

## On the Van der Pol oscillator: An overview

Livija Cveticanin<sup>1,a</sup>

<sup>1</sup>Faculty of Technical Sciences, 21000 Novi Sad, Trg D. Obradovica 6, Serbia

<sup>a</sup>[cveticanin@uns.ac.rs](mailto:cveticanin@uns.ac.rs)

**Keywords:** Van der Pol oscillator, Limit cycle motion, Rayleigh oscillator, Self-sustained vibrations.

**Abstract.** In this paper an overview of the self-sustained oscillators is given. The standard van der Pol and the Rayleigh oscillators are considered as basic ones. The cubic nonlinear term of Duffing type is included. The special attention is given to the various complex systems based on the Rayleigh's and van der Pol's oscillator which are extended with the nonlinear oscillators of Duffing type and also excited with a periodical force. The connection is with the linear elastic force or with linear damping force. The objectives for future investigation are given in this matter.

### Van der Pol oscillator

The van der Pol oscillator was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol (1889-1959) while he was working at Philips. Namely, at that time the radio and the vacuum tube technology were developed. Diode, triode or tetras were involved into electrical circuits forming an oscillatory system. In Fig.1 the electrical circuits with triode (a) and diode (b) are shown.

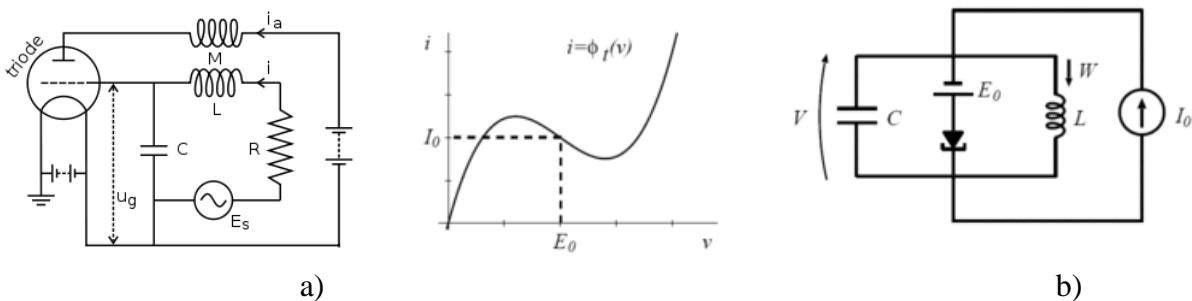


Fig.1. a) Electrical circuit with a triode, b) Electrical circuit with a tunnel diode and the current-voltage diagram of the vacuum tube.

The electrical circuit in Fig.1.a contains: a triode, a resistor  $R$ , a capacitor  $C$ , a coupled inductor-set with self inductance  $L$  and mutual inductance  $M$ . In the serial RLC circuit there is a current  $i$ , and towards the triode anode ("plate") a current  $i_a$ , while there is a voltage  $u_g$  on the triode control grid. The system is forced by an AC voltage source  $E_s$ . Van der Pol considered the oscillations of the electrical circuit and published the experimental results in 1926 [1]. The oscillator is named the van der Pol oscillator.

After Reona (Leo) Esaki (1925 - ) invented the tunnel diode in 1957, making the van der Pol oscillator with electrical circuits has become much simpler (Fig.1.b).

The mathematical model of the electrical circuit with the tunnel diode with input-output relation  $i = \phi(V)$ , shown in Fig.1.b, is as follows

$$\dot{V} = \frac{1}{C}(-\phi(V) - W), \quad \dot{W} = \frac{1}{L}V \quad (1)$$

i.e.,

$$\ddot{V} + \frac{1}{C} \phi'(V) \dot{V} + \frac{1}{LC} V = 0, \quad (2)$$

where  $\phi'(V) = d\phi/dV$ . For the case when the input-output relation of the tunneled diode is a cubic function  $i = \phi(V) = \gamma V^3 - \alpha V$ , the relation (2) transforms into

$$\ddot{V} + \frac{1}{LC} V + \left( \frac{3\gamma}{C} V^2 - \frac{\alpha}{C} \right) \dot{V} = 0. \quad (3)$$

Introducing a new variable  $x = V \sqrt{3\gamma/\alpha}$ , the relation (3) transforms into

$$\ddot{x} + c_1^2 x + \mu(x^2 - 1)\dot{x} = 0, \quad (4)$$

where  $c_1^2 = 1/LC$  and  $\mu = \alpha \sqrt{L/C} > 0$  are positive parameters. The model (4) is often called “the standard van der Pol equation”. It is worthy to say that the same mathematical model (4) corresponds to the electrical circuit shown in Fig.1.a, but also to a significant number of systems in biology, chemistry, economics, and mechanics. In dynamics, the Van der Pol oscillator is a non-conservative system with the non-negative damping:  $x$  is the position coordinate which is a function of the time, and  $\mu$  is a parameter indicating the nonlinearity and the strength of the damping. The following interesting regimes of the unforced oscillator are:

