi < 1$.

```
a <- 6
b <- 4

likelihood <- function(param){
  singlelikelihoods = dbeta(param, a, b, log = T)
}

prior <- function(param){
  pr = dunif(param, min=0, max=1, log = T)
  return(pr)
}

posterior <- function(param){
  return (exp(likelihood(param) + prior(param)))
}
```

Posterior function can be calculated by multiplying likelihood function and prior function. However, we used log in the likelihood function and the prior function, so we need to convert them by putting *exp* in the posterior function.

```
metropolis_MCMC <- function(startvalue, iterations, c){
  chain = array(dim = c(iterations+1,1))
  chain[1,] = startvalue
  for (i in 1:iterations){
    proposal = rbeta(1, c*chain[i,], c*(1-chain[i,]))
    probab = (posterior(proposal)*dbeta(chain[i,],c*proposal,c*(1-proposal)))/
      (posterior(chain[i,])*dbeta(proposal, c*chain[i,], c*(1-chain[i,])))
    if (runif(1) < probab){
      chain[i+1,] = proposal
    }else{
      chain[i+1,] = chain[i,]
    }
  }
  return(chain)
}
```

After implementing all the functions that are required to obtain a chain, then we can finally define metropolis_MCMC function that can yield a chain. Note that

$$\phi_{new} = \phi_{proposal} \text{ with probability } \alpha(\phi_{old}, \phi_{new})$$

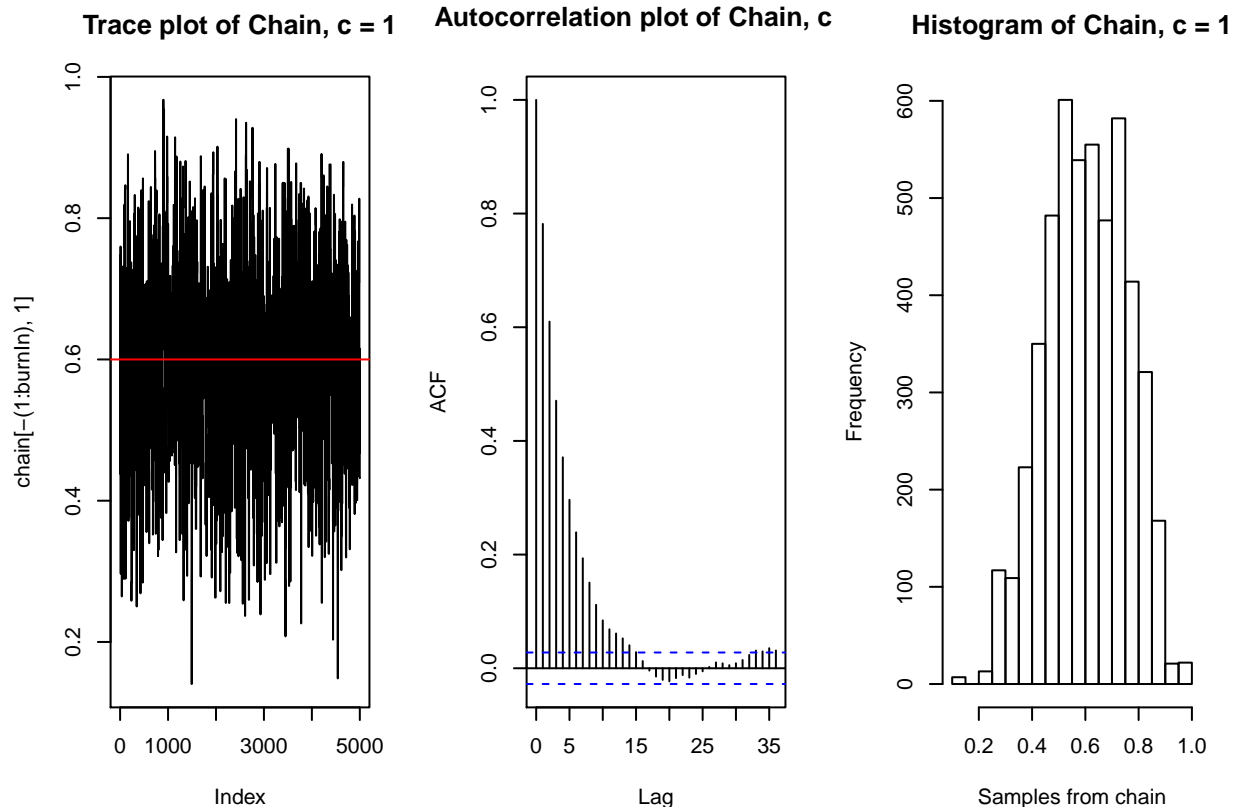
$$= \phi_{old} \text{ with probability } 1 - \alpha(\phi_{old}, \phi_{new})$$

where $\alpha(\phi_{old}, \phi_{new}) = \min(1, \frac{p(\phi_{new})q(\phi_{old}|\phi_{new})}{p(\phi_{old})q(\phi_{new}|\phi_{old})})$. Here, $p(\phi)$ is the posterior function, and $q(\phi_{new}|\phi_{old})$ is our proposal functions that follows $Beta(c\phi_{old}, c(1 - \phi_{old}))$. Since Beta distribution is asymmetric, the probability form cannot be simplified to the ratio of posterior functions. Hence, we need to calculate the probability by using posterior functions and proposal functions.

