Hidden SU(2) symmetry in spin-1 XY model

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1 Onsite SU(2)

For spin-1 model, we can define another onsite SU(2) operator:

$$\begin{cases} \tilde{s}^{\pm} \equiv \frac{1}{2}\left(S^{\pm}\right)^2 \\ \tilde{s}^{z} \equiv \frac{1}{2}S^{z} \end{cases}, \tag{1.1}$$

where $S_j^{\pm}$ is the regular $S=1$ spin operator. For spin-1:

$$\begin{aligned} \left[\tilde{s}^{+}, \tilde{s}^{-}\right] &= 2\tilde{s}^{z}, && (1.2)\\ \left[\tilde{s}^{z}, \tilde{s}^{\pm}\right] &= \pm\tilde{s}^{\pm}, && (1.3)\\ \left\{\tilde{s}^{+}, \tilde{s}^{-}\right\} &= \left(S^{z}\right)^2. && (1.4) \end{aligned}$$

The Casimir invariant for this SU(2) is

$$C_2 = \frac{1}{2}\left(\tilde{s}^{+}\tilde{s}^{-} + \tilde{s}^{-}\tilde{s}^{+}\right) + \left(\tilde{s}^{z}\right)^2 = \frac{3}{4}\left(S^{z}\right)^2. \tag{1.5}$$

So we have

$$\left[\left(S^{z}\right)^2, \tilde{s}^{\pm}\right] = 0. \tag{1.6}$$

Note that

$$\begin{aligned} \left[\tilde{s}_j^{+}, S_j^{-}\right] &= S_j^{-}\left(1 - 2\left(S_j^{z}\right)^2\right), && (1.7)\\ \left[\tilde{s}_j^{-}, S_j^{+}\right] &= S_j^{+}\left(1 - 2\left(S_j^{z}\right)^2\right), && (1.8)\\ \left\{\tilde{s}_j^{+}, S_j^{-}\right\} &= S_j^{-}, && (1.9)\\ \left\{\tilde{s}_j^{-}, S_j^{+}\right\} &= S_j^{+}. && (1.10) \end{aligned}$$

2 Chain operator

Define a chain operator

$$U_j = \prod_{l=1}^{j-1}\left(1 - 2\left(S_l^{z}\right)^2\right). \tag{2.1}$$

Note that

$$S_j^{\pm} U_k = \begin{cases} -U_k S_j^{\pm} & j < k \\ U_k S_j^{\pm} & j \geq k \end{cases}. \tag{2.2}$$

We introduce new operators

$$s_j^{\pm} = \tilde{s}_j^{\pm} U_j, \;\; s_j = \tilde{s}_j. \tag{2.3}$$

Since $\left[U_j, s_k^{\pm}\right] = 0$, $\left\{s_j^z, s_j^{\pm}\right\}$ still satisfies su(2) algebra, and we can further define a global operator:

$$s_T^{\pm} = \sum_{j=1}^{L} s_j^{\pm}, \;\; s_T^z = \sum_{j=1}^{L} s_j^z, \tag{2.4}$$

which also satisfies su(2) algebra.

3 Commutation relation

The Hamiltonian for spin-1 XY model (with open boundary) is:

$$H_{XY} = \frac{1}{2} \sum_{j=1}^{L-1} \left(S_j^+ S_j^- + S_j^- S_j^+ \right). \tag{3.1}$$

We first note that for $k \neq j, j+1$:

$$\left[s_k^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] = 0. \tag{3.2}$$

While for $k = j$:

$$\begin{aligned} \left[s_j^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] &= S_{j+1}^+ \left[\tilde{s}_j^+, S_j^-\right] U_j & (3.3) \\ &= S_j^+ S_{j+1}^+ U_{j+1}, & (3.4) \end{aligned}$$

and for $k = j+1$:

$$\begin{aligned} \left[s_{j+1}^+, S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right] &= -S_j^+ \left\{\tilde{s}_{j+1}^+, S_{j+1}^-\right\} U_{j+1} & (3.5) \\ &= -S_j^+ S_{j+1}^+ U_{j+1}. & (3.6) \end{aligned}$$

In this way

$$\left[s_T^+, H_{XY}\right] = 0. \tag{3.7}$$

Similarly, we can show

$$\left[s_T^-, H_{XY}\right] = 0. \tag{3.8}$$