1. When  $\mu = 0$ , i.e. there is no damping function, the equation becomes:  $\ddot{x} + c_1^2 x = 0$ . This is a form of the simple harmonic oscillator and there is always conservation of energy.
2. When  $x(0)$  and  $\dot{x}(0)$  are small, the quadratic term is negligible and the system becomes a linear differential equation with negative damping. The fixed point  $x=0$ ,  $\dot{x}=0$  is an unstable focus.
3. When  $x$  is large, the quadrate term is dominant and the damping becomes positive. Two cases may occur depending on the value of the initial value  $x(0)$ . Introducing the new variable  $y = x - x^3/3 - \dot{x}/\mu$ , the Liénard transformation [2] of Eq. (4) is:

$$\dot{x} = \mu \left( x - \frac{x^3}{3} - y \right), \quad \dot{y} = \frac{c_1^2 x}{\mu}. \quad (5)$$

Analyzing the two coupled first order differential equations (5) we conclude that for significant values of  $x$  the limit cycle motion appears [3]. Namely:

- a) If  $x(0)$  is far from zero but in the neighborhood of the amplitude of the limit cycle independently of the initial conditions, the motion tends to self-sustained vibrations. In Fig.2 the  $x$ - $t$  diagram (a) and also the limit curve (b) in  $x$ - $y$  plane are plotted.
- b) If the system is away from the limit curve, the velocity in  $x$  direction is much higher than in  $y$  one, and the system moves quickly in horizontal direction while the motion along the curve is slow. The system has stable limit cycle (Fig.3). The period of oscillation is approximately  $\mu$ . For the electric circuit of van der Pol type with the triode the parameter is  $\mu = RC$ . The  $RC$  is the time constant of relaxation in the circuit. Van der Pol found this stable oscillations, which he called relaxation oscillations and are now known as limit cycles, in electrical circuits employing vacuum tube. When these circuits are driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it. The properties of this oscillation are the slow asymptotic behavior and then the sudden jump from one to other value.

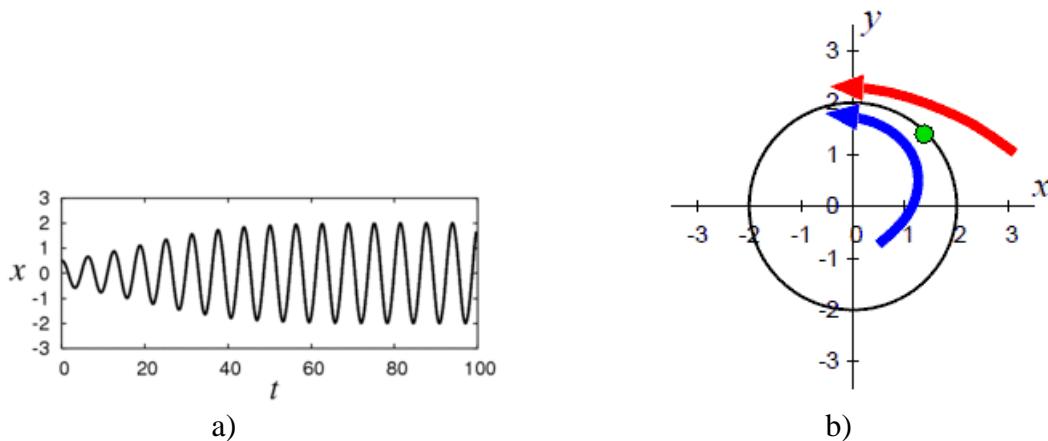


Fig.2. a)  $x$ - $t$  diagram and b)  $y$ - $x$  curve for  $\mu=0.1$  and initial conditions  $x(0)=0.5$  and  $\dot{x}(0)=0$ .

Analyzing equation (4), it is obvious that the parameter  $c_1^2$  controls how much voltage is injected into the system. The parameter  $\mu$  controls the way in which voltage is established across the elements of the system. Think of the oscillator as having two distinct phases: a slow recovery phase and a fast relaxed phase (that is in fact where the term relaxation comes from: vacuum tubes quickly release or relax their voltage after slowly building up tension.) The variable  $\mu$  controls the rate at which the slow build up occurs. As the parameter is decreased, the voltage builds up more slowly, making the slow phase take longer and hence slowing down the oscillator. The van der Pol oscillator is believed to be the first relaxation oscillator.