Next, we can finally get the chain using metropolis_MCMC function with the startvalue generated from Uniform(0,1). We also generate random values from Beta(6,4) for the comparison.

```
startvalue <- runif(1, 0, 1)
chain=metropolis_MCMC(startvalue, 10000, c=1)
test <- rbeta(10000, 6, 4)
```

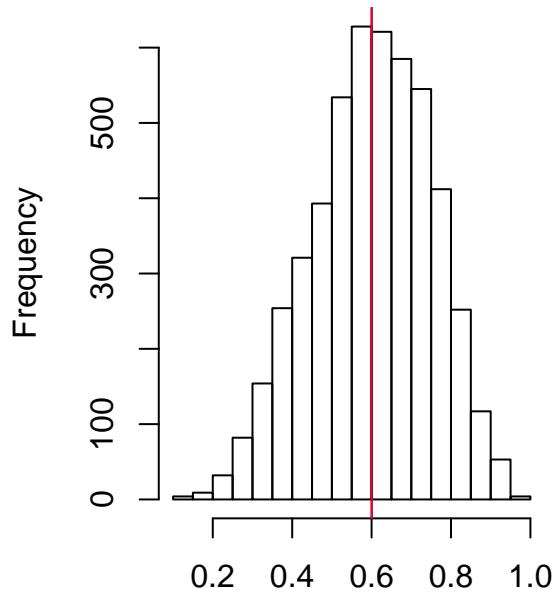
```
# Performance of the sampler: c = 1 with BurnIn
par(mfrow=c(1,3))
test2 <- test[5001:10000]
burnIn = 5000
acceptance = 1-mean(duplicated(chain[-(1:burnIn),]))
plot(chain[-(1:burnIn),1], type='l', main = "Trace plot of Chain, c = 1")
abline(h=0.6, col="red")
acf(chain[-(1:burnIn),1], main = "Autocorrelation plot of Chain, c = 1")
hist(chain[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 1")
```



```
# Histogram of the target distribution Beta(6,4) vs. the Chain with burnIn
par(mfrow=c(1,2))
hist(test2, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
abline(v=mean(test2), col="blue")
abline(v=0.6, col="red")
hist(chain[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 1")
```

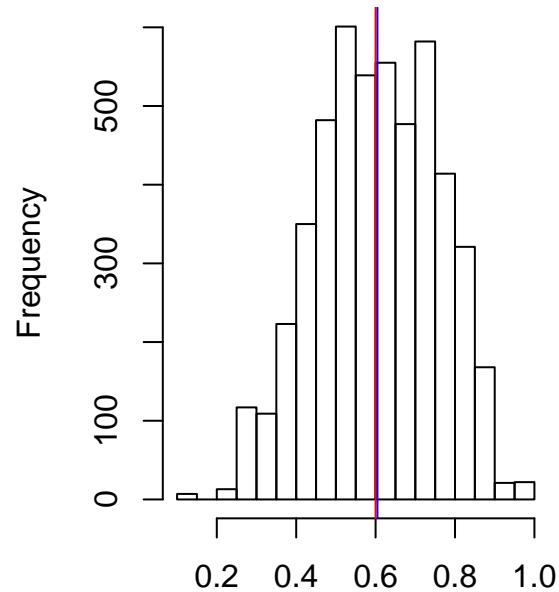
```
abline(v = mean(chain[-(1:burnIn),1]), col="blue")
abline(v=0.6, col="red")
```

Histogram of Beta(6, 4)



Samples from Beta(6, 4)

Histogram of Chain, c = 1



Samples from chain

```
# Kolmogorov-Smirnov statistic
```

```
ks.test(chain[-(1:burnIn),1], test2)
```

```
## Warning in ks.test(chain[-(1:burnIn), 1], test2): p-value will be
```

```
## approximate in the presence of ties
```

```
##
```

```
## Two-sample Kolmogorov-Smirnov test
```

```
##
```

```
## data: chain[-(1:burnIn), 1] and test2
```

```
## D = 0.03813, p-value = 0.001392
```

```
## alternative hypothesis: two-sided
```

Here, we evaluate the performance of the sampler when $c = 1$ without burn in. The first plot is the trace plot, showing the trace of our chain. We can check that the trace plot does not have any trends. /# The auto correlation

