Quantum Geometric Tensor

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I. DISTANCE OF TWO QUANTUM STATES

Consider a family of parameter-dependent Hamiltonian $H(\lambda)$ smoothly depend on $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. The eigen system of the Hamiltonian is $\{E_n(\lambda), \psi_n(\lambda)\}$. A somple definition of the distance between two infinitesimally-varied states is:

$$ds^2 \equiv \|\psi(\lambda + d\lambda) - \psi(\lambda)\|^2 = \langle \partial_\mu \psi | \partial_\nu \psi \rangle d\lambda^\mu d\lambda^\nu = (\gamma_{\mu\nu} + i\sigma_{\mu\nu}) d\lambda^\mu d\lambda^\nu. \tag{1}$$

Because the matrix $\langle \partial_\mu \psi | \partial_\nu \psi \rangle$ is Hermitian,

$$\gamma_{\mu\nu} = \gamma_{\nu\mu}, \ \sigma_{\mu\nu} = -\sigma_{\nu\mu}. \tag{2}$$

The $\sigma_{\mu\nu} d\lambda^\mu d\lambda^\nu$ term vanishes due to the antisymmetry of $\sigma_{\mu\nu}$ and symmetry of $d\lambda^\mu d\lambda^\nu$. So that the distance is

$$ds^2 = \gamma_{\mu\nu} d\lambda^\mu d\lambda^\nu. \tag{3}$$

However, the $\gamma_{\mu\nu}$ is not invariant under the U(1) gauge transformation:

$$|\psi\rangle \longrightarrow e^{i\alpha(\lambda)} |\psi\rangle, \tag{4}$$

which introduce new term

$$\gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} - \beta_\mu \partial_\nu \alpha - \beta_\nu \partial_\mu \alpha + \partial_\mu \alpha \partial_\nu \alpha, \tag{5}$$

where $\beta_\mu = i\langle \psi(\lambda) | \partial_\mu \psi \rangle$ is the Berry connection. We can use the Berry connection to define a gauge-invariant metric:

$$g_{\mu\nu}(\lambda) \equiv \gamma_{\mu\nu}(\lambda) - \beta_\mu(\lambda)\beta_\nu(\lambda). \tag{6}$$

Physically, the metric $\gamma_{\mu\nu}$ measures the distance between the "bare" states in Hilbert space, while the metric $g_{\mu\nu}$ measures the distance of rays in projected Hilbert space $\mathcal{PH} = \mathcal{H}/U(1)$. For simplicity, we define *Quantum Geometric Tensor*(QGT) $Q_{\mu\nu}$ as

$$Q_{\mu\nu}(\lambda) \equiv \langle \partial_\mu \psi(\lambda) | \partial_\nu \psi(\lambda) \rangle - \langle \partial_\mu \psi(\lambda) | \psi(\lambda) \rangle \langle \psi(\lambda) | \partial_\nu \psi(\lambda) \rangle. \tag{7}$$

The metrics previously defined are related to QGT by

$$g_{\mu\nu} = \mathrm{Re}\ Q_{\mu\nu}, \ \sigma_{\mu\nu} = \mathrm{Im}\ Q_{\mu\nu}. \tag{8}$$

II. INNER PRODUCT OF TWO NEARBY STATES

Now consider the inner product of two near by states $|\langle \psi(\lambda + d\lambda) | \psi(\lambda) \rangle|$. Since

$$\langle \psi(\lambda + d\lambda) | \psi(\lambda) \rangle = 1 + i\beta_\mu(\lambda) + \frac{1}{2} \mathrm{Re} \langle \partial_\mu \partial_\nu \psi(\lambda) | \psi(\lambda) \rangle d\lambda^\mu d\lambda^\nu. \tag{9}$$

Note that in the third term, we ignore the imaginary part for the same reason as for $\sigma_{\mu\nu}$. Also, since $\langle \partial_\mu | \psi \rangle$ is purely imaginary, differentiate by ν on both side and take the real part, we have

$$\mathrm{Re} \langle \partial_\mu \partial_\nu \psi(\lambda) | \psi(\lambda) \rangle = -\mathrm{Re} \langle \partial_\mu \psi(\lambda) | \partial_\nu \psi(\lambda) \rangle = -\gamma_{\mu\nu}. \tag{10}$$

The absolute value of the inner product is then:

$$|\langle \psi(\lambda + d\lambda) | \psi(\lambda) \rangle| = 1 - \frac{1}{2} (\gamma_{\mu\nu}(\lambda) - \beta_\mu(\lambda)\beta_\nu(\lambda)) d\lambda^\mu d\lambda^\nu = 1 - \frac{1}{2} g_{\mu\nu} d\lambda^\mu d\lambda^\nu. \tag{11}$$

III. RELATION WITH BERRY CURVATURE

For adiabatic system, $H(\lambda)|\psi_0(\lambda)\rangle = E_0(\lambda)|\psi_0(\lambda)\rangle$. Take derivative with respect to μ and calculate inner product with $|\psi_n\rangle (n \neq 0)$:

$$\langle\psi_n|\partial_\mu\psi_0\rangle = \frac{\langle\psi_n|\partial_\mu H|\psi_0\rangle}{E_0 - E_n},\ n \neq 0. \tag{12}$$

The QGT is

$$\begin{aligned} Q_{\mu\nu} &= \langle\partial_\mu\psi_0|(1 - |\psi_0\rangle\langle\psi_0|)|\partial_\nu\psi_0\rangle \\ &= \sum_{n\neq 0}\langle\partial_\mu\psi_0|\psi_n\rangle\langle\psi_n|\partial_\nu\psi_0\rangle \\ &= \sum_{n\neq 0}\frac{\langle\psi_0|\partial_\mu H|\psi_n\rangle\langle\psi_n|\partial_\nu H|\psi_0\rangle}{(E_0 - E_n)^2}. \end{aligned} \tag{13}$$

Note that the Berry curvature is

$$F_{\mu\nu} = \partial_{[\mu,}\beta_{\nu]} = i(Q_{\mu\nu} - Q_{\nu\mu}) = -2\mathrm{Im}Q_{\mu\nu}. \tag{14}$$

So that the imaginary part of QGT is proportional to Berry curvature:

$$Q_{\mu\nu} = g_{\mu\nu} - \frac{i}{2}F_{\mu\nu}. \tag{15}$$