Extremal Black Hole Simulations: 3+1 Formalism

Jay Motka

Photo Credits: NASA

Outline

- Motivation
- Literature: 3+1 Formalism & Near Kerr-limit BH Simulation
- My Insight: Near-extremal Charged BH Simulation
- Summary

Weak Cosmic Censorship Conjecture (WCCC):

- Kerr metric event horizons:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

- Reissner-Nordström (RN) metric event horizons:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

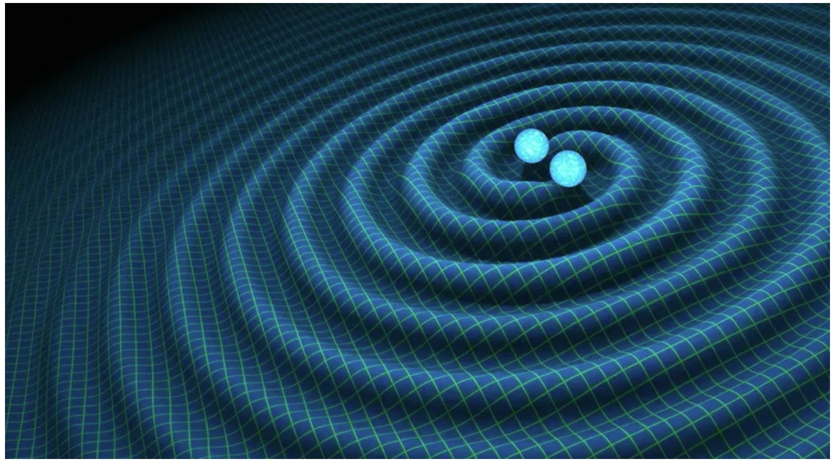
- WCCC:

"A spacetime singularity is always hidden inside of a black hole's event horizon."

Motivation: 

- We can test it using simulations!

- One scenario: Merger of two near-extremal charged black holes



Credits: space.com

Motivation

- Furthermore, near-extremal BH are interesting cases of physics in themselves.
- Near-extremal charged BH has high-gravity & high-electromagnetic regime!
- Simulations could be used as a background to test interesting physics.

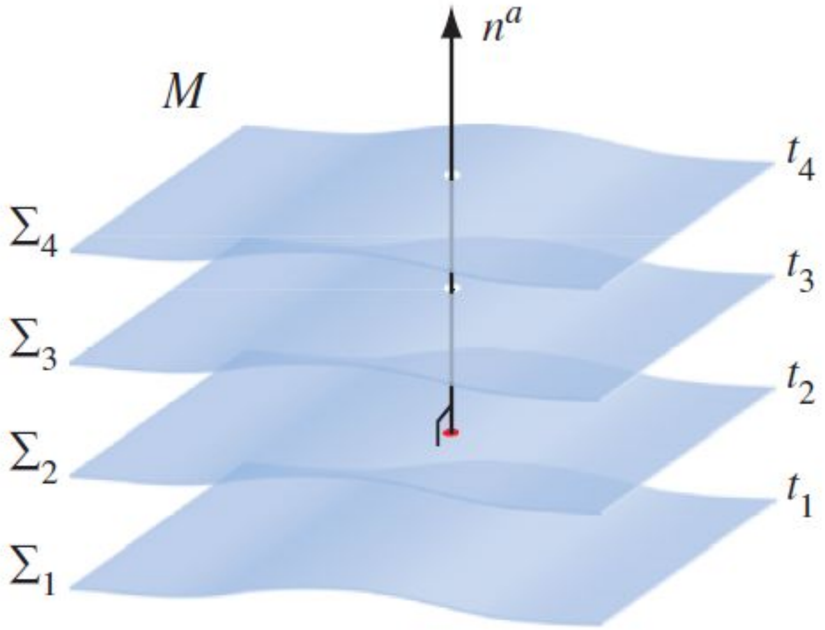
3+1 Formalism (Baumgarte & Shapiro, 2010)

- Numerical relativity is traditional initial condition (Cauchy) problem!
- Einstein equation: 2nd order PDEs
- Initial Conditions: $g_{\mu\nu}$ & $\partial_t g_{\mu\nu}$

- Formulating Cauchy problem in GR:

3-spatial dimension and 1-temporal dimension can be separated

Foliations of Spacetime



Credits: Baumgarte & Shapiro, 2010

- M: 4D spacetime manifold
- Σ_i: 3D spatial slices (hypersurfaces) at time t_i, level surfaces of scalar function t
- n^a: Normalized, timelike unit normal