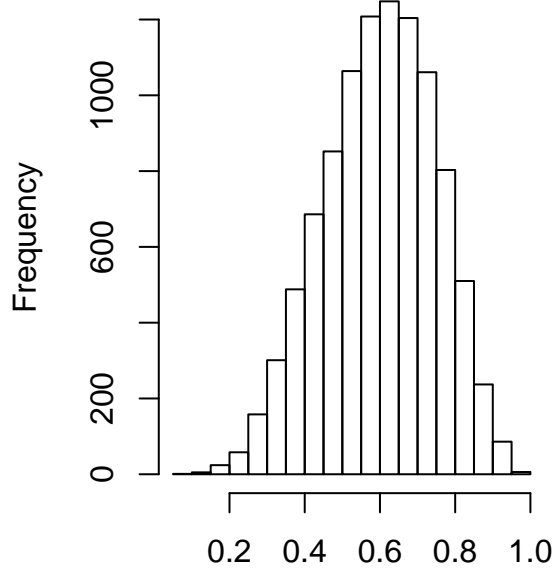
```
# Histogram of the chain vs. the target distribution Beta(6,4) without BurnIn
```

```
par(mfrow=c(1,2))
```

```
hist(test, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
```

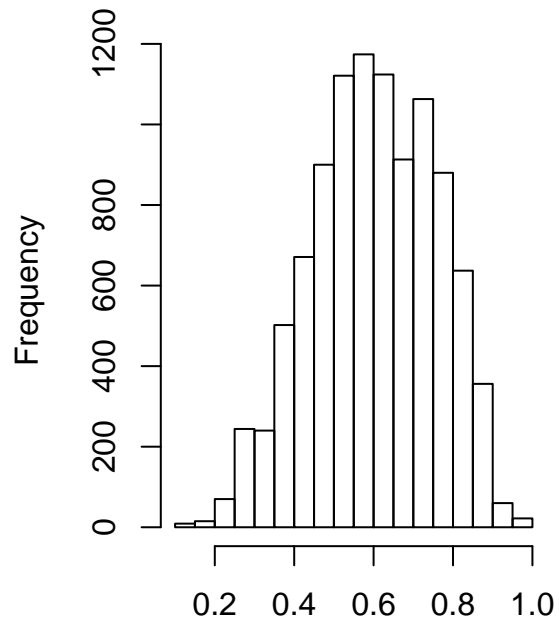
```
hist(chain, xlab = "Samples from Chain", main = "Histogram of Chain, c = 1")
```

Histogram of Beta(6, 4)



Samples from Beta(6, 4)

Histogram of Chain, c = 1

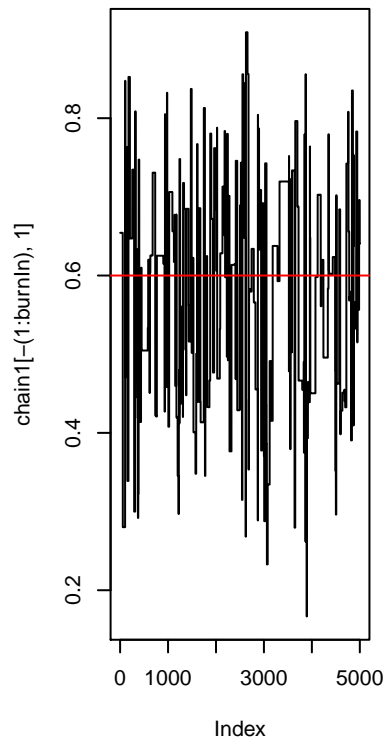


Samples from Chain

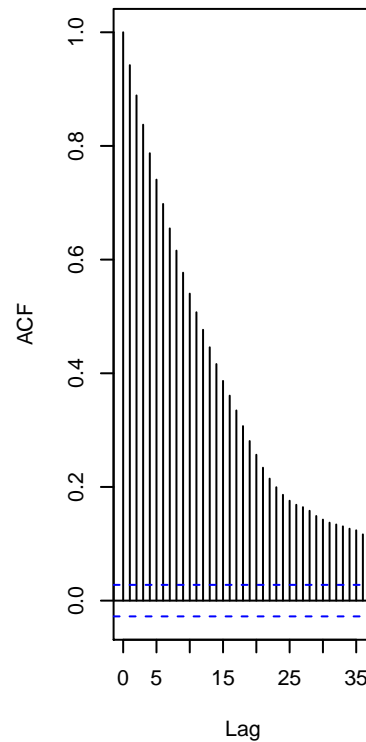
```
# c = 0.1 / c = 2.5 / c = 10
chain1=metropolis_MCMC(startvalue, 10000, c=0.1)
chain2=metropolis_MCMC(startvalue, 10000, c=2.5)
chain3=metropolis_MCMC(startvalue, 10000, c=10)

# Performance of the sampler: c = 0.1
par(mfrow=c(1,3))
acceptance1 = 1-mean(duplicated(chain1[-(1:burnIn),]))
plot(chain1[-(1:burnIn),1], type='l', main = "Trace plot of Chain, c = 0.1")
abline(h=0.6, col="red")
acf(chain1[-(1:burnIn),1], main = "Autocorrelation plot of Chain, c = 0.1")
hist(chain1[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 0.1")
```