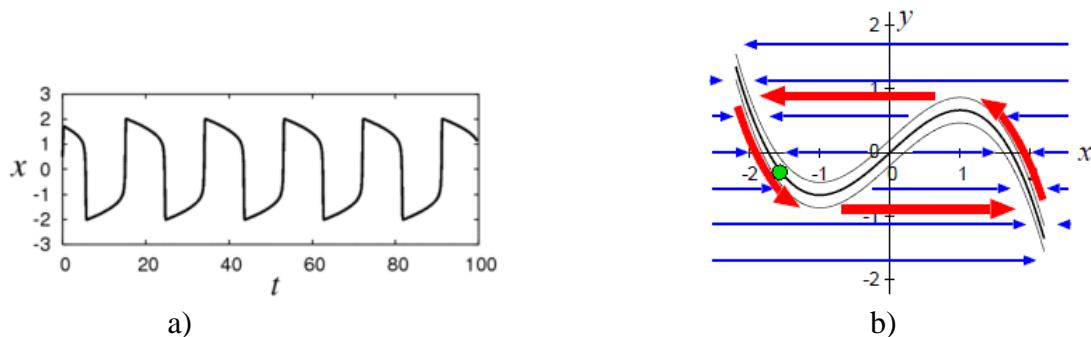


Fig.3. a)  $x$ - $t$  diagram and b)  $x$ - $y$  curve for  $\mu=10$  and initial conditions  $x(0)=0.5$  and  $\dot{x}(0)=0$ .

Balthazar van der Pol and his colleague van der Mark were the first to model the electric activity of the heart and of the human heartbeat based on the relaxing oscillations (see [4]). Using these results the cardiac pacemakers are modeled by a minor change in the van der Pol equation [5].

In Fig.4, the evolution of the limit cycle in the phase plane is plotted. Notice, the limit cycles begin as a circle and, with varying  $\mu$ , becomes increasingly sharp.

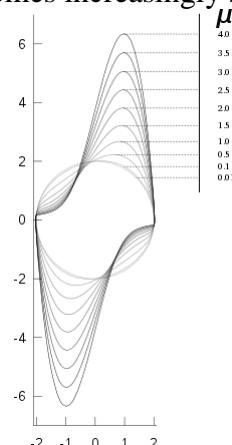


Fig.4. Phase portrait of the unforced Van der Pol oscillator, showing a limit cycle and the direction field.

### Rayleigh's oscillator

The famous British mathematical physicist Lord Rayleigh (John William Strutt, 1842–1919) was one of the first scientists who were dealing with the problem of limit cycle motion. Namely, he introduced the nonlinear velocity damping to model of oscillations of a clarinet reed [6]. The so called Rayleigh model has the form [7]

$$\ddot{y} + \mu(\dot{y}^2 - 1)\dot{y} + c_1^2 y = 0 \quad (6)$$

where  $\mu$  is the coefficient of damping and  $c_1^2$  is a positive constant. Introducing the new variable  $x = \dot{y}/\sqrt{3}$  it is evident that the relation equation (6) transforms into the van der Pol equation (4). Forming the Rayleigh electrical circuit with model equation (6), it is seen that it is an oscillator much like the van der Pol one. The key difference between the two circuits is that as voltage increases, the van der Pol oscillator increases in frequency while the Rayleigh oscillator increases in amplitude. Like the van der Pol oscillator,  $c_1^2$  controls how much voltage is injected into the system.  $\mu$  controls the way in which the voltage is established across the elements of the system. In Fig.5. we see outward and inward spiral trajectories converging to a “limit cycle” solution that corresponds to periodic oscillations of the reed.

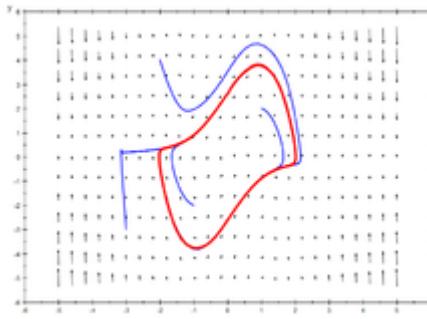


Fig.5. Trajectories in the Rayleigh oscillator

The Rayleigh oscillator is of real physical interest, since it might be useful to model many physical and engineering systems, and in the context of chemical and biological oscillators.

### Forced van der Pol oscillators

The van der Pol and also the Rayleigh oscillators are often driven by an excitation force. The forced, or driven, van der Pol oscillator takes the 'original' function and adds a driving function  $F\sin(\Omega t)$  to give a differential equation of the form:

$$\ddot{x} + c_1^2 x + \mu(x^2 - 1)\dot{x} = F \sin(\Omega t), \quad (7)$$

where  $F$  is the amplitude and  $\Omega$  is the frequency of the excitation force. This oscillator was modeled by an electrical circuit and was investigated by van der Pol and his colleague, van der Mark, and they reported in September 1927 in [8] that at certain drive frequencies  $\Omega$  an irregular noise was heard. This irregular noise was always heard near the natural entrainment frequencies. This was one of the first discovered instances of deterministic chaos [9]. In Fig.6. the time history diagram for chaotic motion is plotted.