Initial Conditions

- We can compute 3D spatial Riemann tensor using “3-metric”:

$$\boxed{\gamma_{ab} = g_{ab} + n_a n_b.}$$

1st Initial Condition

- But that 3-Riemann tensor does not tell us about all the information contained in 4-Reimann tensor.
- It tells us only intrinsic curvature to the slice Σ.
- Extrinsic curvature (projection of gradients of normal vector to Σ):

$$\boxed{K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_{(c} n_{d)} = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d.}$$

(Related to $\partial_t \gamma_{ab}$)

2nd Initial Condition

Constraint & Evolution Equations

- Hamiltonian Constraint:

$$\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi\rho,}$$

$\rho \equiv n_a n_b T^{ab}$
(energy density)

- Momentum Constraint:

$$\boxed{D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i,}$$

$S_a \equiv -\gamma^b_{\ a} n^c T_{bc}$
(momentum density)

- Evolution Equations:

$$\boxed{\begin{aligned}\partial_t K_{ij} = &-D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij}) - 8\pi\alpha(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)) \\ &+ \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k,\end{aligned}}$$

$$\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i.}$$

Simulation of Near Kerr-limit BH (Liu et al., 2019)

- Liu et al. (2019) simulated spinning BH near the Kerr-limit:

$$a/M \to 1$$

- They used 3+1 formalism & recipe called moving puncture method.
- They used a correction to the traditional quasi-isotropic coordinates to avoid simulation errors.

- Results: $\boxed{a/M = 0.99}$

My thoughts: Simulation of Near-extremal Charged BH?

- Traditionally used quasi-isotropic coordinates:

$$r = r_+ cosh^2(\eta/2) - r_- sinh^2(\eta/2)$$

$$r_{qis} = \frac{\sqrt{M^2 - Q^2}}{2} e^{\eta}$$

- At extremal limit, these quasi-isotropic coordinates vanishes to 0.
- This could lead to simulation errors since it is extremally hard to resolve near zero scale.
- Same problem occurred in Liu et al. (2019).

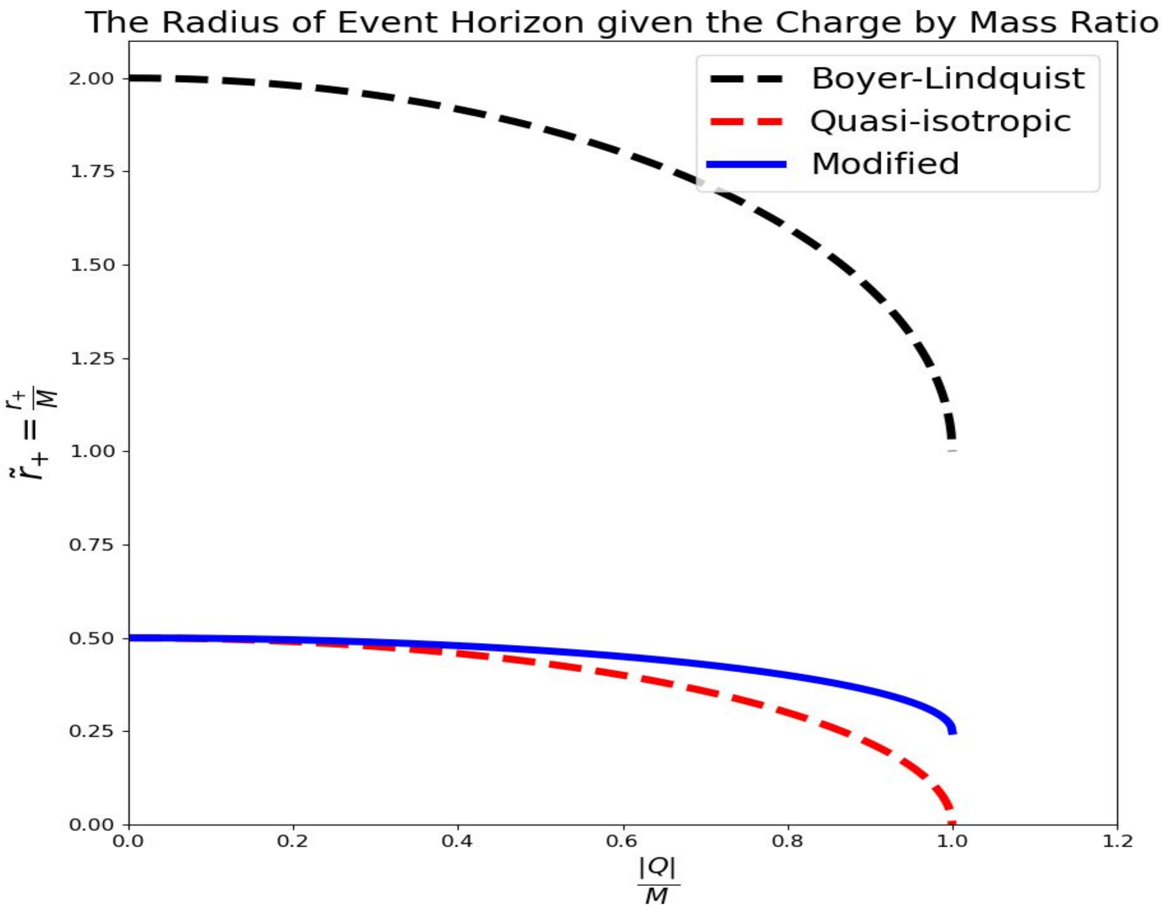
Tweaking the coordinates

- They resolved it by multiplying a correction term to the quasi-isotropic coordinates.
- We can take an analogous term and do the same!