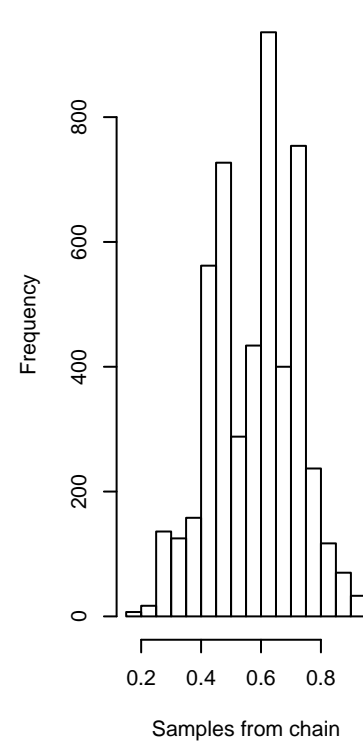
Trace plot of Chain, c = 0.1



Autocorrelation plot of Chain, c =

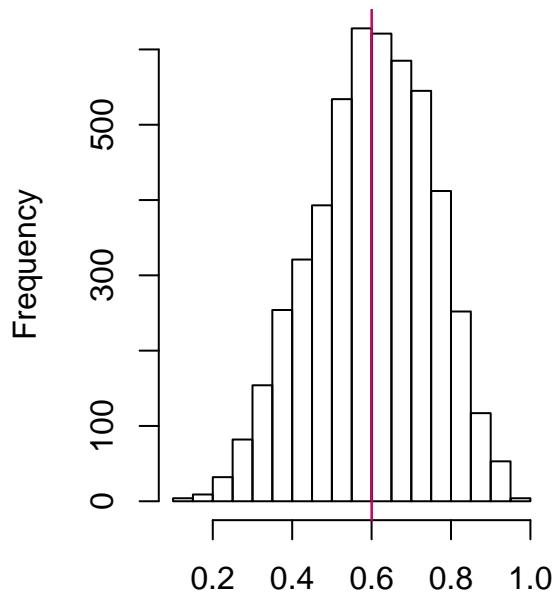


Histogram of Chain, c = 0.1



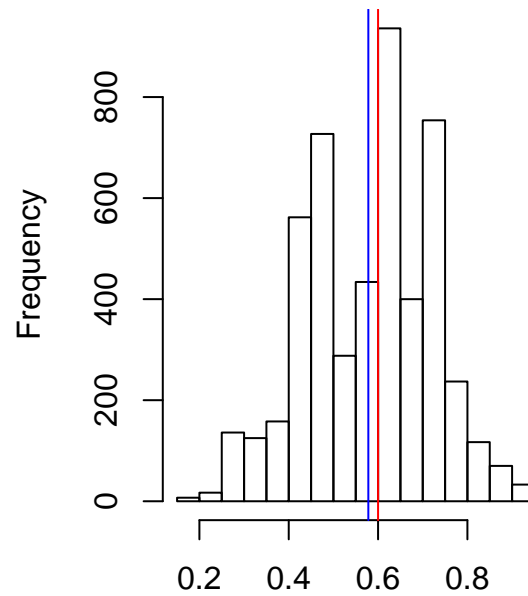
```
# Histogram of the target distribution Beta(6,4) vs. the Chain after burnIn
par(mfrow=c(1,2))
hist(test2, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
abline(v=mean(test2), col="blue")
abline(v=0.6, col="red")
hist(chain1[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 0.1")
abline(v = mean(chain1[-(1:burnIn),1]), col="blue")
abline(v=0.6, col="red")
```

Histogram of Beta(6, 4)



Samples from Beta(6, 4)