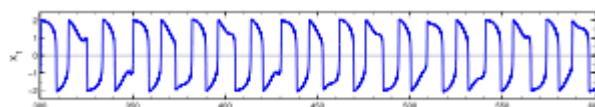


Fig.6. Chaotic behavior in the van der Pol oscillator with sinusoidal forcing for  $F=1.2$ ,  $\Omega=\pi/5$  and  $\mu=8.53$ .

Since that time a significant number of investigations have been done on the problem. The extensive studies given in [3] and [10] – [13], devoted to the van der Pol oscillator have revealed that it possesses a rich dynamic behavior especially when submitted to a sinusoidal excitation: frequency entrainment, chaotic behavior with period doubling, etc.

### Non-linear van der Pol and Rayleigh oscillators

As it is well known, the most of the phenomena in physics, biology, economics, etc., have nonlinear properties. For example, the elements of the electrical circuit have nonlinear characteristics which can be realized using operational amplifiers. They are proper elements to study nonlinear phenomena in physical systems experimentally. The aforementioned van der Pol i.e. Rayleigh oscillators are with nonlinear damping but with a linear displacement. Unfortunately, the assumption of the phenomenological model of the elastic restitution force to be linear with respect to the displacement is not correct. This is at this point where the words of Pippard referred in [14] have to be recalled:

*There is something of a tendency among physicists to try to reduce everything to linearity....., reality may not always conform to what might wish, rather more so with the damping forces than with the restoring force in small-amplitude vibrations.*

According to this suggestion, the non-linear restoring force model is included into the standard van der Pol and Rayleigh oscillators. The Rayleigh oscillator equation (6), which is one canonical example of self-excited systems, is simply generalized by addition of cubic displacement term and of the excitation force. The resultant equation of motion is given by

$$\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} - x + x^3 = F \cos \Omega t \quad (8)$$

where  $\mu$  is a real parameter,  $F$  and  $\Omega$  are respectively the amplitude and the frequency of the external perturbation. The model is analyzed by Siewe et al. [15].

The present approach can be used to generalize model of magneto-rheological dampers in novel studies of their influence on vehicle dynamics [16]. To reduce harmful vibrations one can consider application of dampers composed of Duffing oscillator with the Rayleigh damping.

In the [17] the standard van der Pol oscillator is extended with a Duffing type cubic restoring force

$$\ddot{x} - \dot{x}(1 - x^2) + x + x^3 = \sigma_1 \xi_1(t)x + \sigma_2 \xi_2(t) \quad (9)$$

where the excitation forces  $\xi_1(t)$  and  $\xi_2(t)$  are stochastic. The dissipative force in equation (9) is nonlinear, and the main feature of the oscillator is to be a self-excited system like the van der Pol oscillator.

In [18] the attention is focused on the Duffing-Rayleigh oscillator subject to harmonic and stochastic white noise excitations which is described by

$$\ddot{x} + \mu(1 + \frac{\beta}{c_1^2} \dot{x}^2)\dot{x} + c_1^2(1 + \varepsilon x^2)x = \sqrt{c_1^2} F \sin \Omega t + \sqrt{c_1^2} \sigma_2 \xi \quad (10)$$

where  $c_1 > 0$  is the natural circular frequency and  $\mu$  is the coefficient of linear damping,  $\beta$  and  $\varepsilon$  are the nonlinearity parameters,  $F$  and  $\Omega$  are intensity and frequency of the harmonic excitation, respectively, and  $\sigma_2$  is the intensity of the additive white noise  $\xi$ .

However, Alfred - Marie Liénard (1869-1958) made the most wide generalization of the van der Pol equation. The Liénard - van der Pol equation is a second order differential equation [2]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad (11)$$

where  $f$  and  $g$  are continuously differentiable functions on  $\mathbf{R}$ , with  $g$  an odd function and  $f$  an even function. Liénard gave the theorem which guarantees the uniqueness and the existence of a limit cyclic solution for equation (11). If  $g(x) > 0$  for all  $x > 0$ ,  $\lim_{x \rightarrow \infty} F(x) = \lim_{x \rightarrow \infty} \int_0^x f(x)dx = \infty$ , and the function  $F(x)$  has exactly one positive root at some value  $p$  which satisfies the requirement that  $F(x)$  is negative for  $0 < x < p$  and positive and monotonic for  $x > p$ , a unique and stable limit cycle surrounding the origin exist.

