1 5.3.16

First we want to find the characteristic polynomial of

$$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$$

So we need $det(A - \lambda I)$ ie

$$\begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -3 \\ 2 & 5-\lambda \end{vmatrix} + 4 \begin{vmatrix} -1 & -3 \\ 1 & 5-\lambda \end{vmatrix} - 6 \begin{vmatrix} -1 & -\lambda \\ 1 & 2 \end{vmatrix}$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = (\lambda - 1)(\lambda - 2)^2$$

So it is left to determine the basis for the eigenspaces
$\lambda = 1$

$$\begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 5-1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$
$R_3 : R_1 + R_3$
$R_2 : R_2 + R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 : \frac{-R_2}{2}$
$R_3 : R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so we get $\begin{matrix} x_2 = -x_3 \\ x_1 = 6x_3 - x_2 = 7x_3 \end{matrix}$ Hence the set

$\left\{ \begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda = 1$.

for $\lambda = 2$ we have

$$\begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$$

Row reduction clearly yields

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence a basis for the eigenspace of $\lambda = 2$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Hence A is similar to the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

with $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 7 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2 5.3.24

No, by theorem 7 (b) we must have the sum of the dimensions of the eigenspaces is equal to 3, but in this case it is equal to 2.

3 5.3.28

If A is an $n \times n$ matrix with n linearly independent eigenvectors then by the diagonalization theorem 5 A is similar to a diagonal matrix D with $A = PDP^{-1}$. Thus $A^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D P^T$ which implies that A^T is similar to a diagonal matrix which implies by theorem 5 again that A^T has n linearly independent eigenvectors.

4 5.3.32

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. This matrix clearly has eigenvalues 0 and 1. For $\lambda = 0$ it has eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and for $\lambda = 1$ it has eigenvector

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. These eigenvectors are linearly independent which implies that A is diagonalizable but A is clearly not invertible.