Histogram of Chain, c = 0.1



Samples from chain

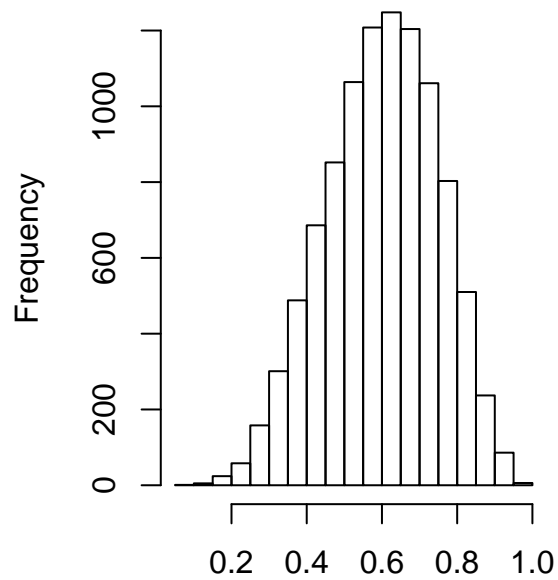
```
# Kolmogorov-Smirnov statistic
ks.test(chain1[-(1:burnIn),1], test2)

## Warning in ks.test(chain1[-(1:burnIn), 1], test2): p-value will be
## approximate in the presence of ties

##
## Two-sample Kolmogorov-Smirnov test
##
## data: chain1[-(1:burnIn), 1] and test2
## D = 0.11892, p-value < 2.2e-16
## alternative hypothesis: two-sided

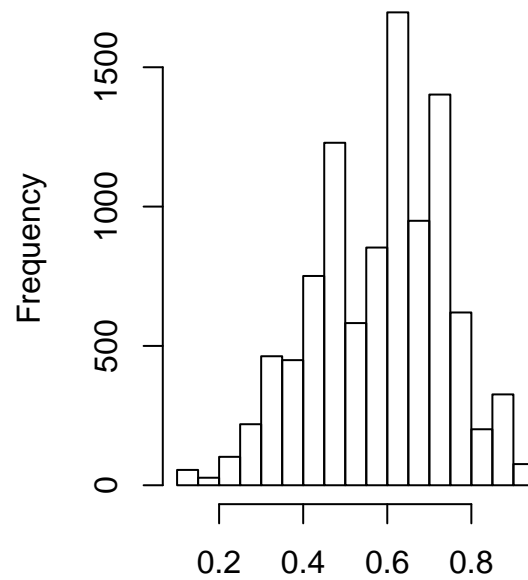
# Comparing the histogram of the chain with the target distribution Beta(6,4)
par(mfrow=c(1,2))
hist(test, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
hist(chain1, xlab = "Samples from chain", main = "Histogram of Chain, c = 0.1")
```

Histogram of Beta(6, 4)



Samples from Beta(6, 4)

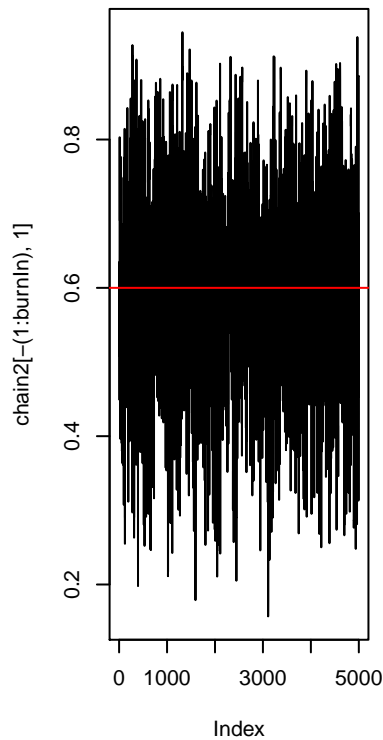
Histogram of Chain, c = 0.1



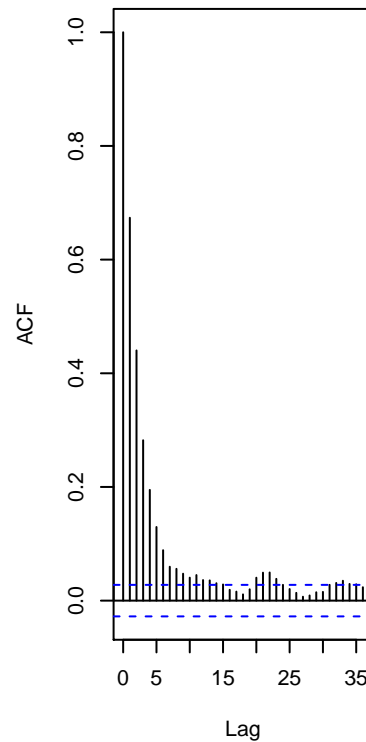
Samples from chain

```
# Performance of the sampler: c = 2.5
par(mfrow=c(1,3))
acceptance2 = 1-mean(duplicated(chain2[-(1:burnIn),]))
plot(chain2[-(1:burnIn),1], type='l', main = "Trace plot of Chain, c = 2.5")
abline(h=0.6, col="red")
acf(chain2[-(1:burnIn),1], main = "Autocorrelation plot of Chain, c = 2.5")
hist(chain2[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 2.5")
```