The other generalization given by Liénard is the mixed Rayleigh-Liénard oscillator

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \quad (12)$$

This equation, which is often taken as the typical example of nonlinear self-excited vibration problem, can be used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. It can also be used to model certain mechanical systems which contain the nonlinear damping coefficients and the restoring force or stiffness. Thus, Ding and Leung [19] determined the number of limit cycles for the special case of Rayleigh-Liénard equation (12), where

$$g(x) = x + \varepsilon ax^3, f(x, \dot{x}) = \lambda + \left( \sum_{i=1}^n b_{2i} x^{2i} + c\dot{x}^2 \right) \quad (13)$$

where  $\varepsilon$  is a small positive parameter,  $f$  is a weak nonlinear damping function in the form of an even degree polynomial, while  $g$  is an odd degree polynomial.  $a$ ,  $c$  and  $b_{2i}$  are constants, and  $\lambda$  is a parameter. The corresponding differential equation is

$$\ddot{x} + x = \varepsilon[-ax^3 + (\lambda + \sum_{i=1}^n b_{2i} x^{2i} + c\dot{x}^2)\dot{x}] \quad (14)$$

In [19] the influence of highly nonlinear terms is investigated, while in [20] only the damping coefficient  $n=7$  is studied.

Akulenko [21] calculated the initial value of velocity which determines the self-sustained oscillations of the system of equation (12) with

$$f(x) = cx|x|^{\alpha-1}, g(x, \dot{x}) = -k|x|^\beta |\dot{x}|^{\gamma-1} + l|x|^\delta |\dot{x}|^{\sigma-1} \quad (15)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  and  $\sigma$  are positive integers and  $c$ ,  $k$  and  $l$  are coefficients of elasticity of the restoring force  $f$  and of the damping force  $g$ , respectively. The considered differential equation is

$$\ddot{x} + \varepsilon(-|x|^\beta |\dot{x}|^{\gamma-1} + |x|^\delta |\dot{x}|^{\sigma-1})\dot{x} + x|x|^{\alpha-1} = 0 \quad (16)$$

where  $\varepsilon > 0$  is the self-sustained factor of the self-sustained oscillations. Akulenko et al. also investigated in detail the self-sustained oscillations of the equation [22]

$$\ddot{x} - \varepsilon(1 - |x|^2)\dot{x} + x|x|^{\alpha-1} = 0, \quad (17)$$

for  $\alpha = 3$  and  $5$ . Cveticanin [23] treated the damped van der Pol oscillator

$$\ddot{x} + k\dot{x} + c_\alpha^2 x|x|^{\alpha-1} = \varepsilon(1 - |x|^2)\dot{x}, \quad (18)$$

where  $\alpha$  is a positive rational number (integer or non-integer). The same author treated the special type of van der Pol equation with time variable parameter [24]

$$m(\tau)\ddot{x} + c_\alpha^2 x|x|^{\alpha-1} = \varepsilon(b - cx^2)\dot{x} + \dot{m}(\tau)\dot{x}, \quad (19)$$

where  $\tau = \varepsilon t$  is the slow time and  $\varepsilon$  is a small positive parameter. The influence of the reactive force due to mass variation is treated.

Nagumo, et al. [25] and Fitzhugh and [26] described the self-sustained oscillations in a neuron combining the van der Pol and Rayleigh damping terms. The obtained equation is usually called Fitzhugh – Nagumo equation

$$\ddot{w} + \varepsilon\gamma\dot{w} + \varepsilon v(v - \theta)(v - 1) + \varepsilon w = \varepsilon\omega \quad (20)$$

where

$$\varepsilon v = \dot{w} + \varepsilon\gamma w. \quad (21)$$

Equation (20) describes the action potentials of neurons [25], [26]. A neuron, a nerve cell, communicates with other neurons by sending them pulse-like electrical signals. When a neuron receives enough electrical input from its neighbours it will fire, sending off its own electrical pulse. Once fired, the neuron must recover before firing again. This entire sequence (integrate, fire, rest) is given by equations (20) and (21).  $v$  is voltage and  $w$  is recovery of voltage. This equation is applied in physical as well as biological sciences. The same model has also been utilised in seismology to model the two plates in a geological fault. Like all relaxation oscillators, this oscillator has a slow accrual phase and a fast release phase. The variable parameter  $\varepsilon$  represents the coupling between the slow and fast phases. As epsilon increases, so does frequency.