- Multiply $r_{qis} = \frac{\sqrt{M^2 - Q^2}}{2} e^{\eta}$ with:

$$\lambda = \frac{e^{-\eta}}{\sqrt{M^2 - Q^2}} \left[r - \frac{r_+}{2} + \sqrt{r(r_+ - r_-)} sinh(\eta/2) \right]$$

Modified Event Horizon Radial Coordinates



Initial Conditions

- Spatial metric:

$$\gamma_{11} = \frac{1}{\Delta'}\left(1 - \frac{r_+^2}{16R^2}\right)^2,$$
$$\gamma_{22} = R^2\left(1 + \frac{r_+}{4R}\right)^4,$$
$$\gamma_{33} = R^2\left(1 + \frac{r_+}{4R}\right)^4 sin^2(\theta),$$
$$\gamma_{ij} = 0\,;\, i \neq j.$$

- Extrinsic Curvature:

$$K_{ij} = 0.$$

- They satisfy both the Hamiltonian and the momentum constraint.

Summary

- 3+1 formalism is an excellent and easier way for numerical simulation of spacetime.
- We found the initial conditions satisfying both constraints using the modified coordinates (to avoid inaccuracies).
- By applying the evolution equations, we can, in future, simulate the near-extremal charged black holes!

Thank you!

Back up slides ahead…

Charged BH

- Reissner-Nordström (RN) metric:

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

$$\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

- RN metric event horizons:

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

- Extremal limit:

$$|Q| > M$$

Unit Normal Vector

- We can construct a1-form on Σ: $\Omega_a = \nabla_a t$
- Its norm is: $\|\Omega^2\| = -\frac{1}{\alpha^2}$
- α is called “lapse function”.
- It measures how much proper time elapses between neighboring time slices along the normal vector Ω^a.

$$n^a = -g^{ab}\alpha\Omega_b$$

Projection Tensors

- Projection tensor along unit normal:

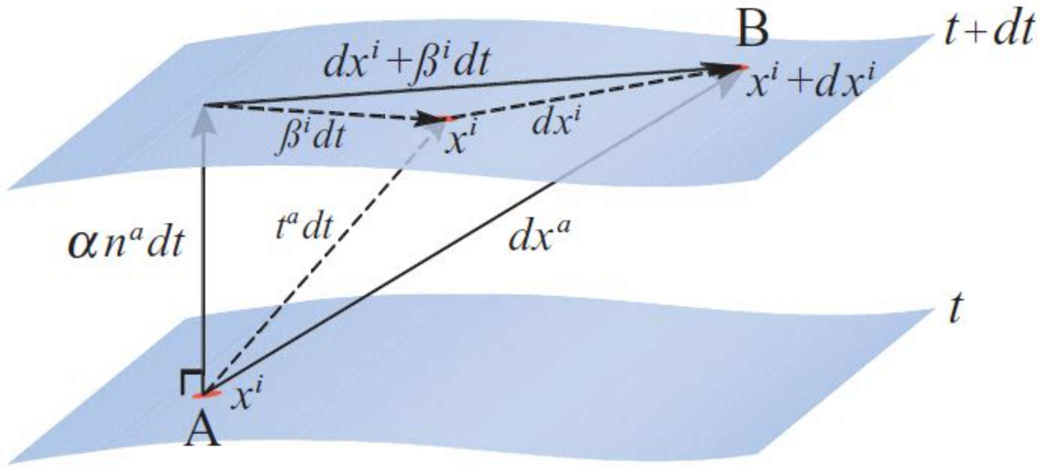
$$N^a_{\ b} \equiv -n^a n_b$$

- Projection tensor along the spatial hypersurface:

$$\gamma^a_{\ b} = g^a_{\ b} + n^a n_b = \delta^a_{\ b} + n^a n_b$$

- Using these we can decompose any tensor into 3+1 form.

Shift Vector



Credits: Baumgarte & Shapiro, 2010

- Define a vector that connects the same spatial coordinates in neighboring slices:

$$t^a = \alpha n^a + \beta^a$$

- β^a is spatial "shift vector".