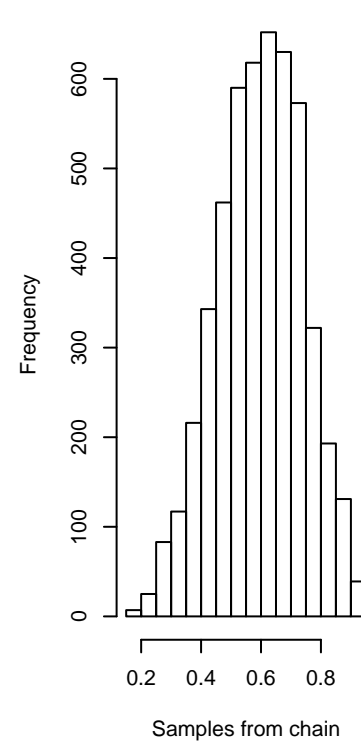
Trace plot of Chain, c = 2.5



Autocorrelation plot of Chain, c =



Histogram of Chain, c = 2.5



```
# Histogram of the target distribution Beta(6,4) vs. the Chain after burnIn
hist(test2, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
abline(v=mean(test2), col="blue")
abline(v=0.6, col="red")
hist(chain2[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 2.5")
abline(v = mean(chain2[-(1:burnIn),1]), col="blue")
abline(v=0.6, col="red")

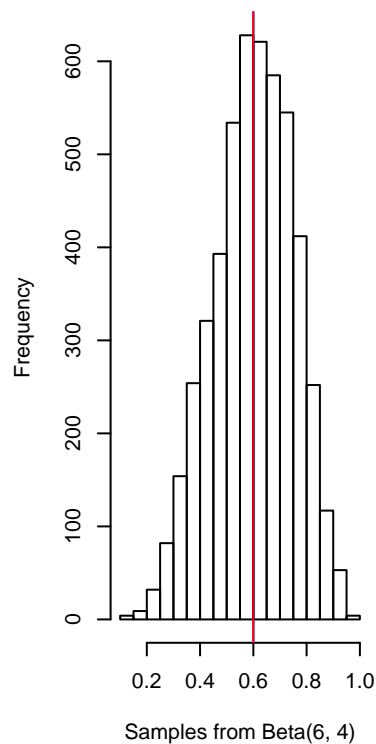
# Kolmogorov-Smirnov statistic
ks.test(chain2[-(1:burnIn),1], test2)

## Warning in ks.test(chain2[-(1:burnIn), 1], test2): p-value will be
## approximate in the presence of ties

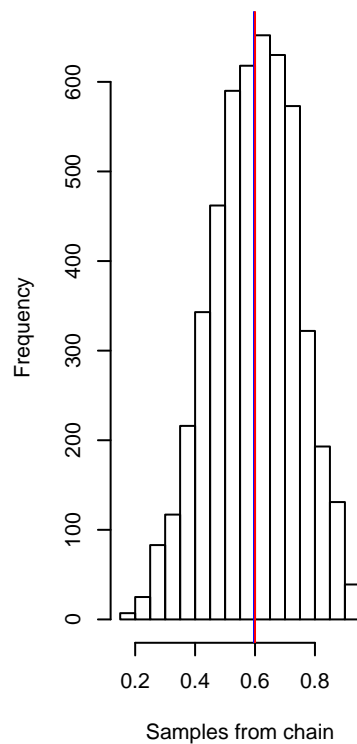
##
## Two-sample Kolmogorov-Smirnov test
##
## data: chain2[-(1:burnIn), 1] and test2
## D = 0.039036, p-value = 0.0009814
## alternative hypothesis: two-sided

# Comparing the histogram of the chain with the target distribution Beta(6,4)
par(mfrow=c(1,2))
```


Histogram of Beta(6, 4)