## Coupled oscillators

To give a more realistic description of the systems and their behavior, the models have to be more complex and to include various types of nonlinear oscillators. Three of them are fundamental: the Duffing, the van der Pol and the Rayleigh oscillators. These oscillators are available in the extensive examination of a number of dynamical features which are embedded in the physical systems. Thus, the van der Pol oscillator and damped Duffing oscillator stand as paradigms of nonlinear oscillators: the former is the limit cycle prototype and the later is the strange attractor prototype [27]. At the other side, the Duffing - Rayleigh systems can occur in physical and electro-mechanical devices [28], [29] or for some chemical or biological oscillators and also in engineering [16] for example vehicle vibrations [30].

Concerning the coupling between a van der Pol oscillator and a Duffing oscillator, three basic schemes can be listed [27]:

- Gyroscopic coupling – coupling through accelerations,
- Dissipation coupling – coupling through velocity
- Elastic coupling – coupling through solutions as it is discussed in [31] and [32].

The case of elastic coupling is given in the form

$$\begin{aligned} \ddot{x} - \mu(1-x^2)\dot{x} + x - k(y-x) &= 0 \\ \ddot{y} + \alpha\dot{y} - y + \varepsilon y^3 - k(x-y) &= 0 \end{aligned}, \quad (22)$$

and

$$\begin{aligned}\ddot{x} + \alpha\dot{x} + \Omega_1^2 x + \varepsilon x^3 - k(y - x) &= 0, \\ \ddot{y} - \mu(1 - y^2)\dot{y} + \Omega_2^2 y - k(x - y) &= 0\end{aligned}\tag{23}$$

where  $\mu$  and  $\varepsilon$  are positive parameters controlling the nonlinearities of the model,  $\alpha$  accounts for the dissipation,  $\Omega_1$  and  $\Omega_2$  are the linear frequencies of the Duffing resonator, while  $k$  represents the coupling strength discussed in [31] and [32], respectively. If the coupling stiffness  $k$  diminishes to zero in the above equations, the equations uncouple to an autonomous Duffing resonator and a van der Pol oscillator. Comparing the two systems of equations (22) and (23) it is evident that in [32] the Duffing oscillator with both positive coefficients is discussed, while in [31] the linear term is negative. Such an assumption gives the significant difference in the behavior of the system especially for the chaotic motion. In the paper [32] the residue harmonic balance analysis is assumed for solving the system of two coupled differential equations.

In [33] the dissipative coupled standard van der Pol oscillator with a Duffing oscillator with another oscillator of the same type is considered. The mathematical model is

$$\begin{aligned}\ddot{x} + x + \beta x^3 - (\lambda_1 - x^2)\dot{x} + \mu(\dot{x} - \dot{y}) &= 0, \\ \ddot{y} + (1 + \delta)y + \beta y^3 - (\lambda_2 - y^2)\dot{y} - \mu(\dot{x} - \dot{y}) &= 0,\end{aligned}\tag{24}$$

where  $\lambda_1$  and  $\lambda_2$  are parameters of bifurcation,  $\delta$  is the frequency mismatch between the autonomous second and first oscillators and  $\mu$  is the coefficient of dissipative coupling.  $\beta$  is the addition coefficient of the nonlinearity. The synchronization for the parameter  $\delta$  is investigated.

Qian et al. [34] extended the model by assuming a linear elastic and a linear damping coupling for two oscillators: van der Pol and Duffing ones. The extended homotopy analysis method is applied to derive the accurate approximate analytical solutions for the two degree of freedom coupled van der Pol –Duffing oscillator:

$$\begin{aligned}\ddot{x} - (\lambda_1 - x^2)\dot{x} + (1 + \frac{\Delta}{2})x + \beta x^3 + \varepsilon(x - y) + \mu(\dot{x} - \dot{y}) &= 0 \\ \ddot{y} - (\lambda_2 - y^2)\dot{y} + (1 + \frac{\Delta}{2})y + \beta y^3 + \varepsilon(y - x) + \mu(\dot{y} - \dot{x}) &= 0\end{aligned}\tag{25}$$

$\lambda_1$  and  $\lambda_2$  are bifurcation parameters,  $\Delta$  is the detuning parameter that is proportional to the difference of the oscillator frequencies,  $\beta$  is the non-isochronisms parameter that relies on the oscillation frequency and amplitude,  $\varepsilon$  and  $\mu$  are the coupling inertial and dissipative parameters. The system of van der Pol –Duffing oscillators connected with a nonlinear elastic force of cubic order is also investigated (see [35]):

$$\begin{aligned}\ddot{x} - \varepsilon(1 - x^2)\dot{x} + (x + x^3) &= x^2(\alpha x + \beta y) + y^2(\gamma x + \delta y), \\ \ddot{y} - \varepsilon(1 - y^2)\dot{y} + (y + y^3) &= -x^2(\alpha x + \beta y) - y^2(\gamma x + \delta y),\end{aligned}\tag{26}$$

In the paper the extended homotopy analysis method is applied which is also used for analytical consideration of the limit cycle motion of the system with delayed amplitude limitation [36]

$$\begin{aligned}\ddot{x} + x &= 2\varepsilon[(1 - z)\dot{x} - \dot{z}x] \\ \mu\dot{z} + z &= x^2\end{aligned}\tag{27}$$

where  $\mu$  is the delay time,  $\varepsilon$  is a constant and  $x(t)$  and  $z(t)$  are two unknown functions to be determined.

## Future investigation

Based on the overview on the van der Pol and Rayleigh oscillators and their extension by connecting them with other types of oscillators (mainly Duffing's one), it is evident that it gives an improvement in comparison to the standard van der Pol and Rayleigh systems but fail to explain the real systems. It requires the extension of the model with pure nonlinear terms of any rational order and also the introduction of the damping terms of nonlinear polynomial type of integer or noninteger order. It is believed that the model would give more accurate description of most of the problems in mechanics, electro techniques (engineering), including biology, physics, chemistry and economics. The improvement of the solution procedures for the nonlinear differential equations is also necessary.

## Summary

In this paper the standard van der Pol and Rayleigh oscillators are considered. The historical development of the knowledge in self-sustained oscillations and limit cycle motion is given. The two standard oscillators are coupled with Duffing ones and various excitation forces are added. The obtained models are suitable for description and explanation of a significant number of phenomena in engineering, physics, biology, chemistry, and economic.

## Acknowledgement

The investigation was supported by the Provincial Secretariat for Science and Technological Development, Autonomous Province of Vojvodina (Proj. No 114-451-2094/2011) and Ministry of Science of Serbia (Proj. No. ON 174028 and IT 41007).

## References

- [1] B. van der Pol, On relaxation-oscillations. The London, Edinburgh and Dublin Philosophical Magazine & Journal of Science **2** (7) (1926) 901-912.
- [2] A. Lienard, Etude des oscillations entretenues. Rev. Generale de l'Electricite **23** (1928) 946-954.
- [3] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations. New York: Wiley, 1979.
- [4] B. van der Pol, J. van der Mark, The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. Phil. Mag. Suppl. **6** (1928) 763-775.
- [5] H.G. Narm, A new state observer for two coupled van der Pol oscillators, Int. J. of Control. Automation and Systems **9** (2011) 420-414.
- [6] J. Rayleigh, The Theory of Sound. New York: Dover, 1945.
- [7] M.R. Navabi, M.S.M. Deghghan, Numerical solution of the controlled Rayleigh nonlinear oscillator by the direct spectral method. J. of Vibr. and Control **14** (6) (2008) 795-806.
- [8] B. van der Pol, J. Van der Mark, Frequency demultiplication. Nature **120** (1927) 363-364.
- [9] J. M. Thompson, H.B. Stewart, Nonlinear Dynamics and Chaos. New York: John Wiley and Sons, 1986.
- [10] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Field. New York: Springer, 1983.
- [11] C. Partliz, W. Lauterborn, Period doubling cascade and devil staircases of the driven van der Pol oscillator. Phys. Rev. A **36** (1987) 1428-1434.
- [12] A. Venkatesan, M. Lakshmanan, Bifurcation and chaos in the double -well Duffing-van der Pol oscillator: numerical and analytical studies. Phys. Rev. E **56** (1997) 6321-6330.