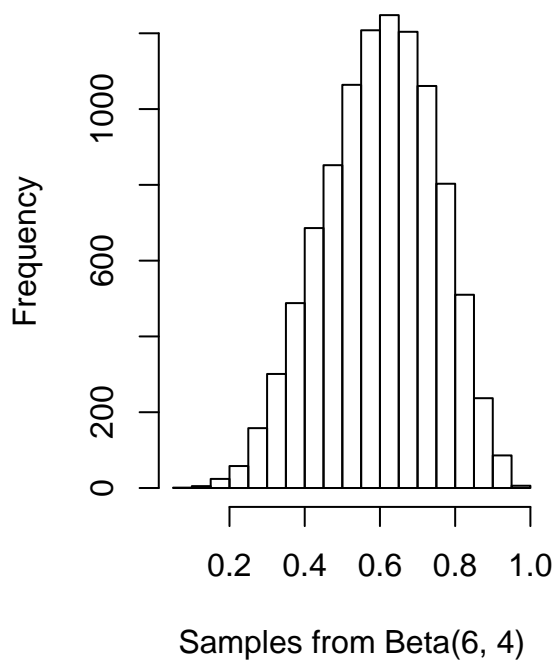


Histogram of Chain, c = 2.5

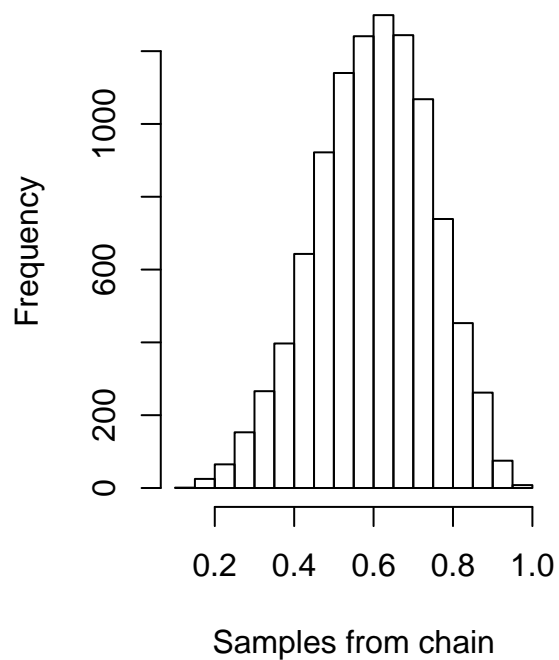


```
hist(test, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")  
hist(chain2, xlab = "Samples from chain", main = "Histogram of Chain, c = 2.5")
```

Histogram of Beta(6, 4)

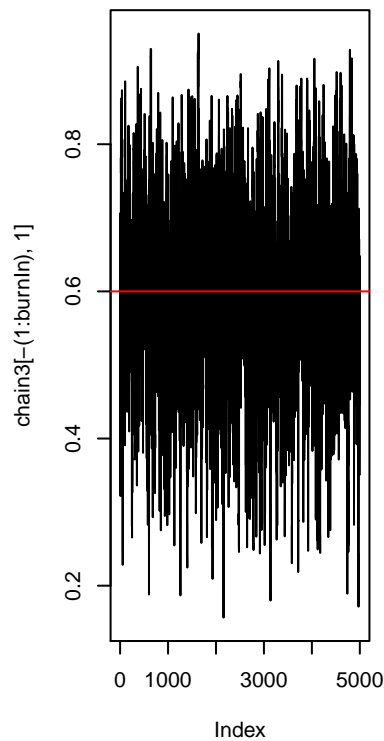


Histogram of Chain, c = 2.5

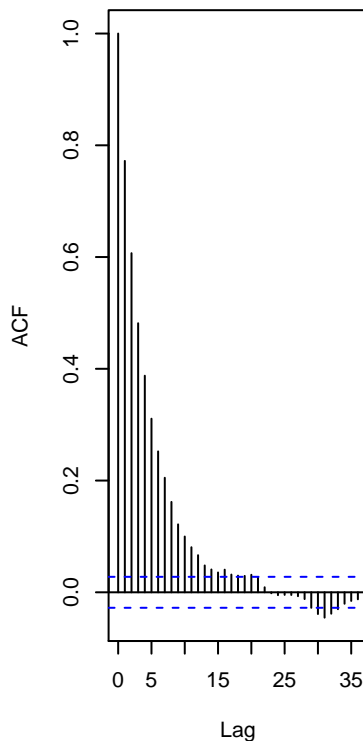


```
# Performance of the sampler: c = 10
par(mfrow=c(1,3))
acceptance3 = 1-mean(duplicated(chain3[-(1:burnIn),]))
plot(chain3[-(1:burnIn),1], type='l', main = "Trace plot of Chain, c = 10")
abline(h=0.6, col="red")
acf(chain3[-(1:burnIn),1], main = "Autocorrelation plot of Chain, c = 10")
hist(chain3[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 10")
```

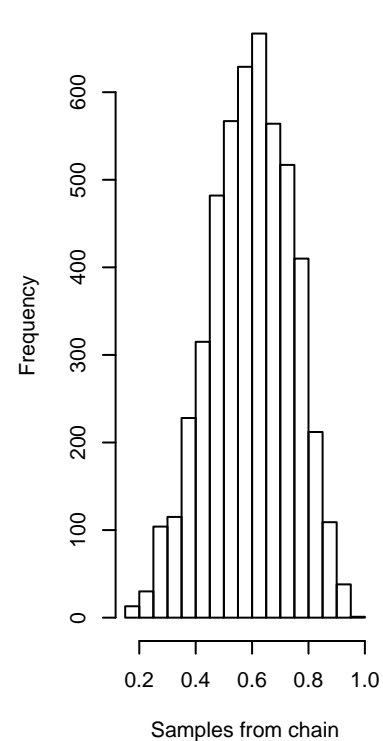
Trace plot of Chain, c = 10



Autocorrelation plot of Chain, c =



Histogram of Chain, c = 10



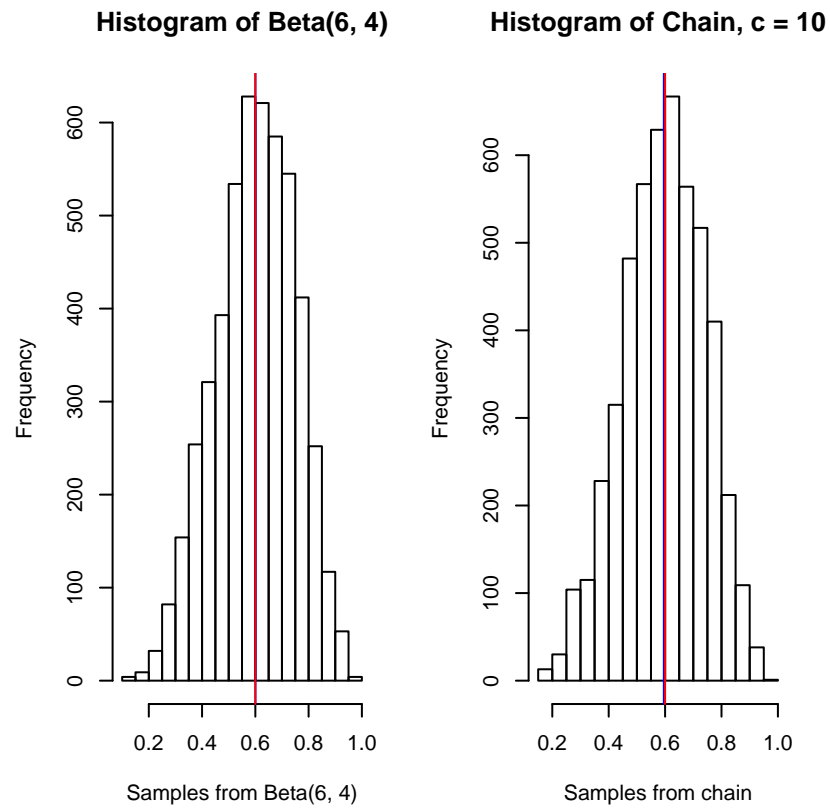
```
# Histogram of the target distribution Beta(6,4) vs. the Chain after burnIn
hist(test2, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
abline(v=mean(test2), col="blue")
abline(v=0.6, col="red")
hist(chain3[-(1:burnIn),1], xlab = "Samples from chain", main = "Histogram of Chain, c = 10")
abline(v = mean(chain3[-(1:burnIn),1]), col="blue")
abline(v=0.6, col="red")
```

```
# Kolmogorov-Smirnov statistic
ks.test(chain3[-(1:burnIn),1], test2)
```

```
## Warning in ks.test(chain3[-(1:burnIn), 1], test2): p-value will be
## approximate in the presence of ties
```

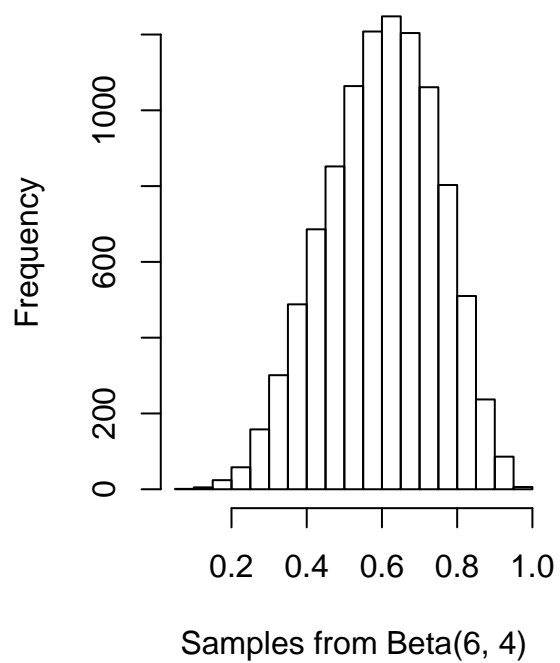
```
##
## Two-sample Kolmogorov-Smirnov test
##
## data: chain3[-(1:burnIn), 1] and test2
## D = 0.028686, p-value = 0.03265
## alternative hypothesis: two-sided
```

```
# Comparing the histogram of the chain with the target distribution Beta(6,4)
par(mfrow=c(1,2))
```



```
hist(test, xlab = "Samples from Beta(6, 4)", main = "Histogram of Beta(6, 4)")
hist(chain3, xlab = "Samples from chain", main = "Histogram of Chain, c = 10")
```

Histogram of Beta(6, 4)



Histogram of Chain, c = 10