- [13] J.C. Chedjou, H.B. Fotsin, P. Woafo, S. Domngang, Analog simulation of the dynamics of a van der Pol oscillator coupled to a Duffing oscillator. *IEEE Trans. Circuits Syst. I Fudam. Theory Application* **48** (2001) 748-756.
- [14] M.S. Siewe, H. Cao, M.A.F. Sanjuan, Effect of nonlinear dissipation on the basin boundaries of a driven two-well Rayleigh-Duffing oscillator. *Chaos, Solitons & Fractals* **39** (2009) 1092-1099.
- [15] M.S. Siewe, C. Tchawoua, P. Woafo, Melnikov chaos in a periodically driven Raylegy-Duffing oscillator. *Mech. Research Commun.* **37** (2010) 363-368.
- [16] S. Li, S. Yang, W. Guo, Investigation of chaotic motion in histeric non-linear suspension system with multi-frequency excitations. *Mech. Research Commun.* **31** (2004) 229-.
- [17] Q. He, W. Xu, H. Rong, T. Fang, Stochastic bifurcation in Duffing-Van der Pol oscillators. *Physica A* **338** (2004) 319-334.
- [18] W.X. Xie, W.Xu, L. Cai, Path integration of the Duffing-Rayleigh oscillator subject to harmonic and stochastic excitations. *Appl. Math. Comput.* **171** (2005) 870-884.
- [19] Q. Ding, A.Y.T. Leung, The number of limit cycle bifurcation diagrams for the generalized mixed Rayleigh-Liénard oscillator. *J. of Sound and Vibr.* **322** (2009) 393-400.
- [20] S. Lynch, C.J. Christopher, Limit cycles in highly non-linear differential equations. *J. of Sound and Vibr.* **224** (1999) 505-517.
- [21] L.D. Akulenko, L.I. Korovina, S.A. Kumakshev, S.V. Nesterov, Self-sustained oscillations of Rayleigh and van der Pol oscillators with moderately large feedback factors. *J. of Appl. Math. Mech.* **68** (2004) 241-248.
- [22] L.D. Akulenko, L.I. Korovina, S.V. Nesterov, Self-sustained oscillations of a highly non-linear system. *Izvestija Rossija Akademija Nauk MTT* **3** (2002) 42-48.
- [23] L. Cveticanin, On the truly nonlinear oscillator with positive and negative damping. Submitted for publication
- [24] L. Cveticanin, Van der Pol oscillator with time-variable parameters. *Acta Mech.* **224** (5) (2013) 945-955.
- [25] J. Nagumo, S. Arimoto, and S. Yoshizawa, An active pulse transmission line simulating 1214-nerve axons. *Proceedings IRL* **50** (1960), pp. 2061-2070.
- [26] R. Fitzhugh, Impulses and physiological states in theoretical models of nerve membranes. *1182-Biophysics Journal* **1** (1961), pp. 445-466.
- [27] J. Kengne, J.C. Chedjou, G. Kenne, K. Kyamakya, G.H. Kom, Analog circuit implementation and synchronization of a system consisting of a van der Pol oscillator linearly coupled to a Duffing oscillator. *Nonlinear Dynamics* **70** (2012) 2163 -2173
- [28] R. Yamapi, P. Woafo, Dynamics and synchronization of coupled self-sustained electromechanical devices. *J. of Sound and Vibr.* **285** (2005) 1151-1170.
- [29] J.C. Chedjou, K. Kyamakya, I. Moussa, H.-P. Kuchenbecker, W. Mathis, Behavior of a self sustained electromechanical transducer and routes to chaos. *J. Vibr. Acoustics* **128** (2008) 282-293.
- [30] G. Litak, M.I. Borowiec, K. Friswell, Chaotic vibration of a quarter-car model excited by the road surface profile. *Commun. in Nonlinear Sci. and Numer. Simul.* **13** (2008) 1373-1383.
- [31] A.P. Kuznetsov, J.P. Roman, Properties of synchronization in the systems of non-identical coupled van der Pol and van der Pol-Duffing oscillators. Broadband synchronization, *Physica D* **238** (2009) 1499-1506.

- [32] A.Y.T. Leung, Z. Guo, H.X. Yang, Residue harmonic balance analysis for the damped Duffing resonator driven by a van der Pol oscillator. *Int. J. of Mech. Sci.* **63** (2012) 59-65.
- [33] A.P. Kuznetsov, N.V. Stankevich, L.V. Turukina, Coupled van der Pol-Duffing oscillators: phase dynamics and structure synchronization. *Physica D* **238** (2009) 1203-1215.
- [34] Y.H. Qian, W. Zhang, B.W. Lin, S.K. Lai, Analytical approximate periodic solutions for two-degree-of-freedom coupled van der Pol-Duffing oscillators by extended homotopy analysis method. *Acta Mech.* **219** (2011) 1-14.
- [35] Y.H. Qian, C.M. Duan, S.M. Chen, S.P. Chen, Asymptotic analytical solutions of the two-degree-of-freedom strongly nonlinear van der Pol oscillators with cubic couple terms using extended homotopy analysis method. *Acta Mech.* **223** (2012) 237-255.
- [36] A.K. Eigoli, M. Khodabakhsh, A homotopy analysis method for limit cycle of the van der Pol oscillator with delayed amplitude limiting. *Appl. Math. Comput.* **217** (2011) 9404-9411.

**Acoustics & Vibration of Mechanical Structures**

10.4028/www.scientific.net/AMM.430

**On the Van Der Pol Oscillator: an Overview**

10.4028/www.scientific.net/AMM.430.3

**DOI References**

- [8] B. van der Pol, J. Van der Mark, Frequency demultiplication. *Nature* 120 (1927) 363-364.  
<http://dx.doi.org/10.1038/120363a0>
- [31] A.P. Kuznetsov, J.P. Roman, Properties of synchronization in the systems of non-identical coupled van der Pol and van der Pol-Duffing oscillators. *Broadband synchronization, Physica D* 238 (2009) 1499-1506.  
<http://dx.doi.org/10.1016/j.physd.2009.04.016>